COMPACTNESS METHODS FOR CERTAIN DEGENERATE ELLIPTIC SYSTEMS

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1. Introduction

Recently, the author was able to show the $C^{1,\alpha}$ regularity for p-harmonic functions (See [W]) using a compactness method originated by E. De Giorgi and F. Almgren. Here we show that its modification yields the interior $C^{1,\alpha}$ regularity for p-systems.

Let us recall the definition of p-systems. We shall use standard notations. Let x be a point in $\mathbb{R}^n$, $|x|$ the norm of x, $|S|$ the norm of a matrix S and $|E|$ the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$. Let $B_r = \{x : |x| \leq r\}$. Let Ω be a bounded, smooth, open subset of $\mathbb{R}^n$. Let $g \in W^{1-\frac{1}{p},p}(\partial\Omega, \mathbb{R}^N)$, $1 < p < +\infty$, and let

$$I_p(v) = \int_\Omega |\nabla v|^p dx, \tag{1.1}$$

where v is a for vector valued function $\Omega \to \mathbb{R}^N$. Then the minimizer u of I_p, i.e.,

$$I_p(u) = \min_{v|_{\partial\Omega}=g} I_p(v) \tag{1.2}$$

is called a solution of the p-system. Clearly, we have in the sense of distributions

$$L_p(u^\alpha) =: \nabla \cdot (|\nabla u|^{p-2}\nabla u^\alpha) = 0, \quad \alpha = 1, 2, \ldots, N \tag{1.3}$$

for all the components of $u = (u^1, u^2, \cdots, u^N)$. Notice that (1.3) is a degenerate elliptic system.

$C^{1,\alpha}$ regularity for p-systems is well known. K. Uhlenbeck proved it for the case $p \geq 2$ (See [Uh]) and P. Tolksdorff proved it for $1 < p < \infty$ (See [T]). Our proof of the $C^{1,\alpha}$ regularity, for $1 < p < \infty$, is easier than theirs. It shows how the degeneracy and the nondegeneracy for p-systems are separated. Our exposition is motivated by the simplicity of the approach and a unified understanding of the regularity theory.

Our main theorem is the following.

Theorem 1.1. *Let u be a weak solution of (1.3). Suppose $\|\nabla u\|_{L^p} \leq 1$ in $B_2 \subset \Omega$. Then $\|u\|_{C^{1,\alpha}(B_1)} \leq C(n,p)$ for some $\alpha > 0$.*

The proof of Theorem 1.1 is based on two approximation lemmas, according to the size of the slope of u. If u has small slope in a set of positive measure, then u is smaller in a smaller ball. If u has a nondegenerate slope in a set of large measure, then u has a nondegenerate slope at 0. Actually, u will stay nondegenerate since the operator L_p is close to a smooth uniformly elliptic system.

2. Perturbation of Equations

As in [U] and [E], we will consider the regularized minimization problem. This will justify an equation for ∇u as in the following Smoothness Lemma. First we regularize the solution by minimizing the functional

$$I_p^\epsilon(v) = \int_\Omega (\epsilon + |\nabla v|^2)^{\frac{p}{2}} \, dx, \qquad 1 \geq \epsilon > 0 \tag{2.1}$$

over all functions $v \in W^{1,p}(\Omega)$ with $v = g$ on $\partial\Omega$. Let u_ϵ be the minimizer. Clearly u_ϵ is a weak solution of

$$L_p^\epsilon u = \nabla \cdot \left((\epsilon + |\nabla u_\epsilon|^2)^{\frac{p-2}{2}} \nabla u_\epsilon \right) = 0. \tag{2.2}$$

One of the advantages of (2.1) is that the functional is smooth.

It is standard that (2.2), with the boundary condition $u_\epsilon = g$ on $\partial\Omega$, has a unique solution u_ϵ for fixed $\epsilon > 0$, and $u_\epsilon \to u$ uniformly as $\epsilon \to 0$ (See Lemma 2.1). We will show uniform $C^{1,\alpha}$ estimates for u_ϵ, for small ϵ. Theorem 1.1 follows from this. For simplicity, we will drop the index ϵ on u_ϵ in the exposition.

Smoothness Lemma. *∇u_ϵ has weak derivatives in L^2_{loc}.*

This is standard for weak solutions. We invoke this from [Uh] on page 223, 227 and 228 where the proof is given by employing the device of quotients of finite differences of L. Nirenberg.

Lemma 2.1. *Let u be a solution of (2.2) in $B_1(0)$. There is a uniformly elliptic operator*

$$\sum_{1\le i,j\le n} \frac{\partial}{\partial x_j}(c_{ij}(x)\frac{\partial}{\partial x_i}\cdot),$$

such that $(\epsilon+|\nabla u|^2)^{\frac{p}{2}}$ is a subsolution. Moreover

$$\|\nabla u\|_{L^\infty(B_{\frac{1}{2}})} \le C(n,p)(\|\nabla u\|_{L^p(B_1)}+1) \tag{2.3}$$

where C is independent of ϵ for $\epsilon>0$.

Proof. Let

$$A_{ij}^{\alpha\beta} = \left(\epsilon+|\nabla u|^2\right)^{\frac{p}{2}-1}\left(\delta_{ij}\delta_{\alpha\beta}+(p-2)\frac{u_i^\alpha u_j^\beta}{\epsilon+|\nabla u|^2}\right) = \left(\epsilon+|\nabla u|^2\right)^{\frac{p}{2}-1}a_{ij}^{\alpha\beta}.$$

Clearly $a_{ij}^{\alpha\beta}$ is uniformly elliptic:

$$\lambda\xi_i^\alpha\xi_i^\alpha \le a_{ij}^{\alpha\beta}\xi_i^\alpha\xi_j^\beta \le \Lambda\xi_i^\alpha\xi_i^\alpha. \tag{2.4}$$

By the Smoothness Lemma, $(\epsilon+|\nabla u|^2)^{\frac{p}{2}-1}\nabla u$ has weak derivatives. Then differentiate (2.2) with respect to x_k, we have, in weak sense :

$$\partial_i\left(\left(\epsilon+|\nabla u|^2\right)^{\frac{p}{2}-1}\left(\delta_{ij}\delta_{\alpha\beta}+(p-2)\frac{u_i^\alpha u_j^\beta}{\epsilon+|\nabla u|^2}\right)u_{jk}^\beta\right)=0\ .$$

As in [Uh], we have, for $\nu=(\epsilon+|\nabla u|^2)^{\frac{p}{2}}$:

$$L\nu = (a_{ij}^{\alpha\alpha}\nu_j)_i = (pA_{ij}^{\alpha\beta}u_{kj}^\beta u_k^\alpha)_i = pA_{ij}^{\alpha\beta}u_{kj}^\alpha u_{ki}^\beta \ge 0\ . \tag{2.5}$$

By a maximum principle for subsolutions of uniformly elliptic operators [Theorem 8.17, GT], we obtain

$$\|\nabla u\|^p_{L^\infty(B_{\frac{1}{2}})} \le \|\nu\|_{L^\infty(B_{\frac{1}{2}})} \le C\|\nu\|_{L^{\epsilon_0}(B_{\frac{3}{4}})} \le C(\|\nabla u\|^p_{L^p(B_{\frac{3}{4}})}+1).$$

Remark: As pointed out by the referee, Lemma 2.1 follows from the paper by F. Duzaar and M. Fuchs, Asymptotic Analysis 5(1991) 129-144.

Lemma 2.2 [Approximation Lemma A]. *For any γ, there exists an $\eta(\gamma)>0$ such that, if*

$$\begin{cases} L_p^\epsilon u = 0 & \text{in } B_1 \\ |\nabla u|\le 1 \end{cases}$$

for some $0 < \epsilon \leq 1$ *and*

$$|\{x \in B_1 : |\nabla u(x)| \leq 1 - \eta\}| \leq \eta |B_1|,$$

then there is a vector valued harmonic function h*, i.e. whose components are harmonic functions, such that*

$$\|u - h\|_{L^\infty(B_{\frac{1}{2}})} \leq \gamma. \tag{2.6}$$

Proof. We prove it by contradiction. If Lemma 2.2 were false, then there would exist u_n and $\gamma_0 > 0$ such that

$$\begin{cases} L_p^{\epsilon_n} u_n = 0 \\ |\nabla u_n| \leq 1 \\ |D_n| \leq \dfrac{1}{2^n}|B_1| \quad \text{where} \quad D_n = \{x \in B_1 : |\nabla u_n(x)| \leq 1 - \dfrac{1}{2^n}\} \\ u_n(0) = 0, \end{cases}$$

but, we have $\|u_n - h\|_{L^\infty(B_{1/2})} \geq \gamma_0 > 0$ for each harmonic h. We may suppose that there is a Lipschitz v such that,

$$\epsilon_n \to \epsilon_0 \geq 0,$$

$$u_n \to v \in L^\infty(B_1)$$

$$\nabla u_n \to \nabla v \in L^\infty(B_1) \quad \text{ in the weak* topology}$$

$$|\nabla u_n| \to 1 \quad a.e.$$

To get a contradiction, we show that v is harmonic. Take $g \in C_0^\infty(B_1)$,

$$\int \nabla g \big(\epsilon_n + |\nabla u_n|^2\big)^{\frac{p-2}{2}} \nabla u_n = 0 \ .$$

Hence,

$$\int_{B_1 - \cup_{k\geq n} D_k} \nabla g \big(\epsilon_n + |\nabla u_n|^2\big)^{\frac{p-2}{2}} \nabla u_n + \int_{\cup_{k\geq n} D_k} \nabla g \big(\epsilon_n + |\nabla u_n|^2\big)^{\frac{p-2}{2}} \nabla u_n = 0.$$

Clearly the first term in the above identity tends to

$$(\epsilon_0 + 1)^{\frac{p-2}{2}} \int \nabla g \nabla v$$

and the second term goes to 0 since the integrand is bounded and the fact that $|\cup_{k\geq n} D_k| \leq \frac{1}{2^{n-1}}$. This shows that v is harmonic. $\square$

Now we study perturbations of solutions when they are nondegenerate.

Lemma 2.3 [Approximation Lemma B]. *For any $\tau > 0$, there exists a $\sigma(\tau) > 0$ such that, if u is a solution of the p-system with $|\nabla u| \leq 1$ in B_1 and $\int_{B_1} |\nabla u - \nabla L|^2 \leq \sigma(\tau)$ with some linear function L such that $|1 - |\nabla L|| \leq \frac{1}{4}$, then there is a solution v of a uniformly elliptic system with constant coefficients such that*

$$\int_{B_{\frac{1}{2}}} |u - L - v|^2 \leq \tau \int_{B_1} |\nabla u - \nabla L|^2, \tag{2.7}$$

where $\sigma(\tau)$ is uniform for p lying in compact subsets of $(1, +\infty)$ and for ϵ small.

Moreover,

$$\int_{B_{\frac{1}{2}}} |\nabla v|^2 \leq C \int_{B_1} |\nabla u - \nabla L|^2 \tag{2.8}$$

$$|D^2 v(x)|^2 \leq C \int_{B_1} |\nabla u - \nabla L|^2 \textit{ for } |x| \leq \frac{1}{4}. \tag{2.9}$$

Proof. (2.8) follows from (2.7) by the interior estimates for v and Poincaré's inequality

$$\int_{B_1} |u - L - \frac{1}{|B_1|} \int_{B_1} (u - L)|^2 \leq C \int_{B_1} |\nabla u - \nabla L|^2.$$

Similarly (2.9) follows from standard interior estimates for solutions of linear elliptic systems. We refer the reader to Theorem 3.3 on Page 89 in [G] for further details.

The rest of the proof is again a standard compactness argument. We have a couple of observations first. Let $G(S) = (\epsilon + |S|^2)^{\frac{p-2}{2}} S$. Then G is smooth away from 0. Hence, we have

$$|G(S) - G(\nabla L) - DG(\nabla L)(S - \nabla L)| \leq C|S - \nabla L|^2$$

for $\frac{1}{2} \leq |S| \leq 2$, where $DG(\nabla L)$ is the derivative of G at ∇L.

Therefore,

$$|G(\nabla u) - G(\nabla L) - DG(\nabla L)(\nabla u - \nabla L)| \leq C|\nabla u - \nabla L|^2 + C\chi_{\{x \in B_1 : |\nabla u| \leq \frac{1}{2}\}}.$$

Moreover, since $|\nabla L| \geq \frac{3}{4}$,

$$|\{x \in B_1 : |\nabla u| \leq \frac{1}{2}\}| \leq C \int_{B_1} |\nabla u - \nabla L|^2.$$

Clearly $u - L$ satisfies the equation

$$\nabla \cdot (DG(\nabla L)(\nabla u - \nabla L)) = -\nabla \left(G(\nabla u) - G(\nabla L) - DG(\nabla L)(\nabla u - \nabla L) \right).$$

Now, we estimate the error term of the above equation. We notice that

$$\int_{B_1} |G(\nabla u)-G(\nabla L)-DG(\nabla L)(\nabla u-\nabla L)| \leq \int_{B_1} |\nabla u-\nabla L|^2+C|\{x \in B_1 : |\nabla u| \leq \frac{1}{2}\}|.$$

The second term on the right hand side of the above inequality can be estimated by a constant multiple of the first term.

Now, let us prove the lemma by contradiction.

Suppose for the contrary that there is an $\epsilon_0 > 0$ such that: for any positive integer k, there are solutions of the p-system u_k such that $|\nabla u_k| \leq 1$ in B_1 and $\int_{B_1} |\nabla u_k - \nabla L_k|^2 \leq \frac{1}{k}$ and $|1 - |\nabla L_k|| \leq \frac{1}{4}$ and for any solution v of linear elliptic systems, we have

$$\int_{B_{\frac{1}{2}}} |u_k - L_k - v|^2 \geq \epsilon_0 \int_{B_1} |\nabla u_k - \nabla L_k|^2.$$

We may suppose $S_k =: \nabla L_k \to S_0 =: \nabla L_0$ such that $|1 - |\nabla L_0|| \leq \frac{1}{4}$. Let

$$w_k(x) = \frac{u_k - L_k}{(\int_{B_1} |\nabla u_k - S_k|^2)^{\frac{1}{2}}}.$$

Then

$$\int_{B_{\frac{1}{2}}} |w_k - v|^2 \geq \epsilon_0$$

for any v which is the solution of

$$\nabla \cdot (DG(S_0)\nabla v) = 0.$$

We also suppose $w_k \to w_0 \in W^{1,2}_{loc}(B_1)$ weakly in $W^{1,2}_{loc}(B_1)$ and $w_k \to w_0$ strongly in $L^2_{loc}(B_1)$.

For any smooth function $\varphi(x)$ with compact support in B_1, we have

$$\int_{B_1} \nabla\varphi DG(S_k)\nabla w_k = -\frac{\int_{B_1} \nabla\varphi \left(G(\nabla u_k) - G(S_k) - DG(S_k)(\nabla u_k - S_k)\right)}{(\int_{B_1} |\nabla u_k - S_k|^2)^{\frac{1}{2}}}.$$

Clearly the left hand side of the above indentity goes to

$$\int_{B_1} \nabla\varphi DG(S_0)\nabla w_0$$

and the right hand side of the above identity is dominated by $(\int |\nabla u_k - S_k|^2)^{1/2} \leq 1/\sqrt{k}$ and then goes to 0.

Therefore w_0 is a solution of a linear system, which is a contradiction.

3. The Separation of Degeneracy and Nondegeneracy

Lemma 3.1. *Let $G(S) = (\epsilon + |S|^2)^{\frac{p-2}{2}} S$ for $\epsilon \leq 1$. Suppose $\frac{3}{4} \leq |S_0| \leq \frac{5}{4}$. Then there are constants $C > 0$ and $D > 0$ such that for $|T| \leq \frac{5}{4}$ the following inequalities hold:*

$$|G(T) - G(S_0)| \leq C|T - S_0|, \tag{3.1}$$

$$D(T - S_0) \circ (G(T) - G(S_0)) \geq |T - S_0|^2 \tag{3.2}$$

Proof. We prove (3.2) only. The proof of (3.1) is easier. First of all, (3.2) is true for $|T - S_0| \leq \frac{1}{2}$. If $|T - S_0| \geq \frac{1}{2}$, we only need to prove $(T - S_0) \circ (G(T) - G(S_0))$ has a lower bound away from 0. This is trivial from the compactness of the set $\{(\epsilon, S) : 0 \leq \epsilon \leq 1, |L| \leq \frac{5}{4}, |L - S_0| \geq \frac{1}{2}\}$.

Now, we have an inequality of the Caccioppoli type.

Lemma 3.2. *Assume a linear function L with $|\nabla L| \leq C$ and $\eta > 0$ such that $|\{x \in B_1 : |\nabla u(x)| \leq 1 - \eta\}| \leq \eta |B_1|$ and $|\nabla u| \leq 1$ in B_1. Then for each $r \leq \frac{1}{2}$,*

$$\int_{B_r} |\nabla(u - L)|^2 \leq C\eta + \frac{C}{r^2} \int_{B_{2r}} |u - L|^2. \tag{3.3}$$

Proof. Let φ be a cut-off function in B_{2r} such that $\varphi = 1$ in B_r, $\varphi = 0$ near ∂B_{2r} and $|\varphi| \leq 1, |\nabla\varphi| \leq 8/r$. Then we have

$$\int_{B_{2r}} \nabla(\varphi^2(u - L)) \cdot (G(\nabla u) - G(\nabla L)) = 0.$$

Consequently, we have

$$\int_{B_{2r}} \varphi^2 \nabla(u - L) \cdot (G(\nabla u) - G(\nabla L)) = -2 \int_{B_{2r}} \varphi(u - L)\nabla\varphi \cdot (G(\nabla u) - G(\nabla L)).$$

By the boundedness of the gradient and Lemma 3.1, we have

$$\int_{B_{2r}} \varphi^2 |\nabla u - \nabla L|^2 \leq C\eta + \frac{1}{D} \int_{B_{2r}} \varphi^2 (\nabla u - \nabla L) \cdot (G(\nabla u) - G(\nabla L)).$$

Similarly for any $\epsilon > 0$, we have

$$|\int_{B_{2r}} \varphi(u-L)\nabla\varphi \cdot (G(\nabla u)-G(\nabla L))| \leq \int_{B_{2r}} \epsilon\varphi^2 |G(\nabla u)-G(\nabla L)|^2 + \frac{1}{4\epsilon}|\nabla\varphi|^2|u-L|^2.$$

and

$$\int_{B_{2r}} \epsilon\varphi^2 |G(\nabla u) - G(\nabla L)|^2 \leq \epsilon\eta C + \epsilon C \int_{B_{2r}} \varphi^2 |\nabla u - \nabla L|^2.$$

Combining the inequalities and taking $\epsilon = \frac{1}{4CD}$, the lemma follows.

Lemma 3.3. *Suppose $u(0) = 0$, $|\nabla u(x)| \le 1$ and $|\{x \in B_1 : |\nabla u(x)| \le 1 - \eta\}| \ge \eta |B_1|$. Then $|\nabla u(x)| \le 1 - \frac{\eta}{C}$ in $B_{\frac{1}{2}}$, where C is a universal constant for small ϵ.*

Proof. Since $(\epsilon+1)^{\frac{p}{2}} - (\epsilon + |\nabla u|^2)^{\frac{p}{2}} \ge 0$ is a supersolution by (2.5), we have (Theorem 8.18, [GT]),

$$\inf_{B_{\frac{1}{2}}} \left((\epsilon+1)^{\frac{p}{2}} - (\epsilon+|\nabla u|^2)^{\frac{p}{2}}\right) \ge \frac{1}{C} \int_{B_1} (1+\epsilon)^{\frac{p}{2}} - (\epsilon+|\nabla u|^2)^{\frac{p}{2}} \ge \frac{1}{C} \eta ((1+\epsilon)^{\frac{p}{2}} - (\epsilon + (1-\eta)^2)^{\frac{p}{2}}).$$

Lemma 3.4 [Concentration of Gradient]. *For any $0 < \delta \le 1$, there is an η such that, if $|\{x \in B_1 : |\nabla u(x)| \le 1 - \eta\}| \le \eta |B_1|$ and $|\nabla u| \le 1$ in B_1, then there is a linear function L such that*

$$\left(\frac{1}{|B_r|} \int_{B_r} |\nabla u - \nabla L|^2\right)^{\frac{1}{2}} \le \delta$$

and $||\nabla L| - 1| \le 2\delta$, where $r > 0$ is a constant depending on δ.

Proof. For any $\gamma > 0$, by Approximation Lemma A, there exists an $\eta(\gamma) > 0$ such that

$$\int_{B_{\frac{1}{2}}} |u(x) - v(x)|^2 \le \gamma,$$

for some v harmonic, provided $\eta \le \eta(\gamma)$. Consequently $\int_{B_{\frac{1}{2}}} |v(x)|^2$ is bounded. Hence by the interior estimates of harmonic functions, we obtain

$$\int_{B_{2r}} |v - L|^2 \le H r^{4+n} |B_1|, \quad \text{for} \quad r \le \frac{1}{8},$$

where $L = (\nabla v)(0)x + v(0)$ and constant H depends only on the interior $C^{1,1}$ estimates for harmonic functions. Hence

$$\int_{B_{2r}} |u - L|^2 \le 2\gamma + 2H r^{4+n} |B_1|.$$

Now applying Lemma 3.2, we obtain

$$\int_{B_r} |\nabla u - \nabla L|^2 \le C\eta + \frac{C}{r^2}(2\gamma + 2Hr^{4+n}|B_1|) \le \delta^2 r^2 |B_r|,$$

provided $Hr^2 \le \frac{\delta^2}{6}$, $\frac{2C\gamma}{r^2} \le \frac{1}{3}\delta^2 r^2$ and $C\eta \le \frac{1}{3}\delta^2 |B_r|$. As before, we have

$$|\{x \in B_r : |\nabla u - \nabla L| \ge \delta\}| \le \frac{\int_{B_r} |\nabla u - \nabla L|^2}{\delta^2} \le \frac{1}{2}|B_r|.$$

Therefore

$$|\{x \in B_r : |\nabla u - \nabla L| \leq \delta\}| \geq \frac{1}{2} r^n |B_1| \geq \frac{1}{2} r^n |B_1| \geq \eta,$$

provided $\eta \leq \frac{1}{2}|B_r|$. We clearly have $||\nabla L| - 1| \leq \delta + \eta \leq 2\delta$ from the geometry, if we take $\eta \leq \delta$.

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. The strategy is the following:

(a) If $|\nabla u(x)| \leq (1-\eta)\|\nabla u\|_{\infty, B_r}$ in a portion of B_r, then $|\nabla u(x)| \leq (1 - \frac{\eta}{C})\|\nabla u\|_{\infty, B_r}$ in $B_{\frac{r}{2}}$, i.e. u is more degenerate.

(b) If $|\nabla u(x)| \geq (1-\eta)\|\nabla u\|_{\infty, B_r}$ in a large portion of the ball B_r, then u is well approximated by a harmonic function. This good approximation makes the harmonic function nondegenerate. Hence u has a nondegenerate slope at 0.

(c) Once u is nondegenerate, it will stay nondegenerate. This is due to the fast geometric convergence of the approximations.

First, we have an inequality of the Caccioppoli type.

Lemma 4.1. *Suppose u is a solution of the p-system with $|\nabla u| \leq 1$ in B_1 and some linear function L_0 such that $|1 - |\nabla L_0|| \leq \frac{1}{4}$. Then*

$$\int_{B_{\frac{1}{2}}} |\nabla u - \nabla L_0|^2 \leq C \int_{B_1} |u - L_0 x|^2. \tag{4.1}$$

Proof. Let $\varphi(x)$ be a cut-off function such that $\varphi = 1$ in $B_{\frac{1}{2}}$ and $\varphi = 0$ outside $B_{\frac{3}{4}}$ and $|\nabla \varphi| \leq 8$.

Let $G(T)$ be as in Lemma 3.1. We have the following,

$$\nabla \cdot (G(\nabla u) - G(\nabla L_0)) = 0.$$

Taking test function $\varphi^2(u - L_0)$, we obtain

$$\int_{B_1} \varphi^2 (\nabla u - \nabla L_0) \circ (G(\nabla u) - G(\nabla L_0)) = -\int_{B_1} 2\varphi(u - L_0)\nabla\varphi(G(\nabla u) - G(\nabla L_0)).$$

Combining (3.1) and (3.2) with Hölder inequality, we obtain

$$\begin{aligned}
\int_{B_1} \varphi^2 |\nabla u - \nabla L_0|^2 \leq & D \int_{B_1} \varphi^2 (\nabla u - \nabla L_0) \circ (G(\nabla u) - G(\nabla L_0)) \\
\leq & 2D \sqrt{\int_{B_1} \varphi^2 |G(\nabla u) - G(\nabla L_0)|^2 \int_{B_1} |\nabla \varphi|^2 |u - L_0|^2} \\
\leq & 2CD \sqrt{\int_{B_1} \varphi^2 |\nabla u - \nabla L_0|^2 \int_{B_1} |u - L_0|^2}.
\end{aligned}$$

The inequality follows by cancelling $\sqrt{\int_{B_1} \varphi^2 |\nabla u - \nabla L_0|^2}$ from the above inequality.

Lemma 4.2. *Suppose u is a solution of (2.2) with $|\nabla u| \leq 1$ in B_1. Then there exists a δ (fixed now) such that if for some matrix S_0, $\int_{B_1} |\nabla u - S_0|^2 \leq \delta$ and $||S_0| - 1| \leq 2\delta$, where $r > 0$ is a constant depending on then u is $C^{1,\alpha}$ at 0 in the $W^{1,2}$ semi-norm, i.e. there exists a constant matrix S such that*

$$\frac{1}{r^n} \int_{B_r} |\nabla u - S|^2 \leq C\delta r^{2\alpha} \text{ for } r \leq 1, \tag{4.2}$$

where $|S|$ and C have a universal upper bound.

Proof. It suffices to prove the following. There exist $r_0 > 0$, depending only on p and n, and constant matrices S_k for $k = 0, 1, 2, ...$, such that

$$\frac{1}{r_0^{kn}|B_1|} \int_{B_{r_0^k}} |\nabla u - S_k|^2 \leq \delta r_0^{2k\alpha} \tag{4.3}$$

with conditions

$$|S_k - S_{k+1}| \leq C\sqrt{\delta} r_0^{k\alpha}. \tag{4.4}$$

We prove it by induction on k. $k = 0$ follows from the condition. Now suppose it is true for k. First we show that S_k stays nondegenerate, i.e. $|1 - |S_k|| \leq \frac{1}{4}$. To see this nondegeneracy, we have

$$|1 - |S_k|| \leq |1 - |S_1|| + (|S_2 - S_1| + \cdots + |S_k - S_{k-1}|) \leq 2\delta + C_1\sqrt{\delta}.$$

Now, consider the blow-up function

$$w(x) = \frac{u(r_0^k x)}{r_0^k}$$

which is a solution for the $p-$system again and satisfies $\int_{B_1} |\nabla w - S_k|^2 \leq \delta r_0^{k\alpha}$, $|\nabla w| \leq 1$ in B_1 with $||S_k| - 1| \leq C\sqrt{\delta} \leq \frac{1}{4}$. By Lemma 2.3, we have that there is a solution of a constant elliptic system v such that,

$$\int_{B_{\frac{1}{2}}} |v - S_k x - w|^2 \leq \tau \int_{B_1} |\nabla w - S|^2,$$

provided $\delta r_0^{k\alpha} \leq \delta \leq \sigma(\tau)$, where $\sigma(\tau)$ is the function in Lemma 2.3. By (2.9) and letting the linear function $L^* = Dv(0)x + v(0)$, we have

$$\frac{1}{|B_{2r}|} \int_{B_{2r}} |v - L^*|^2 \leq C_1 r^4 \int_{B_1} |\nabla w - S_k|^2 \leq C_1 r^4 \delta r_0^{2k\alpha}, \text{ for } r \leq \tfrac{1}{4}$$

with

$$|\nabla L^*| \le C_2(\int_{B_1} |\nabla w - S_k|^2)^{\frac{1}{2}} \le C\sqrt{\delta} r_0^{k\alpha}$$

and C_1, C_2 depending only on the interior estimates of uniformly elliptic systems. Then invoking Lemma 4.1 and letting $S^* = \nabla L^*$, we obtain

$$\begin{aligned}\frac{1}{|B_r|}\int_{B_r} |\nabla w - S_k - S^*|^2 \le & C\frac{r^{-2}}{|B_{2r}|}\int_{B_{2r}} |w(x) - S_k x - L^*(x)|^2 \\ \le & \frac{2Cr^{-2}}{|B_{2r}|}(\int_{B_{2r}} |v(x) - S_k x - w(x)|^2 + \int_{B_{2r}} |v - L^*|^2) \\ \le & \frac{2Cr^{-2}}{|B_{2r}|}(\tau\delta r_0^{2k\alpha} + C_1|B_{2r}|r^2\delta r_0^{2k\alpha}).\end{aligned}$$

Taking $0 < r = r_0 < \frac{1}{4}$ such that

$$2CC_1 r_0^{2(1-\alpha)} \le \frac{1}{2}$$

and then taking $\tau = \frac{r_0^{2(-1+\alpha)}|B_{2r_0}|}{4C}$, we have

$$\frac{1}{|B_{r_0}|}\int_{B_{r_0}} |\nabla w - S_k - S^*|^2 \le \delta r_0^{2(k+1)\alpha}.$$

(From now on, r_0 and τ are fixed. So are σ, δ and η.) Finally scaling back, we have

$$\frac{1}{|B_{r_0^{k+1}}|}\int_{B_{r_0^{k+1}}} |\nabla u - S_{k+1}|^2 \le \delta r_0^{2(k+1)\alpha},$$

where

$$S_{k+1} = S_k + S^*.$$

Clearly S_{k+1} satisfies the requirements. □

Lemma 4.3. *Suppose $u(0) = 0$, $|\nabla u(x)| \le 1$ in B_1 and*

$$|\{x \in B_{\frac{1}{2^i}} : |\nabla u(x)| \le \eta(1-\delta_0)^i\}| \ge \eta |B_{\frac{1}{2^i}}| \text{ for } i = 0, 1, \dots, k. \tag{4.5}$$

Then $|\nabla u(x)| \le (1-\delta_0)^i$ in $B_{\frac{1}{2^i}}$ for $i = 1, 2, \dots, k+1$, where $\delta_0 = \frac{\eta}{C}$ as in Lemma 3.3.

This follows immediately from Lemma 3.3.

Proof of Theorem 1.1. We need only to prove that u is $C^{1,\alpha}$ at 0 in the $W^{1,2}$ seminorm, i.e.,

$$\int_{B_r} |\nabla u - S|^2 \le Cr^{n+2\alpha}$$

for some constant matrix S and $0 < r \leq 1$. Then u is locally $C^{1,\alpha}$ in the classical sense following Theorem 1.3, page 72 in [G]. We start by assuming $u(0) = 0$ and $|\nabla u(x)| \leq 1$. Otherwise, one consider $\dfrac{u - u(0)}{\|\nabla u\|}$, which is also a solution of the p-system. Let k be the minimum such that (4.5) does not hold. If $k = \infty$, then $|\nabla u(x)| \leq (1-\delta_0)^k$ in $B_{\frac{1}{2^k}}(0)$ for any k. It is easy to see that $|u(x)| \leq \frac{2}{1-\delta_0}|x|^{1+\alpha_0}$ for $|x| \leq 1$, and $\alpha_0 = \frac{-\ln(1-\delta_0)}{\ln 2}$. If k is finite, then $|u(x)| \leq \frac{2}{1-\delta_0}|x|^{1+\alpha_0}$ for $1 \geq |x| \geq \frac{1}{2^k}$. However, we are now in the situation of Lemma 3.4 for the solution of the p-system

$$w(x) = \frac{2^k u}{(1-\delta_0)^k}(\frac{x}{2^k}).$$

By Lemma 4.2, there is a linear function $Ax + B$ such that

$$\int_{B_r} |\nabla w(x) - A|^2 \leq C r^{1+\alpha}.$$

Scale back, for $|x| \leq \frac{1}{2^k}$,

$$\int |\nabla u(x) - (1-\delta_0)^k A|^2 \leq C 2^{k\alpha_0}(1-\delta_0)^k r^{1+\alpha_0}.$$

Clearly the above inequality is also true for $1 \geq |x| \geq \frac{1}{2^k}$ due to the fact that $(1-\delta_0)^k = 2^{-k\alpha_0} \leq |x|^{\alpha_0}$.

Final Remark. An extension of the results to a more general class of systems can be carried out using the standard perturbation technique. We refer readers to [M]. At the same time, it is still a challenge to prove $C^{1,\alpha}$ boundary regularity.

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