Numerical Algorithms
https://doi.org/10.1007/s11075-019-00817-4

ORIGINAL PAPER

Efficient methods for nonlinear time fractional diffusion-wave equations and their fast implementations

Jianfei Huang[1] · Dandan Yang[2] · Laurent O. Jay[3]

Received: 9 October 2018 / Accepted: 23 September 2019 / [redacted]

Abstract
Recently, numerous numerical schemes for solving linear time fractional diffusion-wave equations have been developed. However, most of these methods require relatively high smoothness in time and need extensive computational work and large storage due to the nonlocal property of fractional derivatives. In this paper, an efficient scheme and an alternating direction implicit (ADI) scheme are constructed for one-dimensional and two-dimensional nonlinear time fractional diffusion-wave equations based on their equivalent partial integro-differential equations. The proposed methods require weaker smoothness in time compared to the methods based on discretizing fractional derivative directly. They are proved to be unconditionally stable and convergent with first-order of accuracy in time and second order of accuracy in space. Fast implementations of the proposed methods are presented by the sum-of-exponentials (SOE) approximation for the kernel $t^{-2+\alpha}$ on the interval $[\tau, T]$, where $1 < \alpha < 2$. Finally, numerical experiments are carried out to illustrate the theoretical results of our direct schemes and demonstrate their powerful computational performances.

Keywords Time fractional diffusion-wave equations · Nonlinear system · Finite difference schemes · Stability · Convergence · Fast implementations

Mathematics Subject Classification (2010) 65M06 · 65M12 · 35R11

✉ Dandan Yang
ydd423@sohu.com

[1] College of Mathematical Sciences, Yangzhou University, Yangzhou 225002, China

[2] School of Mathematical Science, Huaiyin Normal University, Huaian 223300, China

[3] Department of Mathematics, 14 MacLean Hall, The University of Iowa, Iowa City, IA, 52242, USA

1 Introduction

In this paper, we consider numerical methods for nonlinear time fractional diffusion-wave problems of the following form

$$ {}_0^C D_t^{\alpha} u(X,t) = \Delta u(X,t) + f(X,t,u(X,t)),\ X \in \Omega,\ 0 < t \le T,\ 1 < \alpha < 2, \tag{1.1} $$

with initial conditions

$$ u(X,0) = \phi(X),\ u_t(X,0) = \varphi(X),\ X \in \overline{\Omega}, \tag{1.2} $$

and boundary condition

$$ u(X,t) = \psi(X,t),\ X \in \partial\Omega,\ 0 < t \le T, \tag{1.3} $$

where the spatial variable X can be seen as the one-dimensional $X = x$ or the two-dimensional $X = (x, y)$, Δ is the Laplacian, Ω is the domain of X, and $\partial\Omega$ and $\overline{\Omega}$ are the boundary and the closure of Ω, respectively. $f(X, t, u)$ is a nonlinear function of unknown $u \in \mathbb{R}$ and fulfills a Lipschitz condition with respect to u. $\phi(X)$, $\varphi(X)$, and $\psi(X, t)$ are assumed to be sufficiently smooth functions. ${}_0^C D_t^{\alpha} u$ is the temporal Caputo derivative of order α defined as

$$ {}_0^C D_t^{\alpha} u(X,t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \frac{\partial^2 u(X,s)}{\partial s^2} ds. \tag{1.4} $$

The time fractional diffusion-wave (1.1) possesses the remarkable feature that it can be considered as intermediate between parabolic diffusion equations and hyperbolic wave equations. It has been widely applied in the modeling of anomalous diffusive processes and in the description of viscoelastic damping materials, etc. [4, 16, 18, 22, 29]. However, due to the nonlocal property of fractional derivatives and fractional integrals, it is difficult or even impossible to obtain analytical solutions of time fractional diffusion-wave equations (see [2, 26, 28] for examples). Thus, there has been a growing interest to develop numerical methods for solving time fractional diffusion-wave equations.

In recent years, numerous numerical methods for solving time fractional diffusion-wave equations, especially for the linear ones, have been proposed, discussed, and analyzed. These numerical methods can be classified into two groups. The methods in the first group are constructed from discretizing the Caputo derivative in (1.1) directly (see [1, 12, 17, 30, 32, 35, 36, 42] for examples). In [32], Sun and Wu proposed a $3 - \alpha$ order approximation for the Caputo derivative, constructed a fully discrete difference scheme for linear time fractional diffusion-wave equations, and gave a theoretical analysis. Li et al. in [17] applied a finite difference method in time and a finite element method in space for time-space fractional diffusion-wave equations, and analyzed the semidiscrete and fully discrete numerical approximations. In [30], Mustapha and Schötzau established an hp-version time-stepping discontinuous Galerkin method for fractional diffusion-wave evolution problems and showed that exponential rates of convergence in the number of temporal degrees of freedom were achieved for solutions with initial singularity. In [36], Wang and Vong presented a high-order alternating direction implicit (ADI) finite difference scheme for the two-dimensional time fractional diffusion-wave equation, with temporal and spatial accuracy order equal to two and four, respectively. In [42], Zhang et al. proposed a compact difference approach for spatial discretization and an ADI method for the time-stepping and proved that the scheme was unconditionally stable and H_1-norm convergent with $3-\alpha$ order in time and fourth order in space. Then, a similar ADI scheme was also constructed by Wang et al. in [35]. Fairweather et al. in [12] formulated a numerical method for a two-dimensional time fractional diffusion-wave equation based on orthogonal spline collocation in space and ADI Crank-Nicolson L_1-approximation in time, and proved that this scheme was stable and super-convergent in various norms. In [1], Abbaszadeh and Dehghan proposed an improved meshless method for solving two-dimensional distributed order time fractional diffusion-wave equation and investigated the uniqueness, existence, and stability of the new schemes and obtained an error estimate for the full-discrete schemes. All those methods presented above are based on direct approximations to the Caputo derivative. We remark that all these approximations based on uniform grids often require the high smoothness in time, such as $u(\cdot, t) \in C^3([0, T])$. However, high smoothness in fact does not generally hold in fractional systems due to the singularities of fractional derivatives (see [15, 40] and references therein).

The second group of methods for time fractional diffusion-wave equations is based on the partial integro-differential equations which are equivalent to (1.1) by discretizing the Riemman-Liouville integral [5, 6, 8, 9, 13, 19, 27, 38, 39, 41]. It is well-known that the numerical methods constructed for integral equations are more stable than for the corresponding differential equations and often need less smoothness requirements (see [11] and the references therein). In [13], Huang et al. constructed two finite difference schemes to solve a class of time fractional diffusion-wave equations based on their equivalent partial integro-differential equations, and proved that the proposed two schemes were convergent and stable. Yang et al. in [38] extended the results and constructed a difference scheme with α-order accuracy in time by using the second-order convolution quadrature formula proposed in [24] by Lubich. In [19], Li et al. presented a fast and efficient numerical method to solve a two-dimensional fractional evolution equation by using a second-order difference quotient in space, the backward Euler in time and the first-order convolution quadrature approximating the integral term. Bhrawya et al. in [5] presented a spectral numerical method for solving fractional diffusion-wave equations and fractional wave equations with damping. The proposed method was based on Jacobi tau spectral procedure together with the Jacobi operational matrix for fractional Riemann-Liouville integrals. Zeng in [41] proposed two stable and one conditionally stable finite difference schemes of second order in both time and space for the time fractional super-diffusion equation by means of the fractional trapezoidal rule and the generalized Newton-Gregory formula. Chen and Li in [6] constructed a novel compact finite difference scheme for solving the fractional diffusion-wave equation based on its equivalent integro-differential equation. The product trapezoidal scheme was employed to treat the fractional integral term. In [8], Chen et al. proposed and analyzed a second-order backward differentiation formula ADI difference scheme for the two-dimensional fractional evolution equation based on standard central difference approximation in space and second-order convolution quadrature in time. And then stability and convergence of the proposed difference scheme in the L_2 norm were derived by the energy method. In [9], Dehghan and Abbaszadeh constructed a finite difference-spectral element method for nonlinear fractional evolution equations and discussed their stability and convergence. Yang et al. in [39] devised a high-order numerical scheme for an anomalous diffusion equation based on an equivalent transformation by the use of a smooth operator. The main advantage of this approach was its high convergence rate even though the solution had lower regularity at the starting point. Lyu and Vong in [27] proposed a finite difference scheme with temporal nonuniform mesh for time fractional Benjamin-Bona-Mahony equations with non-smooth solutions. The proposed scheme was a linearized scheme on a nonuniform mesh deduced from some high-order interpolation formulas to the Riemann-Liouville integral.

As is known, the above methods to approximate time fractional derivatives and fractional integrals require the storage of the solution at all previous time steps and the computational complexity of these approximations is $O(N^2)$, where N is the number of time steps. The computational work is large when N is a large number. Jiang et al. in [14] proposed a fast evaluation of the Caputo derivative by the sum-of-exponentials (SOE) approximation and then applied this fast evaluation to solve a time fractional diffusion equation. The resulting fast scheme can greatly reduce the computational cost and the memory and is very suitable for long-time simulations. The idea of this fast evaluation was also extended in [37] to improve the performance of $L_2 - 1_\sigma$ formula.

To the best of our knowledge, there still does not exist fast schemes for solving time fractional diffusion-wave equations based on equivalent partial integro-differential equations, especially for nonlinear problems. Herein, an efficient scheme and an ADI scheme are proposed for one-dimensional and two-dimensional nonlinear time fractional diffusion-wave equations based on equivalent partial integro-differential equations. The proposed methods, constructed by the piecewise product of right and left rectangular quadrature to the Riemann-Liouville integral, only require $u(\cdot, t) \in C^2([0, T])$ in time (note $1 < \alpha < 2$) and can be proved to be unconditionally stable and convergent with first-order accuracy in time and second-order accuracy in space. Fast implementations of the proposed methods are presented based on the SOE approximation to the kernel $t^{-2+\alpha}$ on the interval $[\tau, T]$. The fast schemes obtained only need $O(NN_{\exp})$ computational cost where $N_{\exp}$ is the number of exponentials to numerically calculate the nonlinear time fractional diffusion-wave equations at a fixed spatial point. It is worth mentioning that Lubich's convolution quadratures [24] to the Riemann-Liouville integral cannot be fast evaluated, because these quadratures are based on the generating functions. Additionally, if the high-order piecewise approximations to the Riemann-Liouville integral, such as the product trapezoidal approximation, is used, but it is difficult to prove convergence although the deduced scheme can be fast calculated by SOE technique (see [36]).

The paper is organized as follows. In Section 2, some preparations and useful lemmas are discussed and proved. In Section 3, a finite difference scheme for the one-dimensional nonlinear time fractional diffusion-wave equations is derived, and the unconditional stability and convergence are proved. In Section 4, an ADI scheme is formulated and analyzed for the two-dimensional problems. Fast implementations of discretizing the Riemann-Liouville integral and the above proposed schemes are derived in Section 5. Numerical experiments are carried out in Section 6 to illustrate the theoretical results and demonstrate the performance of our schemes. Finally, concluding remarks are given.

2 Preliminaries

In this section, we present basic definitions, some notations, and important lemmas used throughout the remaining sections of this paper.

Lemma 2.1 *(see* [10, 13]*).* Equation (1.1) *is equivalent to the following partial integro-differential equation,*

$$u_t(X,t) = \varphi(X) + {}_0J_t^{\alpha-1}\Delta u(X,t) + {}_0J_t^{\alpha-1} f(X,t,u), \tag{2.1}$$

where ${}_0J_t^{\beta}$ *is the Riemann-Liouville integral operator of order* β $(0<\beta<1)$ *defined as*

$${}_0J_t^{\beta} g(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s)ds,$$

where $g(t)$ *is a function with certain smoothness.*

Lemma 2.2 *Assume* $f(t) \in C^1([0,T])$*, then for any* $0<\beta<1$*, the following two approximations for the Riemann-Liouville integral hold*

$${}_0J_t^{\beta} f(t_n) = \frac{\tau^{\beta}}{\Gamma(\beta+1)}\sum_{k=1}^{n} c_{n-k}^{\beta} f(t_k) + O(\tau), \tag{2.2}$$

and

$${}_0J_t^{\beta} f(t_n) = \frac{\tau^{\beta}}{\Gamma(\beta+1)}\sum_{k=1}^{n} c_{n-k}^{\beta} f(t_{k-1}) + O(\tau), \tag{2.3}$$

where $t_n = n\tau$*,* n *is an integer and* τ *is a step size.* $c_k^{\beta} = (k+1)^{\beta} - k^{\beta}$ *for the integer* $k \geq 0$.

In Lemma 2.2, the two approximations are constructed by the piecewise product right and left rectangular quadrature to the Riemann-Liouville integral, respectively. These two approximations with first-order accuracy can be used to discretize the Riemann-Liouville integral under a weak smoothness assumption of the integrand $f(t)$, i.e., $f(t) \in C^1([0,T])$. This can provide enough accuracy in most situations of practical scientific computing. In fact, a high smoothness requirement in fractional system is often not available due to the properties of fractional derivatives and fractional integral. Furthermore, (2.2) and (2.3) can be used to implicitly and explicitly discretize the fractional systems with the Riemann-Liouville integral, respectively. Now we only prove (2.2), the proof of (2.3) is similar.

Proof We have

$$\frac{1}{\Gamma(\beta)}\int_{t_{k-1}}^{t_k} (t_n-s)^{\beta-1} ds = \frac{1}{\Gamma(\beta)\beta}\left((t_n-t_{k-1})^{\beta} - (t_n-t_k)^{\beta}\right) = \frac{\tau^{\beta}}{\Gamma(\beta+1)} c_{n-k}^{\beta}.$$

Let us consider the absolute error E between ${}_0J_t^{\beta} f(t_n)$ and $\frac{\tau^\beta}{\Gamma(\beta+1)} \sum\limits_{k=1}^{n} c_{n-k}^{\beta} f(t_k)$; we have

$$
\begin{aligned}
E &= \Big| \frac{1}{\Gamma(\beta)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_n - s)^{\beta-1} f(s) ds - \frac{1}{\Gamma(\beta)} \sum_{k=1}^{n} f(t_k) \int_{t_{k-1}}^{t_k} (t_n - s)^{\beta-1} ds \Big| \\
&= \Big| \frac{1}{\Gamma(\beta)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_n - s)^{\beta-1} [f(s) - f(t_k)] ds \Big| \\
&= \Big| \frac{1}{\Gamma(\beta)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_n - s)^{\beta-1} [(s - t_k) f'(\xi_k)] ds \Big|,
\end{aligned}
$$

where ξ_k is a real number between s and t_k. Note that for $f(t) \in C^1([0, T])$, from the above formula, we have

$$
E \le \frac{C\tau}{\Gamma(\beta)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_n - s)^{\beta-1} ds = \frac{C\tau}{\Gamma(\beta)} \int_0^{t_n} (t_n - s)^{\beta-1} ds = \frac{C t_n^{\beta} \tau}{\Gamma(\beta+1)}.
$$

This completes the proof. □

The following lemma, concerning the non-negative character of certain real quadratic forms with convolution structure, plays an important role in our stability and convergence analysis.

Lemma 2.3 *(see [23]). Let $\{\omega_p\}_{p=0}^{\infty}$ be a monotonously decreasing sequence of non-negative real numbers with property $\omega_{p+1} + \omega_{p-1} \ge 2\omega_p$ $(p \ge 1)$, then for any positive integer K and real vector $(V_1, V_2, \cdots, V_K) \in \mathbb{R}^K$, it holds that*

$$
\sum_{n=1}^{K} \left(\sum_{p=0}^{n-1} \omega_p V_{n-p} \right) V_n \ge 0. \tag{2.4}
$$

Actually, the inequality (2.4) is one of the key ingredients in the numerical analysis of the partial integro-differential equation (see [34]). An alternative way to obtain (2.4) is that if $\{\omega_p\}_{p=0}^{\infty}$ is a sequence of real numbers such that $\widehat{a}(z) = \sum\limits_{n=0}^{\infty} \omega_n z^n$ is analytic in the open unit disk $D = \{z \in \mathcal{C} : |z| < 1\}$ and $\mathrm{Re}\,(\widehat{a}(z)) \ge 0$ for $z \in D$, see [8] and [25] for examples. Obviously, it may be easier to check if the weights c_n^{β} in Lemma 2.2 meet the conditions in Lemma 2.3. Thus, we have

Lemma 2.4 *For $0 < \beta < 1$, the weights $\{c_n^{\beta}\}_{n=0}^{\infty}$ defined in Lemma 2.2 form a real monotonously decreasing positive sequence with $c_{n+1}^{\beta} + c_{n-1}^{\beta} \ge 2c_n^{\beta}$, $(n \ge 1)$.*

Proof Rewrite c_n^{β} as

$$
c_n^{\beta} = (n+1)^{\beta} - n^{\beta} = \beta \int_n^{n+1} t^{\beta-1} dt.
$$

From this expression, it can be deduced that $\{c_n^\beta\}_{n=0}^\infty$ form a real monotonously decreasing positive sequence.

To prove $c_{n+1}^\beta + c_{n-1}^\beta \geq 2c_n^\beta$, $(n \geq 1)$, we note that $c_{n+1}^\beta + c_{n-1}^\beta - 2c_n^\beta = (c_{n+1}^\beta - c_n^\beta) - (c_n^\beta - c_{n-1}^\beta)$. Thus, the result can be achieved if $\{c_{n+1}^\beta - c_n^\beta\}_{n=0}^\infty$ is an increasing sequence. Namely, we only need to prove $\{(n+2)^\beta - 2(n+1)^\beta + n^\beta\}_{n=0}^\infty$ is increasing. Let $f(x) = (x+2)^\beta - 2(x+1)^\beta + x^\beta$, then this result is true due to the fact that

$$\begin{aligned} f'(x) &= \beta(x+2)^{\beta-1} - 2\beta(x+1)^{\beta-1} + \beta x^{\beta-1} \\ &= \beta\left[(x+2)^{\beta-1} - 2(x+1)^{\beta-1} + x^{\beta-1}\right] \\ &= \beta(\beta-1)\left[\int_{x+1}^{x+2} t^{\beta-2}dt - \int_{x}^{x+1} t^{\beta-2}dt\right] > 0. \end{aligned}$$

The proof is completed. □

Lemma 2.5 *(see [33]). Suppose $f(t) \in C^2([t_{n-1}, t_n])$, then the following approximation with integral remainder holds*

$$\delta_t f(t_n) = \frac{f(t_n) - f(t_{n-1})}{\tau} = f'(t_n) - \frac{\tau}{2}\int_0^1 f''(t_n - s\tau)(1-s)ds. \tag{2.5}$$

Moreover, if $g(x) \in C^4([x_{i-1}, x_{i+1}])$ and $\xi(s) = g^{(4)}(x_i + sh) + g^{(4)}(x_i - sh)$, then

$$\delta_x^2 g(x_i) = \frac{g(x_{i+1}) - 2g(x_i) + g(x_{i-1})}{h^2} = g''(x_i) + \frac{h^2}{24}\int_0^1 \xi(s)(1-s)^3 ds. \tag{2.6}$$

t_n and τ are defined in Lemma 2.2, and $x_i = ih$ is a point with the integer i and the step size h.

To implement a fast evaluation of the Riemann-Liouville integral, the following lemma is quite helpful. It establishes an error estimate of the SOE approximation to the kernel $t^{-2+\alpha}$ with $1 < \alpha < 2$ on the interval $[\tau, T]$.

Lemma 2.6 *(see [14]). For the power function $t^{-\beta}(0 < \beta < 1)$, the following sum-of-exponentials approximation holds*

$$\left| t^{-\beta} - \sum_{i=1}^{N_{\exp}} \omega_i e^{-s_i t} \right| \leq \epsilon, \ t \in [\tau, T], \tag{2.7}$$

where s_i and ω_i are the nodes and weights of the Gaussian quadrature, ϵ is the absolute error, and $N_{\exp} = O\left(\log\frac{1}{\epsilon}\left(\log\log\frac{1}{\epsilon} + \log\frac{T}{\tau}\right) + \log\frac{1}{\tau}\left(\log\log\frac{1}{\epsilon} + \log\frac{1}{\tau}\right)\right)$.

3 A direct scheme for the one-dimensional problem

3.1 Derivation of the direct scheme

For the one-dimensional problem with $\Omega = (0, L)$ and $X = x$, we introduce the spatial step size $h = \frac{L}{M}$ with a positive integer M and the spatial grid $x_i = ih,\ i = 0, 1, \cdots, M$. Similarly, the temporal step size $\tau = \frac{T}{N}$ with a positive integer N and the temporal grid $t_n = n\tau,\ n = 0, 1, \cdots, N$ are defined, respectively. Then, we consider (2.1) at the point (x_i, t_n), and let $u_i^n = u(x_i, t_n)$, $\varphi_i = \varphi(x_i)$ and $f_i^n(u_i^n) = f(x_i, t_n, u(x_i, t_n))$, namely

$$\frac{\partial u(x_i, t)}{\partial t}|_{t=t_n} = \varphi_i + {}_0J_{t_n}^{\alpha-1}\frac{\partial^2 u(x, t)}{\partial x^2}|_{x=x_i} + {}_0J_{t_n}^{\alpha-1} f(x_i, t, u(x_i, t)).$$

To avoid solving a nonlinear system, we apply (2.2) and (2.3) of Lemma 2.2 to discretize the first integral and the second integral in the right-hand side of the above equation respectively, and use (2.5) and (2.6) of Lemma 2.5 to approximate the terms $\frac{\partial u(x_i,t)}{\partial t}$ and $\frac{\partial^2 u(x,t_n)}{\partial x^2}$ in the above equation respectively, then we get

$$\delta_t u_i^n = \varphi_i + \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\delta_x^2 u_i^k + \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}\sum_{k=1}^{n} c_{n-k}^{\alpha-1} f_i^{k-1}(u_i^{k-1}) + (R_1^{\alpha-1})_i^n, \quad (3.1)$$

where the local truncation error $\left(R_1^{\alpha-1}\right)_i^n$ can be represented as

$$\begin{aligned}\left(R_1^{\alpha-1}\right)_i^n &= -\frac{\tau}{2}\int_0^1 (1-s)\frac{\partial^2 u(x_i, t_n - s\tau)}{\partial t^2}ds + O(\tau) + \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\left(O\left(h^2\right)\right)\\ &= O(\tau) + t_n^{\alpha-1}\left(O\left(h^2\right)\right) = O\left(\tau + h^2\right).\end{aligned}$$

Multiplying (3.1) by τ, omitting the small term $\tau\left(R_1^{\alpha-1}\right)_i^n$, and replacing u_i^n with its numerical approximation U_i^n, one can get the following direct scheme for the one-dimensional problem (1.1),

$$\begin{aligned}U_i^n &= U_i^{n-1} + \tau\varphi_i + \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\delta_x^2 U_i^k\\ &\quad + \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{k=1}^{n} c_{n-k}^{\alpha-1} f_i^{k-1}\left(U_i^{k-1}\right),\ 1 \le n \le N, \quad (3.2)\end{aligned}$$

with initial values $U_i^0 = \phi(x_i),\ 0 \le i \le M$, and boundary values $U_0^n = \psi(0, t_n),\ U_M^n = \psi(L, t_n)$.

3.2 Analysis of the direct scheme

In this subsection and in the sequel, the symbol C denotes a generic positive constant, whose value may be different from one line to another and is independent of discretization parameters.

The convergence and stability results of (3.2) will be considered in this subsection. To do this, let us first define a grid function space

$$V_0 = \{v_i \mid i = 0, 1, \ldots, M, \text{and } v_0 = v_M = 0\},$$

and the following inner product and norms for any grid function $v, w \in V_0$,

$$\langle v, w \rangle = h \sum_{i=1}^{M-1} v_i w_i, \ \|v\| = \sqrt{\langle v, v \rangle}, \ \|\delta_x v\| = \sqrt{h \sum_{i=1}^{M} |\delta_x v_i|^2}.$$

Now we prove the following convergence result for the direct scheme (3.2).

Theorem 3.1 *Suppose $u(x,t) \in C_{x,t}^{4,2}(\Omega \times [0,T])$, and let $u(x_i, t_n)$ and U_i^n be the exact and numerical solutions at the point (x_i, t_n), respectively. Then for sufficiently small τ and h, and for $1 \leq n \leq N$, we get*

$$\|u^n - U^n\| \leq C\left(\tau + h^2\right).$$

Proof Subtracting (3.2) from (3.1), we obtain the following error equation

$$\begin{aligned} e_i^n - e_i^{n-1} &= \frac{\tau^\alpha}{\Gamma(\alpha)} \sum_{k=1}^{n} c_{n-k}^{\alpha-1} \delta_x^2 e_i^k + \frac{\tau^\alpha}{\Gamma(\alpha)} \sum_{k=1}^{n} c_{n-k}^{\alpha-1} \\ &\quad \times \left[f_i^{k-1}\left(u_i^{k-1}\right) - f_i^{k-1}\left(U_i^{k-1}\right)\right] + \left(r_1^{\alpha-1}\right)_i^n, \end{aligned}$$

where $e_i^n = u_i^n - U_i^n$, and $\left(r_1^{\alpha-1}\right)_i^n$ is the local truncation error which can be bounded by $C\left(\tau^2 + \tau h^2\right)$ with a positive constant C. Then, multiplying both sides of the above equation by he_i^n and summing over $1 \leq i \leq M-1$, we get

$$\begin{aligned} \|e^n\|^2 - \langle e^{n-1}, e^n \rangle &= \frac{\tau^\alpha}{\Gamma(\alpha)} \sum_{k=1}^{n} c_{n-k}^{\alpha-1} \langle \delta_x^2 e^k, e^n \rangle + \frac{\tau^\alpha}{\Gamma(\alpha)} \sum_{k=1}^{n} c_{n-k}^{\alpha-1} \langle f_i^{k-1}\left(u_i^{k-1}\right) \\ &\quad - f_i^{k-1}\left(U_i^{k-1}\right), e^n \rangle + \langle \left(r_1^{\alpha-1}\right)^n, e^n \rangle. \end{aligned}$$

Applying Cauchy-Schwarz inequality to the second term of the left-hand side of the above equality, we deduce that

$$\begin{aligned} \frac{\|e^n\|^2 - \|e^{n-1}\|^2}{2} &\leq \frac{\tau^\alpha}{\Gamma(\alpha)} \sum_{k=1}^{n} c_{n-k}^{\alpha-1} \langle \delta_x^2 e^k, e^n \rangle + \frac{\tau^\alpha}{\Gamma(\alpha)} \sum_{k=1}^{n} c_{n-k}^{\alpha-1} \langle f_i^{k-1}\left(u_i^{k-1}\right) \\ &\quad - f_i^{k-1}\left(U_i^{k-1}\right), e^n \rangle + \langle \left(r_1^{\alpha-1}\right)^n, e^n \rangle. \end{aligned}$$

Summing over n from 1 to K yields

$$\frac{\|e^K\|^2-\|e^0\|^2}{2} \leq \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{n=1}^{K}\sum_{k=1}^{n}c_{n-k}^{\alpha-1}\langle\delta_x^2 e^k, e^n\rangle + \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{n=1}^{K}\sum_{k=1}^{n}c_{n-k}^{\alpha-1}\langle f_i^{k-1}\left(u_i^{k-1}\right)$$
$$-f_i^{k-1}\left(U_i^{k-1}\right), e^n\rangle + \sum_{n=1}^{K}\langle\left(r_1^{\alpha-1}\right)^n, e^n\rangle. \tag{3.3}$$

Let us consider the first term of the right-hand side of (3.3); we have

$$\begin{aligned}\frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{n=1}^{K}\sum_{k=1}^{n}c_{n-k}^{\alpha-1}\langle\delta_x^2 e^k, e^n\rangle &= -\frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{n=1}^{K}\sum_{k=1}^{n}c_{n-k}^{\alpha-1}\langle\delta_x e^k, \delta_x e^n\rangle \\ &= -\frac{\tau^\alpha h}{\Gamma(\alpha)}\sum_{i=1}^{M}\sum_{n=1}^{K}\left(\sum_{k=1}^{n}c_{n-k}^{\alpha-1}\delta_x e_i^k\right)\delta_x e_i^n.\end{aligned}$$

According to Lemma 2.3, it is clear that the above term is negative. Then by using the Lipschitz condition of f with respect to u, (3.3) becomes

$$\|e^K\|^2 \leq \frac{2L\tau^\alpha}{\Gamma(\alpha)}\sum_{n=1}^{K}\sum_{k=1}^{n}c_{n-k}^{\alpha-1}\|e^{k-1}\|\|e^n\| + C\sum_{n=1}^{K}(\tau^2+\tau h^2)\|e^n\|, \tag{3.4}$$

where L is the Lipschitz constant. Assume $\|e^M\| = \max\limits_{1\leq k\leq N}\|e^k\|$, then (3.4) becomes

$$\|e^M\| \leq \frac{2L\tau^\alpha}{\Gamma(\alpha)}\sum_{n=1}^{M}\sum_{k=1}^{n}c_{n-k}^{\alpha-1}\|e^{k-1}\| + C(\tau+h^2). \tag{3.5}$$

Exchanging the orders of two summations in (3.5), and noting the sequence $\{c_{n-k}^{\alpha-1}\}$ is positive and $\tau^{\alpha-1}\sum\limits_{k=1}^{n}c_{n-k}^{\alpha-1} = t_n^{\alpha-1}$, we get

$$\tau^\alpha\sum_{n=1}^{M}\sum_{k=1}^{n}c_{n-k}^{\alpha-1}\|e^{k-1}\| = \tau\sum_{k=1}^{M}\left(\tau^{\alpha-1}\sum_{n=k}^{M}c_{n-k}^{\alpha-1}\right)\|e^{k-1}\| \leq \tau C\sum_{k=1}^{M}\|e^{k-1}\|.$$

Thus, (3.5) becomes

$$\|e^M\| \leq \tau C\sum_{k=1}^{M}\|e^{k-1}\| + C\left(\tau+h^2\right).$$

After applying the discrete Gronwall's lemma to this inequality, we can conclude that

$$\|e^M\| \leq C\left(\tau+h^2\right),$$

and this completes the proof. □

Now let us prove that the difference scheme (3.2) is unconditionally stable to the initial values ϕ and φ, and the inhomogeneous term f in the L_2-norm.

Theorem 3.2 *Suppose the grid function $\{U_i^n | 0 \le i \le M, 0 \le n \le N\}$ is the solution of the difference scheme (3.2), and let $U_0^n = U_M^n = 0$. Then, for $1 \le K \le N$, it holds that*

$$\|U^K\| \le C\left(\|\phi\| + \|\varphi\| + \max_{0\le k\le N}\|f^k\|\right). \tag{3.6}$$

Proof Multiplying both sides of (3.2) by hU_i^n and summing over $1 \le i \le M-1$, we get

$$\begin{aligned}\langle U^n - U^{n-1}, U^n\rangle &= \tau\langle\varphi, U^n\rangle + \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\langle\delta_x^2 U^k, U^n\rangle \\ &\quad + \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\langle f^{k-1}, U^n\rangle.\end{aligned}$$

From

$$\langle U^n - U^{n-1}, U^n\rangle = \|U^n\|^2 - \langle U^{n-1}, U^n\rangle,$$

and applying Cauchy-Schwarz inequality to the second term on the right-hand side, we can obtain the following inequality

$$\begin{aligned}\frac{\|U^n\|^2 - \|U^{n-1}\|^2}{2} &\le \tau\langle\varphi, U^n\rangle + \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\langle\delta_x^2 U^k, U^n\rangle \\ &\quad + \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\langle f^{k-1}, U^n\rangle.\end{aligned}$$

Adding up the above inequalities for n from 1 to K yields

$$\begin{aligned}\frac{\|U^K\|^2 - \|U^0\|^2}{2} &\le \tau\sum_{n=1}^{K}\langle\varphi, U^n\rangle + \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{n=1}^{K}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\langle\delta_x^2 U^k, U^n\rangle \\ &\quad + \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{n=1}^{K}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\langle f^{k-1}, U^n\rangle.\end{aligned} \tag{3.7}$$

According to Lemma 2.3, it can be checked that the second term of the right-hand side of (3.7) is negative, thus (3.7) becomes

$$\|U^K\|^2 \le \|U^0\|^2 + 2\tau\sum_{n=1}^{K}\|\varphi\|\|U^n\| + \frac{2\tau^\alpha}{\Gamma(\alpha)}\sum_{n=1}^{K}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\|f^{k-1}\|\|U^n\|.$$

Using Young's inequality, we have

$$\begin{aligned}\|U^K\|^2 &\le \|U^0\|^2 + \tau\sum_{n=1}^{K}\|\varphi\|^2 + \tau\sum_{n=1}^{K}\|U^n\|^2 \\ &\quad + \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{n=1}^{K}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\|f^{k-1}\|^2 + \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{n=1}^{K}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\|U^n\|^2.\end{aligned}$$

Note that $\frac{\tau^{\alpha-1}}{\Gamma(\alpha)}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}$ is bounded. Thus, for a sufficiently small τ, this inequality becomes

$$\|U^K\|^2 \le C\|U^0\|^2 + \tau C\sum_{n=1}^{K}\|\varphi\|^2 + \tau^\alpha C\sum_{n=1}^{K}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\|f^{k-1}\|^2 + \tau C\sum_{n=1}^{K-1}\|U^n\|^2.$$

By the discrete Gronwall's lemma, we obtain

$$\|U^K\|^2 \le C\|U^0\|^2 + \tau C\sum_{n=1}^{K}\|\varphi\|^2 + \tau^\alpha C\sum_{n=1}^{K}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\|f^{k-1}\|^2.$$

Due to $\tau\sum_{n=0}^{N-1} 1 = T$, we obtain

$$\|U^K\|^2 \le C\left[\|\phi\|^2 + T\|\varphi\|^2 + T^\alpha \max_{0\le k\le N}\|f^k\|^2\right],$$

and this completes the proof. □

4 An ADI scheme for the two-dimensional problem

4.1 Derivation of the ADI scheme

For the two-dimensional problem with $\Omega = (0, L_x)\times(0, L_y)$ and $X = (x, y)$, we introduce the spatial step sizes $h_x = \frac{L_x}{M_x}$ with a positive integer M_x and $h_y = \frac{L_y}{M_y}$ with a positive integer M_y in x and y directions, respectively. And the spatial grids are $(x_i, y_j) = (ih_x, jh_y)$, $i = 0, 1, \cdots, M_x$, $j = 0, 1, \cdots, M_y$. To discretize (2.1), we consider (2.1) at the point (x_i, y_j, t_n), and let $u_{ij}^n = u(x_i, y_j, t_n)$, $\varphi_{ij} = \varphi(x_i, y_j)$ and $f_{ij}^n(u_{ij}^n) = f(x_i, y_j, t_n, u(x_i, y_j, t_n))$, then we get

$$\frac{\partial u(x_i, y_j, t)}{\partial t}|_{t=t_n} = \varphi_{ij} + {}_0J_{t_n}^{\alpha-1}\Delta u(x, y, t)|_{x=x_i, y=y_j} + {}_0J_{t_n}^{\alpha-1} f(x_i, y_j, t, u(x_i, y_j, t)).$$

In the above equation, we apply (2.2) and (2.3) of Lemma 2.2 to discretize the first integral and the second integral in the right-hand side respectively, use (2.5) to discretize the term $\frac{\partial u(x_i,t)}{\partial t}$, and adopt (2.6) to approximate the terms $\frac{\partial^2 u(x,y,t_n)}{\partial x^2}$ and $\frac{\partial^2 u(x,y,t_n)}{\partial y^2}$ in the Laplacian Δ, then we obtain the following linear difference equation

$$\begin{aligned}\delta_t u_{ij}^n &= \varphi_{ij} + \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\left(\delta_x^2 + \delta_y^2\right)u_{ij}^k + \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}\sum_{k=1}^{n} c_{n-k}^{\alpha-1} f_{ij}^{k-1}\left(u_{ij}^{k-1}\right)\\ &\quad + \left(R_2^{\alpha-1}\right)_{ij}^n,\end{aligned} \tag{4.1}$$

where the local truncation error is $\left(R_2^{\alpha-1}\right)_{ij}^n = O\left(\tau + h_x^2 + h_y^2\right)$.

Multiplying (4.1) by τ, then to construct the ADI scheme and to ensure the first-order accuracy in time, a small term $\frac{\tau^{2\alpha}}{\Gamma^2(\alpha)}\delta_x^2\delta_y^2 u_{ij}^n$ with $2\alpha > 2$ is added to both sides of (4.1), and we have

$$\begin{aligned} & u_{ij}^n - \frac{\tau^\alpha}{\Gamma(\alpha)}\left[\delta_x^2 u_{ij}^n + \delta_y^2 u_{ij}^n\right] + \frac{\tau^{2\alpha}}{\Gamma^2(\alpha)}\delta_x^2\delta_y^2 u_{ij}^n \\ &= u_{ij}^{n-1} + \tau\varphi_{ij} + \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{k=1}^{n-1} c_{n-k}^{\alpha-1}\left[\delta_x^2 u_{ij}^k + \delta_y^2 u_{ij}^k\right] \\ &\quad + \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{k=1}^{n} c_{n-k}^{\alpha-1} f_{ij}^{k-1}\left(u_{ij}^{k-1}\right) + \tau\left(R_2^{\alpha-1}\right)_{ij}^n . \end{aligned} \tag{4.2}$$

In (4.2), omitting the small term $\tau\left(R_2^{\alpha-1}\right)_{ij}^n$, and replacing the term u_{ij}^n with its numerical approximation U_{ij}^n, one can get the following direct scheme for the two-dimensional problem (1.1),

$$\begin{aligned} & \left(1 - \frac{\tau^\alpha}{\Gamma(\alpha)}\delta_x^2\right)\left(1 - \frac{\tau^\alpha}{\Gamma(\alpha)}\delta_y^2\right) U_{ij}^n \\ &= U_{ij}^{n-1} + \tau\varphi_{ij} + \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{k=1}^{n-1} c_{n-k}^{\alpha-1}\left[\delta_x^2 U_{ij}^k + \delta_y^2 U_{ij}^k\right] \\ &\quad + \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{k=1}^{n} c_{n-k}^{\alpha-1} f_{ij}^{k-1}\left(U_{ij}^{k-1}\right) \end{aligned} \tag{4.3}$$

with initial values $U_{ij}^0 = \phi(x_i, y_j)$, $(x_i, y_j) \in \overline{\Omega}$, and boundary values $U_{ij}^n = \psi(x_i, y_j, t_n)$, $(x_i, y_j) \in \partial\Omega$.

Following the Peaceman-Rachford strategy, we introduce intermediate variables $U_{ij}^* = \left(1 - \frac{\tau^\alpha}{\Gamma(\alpha)}\delta_y^2\right) U_{ij}^n$. Then, the numerical solutions U_{ij}^n can be obtained by solving two sets of independent one-dimensional linear problems. Thus, the direct ADI scheme is presented as follows. For fixed $j \in \{1, 2, \ldots, M_y - 1\}$, we solve the following system to get $\{U_{ij}^*\}$ for $1 \le i \le M_x - 1$,

$$\begin{cases} \left(1-\frac{\tau^\alpha}{\Gamma(\alpha)}\delta_x^2\right) U_{ij}^* = U_{ij}^{n-1} + \tau\varphi_{ij} + \frac{\tau^\alpha}{\Gamma(\alpha)}\sum\limits_{k=1}^{n-1} c_{n-k}^{\alpha-1}\left[\delta_x^2 U_{ij}^k + \delta_y^2 U_{ij}^k\right] + \frac{\tau^\alpha}{\Gamma(\alpha)}\sum\limits_{k=1}^{n-1} c_{n-k}^{\alpha-1} f_{ij}^{k-1}\left(U_{ij}^{k-1}\right), \\ U_{0j}^* = \left(1-\frac{\tau^\alpha}{\Gamma(\alpha)}\delta_y^2\right) U_{0j}^n,\ U_{M_x j}^* = \left(1-\frac{\tau^\alpha}{\Gamma(\alpha)}\delta_y^2\right) U_{M_x j}^n. \end{cases}$$

Once $\{U_{ij}^*\}$ is available, we alternate the spatial direction to solve the following system for fixed $i \in \{1, 2, \ldots, M_x - 1\}$,

$$\begin{cases} \left(1 - \frac{\tau^\alpha}{\Gamma(\alpha)}\delta_y^2\right) U_{ij}^n = U_{ij}^*,\ 1 \le j \le M_y - 1, \\ U_{i0}^n = \psi(x_i, y_0, t_n),\ U_{iM_y}^n = \psi(x_i, y_{M_y}, t_n). \end{cases}$$

4.2 Analysis of the ADI scheme

Now let us establish the convergence and stability results for (4.3). We first define a grid function space

$$V_0 = \{v_{ij} \mid 0 \le i \le M_x, 0 \le j \le M_y, \text{and } v_{ij} = 0, (x_i, y_j) \in \partial\Omega\}.$$

For any grid function $v, w \in V_0$, we can introduce the following inner product and norms,

$$\langle v, w\rangle = h_x h_y \sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} v_{ij} w_{ij}, \ \|v\| = \sqrt{\langle v, v\rangle},$$

$$\|\delta_x v\| = \sqrt{h_x h_y \sum_{i=1}^{M_x} \sum_{j=1}^{M_y-1} |\delta_x v_{i,j}|^2}, \ \|\delta_y v\| = \sqrt{h_x h_y \sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y} |\delta_y v_{i,j}|^2},$$

and

$$\|\delta_x \delta_y v\| = \sqrt{h_x h_y \sum_{i=1}^{M_x} \sum_{j=1}^{M_y} |\delta_x \delta_y v_{i,j}|^2}.$$

Next we can prove the following convergence result for the difference scheme (4.3).

Theorem 4.1 *Suppose $u(x, y, t) \in C^{4,4,2}_{x,y,t}(\overline{\Omega} \times [0, T])$, and let $u(x, y, t)$ be the exact solution and $\{U^n_{ij} | 0 \le i \le M_x, 0 \le j \le M_y, 1 \le n \le N\}$ be the solution of the difference scheme (4.3). Then, for $1 \le n \le N$, it holds that*

$$\|U^n - u^n\| \le C\left(\tau + h_x^2 + h_y^2\right).$$

Proof Subtracting (4.3) from (4.2), we have

$$e^n_{ij} + \frac{\tau^{2\alpha}}{\Gamma^2(\alpha)}\delta_x^2\delta_y^2 e^n_{ij} = e^{n-1}_{ij} + \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{k=1}^{n} c^{\alpha-1}_{n-k}\left(\delta_x^2+\delta_y^2\right)e^k_{ij}$$
$$+\frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{k=1}^{n} c^{\alpha-1}_{n-k}\left[f^{k-1}_{ij}\left(u^{k-1}_{ij}\right) - f^{k-1}_{ij}\left(U^{k-1}_{ij}\right)\right] + \left(r_2^{\alpha-1}\right)^n_{ij},$$

where $e^n_{ij} = u^n_{ij} - U^n_{ij}$, and $\left(r_2^{\alpha-1}\right)^n_{ij}$ is the local truncation error $\left(r_2^{\alpha-1}\right)^n_{ij} = O\left(\tau^2 + \tau h_x^2 + \tau h_y^2\right)$. Then, multiplying both sides of the above equation by $h_x h_y e^n_{ij}$

and summing over $1 \le i \le M_x - 1, 1 \le j \le M_y - 1$, we have the following result

$$\begin{aligned}\|e^n\|^2 + \frac{\tau^{2\alpha}}{\Gamma^2(\alpha)}\|\delta_x\delta_y e^n\|^2 &= \langle e^{n-1}, e^n\rangle + \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\left(\delta_x^2+\delta_y^2\right)\langle e^k, e^n\rangle \\ &\quad + \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\langle\left[f^{k-1}\left(u^{k-1}\right) - f^{k-1}\left(U^{k-1}\right)\right], \\ &\quad e^n\rangle + \langle\left(r_2^{\alpha-1}\right)^n, e^n\rangle.\end{aligned}$$

Omitting the non-negative term $\frac{\tau^{2\alpha}}{\Gamma^2(\alpha)}\|\delta_x\delta_y e^n\|^2$, and applying Cauchy-Schwarz inequality to the first term of the right-hand side of the above equality, we can obtain the inequality

$$\begin{aligned}\|e^n\|^2 &\le \frac{\|e^n\|^2 + \|e^{n-1}\|^2}{2} + \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\langle\left(\delta_x^2+\delta_y^2\right)e^k, e^n\rangle \\ &\quad + \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\langle\left[f^{k-1}\left(u^{k-1}\right) - f^{k-1}\left(U^{k-1}\right)\right], e^n\rangle + \langle\left(r_2^{\alpha-1}\right)^n, e^n\rangle.\end{aligned}$$

Summing over n from 1 to K yields

$$\begin{aligned}\frac{\|e^K\|^2 - \|e^0\|^2}{2} &\le \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{n=1}^{K}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\langle\left(\delta_x^2+\delta_y^2\right)e^k, e^n\rangle \\ &\quad + \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{n=1}^{K}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\langle\left[f^{k-1}\left(u^{k-1}\right) - f^{k-1}\left(U^{k-1}\right)\right], e^n\rangle \\ &\quad + \sum_{n=1}^{K}\langle\left(r_2^{\alpha-1}\right)^n, e^n\rangle. \end{aligned} \tag{4.4}$$

Let us consider the first term of the right-hand side of the above inequality, namely

$$\begin{aligned}&\frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{n=1}^{K}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\langle(\delta_x^2+\delta_y^2)e^k, e^n\rangle \\ &= \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{n=1}^{K}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\langle\delta_x^2 e^k, e^n\rangle + \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{n=1}^{K}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\langle\delta_y^2 e^k, e^n\rangle \\ &= -\frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{n=1}^{K}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\langle\delta_x e^k, \delta_x e^n\rangle - \frac{\tau^\alpha}{\Gamma(\alpha)}\sum_{n=1}^{K}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\langle\delta_y e^k, \delta_y e^n\rangle.\end{aligned}$$

According to Lemma 2.3, the above term is negative. And after using the Lipschitz condition of f with respect to u, (4.4) becomes

$$\|e^K\|^2 \le \frac{2L\tau^\alpha}{\Gamma(\alpha)}\sum_{n=1}^{K}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\|e^{k-1}\|\|e^n\| + C\sum_{n=1}^{K}(\tau^2 + \tau h_x^2 + \tau h_y^2)\|e^n\|,$$

where L is the Lipschitz constant. Assume $\|e^M\| = \max_{0\le k\le N} \|e^k\|$, then it becomes

$$\|e^M\| \le \frac{2L\tau^\alpha}{\Gamma(\alpha)} \sum_{n=1}^{M}\sum_{k=1}^{n} c_{n-k}^{\alpha-1}\|e^{k-1}\| + C(\tau + h_x^2 + h_y^2).$$

According to the same technique as for dealing with (3.5), we can achieve

$$\|e^M\| \le C\left(\tau + h_x^2 + h_y^2\right),$$

and the proof is completed. □

Now it can be deduced that the difference scheme (4.3) is stable to the initial values ϕ and φ, and the inhomogeneous term f in the L_2-norm. The proof of the following stability is very similar to the proofs of Theorems 3.2 and 4.1, thus we omit it here.

Theorem 4.2 *Suppose the grid function $\{U_{ij}^n | 0 \le i \le M_x, 0 \le j \le M_y, 0 \le n \le N\}$ is the solution of difference scheme (4.3), and let $U_{ij}^n = 0$ when (x_i, y_j) is on $\partial\Omega$. Then, for $1 \le K \le N$, it holds that*

$$\|U^K\| \le C\left(\|\phi\| + \|\varphi\| + \max_{0\le k\le N} \|f^k\|\right). \tag{4.5}$$

5 Fast implementations of the proposed schemes

As is known that the main time-consuming computation in schemes (3.2) and (4.3) is to calculate the summations, which are generated from the numerical approximations by (2.2) and (2.3) for the Riemann-Liouville integral. In this section, we will extend the idea of fast evaluation for the Caputo derivative in [14] to the fast calculations of approximations (2.2) and (2.3) without losing accuracy such that the direct schemes (3.2) and (4.3) can be fast implemented. Obviously, we can split the Riemann-Liouville integral into the following two parts, namely

$$\begin{aligned}\frac{1}{\Gamma(\beta)}\int_0^{t_n}(t_n - t)^{\beta-1} f(t)dt &= \frac{1}{\Gamma(\beta)}\int_0^{t_{n-1}}(t_n - t)^{\beta-1} f(t)dt \\ &+ \frac{1}{\Gamma(\beta)}\int_{t_{n-1}}^{t_n}(t_n - t)^{\beta-1} f(t)dt.\end{aligned} \tag{5.1}$$

The second term in the right-hand side of (5.1) is the local part; we can apply the product left rectangular quadrature or the product right rectangular quadrature, the same idea of deducing (2.2) and (2.3), to numerically approximate it. The first term in the right-hand side of (5.1), usually called the history part, is deduced due to the nonlocal property of the Riemann-Liouville integral. Essentially, we only need to focus on the fast calculations of the history part for the fast implementations of schemes (3.2) and (4.3). According to Lemma 2.6, the SOE technique can be applied to give a high-accuracy approximation to the power function $(t_n - t)^{\beta-1}$ in the interval $[0, t_{n-1}]$, thus

$$\begin{aligned}\frac{1}{\Gamma(\beta)}\int_0^{t_{n-1}}(t_n-t)^{\beta-1}f(t)dt &\approx \frac{1}{\Gamma(\beta)}\int_0^{t_{n-1}}\sum_{i=1}^{N_{\exp}}\omega_i e^{-s_i(t_n-t)}f(t)dt\\ &= \frac{1}{\Gamma(\beta)}\sum_{i=1}^{N_{\exp}}\omega_i\int_0^{t_{n-1}}e^{-s_i(t_n-t)}f(t)dt\\ &\triangleq \frac{1}{\Gamma(\beta)}\sum_{i=1}^{N_{\exp}}\omega_i U_{hist,i}(t_n),\end{aligned} \tag{5.2}$$

where $U_{hist,i}(t_n) = \int_0^{t_{n-1}} e^{-s_i(t_n-t)}f(t)dt$. It is clear that $U_{hist,i}(t_n) = 0$ for $n = 1$. For $n \geq 2$, then the following recurrence relation, which only needs $O(1)$ computational cost to get $U_{hist,i}(t_n)$, holds

$$\begin{aligned}U_{hist,i}(t_n) &= e^{-s_i\tau}\int_0^{t_{n-2}}e^{-s_i(t_{n-1}-t)}f(t)dt + \int_{t_{n-2}}^{t_{n-1}}e^{-s_i(t_n-t)}f(t)dt\\ &= e^{-s_i\tau}U_{hist,i}(t_{n-1}) + \int_{t_{n-2}}^{t_{n-1}}e^{-s_i(t_n-t)}f(t)dt.\end{aligned} \tag{5.3}$$

For the integral term in the right-hand side of (5.3), one may use the product trapezoidal technique to compute it, namely

$$\int_{t_{n-2}}^{t_{n-1}}e^{-s_i(t_n-t)}f(t)dt \approx \frac{(e^{-s_i\tau}-e^{-2s_i\tau})}{s_i}\cdot\frac{(f(t_{n-2})+f(t_{n-1}))}{2}.$$

Based on the above derivations, the computational work for the fast calculation of the Riemann-Liouville integral is $O(N_{\exp})$ for a fixed n, while the direct calculations, such as (2.2) and Lubich's approximations ([24]), require $O(n)$ work. Thus, the fast implementation of scheme (3.2) by using the fast calculation to the Riemann-Liouville integral and the fast ADI scheme have a computational cost of $O(MNN_{\exp})$ and $O(M_xM_yNN_{\exp})$ respectively, while the direct scheme (3.2) and the direct ADI scheme have a computational cost of $O(MN^2)$ and $O(M_xM_yN^2)$ respectively. Obviously, compared to the direct schemes, the fast schemes have the significant advantage in computational efficiency when N is large.

Remark 5.1 The SOE technique can be applied to fast calculate the approximations for fractional derivatives and fractional integrals obtained by the idea of piecewise approximations. Lubich's six approximations in [24] cannot be fast evaluated by using the SOE technique because these approximations are based on some generating functions. Additionally, if we use the product trapezoidal approximation to discretize the Riemann-Liouville integral, however, the convergence and stability of the deduced scheme cannot be proved (see [3, 34, 36] for examples); the reason is that the positivity of the deduced convolution quadratic form cannot be achieved.

6 Numerical experiments

In this section, two numerical examples of nonlinear time fractional diffusion-wave equations with different dimensions are presented to verify the theoretical results and demonstrate the performance of our new schemes. One can observe that the fast schemes have overwhelming superiorities over the direct schemes on the computational cost. All of the computations are performed by using a MATLAB(R2012a) subroutine on a computer (Dell Optiplex 5040) with the Intel(R) Core(TM) i5-6500 CPU 3.20GHz and 4G RAM.

Example 6.1 We consider the following one-dimensional nonlinear time fractional diffusion-wave equation

$$\begin{aligned} {}_0^C D_t^{\alpha} u(x,t) &= \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t,u), \;\; 0 \le x \le \pi, \;\; 0 < t \le 1, \\ u(x,0) &= \sin(x), \;\; u_t(x,0) = 0, \;\; 0 \le x \le 1, \\ u(0,t) &= u(\pi,t) = 0, \;\; 0 < t \le 1. \end{aligned}$$

The exact solution of this problem is

$$u(x,t) = \left(1 + t^{\alpha+1}\right)\sin(x).$$

The function $f(x,t,u)$ in the right-hand side of the above equation is

$$f(x,t,u) = u^2 + \left(\Gamma(\alpha+2)t + 1 + t^{\alpha+1}\right)\sin(x) - \left[(1 + t^{\alpha+1})\sin(x)\right]^2.$$

In this one-dimensional example, we use the L_2-norm errors

$$e(\tau,h) = \sqrt{h\sum_{j=0}^{M} |u_i^N - U_i^N|^2},$$

where u_i^N and U_i^N are the exact solution and the numerical solution at grid point (x_i, t_N), respectively.

Table 1 For $h = \pi/3000$, the errors and CPU times (second) for different τ, and numerical convergence orders in time of the direct scheme (3.2) for Example 6.1

τ	$\alpha = 1.3$		$\alpha = 1.5$		$\alpha = 1.8$		CPU time
	Error	Order	Error	Order	Error	Order	Mean ± sd
1/5	3.811e−1		2.849e−1		1.807e−1		0.52 ± 0.01
1/10	1.957e−1	0.961	1.308e−1	1.123	7.445e−2	1.279	1.61 ± 0.05
1/20	9.517e−2	1.040	5.905e−2	1.147	3.143e−2	1.244	5.82 ± 0.07
1/40	4.523e−2	1.073	2.696e−2	1.131	1.403e−2	1.164	23.04 ± 0.02
1/80	2.139e−2	1.081	1.254e−2	1.104	6.541e−3	1.101	92.39 ± 0.52
1/160	1.014e−2	1.076	5.944e−3	1.077	3.140e−3	1.059	369.54 ± 1.19

Table 2 For $h = \pi/3000$, the errors and CPU times (second) for different τ, and numerical convergence orders in time of the fast scheme for Example 6.1

τ	$\alpha = 1.3$		$\alpha = 1.5$		$\alpha = 1.8$		CPU time
	Error	Order	Error	Order	Error	Order	Mean $\pm$ sd
1/5	2.796e−1		1.873e−1		1.160e−1		0.24 $\pm$ 0.01
1/10	1.295e−1	1.110	8.357e−2	1.164	5.402e−2	1.103	0.34 $\pm$ 0.02
1/20	6.175e−2	1.069	3.880e−2	1.107	2.618e−2	1.045	0.43 $\pm$ 0.01
1/40	3.027e−2	1.029	1.866e−2	1.056	1.296e−2	1.014	0.64 $\pm$ 0.01
1/80	1.436e−2	1.075	8.771e−3	1.089	6.106e−3	1.086	1.08 $\pm$ 0.01
1/160	7.226e−3	0.991	4.325e−3	1.020	2.992e−3	1.029	1.97 $\pm$ 0.02

Tables 1 and 2 are computed by the direct scheme (3.2) and the fast scheme for Example 6.1, respectively. Specifically, we take three different values of fractional index α, i.e., $\alpha = 1.3, 1.5, 1.8$, and set $h = \pi/3000$, a value small enough such that the spatial discretization errors are negligible as compared with the temporal errors, and choose different time step size to observe the CPU time and to obtain the numerical convergence orders in time. We can check that these numerical convergence orders of the direct scheme and of the fast scheme approach 1 and are thus consistent with our theoretical analysis. The average CPU time, expressed as the mean time (mean) $\pm$ the standard deviation (sd) for three different α, is listed in the last column. The results of CPU time demonstrate that the fast scheme has an overwhelming performance over the direct scheme, especially for large integer N.

On the other hand, we check the numerical convergence orders and CPU time in space of the direct scheme and the fast scheme in Tables 3 and 4, respectively. Specifically, we take sufficiently small $\tau = 0.00025$, and choose different α and spatial step size to obtain the numerical errors, convergence orders and CPU time in space. As expected, the spatial numerical convergence order is 2 for all scenarios. Furthermore, one can see that the CPU time of the fast scheme is extremely less than that's of the direct scheme due to the small τ or the large N.

Table 3 When τ=0.00025, the errors and CPU times (second) for different h, and numerical convergence orders in space of the direct scheme for Example 6.1

h	$\alpha = 1.3$		$\alpha = 1.5$		$\alpha = 1.8$		CPU time
	Error	Order	Error	Order	Error	Order	Mean $\pm$ sd
$\pi/5$	6.575e−2		4.782e−2		3.158e−2		85.62 $\pm$ 0.54
$\pi/10$	1.584e−2	2.053	1.165e−2	2.037	7.762e−3	2.025	85.66 $\pm$ 0.28
$\pi/20$	3.728e−3	2.087	2.790e−3	2.062	1.864e−3	2.058	87.09 $\pm$ 0.41
$\pi/40$	9.378e−4	1.991	6.837e−4	2.029	4.611e−4	2.015	90.90 $\pm$ 0.28

Table 4 When τ=0.00025, the errors and CPU times (second) for different h, and numerical convergence orders in space of the fast scheme for Example 6.1

h	$\alpha = 1.3$		$\alpha = 1.5$		$\alpha = 1.8$		CPU time
	Error	Order	Error	Order	Error	Order	Mean $\pm$ sd
$\pi/5$	6.732e−2		4.928e−2		3.286e−2		0.35 $\pm$ 0.01
$\pi/10$	1.735e−2	1.956	1.238e−2	1.993	8.123e−3	2.016	0.38 $\pm$ 0.01
$\pi/20$	4.121e−3	2.074	3.008e−3	2.041	1.922e−3	2.080	0.43 $\pm$ 0.02
$\pi/40$	1.017e−3	2.019	7.589e−4	1.987	4.897e−4	1.972	0.53 $\pm$ 0.01

Example 6.2 Let us consider the following two-dimensional nonlinear time fractional diffusion-wave equation with a prescribed exact solution $u(x, y, t) = \left(t + t^{\alpha+1}\right)\sin(x)\sin(y)$, and $\Omega = (0, \pi) \times (0, \pi)$,

$$ {}_0^lD_t^\alpha u(x, y, t) = \Delta u(x, y, t) + f(x, y, t, u), \ (x, y) \in \Omega, \ 0 < t \le 1. $$

The initial conditions are $u(x, y, 0) = 0$, $u_t(x, y, 0) = \sin(x)\sin(y)$, $(x, y) \in \overline{\Omega}$, and the boundary condition is $u(x, y, t) = 0$, $(x, y) \in \partial\Omega$. The right-hand side nonlinear driving term is constructed as

$$ \begin{aligned} f(x, y, t, u) = & \left[2\left(t + t^{\alpha+1}\right) + \Gamma(\alpha+2)t\right]\sin(x)\sin(y) \\ & + \sin\left[\left(t + t^{\alpha+1}\right)\sin(x)\sin(y)\right] - \sin(u). \end{aligned} $$

In this example, we use the same step size h in each spatial direction, i.e., $h_x = h_y = h$, and compute the L_2 norm errors of the numerical solution,

$$ e(\tau, h) = h\sqrt{\sum_{i=0}^{M_x}\sum_{j=0}^{M_y}|u_{ij}^N - U_{ij}^N|^2}, $$

Table 5 For $h = \pi/500$ and $\alpha = 1.5$, the errors and CPU times for different τ, and numerical convergence orders in time of the direct ADI scheme and the fast ADI scheme for Example 6.2

τ	Direct scheme			Fast scheme		
	Error	Order	CPU time	Error	Order	CPU time
1/5	5.889e−1		10.57	3.005e−1		12.84
1/10	2.856e−1	1.044	33.07	1.424e−1	1.078	28.91
1/20	1.393e−1	1.036	115.62	6.917e−2	1.041	58.90
1/40	6.839e−2	1.026	456.48	3.179e−2	1.122	117.73
1/80	3.378e−2	1.018	1754.13	1.561e−2	1.026	239.59

Table 6 For $\tau = 0.00025$ and $\alpha = 1.5$, the errors and CPU times for different τ, and numerical convergence orders in space of the direct ADI scheme and the fast ADI scheme for Example 6.2

h	Direct scheme			Fast scheme		
	Error	Order	CPU time	Error	Order	CPU time
$\pi/5$	3.224e−2		377.38	3.405e−2		2.17
$\pi/10$	7.103e−3	2.182	876.23	8.010e−3	2.088	5.17
$\pi/20$	1.717e−3	2.048	1872.12	1.942e−3	2.044	11.84
$\pi/40$	4.335e−4	1.986	3815.71	4.985e−4	1.962	29.35

where u_{ij}^N and U_{ij}^N represent the exact solution and the numerical solution at grid point (x_i, y_j, t_N), respectively.

For fixed $\alpha = 1.5$ (actually, the results for any $1 < \alpha < 2$ are similar), we have checked the temporal numerical convergence orders and spatial numerical convergence orders of the direct ADI scheme and the fast ADI scheme for Example 6.2 in Tables 5 and 6, respectively. We can observe that the direct ADI scheme and the fast ADI scheme have similar accuracies and numerical convergence orders, i.e., first-order accuracy in time and second-order accuracy in space. However, the fast ADI scheme has better computational performance than the direct ADI scheme.

7 Concluding remarks

In this paper, an efficient scheme and an ADI scheme are constructed for one-dimensional and two-dimensional nonlinear time fractional diffusion-wave equations based on their equivalent partial integro-differential equations, respectively. Fast implementations of these two proposed schemes are achieved by using the SOE approximation technique. The proposed direct schemes and fast schemes only require a weak smoothness assumption in time and can be proved to be unconditionally stable and convergent with first-order accuracy in time and second-order accuracy in space. Numerical experiments illustrate the theoretical results and show that the fast schemes have an overwhelming better computational performance over the direct schemes and are very suitable for long-time simulations. Furthermore, we can conclude that the idea of the proposed fast schemes can be extended to any numerical methods for fractional systems which are constructed based on the piecewise approximation to fractional derivatives or fractional integrals. Finally, it should be mentioned that the optimal convergence order of the numerical schemes based on the graded mesh (see [7, 20, 21, 31] for examples) cannot be achieved, when these schemes are used for solving multi-term fractional differential equations. Because the mesh parameter of the graded mesh depends on the fractional index. However, our scheme and its fast implement can be easily extended to solve the multi-term fractional ones.

Funding information This research is financially supported by the National Natural Science Foundation of China (Grant Nos. 11701502 and 11426141).

References

1. Abbaszadeh, M., Dehghan, M.: An improved meshless method for solving two-dimensional distributed order time-fractional diffusion-wave equation with error estimate. Numer. Algor. **75**, 173–211 (2017)
2. Agrawal, O.P.: Solution for a fractional diffusion-wave equation defined in a bounded domain. Nonlinear Dyn. **29**, 145–155 (2002)
3. Arshad, S., Huang, J.F., Khaliq, A.Q.M., Tang, Y.F.: Trapezoidal scheme for time-space fractional diffusion equation with Riesz derivative. J. Comput. Phys. **350**, 1–15 (2017)
4. Becker-Kern, P., Meerschaert, M.M., Scheffler, H.P.: Limit theorem for continuous-time random walks with two time scales. J. Appl. Probab. **41**, 455–466 (2004)
5. Bhrawya, A.H., Doha, E.H., Baleanu, D., Ezz-Eldien, S.S.: A spectral tau algorithm based on Jacobi operational matrix for numerical solution of time fractional diffusion-wave equations. J. Comput. Phys. **293**, 142–156 (2015)
6. Chen, A., Li, C.P.: Numerical solution of fractional diffusion-wave equation. Numer. Func. Anal. Opt. **37**(1), 19–39 (2016)
7. Chen, H., Stynes, M.: Error analysis of a second-order method on fitted meshes for a time-fractional diffusion problem. J. Sci. Comput. **79**, 624–647 (2019)
8. Chen, H.B., Xu, D., Peng, Y.L.: A second order BDF alternating direction implicit difference scheme for the two-dimensional fractional evolution equation. Appl. Math. Model. **41**, 54–67 (2017)
9. Dehghan, M., Abbaszadeh, M.: Spectral element technique for nonlinear fractional evolution equation, stability and convergence analysis. Appl. Numer. Math. **119**, 51–66 (2017)
10. Diethelm, K.: The analysis of fractional differential equations. Springer, Berlin (2010)
11. Dutt, A., Greengard, L., Rokhlin, V.: Spectral deferred correction methods for ordinary differential equations. BIT Numer. Anal. **40**(2), 241–266 (2000)
12. Fairweather, G., Yang, X.H., Xu, D., Zhang, H.X.: An ADI Crank-Nicolson orthogonal spline collocation method for the two-dimensional fractional diffusion-wave equation. J. Sci. Comput. **65**, 1217–1239 (2015)
13. Huang, J.F., Tang, Y.F., Vázquez, L., Yang, J.Y.: Two finite difference schemes for time fractional diffusion-wave equation. Numer. Algor. **64**, 707–720 (2013)
14. Jiang, S.D., Zhang, J.W., Zhang, Q., Zhang, Z.M.: Fast evaluation of the Caputo fractional derivative and its applications to fractional diffusion equations. Commun. Comput. Phys. **21**(3), 650–678 (2017)
15. Jin, B.T., Li, B.Y., Zhou, Z.: Numerical analysis of nonlinear subdiffusion equations. SIAM J. Numer. Anal. **56**(1), 1–23 (2018)
16. Koeller, R.C.: Application of fractional calculus to the theory of viscoelasticity. J. Appl. Mech. **51**, 229–307 (1984)
17. Li, C.P., Zhao, Z.G., Chen, Y.Q.: Numerical approximation of nonlinear fractional differential equations with subdiffusion and superdiffusion. Comput. Math. Appl. **62**, 855–875 (2011)
18. Li, C.P., Zeng, F.H.: Numerical methods for fractional calculus. Chapman and Hall/CRC, Boca Raton (2015)
19. Li, L.M., Xu, D.: Alternating direction implicit-Euler method for the two-dimensional fractional evolution equation. J. Comput. Phys. **236**, 157–168 (2013)
20. Liao, H.L., Li, D., Zhang, J.: Sharp error estimate of the nonuniform L1 formula for linear reaction-subdiffusion equations. SIAM J. Numer. Anal. **56**, 1112–1133 (2018)
21. Liao, H.L., Yan, Y.G., Zhang, J.W.: Unconditional convergence of a fast two-level linearized algorithm for semilinear subdiffusion equations. J. Sci. Comput. **80**, 1–25 (2019)
22. Lodge, A.S., Renardy, M., Nohel, J.A.: Viscoelasticity and rheology. Academic Press, New York (1985)
23. Lopze-Marcos, J.C.: A difference scheme for a nonlinear partial integrodifferential equation. SIAM J. Numer. Anal. **27**(1), 20–31 (1990)
24. Lubich, C.H.: Discretized fractional calculus. SIAM J. Math. Anal. **17**(3), 704–719 (1986)
25. Lubich, C.h., Sloan, I.H., Thomée, V.: Nonsmooth data error estimates for approximations of an evolution equation with a positive-type memory term. Math. Comput. **65**(213), 1–17 (1996)
26. Luchko, Y., Mainardi, F.: Some properties of the fundamental solution to the signalling problem for the fractional diffusion-wave equation. Cent. Eur. J. Phys. **11**, 666–675 (2013)
27. Lyu, P., Vong, S.: A high-order method with a temporal nonuniform mesh for a time-fractional Benjamin-Bona-Mahony equation. J. Sci. Comput. **80**, 1607–1628 (2019)

28. Mainardi, F.: The fundamental solutions for the fractional diffusion-wave equation. Appl. Math. Lett. **9**, 23–28 (1996)
29. Metzler, R., Nonnenmacher, T.F.: Space- and time-fractional diffusion and wave equations, fractional Fokker-Planck equations, and physical motivation. Chem. Phys. **284**, 67–90 (2002)
30. Mustapha, K., Schötzau, D.: Well-posedness of hp-version discontinuous Galerkin methods for fractional diffusion wave equations. IMA J. Numer. Anal. **34**(4), 1426–1446 (2014)
31. Stynes, M., O'Riordan, E., Gracia, J.L.: Error analysis of a finite difference method on graded meshes for a time-fractional diffusion equation. SIAM J. Numer. Anal. **55**, 1057–1079 (2017)
32. Sun, Z.Z., Wu, X.N.: A fully discrete difference scheme for a diffusion-wave system. Appl. Numer. Math. **56**, 193–209 (2006)
33. Sun, Z.Z.: The method of order reduction and its application to the numerical solutions of partial differential equations. Science Press, Beijing (2009)
34. Tang, T.: A finite difference scheme for partial integro-differential equations with a weakly singular kernel. Appl. Numer. Math. **11**, 309–319 (1993)
35. Wang, Y.M., Wang, T.: A compact locally one-dimensional method for fractional diffusion-wave equations. J. Appl. Math. Comput. **49**, 41–67 (2015)
36. Wang, Z.B., Vong, S.: A high-order ADI scheme for the two-dimensional time fractional diffusion-wave equation. Int. J. Comput. Math. **92**(5), 970–979 (2014)
37. Yan, Y.G., Sun, Z.Z., Zhang, J.W.: Fast evaluation of the Caputo fractional derivative and its applications to fractional diffusion equations: a second-order scheme. Commun. Comput. Phys. **22**(4), 1028–1048 (2017)
38. Yang, J.Y., Huang, J.F., Liang, D.M., Tang, Y.F.: Numerical solution of fractional diffusion-wave equation based on fractional multistep method. Appl. Math. Model. **38**(14), 3652–3661 (2014)
39. Yang, Z., Wang, J., Li, Y., Nie, Y.: Effective numerical treatment of sub-diffusion equation with non-smooth solution. Int. J. Comput. Math. **95**(6), 1394–1407 (2018)
40. Yuste, S.B., Quintana-Murillo, J.: A finite difference method with non-uniform timesteps for fractional diffusion equations. Comput. Phys. Commun. **183**(12), 2594–2600 (2012)
41. Zeng, F.H.: Second-order stable finite difference schemes for the time-fractional diffusion-wave equation. J. Sci. Comput. **65**, 411–430 (2015)
42. Zhang, Y.N., Sun, Z.Z., Zhao, X.: Compact alternating direction implicit scheme for the two-dimensional fractional diffusion-wave equation. SIAM J. Numer. Anal. **50**, 1535–1555 (2012)