

Math 217 Take Home Final, Due in my mailbox 4:30pm May 16, 2003. Name _____

1. Let B_1 be the unit disk in R^2 . Show that the following problem has a unique bounded solution of:

$$\begin{cases} \Delta u = 0 & \text{in } B_1 \\ u = 1 & \text{on } \partial B_1^+ \\ u = -1 & \text{on } \partial B_1^- \end{cases} .$$

Discuss $\lim u(x, y)$ along rays to $(\pm 1, 0)$.

2. Let B_1 be the unit ball in R^n . Prove that for any solution u of

$$\begin{cases} \Delta u = f & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1 \end{cases}$$

Then (a) $\|u\|_{L^\infty(B_1)} \leq \frac{1}{2n} \|f\|_{L^\infty(B_1)}$.
(b) $\|u\|_{L^2(B_1)} \leq C \|f\|_{L^2(B_1)}$,

for some constant C .

Identify the best constant C .

Moreover show that there is no constant C so that for any smooth function u with

$$\begin{cases} \Delta u = f & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1 \end{cases}$$

implies $|u(x)| \leq C|f(x)|$ for all x in B_1 .

3. Show that there is a unique radially symmetric solution of

$$\begin{cases} -\Delta u = 1 - \sin(u^2) & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1. \end{cases}$$

4. Let $u(x, t)$ be the weak solution of

$$\begin{cases} u_t - \Delta u = 0 & \text{in } B_1 \times (0, \infty), \\ u = f(x) & \text{on } B_1 \times \{0\} \\ u = 0 & \text{on } \partial B_1 \times (0, 1]. \end{cases}$$

Show that for some universal constant C ,

$$\int_{B_1 \times (0, 1]} u^2 + |\nabla u|^2 \leq C \int_{B_1} f^2$$

and that

$$|u(x, t)| \leq C\varphi_1(x)e^{-\lambda_1 t}\|f\|_{L^2(B_1)}$$

for $t \geq 1$. Here φ_1 is the first eigenfunction of $-\Delta$ with eigenvalue $\lambda_1 > 0$. Is the above inequality true for $t \geq 0$?

5. Let Ω be a bounded smooth domain in R^2 . Show that there are at least three solutions of

$$\begin{cases} -\Delta u = u^3 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

6. Show that for each smooth function $f(x)$, there is a solution of

$$\begin{cases} \operatorname{div}((1 + \chi_{B_1})\nabla u) = 0 & \text{in } B_2 \\ u = f(x) & \text{on } \partial B_2. \end{cases}$$

Here

$$\chi_{B_1}(x) = \begin{cases} 1 & \text{if } x \text{ is in } B_1 \\ 0 & \text{else.} \end{cases}$$

Find how ∇u jumps along ∂B_1 .