22M174/22C174: Optimization techniques.

Homework 6. Due 03/27/13.

1. Consider the problem of finding the minimizer of $f(x) := x^2/2$ or equivalently of finding the zero of simple univariate function $F(x) = x$. Apply Newton's method with "line-search" starting from the point $x_0 = 2$ and use steplengths $\alpha_k := 1/2^{k+1}$. Show that the tests $f(x_{k+1}) < f(x_k)$ and $|F(x_{k+1})| < |F(x_k)|$ are satisfied for all $k \geq 0$, but that $\lim_{k\to\infty} x_k \neq 0$. This is a simple example demonstrating that α_k should never converge to 0 (unless $\|p_k\|$ also converges to 0). It also demonstrates that a poor choice of α_k may cause $\|x_{k+1} - x_k\|$ to converge to 0 even when $\|p_k\|$ does not converge to 0.

2. Consider any function $F : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying

$$
\begin{aligned}
F(x_0) &= \frac{1}{10}\begin{bmatrix} 4\sqrt{3}+3 \\ -4\sqrt{3}+3 \end{bmatrix}, \quad DF(x_0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
F(x_1) &= \frac{1}{5}\begin{bmatrix} -3 \\ 4 \end{bmatrix}, \quad DF(x_1) = \begin{bmatrix} 1 & 1 \\ -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}
\end{aligned}
$$

   for a certain point $x_0 \in \mathbb{R}^2$ and $x_1 := x_0 - (DF(x_0))^{-1}F(x_0) \in \mathbb{R}^2$. Show that the sequence x_k generated by Newton's method starting with x_0 satisfies for $k = 0, 1, 2, \ldots$

$$
x_{k+2} = x_k \qquad \text{and} \qquad \|(DF(x_k))^{-1}F(x_{k+1})\|_2 < \|(DF(x_k))^{-1}F(x_k)\|_2.
$$

3. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be differentiable. Show that for $0 \leq \alpha \leq 1$ and Newton's direction $p_k := -(DF(x_k))^{-1}F(x_k)$, the condition $R(\alpha) \geq \gamma_1$ where

$$
R(\alpha) := \frac{\|AF(x_k)\| - \|AF(x_k + \alpha p_k)\|}{\|AL_k(x_k)\| - \|AL_k(x_k + \alpha p_k)\|}
$$

   and $L_k(x) := F(x_k) + DF(x_k)(x - x_k)$ is equivalent to

$$
\|AF(x_k + \alpha p_k)\| \leq (1 - \alpha\gamma_1)\|AF(x_k)\|.
$$

   Hint: simplify $\|AL_k(x_k)\| - \|AL_k(x_k + \alpha p_k)\|$.