Research Statement

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My research interests lie in topology, specifically the study of piecewise-linear knots. There are many reasons for studying knot theory, including stereochemistry and the topological properties of DNA. One problem to look at when studying piecewise-linear knots is how many straight edges does it take to create a knot.

One can expand the depth of the question above by being more specific. You could make all the edges have the same length. You could make all the angles between the edges be the same. You could make all the edges have the same length and have all the angles be equal. Naturally, the more restrictions you add, the more difficult the question becomes.

Define $e(K)$, the edge index of K, to be the minimum number of straight edges needed to make K. Let $c(K)$ be the minimal number of crossings over all projections of the link. Negami [7] showed that for a link, K, that has neither the Hopf link as a connected sum factor nor a splittable trivial component, $e(K) \leq 2c(K)$. The work of Randell [8] and Meissen [6] have given us the exact edge index that it takes to make the trefoil, figure-eight, and all five, six, and seven-crossing knots. The work of McCabe [5] has given us a much better upper bound for two-bridge knots, a specific class of knots, which is $e(K) \leq c(K) + 3$, where K is a two-bridge knot. The torus knots, denoted $T_{p,q}$, are another class that has been looked at closely. Jin and Kim [4] have shown that $e(T_{p,q}) \leq 2q$ when $2 \leq p < q$, and more specifically, $e(T_{p,q}) = 2q$ when $2 \leq p < q < 2p$.

What follows is a brief description of my research and other topics I would like to explore.

1. Contributions to the study of Piecewise-Linear Links

My research takes a little different approach than that of the people I mentioned above. To obtain my results dealing with PL-links, I use results already known about the arc index of a knot.

Definition: There is an open-book decomposition of the 3-sphere which has open discs as pages and an unknotted circle as the binding. We can think of the 3-sphere as $\mathbb{R}^3 \cup \{\infty\}$ and of the circle as the $z - axis \cup \{\infty\}$. The pages are then half-planes H_θ at the angle θ when the $x - y$ plane has polar coordinates. An **arc presentation** of a link, K, is an embedding of K in finitely many pages of the open-book decomposition so that each of the pages meets K in a single simple arc. The minimum number of pages required to present the link K in this manner is the **arc index** of K, denoted $\alpha(K)$.

In other words, the arc index of a link ($\alpha(K)$) is the number of pages in a book it takes to create a link where each page contains exactly one simple arc. One can represent an arc presentation of a link by two permutations, η and τ. Here η is the order of the pages that you progress through if you follow the link, and τ is the order of where the link hits the binding axis of the book that you have created. We can denote $K = (\eta, \tau)$. Cromwell, Bae, and Park [3, 2] have come up with many very nice theorems. The main theorem [2] that I use is: Given a non-split link $K, \alpha(K) \leq c(K) + 2$. It is easy to see that an arc on a page can be made with two straight edges. So, right away we have $e(K) \leq 2\alpha(K)$. I then look deeper into the arc index to see when you can eliminate edges to reduce this upper bound. Using this technique, I have the following theorem:

$$\text{Given } \alpha(K) \geq 12, \text{ then } e(K) \leq 2\alpha(K) - 5.$$

This then gives the corollary using the theorem from Bae and Park, $e(K) \leq 2c(K) - 1$. Also, using the arc index of a torus knot, I have shown that $e(T_{p,q}) \leq 2p + q - 1$ where $2 \leq p < q$. This then gives us equality for $T_{p,2p+1}$ knots. These two results are for edges that can be any length. This then gives rise to what one can do if you need your edges to be of equal length.

There is not very much theory dealing with equilateral edge index. In my work, I again used the idea of the arc index of a link to find an upper bound for the equilateral edge index, which I will denote $e_=(K)$. Each page has a simple arc. You can find the maximal distance, d, between adjacent entries in τ and make your edge length be $\frac{d+1}{2}$. This way we can have every arc be two sides of an isosceles triangle. We know this is possible by our construction. Then I do one trick to obtain the theorem:

$$\text{For a link, } K, e_=(K) \leq 2\alpha(K) - 1.$$

This gives us the corollary: For a non-split link $K, e_=(K) \leq 2c(K) + 3$.

2. Future Plans

My future goals are always expanding as I see more details about links. I plan to continue working on improving the upper bounds for the edge index and equilateral edge index. I believe that the edge index upper bound can be improved to have a lower constant multiple of $c(K)$.

I am excited to use this topic for undergraduate research projects. This is one of the reasons why I chose this area for my research. I felt that it would be a good source for undergraduates. It is a topic that does not need very much background knowledge to get into the basics, and there are many examples and pictures that can be used to understand the material. Also, the exact index is not known for many knots and links, which would also be a great project for undergraduates, as they can use both technology and hands on approaches to attack the problem. I would also be interested in working with undergraduates in general knot theory. Colin Adams' book [1] is a good source to work from that is geared toward upper level undergraduates.

References

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