Mathematical Existence of Crystal Growth with Gibbs-Thomson Curvature Effects[1]

by Fred Almgren[2] and Lihe Wang[3]

ABSTRACT

This paper introduces and studies a mathematical evolution process which models one type of growth of a crystal as it freezes from a cold melt. The crystal freezes (melts) as rapidly as it can anywhere along its interface where the temperature is below (above) the local freezing temperature so that rate of growth is governed by the rate at which latent heat of fusion can diffuse. The model incorporates general Gibbs-Thomson relations between freezing temperatures and interface surface tension and general heat capacities and conductivities.

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1. Introduction and Main Theorem

We introduce and study a mathematical evolution process which models important and subtle features of one type of growth of a idealized crystal as it freezes from a cold melt.

Our main theorem asserts the mathematical existence of such an evolution in reasonable mathematical spaces. It is the first such theorem for mathematical models of crystal growth which allows: (i) general Gibbs-Thomson relations between interface surface tension and freezing temperatures, (ii) nonisotropic heat conductivities which differ in crystal and melt, (iii) heat capacities which differ in crystal and melt.

When the first author began studying models for crystal growth it was not clear to him that several common assumptions about way crystals may freeze were simultaneously mathematically compatible. It seemed a stringent condition to impose that crystals and temperature should interact in such a way that interface surface tension should exactly balance temperature. One implication of such a requirement is that a temperature maximum principle cannot hold for heat flow. Indeed, on the boundary of a small shrinking crystal with high curvatures, temperatures would have to be low and become even lower as the crystal melts. Similarly, on the boundary of a small shrinking hole in a crystal with large curvatures of the opposite sign temperatures would have to be high and become even higher as the hole freezes. The freezing temperature across a planar interface is what is given in handbooks; we call this temperature $T_\star$. Also it was not clear what such a condition should mean when the evolving crystal was making complicated geometric or topological changes–dendritic fingers, for example, can grow into each other. Finally, equations of heat flow near an evolving crystal are not standard. Indeed, the authors still know no general existence theorem for such equations other than the one proved in this paper.

The mathematical ingredients of the our model are the following:

§ a torus domain Ω; since boundary effects of crystal growth and heat flow are secondary to the main focus of our study it is easiest to work in a flat domain without boundary.

§ constant heat capacities for crystal c_s and melt c_ℓ with $c_s < c_\ell$. The inequality implies the the latent heat of fusion depends on temperature and that the heat equation one must solve is NOT of the form $\overset{\bullet}{T} = \operatorname{div}\bigl(\cdots\nabla T\bigr)$.

§ constant heat conductivities for crystal κ_s and melt κ_ℓ Conductivity in the solid is a positive definite symmetric matrix allowing for the fact that the direction of heat flow in some crystalline materials need not coincide with the direction of negative temperature gradient. In the liquid one would typically take the conductivity matrix to be a multiple of the identity (although this is not required).

§ a convex surface energy density function Φ for determining interface energy and hence interface surface tension. Φ is not required to be differentiable. When Φ is not differentiable, weighted mean curvature is not a local notion, and it is not clear to the authors whether or not the mathematical implications should be translated into Gibbs-Thomson language.

§ a prescribed decreasing Gibbs-Thomson function $H:\mathbb{R}^+ \to \mathbb{R}$ which relates interface temperature and interface surface tension

§ evolving crystals $[\![K_*(t)]\!]$; since we use constructions from geometric measure theory it is convenient to identify crystals as the integral currents $[\![K]\!]$ associated with sets of finite perimeter K in Ω.

§ evolving temperature fields $T_*(x,\,t)$ belonging to $\mathbb{L}_{2,\,loc}\bigl(\mathbb{R}^+,\,\mathbb{H}^1(\Omega)\bigr)$.

§ evolving heat distributions $Q_*(x,\,t)$; since heat is identified with internal energy we require that the integral of Q_* over Ω should not change with time.

With these ingredients we have

Main Theorem *Corresponding to a suitable initial crystal $[\![K_0]\!]$ and heat distribution Q there exist evolving crystals $[\![K_*(t)]\!]$, evolving heat distributions $Q_*(t)$, and associated evolving temperature fields $T_*(t)$ for which the following is true.*

(1) The heat distributions $Q_*(t)$ vary Hölder continuously with exponent $1/2$ in the Monge-Kantorovich metric with $Q_*(0) = Q_0$ [this is Conclusion (2.2) of Theorem 7.2].

(2) The union of the crystal sets $K_*(t)$ is $\mathcal{L}^{n+1}$ measurable [this is Conclusion (3) of Theorem 7.7].

(3) The crystals $[\![K_*(t)]\!]$, heat distributions $Q_*(t)$, and associated temperature fields $T_*(t)$ satisfy the heat equations

$$c_{K_*} T_* = Q_*, \quad \overset{\bullet}{Q}_* = \operatorname{div}\big(\kappa_{K_*} \nabla T_*\big)$$

in the sense of distributions [this is Conclusion (2) of Theorem 7.7].

(4) If the surface energy density Φ is smooth and elliptic, then, for almost every positive time t, the Gibbs-Thomson condition holds in the sense that

$$\frac{d}{ds}\Phi\Big(G_{s\sharp}\partial[\![K_*(t)]\!]\Big)\Big|_{s=0} = \int_{x\in K_*(t)} \operatorname{div}\big(H\big(T_*(x,\,t)\big)\,g_*(x)\big)\;d\mathcal{L}^n x$$

for each smooth initial velocity vector field $g_*\colon \Omega \to \mathbb{R}^n$ and associated deformation $G_s(x) = x + s\,g_*(x)$ [this is Theorem 8.7]. *Additionally, if $n = 2$ then $\partial K_*(t)$ will be a one dimensional Hölder continuously differentiable submanifold of Ω* [this is Conclusion (2) of Theorem 8.8]. *Alternatively, if $n = 3$ then $\partial K_*(t)$ will be the homeomorphic image in Ω of a compact two dimensional manifold without boundary.* [this is Conclusion (3) of Theorem 8.8].

Here

§ $\overset{\bullet}{Q}_*(t)$ is the time derivative of heat,

§ $\nabla T_*(t)$ is the space gradient of temperature,

§ $c_{K_*(t)}$ is the heat capacity function in Ω equal to c_s in $K_*(t)$ and to c_ℓ in $\Omega \sim K_*(t)$,

§ $\kappa_{K_*(t)}$ is the heat conductivity function equal to κ_s in $K_*(t)$ and to κ_ℓ in $\Omega \sim K_*(t)$,

In case $H(T_*(t))$ is $\mathcal{H}^{n-1} \llcorner \partial K_*(t)$ summable (which is not asserted in general), then the Gauss-Green theorem tells us

$$\frac{d}{ds}\Phi\Big(G_{s\sharp}\partial[\![K_*(t)]\!]\Big)\Big|_{s=0} = \int_{x\in\partial K_*(t)} \Big(H\big(T_*(x,\,t)\big)\,\mathbf{n}_{K_*(t)}(x)\Big) \bullet g_*(x)\,d\mathcal{L}^n x$$

in conclusion (4) above; here $\mathbf{n}_{K_*(t)}(x)$ is the unit exterior normal vector to $K_*(t)$ at x. This equation means that $H\big(T_*(x,\,t)\big)\,\mathbf{n}_{K_*(t)}(x)$ is the Φ weighted mean curvature of $\partial K(t)$ in the sense of first variation distributions [AW].

We give an example in Theorem 3.16 of initial conditions which lead to a discontinuity in crystal positions at positive time.

The evolution process of the theorem is the limit as $j \to \infty$ of approximating evolutions $[\![K_j]\!]$, T_j, Q_j associated with time steps Δt_j converging to zero. This approximating evolutions are obtained by alternating a heat flow with fixed crystal for a time Δt_j together with a crystal changing step which occurs with the clock stopped.

The heat flow process. During each time interval the crystal $K = K_j$ remains constant while temperatures fields T_j and heat distributions Q_j evolve according to the heat equations

$$c_K T_j = Q_j, \quad \overset{\bullet}{Q}_j = \operatorname{div}\big(\kappa_K\,\nabla T_j\big).$$

The existence of such heat flow is established in Theorem 5.1. In Theorem 5.2 we establish a regularity theorem for such heat flow in case the crystal boundary is a Hölder continuously differentiable hypersurface.

The crystal moving step. At the end of a Δt_j time interval, one chooses new a crystal position $[\![L]\!]$ and adjusted heat distribution Q by using general existence theorems from geometric measure theory to minimize the quantity

$$\Phi\big(\partial[\![L]\!]\big) + \int_\Omega c_L F(T)\,d\mathcal{L}^n + \frac{1}{\Delta t^\alpha}\|P - Q\|_*.$$

Existence and properties of the new crystals and temperature fields resulting from the crystal moving step are set forth in Chapter 4. In Theorem 4.5 we show in dimensions 2 and 3, that crystal interfaces are Hölder continuously differentiable hypersurfaces in case the surface energy function Φ is smooth and elliptic. Here

§ $\Phi\big(\partial[\![L]\!]\big)$ is the surface energy of the liquid solid interface.

§ $\int_\Omega c_L F(T)\,d\mathcal{L}^n$ is the bulk energy associated with the crystal $[\![L]\!]$ and the heat distribution Q. L together with Q determine the new temperature T. The bulk energy function $F{:}\mathbb{R}^+ \to \mathbb{R}^+$ is determined by the desired Gibbs-Thomson function H as above. F is convex and has a minimum at the planar freezing temperature $T_\star$.

§ α is a small positive number less than $1/24$. This bound on the size of α is determined by the requirement that the exponent $1/12 - 2\alpha$ of Δt in the conclusion inequality of Theorem 8.3 be a positive number.

§ P is the heat distribution which was present at the end of the heat evolution of the previous time interval.

§ $\|P - Q\|_*$ is the Monge-Kantorovich distance between the heat distributions P and Q; it equals the infimum of the numbers

$$\int_\Omega (Q - P)\varphi \, d\mathcal{L}^n$$

over all test functions φ defined on Ω which satisfy a Lipschitz condition with constant not exceeding one. This is the first time to the knowledge of the authors that a Monge-Kantorovich distance between heat distributions has been used in parabolic problems. Its introduction makes possible major simplifications in the technical details of our arguments.

It is necessary to have some way of moving heat before computing bulk energy in order for this variational scheme to work. Consider the case of water freezing. The bulk energy function $F(T)$ has its minimum at $T_\star$ which for water corresponds to 0°C. Suppose a water-ice interface were at say temperature T_1 corresponding to -0.1°C. If one could freeze a small region of water into ice and use the latent heat of fusion to warm nearly water or ice to nearly 0°C then the bulk energy integral would decrease. This is the main driving mechanism for freezing in our model. If the same heat (internal energy) remained in the region just frozen, the heat of fusion of 80 calories per gram would raise the temperature of the newly frozen ice to a much higher temperature T_2 with $F(T_2) >> F(T_1)$ so that there would be a large bulk energy disadvantage in freezing. Hence, there is a need to find a reasonable way of smearing out the heat of fusion before computing bulk energies. It must be done enough so that crystals do grow reasonably, but not so much that the cumulative effects of the smearing out means that the heat equations are not satisfied in the limit– each smearing out itself is a discontinuity in the heat flow process. Monge-Kantorovich methods seem well adapted to such control. With the Monge-Kantorovich term present in the quantity being minimized, it turns out that the Gibbs-Thomson relation holds exactly after each crystal moving.

There is an appealing conceptual generality as well as various practical advantages in the use of variational schemes to model crystal growth with Gibbs-Thomson effects. By formulating the problem in the geometric measure theory setting for static minimizations one can invoke existence and structure theorems of considerable power and generality and easily work with the complicated geometric configurations such as arise in dendritic structures. Also in this setting one does not have to make specific reference to principal or mean curvatures. This is of theoretical importance because there will be times when these are not classically defined. It is of practical importance in computations because calculation of curvature is notoriously difficult and unreliable.

Such a variational scheme was first suggested to the first author several years ago by R. Almgren when he discovered a conceptual flaw in one of the algorithms the first author had been considering to compute crystal growth. This led the first author within several months to an existence theorem for growth with general surface energies provided heat conductivities and capacities were uniform. Independently, and several months earlier, S. Luckhaus [LS] discovered a similar construction and obtained an existence theorem in case surface energy equaled surface area. These schemes together with that of the present paper are the only ones known to the authors which successfully model crystal growth with Gibbs-Thomson curvature effects. The assumption of uniform heat capacities substantially limited the applicability of the earlier results; the heat capacities of ice and water, for example, differ by a factor of two.

Some of the mathematical assumptions (like constant heat capacities and heat conductivities in both solid and liquid) we make in our model have physical relevance only in a limited range of temperatures about the usual freezing temperature $T_\star$ and are perhaps best thought of as a "linearization" of more complicated, more nonlinear relationships whose exact nature is generally unknown.

The pattern into which our crystal grows is often the result of a competition between a (generally destabilizing) bulk energy pressure to freeze cool liquid (or melt warm solid) and an (generally stabilizing) surface tension to reduce interface energy. Suppose, for example, a crystal were growing into cold melt with nearly uniform velocity along a nearly planar interface and a bump developed along the interface sticking into the cold melt. This bump would have a relative advantage diffusing away heat of fusion; it would also have increased surface tension with associated slightly depressed freezing temperature. If the diffusion advantage prevails then growth along the bump would accelerate perhaps creating a dendrite; this is one form of the "Mullens-Sekerka instability". If surface tension prevailed the bump would shrink. R. Almgren has written code to implement his variational algorithm for "crystal growth" in the plane. His computations and figures nicely illustrate the dependence of the dendritic patterns of growth on the conditions of growth [AR]. Note also the computations of A. Roosen [RA].

In our model, the plot of heat Q as a function of temperature T in the solid (resp. liquid) is a straight line having slope c_s (resp. c_ℓ) since heat capacity is the amount of heat required to raise temperature one unit. Also, in our model, at the freezing temperature $T_\star$. heat energy (internal energy) is greater in the liquid than in the solid and heat energy is liberated when liquid solidifies at this temperature. This means that our two straight line graphs intersect at some temperature less than $T_\star$. We choose our temperature scale so that this temperature is zero by definition. This is substantial mathematical convenience and is not supposed to be associated with absolute zero in the physical world. We do construct our bulk energy function F so that temperatures lower than zero will not occur.

Among the general mathematical models of crystal growth discussed in the overview [CHT] by J. Cahn, C. Handwerker, and J. Taylor is the formula

$$v = M(H + \mathrm{wmc})$$

in which v denotes the surface normal velocity, H is (some reasonable function of the) undercooling below the freezing temperature $T_\star$, wmc is the weighted mean curvature or surface tension of the interface, and M is the mobility of the interface measuring the response of the interface to the driving pressures of undercooling and surface tension, e.g. it make take time for a cold liquid to assemble its atoms into a crystal lattice. This equation also applies to a wide range of other interface motion problems, involving chemical- as well as temperature-dependent bulk driving forces. There seems little general theoretical or experimental knowledge about the form of the mobility function M (only its sign seems determined by classical thermodynamics [GM]). If one takes H identically equal to zero and M identically equal to one, then one obtains a heuristic equation for "motion by weighted mean curvature"; a variational scheme similar to the present one was used by the authors together with J. Taylor in the study of such motion [ATW]. In the present paper the mobility M is infinite; this is interpreted to mean that bulk and interface forces must be in equilibrium across the interface. It is in this sense that we say that freezing and melting occur instantaneously across along the interface. In calculations of crystal growth by various schemes one *de facto* has finite mobility. The present existence theorem suggests that calculations need not become wildly unstable as mobility becomes large.

During the preparation of this paper the first author was a visitor at the Centre for Mathematical Analysis of the The Australian National University and at the Aspen Center for Physics and was a visiting member of the Institute for Advanced Study, Princeton.

2. Ingredients of our model of crystal growth

In this chapter we give formal definitions to establish the mathematical setting in which we study crystal growth. We discuss first variations of surface energy and Gibbs-Thomson relations and set forth the time stepping procedure for constructing approximating evolutions.

2.1 The ambient space. Corresponding to a fixed (large) positive number ρ_* we denote by Ω the Cartesian product (as Riemannian manifolds) of n circles, each of circumference ρ_*. By identification of Ω with the quotient of n dimensional Euclidean space $\mathbb{R}^n$ by ρ_* times the integer lattice we obtain local coordinate functions $x_1, \ldots, x_n$, an inner product $v \bullet w$ between various tangent vectors v and w, orthonormal basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ for tangent spaces, and a distance $|p-q|$ between points p and q in Ω. The diameter of Ω is

$$\operatorname{diam}(\Omega) = \frac{1}{2} n^{\frac{1}{2}} \rho_*.$$

In the obvious way, we measure Ω by Lebesgue's n dimensional measure $\mathcal{L}^n$ and by Hausdorff's $n-1$ dimensional measure $\mathcal{H}^{n-1}$ so that the volume of Ω is

$$|\Omega| := \mathcal{L}^n(\Omega) = \rho_*^n.$$

One reason that we work in Ω rather than in $\mathbb{R}^n$ is to be able to treat questions of heat flow without the complications of boundary effects (equivalently, we are using periodic boundary conditions). Another reason is to avoid questions about the way in which crystallization should occur at the boundary.

When we are studying time evolutions we will use the space-time

$$\Omega \times \mathbb{R}^+$$

with coordinate functions $x_1, \ldots, x_n, t$. For functions T defined on Ω or on $\Omega \times \mathbb{R}^+$ the gradient notation ∇T always refers to (space) gradients computed with respect to coordinates $x_1, \ldots, x_n$ only (not t). For functions Q defined on $\Omega \times \mathbb{R}^+$ the dot notation $\overset{\bullet}{Q}$ always refers to t (time) derivatives.

We denote by $\mathcal{D}(\Omega)$ the real vector space of all infinitely differentiable functions $\varphi\colon \Omega \to \mathbb{R}$. Similarly, for $-\infty \le a < b \le \infty$ we denote by $\mathcal{D}(\Omega \times (a, b))$ the real vector space of all infinitely differentiable functions $\varphi\colon \Omega \times (a, b) \to \mathbb{R}$ having compact support.

2.2 Crystals. The crystals we consider are all integral currents $[\![K]\!]$ associated with $\mathcal{L}^n$ measurable subsets K of Ω having finite perimeter. Currents seem the natural setting for our study since, in particular, sets differing in measure zero are identified. Associated with each such K is its perimeter ∂K consisting those points p at which K has a (measure theoretic) unit exterior normal vector $\mathbf{n}_K(p)$. As currents we can write

$$\partial[\![K]\!] = \mathcal{H}^{n-1} \llcorner \partial K \wedge *\mathbf{n}_K.$$

We fix positive a number $0 < \gamma_\star < 1$ and denote by

$$\mathcal{K}$$

the collection of all $[\![K]\!]$'s for which the following two conditions hold:

2.2.1 One or the other inequalities

$$\mathbb{M}([\![K]\!]) = \mathcal{L}^n(K) \le \gamma_\star|\Omega| \quad \textbf{OR} \quad \mathbb{M}([\![K]\!]) = \mathcal{L}^n(K) \ge \gamma_\star|\Omega|$$

holds for all members of $\mathcal{K}$. See the discussion of initial conditions in Chapter 3. As noted there this assumption is used in Step 1 of Part 3 of the proof of Theorem 6.1 which theorem provides a basic estimate in showing convergence of our approximating evolutions given in 2.11 below.

2.2.2 $\mathbb{M}(\partial K) = \mathcal{H}^{n-1}(\partial K) \le B_\star$; here as a convenient choice we set

$$B_\star = \mathbb{E}_0 / \inf\big\{\Phi(v) \colon |v| = 1\big\}$$

as determined by our surface energy function Φ in 2.6 below and by our initial total energy $\mathbb{E}_0$ in 2.12 below.

By a **crystal** in Ω we mean any member of $\mathcal{K}$. If $[\![K]\!]$ is a crystal then the current $[\![\Omega \sim K]\!]$ is called the **melt of** K.

The metric we use on $\mathcal{K}$ is given by the mass norm (on currents) so that the distance between $[\![K]\!]$ and $[\![L]\!]$ in $\mathcal{K}$ equals

$$\mathbb{M}([\![K]\!] - [\![L]\!]) = \mathcal{L}^n\big(K \triangle L\big).$$

Here

$$K \triangle L = (K \sim L) \cup (L \sim K)$$

is the symmetric difference between K and L. The space $\mathcal{K}$ is compact in the mass topology.

2.3 Heat distributions. By a **heat (internal energy) distribution** in Ω we mean any (equivalence class of) nonnegative function(s) Q in $\mathbb{L}_2(\Omega)$ for which $1/Q$ also belongs to $\mathbb{L}_2(\Omega)$ and for which

$$\int_\Omega Q \, d\mathcal{L}^n = Q_\star|\Omega|.$$

Here $Q_\star$ is a fixed positive number specified in giving initial conditions in Section 3.2 below. We denote by

$$\mathcal{Q}$$

the affine subspace of $\mathbb{L}_2(\Omega)$ of heat distributions. One important metric on $\mathcal{Q}$ associates with heat distributions P and Q in $\mathcal{Q}$ the **Monge-Kantorovich distance**

$$\|P-Q\|_* = \sup\left\{\int_\Omega \varphi\bigl(P-Q\bigr)\,d\mathcal{L}^n : \mathrm{Lip}(\varphi) \le 1\right\}.$$

Here

$$\mathrm{Lip}(\varphi) = \sup\left\{\frac{|\varphi(x)-\varphi(y)|}{|x-y|} : x,\, y \in \Omega \text{ with } x \ne y\right\}$$

is the Lipschitz constant of φ. The review article [RS] by S. T. Rachev gives relations between our definition above and the "Monge optimization problem" and "Kantorovich's mass transference problem".

For any pair P and Q of heat distributions in $\mathcal{Q}$ we have

$$\|P-Q\|_* \le \mathrm{diam}(\Omega)\, Q_\star|\Omega| = 2^{-1} n^{\frac{1}{2}} \rho_* Q_\star |\Omega|.$$

For each positive number M the set

$$\mathcal{Q} \cap \bigl\{Q : \|Q\|_{\mathbf{L}_2} + \|1/Q\|_{\mathbf{L}_2} \le M\bigr\}$$

is compact in the $\|\cdot\|_*$ topology as noted in Appendix B.6. The use of this distance function in a parabolic problem is a novel feature of the present paper. It is also indispensable in our proofs.

2.4 Temperature fields and heat capacities. By a **temperature field** in Ω we mean any (equivalence class of) nonnegative function(s) in $\mathbb{L}_2(\Omega)$. It is an assumption in our model that there is a linear relationship between heat and temperature in both crystal determined by positive coefficients $0 < c_s < c_\ell < \infty$. Corresponding to each $\mathcal{L}^n$ measurable subset K of Ω (in particular to each crystal $[\![K]\!]$ in $\mathcal{K}$) we define

$$c_K : \Omega \to \{c_s,\, c_\ell\}, \quad c_K(x) = \begin{cases} c_s & \text{if } x \in K \\ c_\ell & \text{if } x \in \Omega \sim K \end{cases}.$$

In our model we always assume that the relationship

$$c_K T = Q$$

holds $\mathcal{L}^n$ almost everywhere between temperature T, heat Q, and crystal $[\![K]\!]$. The number c_s is thus the **heat capacity of the solid crystal** while c_ℓ is the **heat capacity of the liquid melt**.

If $c_K T = Q \in \mathcal{Q}$ as above with $T \in \mathbb{H}^1(\Omega)$ then Poincaré's inequality as in appendix B.7 guarantees

$$\int_\Omega T^2 \le \frac{Q_\star^2 |\Omega|^2}{\rho_*^n c_s^2} + \rho_*^2 \int |\nabla T|^2 \, d\mathcal{L}^n.$$

2.5 Heat conductivity and heat equations. We assume that the flow of heat (a "velocity vector field") in response to a temperature gradient (a vector field) is determined by positive definite symmetric conductivity matrices κ_s and κ_ℓ. We set

$$\kappa_0 = \sup \big\{ \|\kappa_s\|, \, \|\kappa_\ell\| \big\},$$

(mapping norms) and denote by κ_* the smaller of the two numbers

$$\inf \big\{ v \bullet \kappa_s(v) \colon v \in \mathbb{R}^n \text{ with } |v| = 1 \big\}, \quad \inf \big\{ v \bullet \kappa_\ell(v) \colon v \in \mathbb{R}^n \text{ with } |v| = 1 \big\}$$

and denote by κ^* the larger of the two numbers

$$\sup \big\{ v \bullet \kappa_s(v) \colon v \in \mathbb{R}^n \text{ with } |v| = 1 \big\}, \quad \sup \big\{ v \bullet \kappa_\ell(v) \colon v \in \mathbb{R}^n \text{ with } |v| = 1 \big\}.$$

Corresponding to each crystal $[\![K]\!]$ in $\mathcal{K}$ we define

$$\kappa_K \colon \Omega \to \{c_s, \, c_\ell\}, \quad \kappa_K(x) = \begin{cases} \kappa_s & \text{if } x \in K \\ \\ \kappa_\ell & \text{if } x \in \Omega \sim K. \end{cases}$$

We say that functions $Q(x, t)$ and $T(x, t)$ defined on $\Omega \times [a, b]$ are a **solution to the heat equation with fixed crystal** $[\![K]\!]$ and with initial heat distribution Q_a and final heat distribution Q_b if and only if the equations

$$c_K T = Q, \quad \overset{\bullet}{Q} = \operatorname{div}(\kappa_K \nabla T), \quad Q(\cdot, a) = Q_a, \quad Q(\cdot, b) = Q_b$$

hold in the following sense:

2.5.1 $[\![K]\!]$ is a crystal in $\mathcal{K}$.

2.5.2 The functions Q_a and Q_b and $Q(\cdot, t)$ belong to $\mathcal{Q}$ for each $a < t < b$.

2.5.3 The function Q belongs to $\mathbb{C}^0\big[(a, b), \mathbb{L}_2(\Omega)\big]$.

2.5.4 The function T belongs to $\mathbb{L}_2\big[(a, b), \mathbb{H}^1(\Omega)\big]$; this implies the existence of ∇T in $\mathbb{L}_2\big[(a, b), \mathbb{L}_2(\Omega, \mathbb{R}^n)\big]$.

For each $\varphi \in \mathcal{D}\big(\Omega \times (a,\, b)\big)$ the following two equalities hold

2.5.5 $$0 = \int_{\Omega\times(a,\,b)} \varphi \nabla T + \nabla\varphi T \, d\mathcal{L}^{n+1}$$

(this is actually implied by 2.5.4).

2.5.6 $$0 = \int_{\Omega\times(a,\,b)} \varphi\big(c_{_K} T - Q\big) \, d\mathcal{L}^{n+1}$$

For each $\varphi \in \mathcal{D}\big(\Omega \times \mathbb{R}\big)$ the following equality holds

2.5.7 $$0 = \int_{\Omega\times(a,\,b)} -\overset{\bullet}{\varphi} Q + \kappa_{_K} (\nabla\varphi) \bullet \nabla T \, d\mathcal{L}^{n+1} + \int_{\Omega} \varphi(\cdot,\, b)Q_b - \varphi(\cdot,\, a)Q_a \, d\mathcal{L}^n.$$

The existence, properties, and regularity of such solutions are discussed in Chapter 5.

Similarly, we say that crystals $[\![K(t)]\!]$ and functions $Q(x,\, t)$, and $T(x,\, t)$ defined for $a \le t \le b$ are a **solution to the heat equation with varying crystals** with initial heat distribution Q_a and final heat distribution Q_b if and only if the equations

$$c_{_K} T = Q, \quad \overset{\bullet}{Q} = \operatorname{div}\left(\kappa_{_K} \nabla T\right), \quad Q(\cdot,\, a) = Q_a, \quad Q(\cdot,\, b) = Q_b$$

hold in the following sense:

2.5.8 Each $[\![K(t)]\!]$ is a crystal and the set $\big\{(x,\, t)\colon a < t < b, x \in K(t)\big\}$ is $\mathcal{L}^{n+1}$ measurable.

2.5.9 The function Q belongs to $\mathbb{L}_{2,loc}\big[(a,\, b), \mathbb{L}_2(\Omega)\big]$.

2.5.10 The functions Q_a and Q_b and $Q(\cdot,\, t)$ belong to $\mathcal{Q}$ for each $a \le t \le b$, and the function $[a,\, b] \to \mathcal{Q}$ which maps t to $Q(\cdot,\, t)$ is continuous in the Monge-Kantorovich distance topology with $\|Q_a - Q(\cdot,\, a)\|_* = 0$ and $\|Q_b - Q(\cdot,\, b)\|_* = 0$

2.5.11 The function T belongs to $\mathbb{L}_2\big[(a,\, b), \mathbb{H}^1(\Omega)\big]$; this implies the existence of ∇T in $\mathbb{L}_2\big[(a,\, b), \mathbb{L}_2(\Omega, \mathbb{R}^n)\big]$.

For each $\varphi \in \mathcal{D}\big(\Omega \times (a,\, b)\big)$ the following two equalities hold:

2.5.12 $$0 = \int_{\Omega\times(a,\,b)} \varphi \nabla T + \nabla\varphi T \, d\mathcal{L}^{n+1}$$

(this is actually implied by 2.5.11).

2.5.13 $$0 = \int_{(x,\,t)\in\Omega\times(a,\,b)} \varphi(x,\, t)\big(c_{_{K(t)}}(x)\, T(x,t) - Q(x,\, t)\big) \, d\mathcal{L}^n x d\mathcal{L}^1 t)$$

For each $\varphi \in \mathcal{D}(\Omega \times \mathbb{R})$

2.5.14
$$\begin{aligned} 0 = \int_{(x,\,t)\in\Omega\times(a,\,b)} & - \overset{\bullet}{\varphi}(x,\,t) Q(x,\,t) \\ & + \kappa_{K(t)}(x)\,(\nabla\varphi(x,\,t)) \bullet \nabla T(x,\,t)\, d\mathcal{L}^n x\, d\mathcal{L}^1 t \\ & + \int_{x\in\Omega} \varphi(x,\,b)\, Q_b(x) - \varphi(x,\,a)\, Q_a(x)\, d\mathcal{L}^n x. \end{aligned}$$

The matrix κ_s is thus the **heat conductivity of the solid crystal** while κ_s is the **heat conductivity of the liquid melt**; since $\Omega \sim K$ is a liquid one usually assumes κ_ℓ is a positive multiple of the $n \times n$ identity matrix.

2.6 Surface energy integrands and their integrals. We denote by

$$\Phi \colon \mathbb{R}^n \to \mathbb{R}^+$$

a fixed **parametric surface energy integrand.** This means that Φ is continuous and positively homogeneous of degree one (i.e. $\Phi(\lambda v) = \lambda\Phi(v)$ for each v and each nonnegative Φ). We also assume that Φ is positive on $\mathbb{R}^n$ except at the origin and that Φ is convex.[1] We call Φ **smooth** provided the function Φ is three times Hölder continuously differentiable except at the origin. We call a smooth Φ **elliptic** provided Φ is uniformly convex in the sense that the restriction of Φ to any arc length parametrized line in $\mathbb{R}^n$ not containing the origin has strictly positive second derivatives.

Whenever $[\![K]\!]$ is a crystal in $\mathcal{K}$ we call the number

$$\Phi\big(\partial[\![K]\!]\big) := \int_{x\in\partial K} \Phi\left(\mathbf{n}_K(x)\right)\, d\mathcal{H}^{n-1}x$$

the **integral of Φ over $\partial[\![K]\!]$**.

2.7 First variation vector fields for Φ and Φ weighted mean curvature. Associated with each smooth initial velocity vector field $g \colon \Omega \to \mathbb{R}^n$ is the one parameter of diffeomorphisms

$$G_t \colon \Omega \to \Omega, \quad G_t(x) = x + tg(x)$$

[1] This assumption of convexity is not necessarily true for the surface energy functions of physical crystalline materials. When energies are not convex it is customary to replace the actual energy by the smallest convex energy which is not less that the actual energy and then understand surfaces to have tangent planes in the sense of varifold geometry. The Wulff shape of the convexified energy is the same as that of the original energy.

defined for small times t. Associated with each crystal $[\![K]\!]$ in $\mathcal{K}$ and each such initial velocity vector field g is the **first variation**

$$\frac{d}{dt}\Phi\left(G_{t\sharp}\partial[\![K]\!]\right)\Big|_{t=0}$$

which always exists in case Φ is smooth. It may occur (and will often be the case in this paper) that there is a vector field $\mathbf{H}\colon \partial K \to \mathbb{R}^n$

$$\frac{d}{dt}\Phi\left(G_{t\sharp}\partial[\![K]\!]\right)\Big|_{t=0} = \int_{x\in\partial K} \mathbf{H}(x)\bullet g(x)\, d\mathcal{H}^{n-1}x$$

for every initial velocity vector field g. In that case, $\mathbf{H}$ is defined $\mathcal{H}^{n-1}$ almost uniquely, and one says that $\mathbf{H}$ is the Φ **first variation vector field of** $\partial[\![K]\!]$. When one can write $\mathbf{H} = wmc\,\mathbf{n}_{K}$, the function wmc is called the Φ **weighted mean curvature of** $\partial[\![K]\!]$.[2]

2.8 Gibbs-Thomson relations between curvature and temperature. When we say that a crystal $[\![K]\!]$ in $\mathcal{K}$ and a temperature field T satisfy the **Gibbs-Thomson relation** with Gibbs-Thomson function H we mean that the following is true.

2.8.1 $H\colon\mathbb{R}^+ \to \mathbb{R}$ is a smooth decreasing function,

2.8.2 The vector field $H\big(T(\cdot)\big)\,\mathbf{n}_{K}(\cdot)\colon \partial K \to \mathbb{R}^n$ is the Φ first variation vector field of $\partial[\![K]\!]$. In other words, a Gibbs-Thomson relation holds if and only if weighted mean curvature equals a prescribed function of temperature.

2.9 Gibbs-Thomson functions and bulk energy densities. The object of this paper is to use a variational scheme of minimizations of sums of bulk energies and surface energies in order to obtain solutions to the heat equation with varying crystals for which the Gibbs-Thomson relation between weighted mean curvature and temperature holds at almost every time (at least when Φ is smooth). We seek to do this for any suitable Gibbs-Thomson function H. The nonnegative **bulk energy function** $F(T)$ used in the minimizations is determined by the Gibbs-Thomson function H as a solution to the ordinary differential equation

2.9.1 $$H(T) = (c_{\ell} - c_{s})\Big(F(T) - TF'(T)\Big)$$

[2] In case $\Phi(v) = |v|$ surface energy coincides with surface area. In that case the weighted mean curvature of the unit sphere in $\mathbb{R}^3$ bounding the unit ball as crystal has weighted mean curvature equal to two everywhere. The mean curvature vector field in the sense of differential geometry consists of inward point normal vectors of unit length.

with initial condition

2.9.2 $$F(T_\star) = 0$$

for a positive temperature $T_\star$. In our model the **bulk free energy density of liquid** at temperature T is $c_\ell F(T)$ while the **bulk free energy density of solid** is $c_\ell F(T)$. In the present setup the bulk free energy densities have a unique minimum at temperature $T_\star$ and if the crystal has an interface near a point which is planar, then our sum of bulk and surface energies cannot be decreased by local changes in crystal position. For this reason we call $T_\star$ the **standard freezing temperature** of our crystalline substance Such an equilibrium condition corresponds to the physical definition of standard freezing temperature.

Figure 1 illustrates the relations between heat (internal) energy density and temperature assumed in this model. The slope of line QL is c_ℓ and the slope of line QS is c_s. The standard freezing temperature $T_\star$ is denoted T0 in Figures 1 and 2. The latent heat of fusion at the standard freezing temperature $T_\star$ is thus $(c_\ell - c_s)T_\star$.

Figure 2 illustrates bulk free energies. The curve EL is the graph of the function $c_\ell F(T)$ while the curve ES is the graph of the function $c_s F(T)$.

There are several conditions which we impose on F (correspondingly on H) for the purposes of our mathematical analysis as follows:

2.9.3 F is twice continuously differentiable and there is a positive number c_* such that

$$F''(T) \geq 2c_*$$

for each T, so that F is uniformly convex.

2.9.4 There are numbers $0 < a_* < T_\star < b_* < \infty$ such that

$$F(T) = \begin{cases} T^{-2} & \text{if } 0 < T \leq a_* \\ T^2 & \text{if } b_* \leq T < \infty. \end{cases}$$

These conditions on the behavior of F at extremes of temperature are primarily for mathematical convenience. Indeed, such temperatures are beyond the range at which one would normally wish to think of the model as a representative of physical reality. In order to have a mathematical theory, however, we have to do something; we might as well do something convenient. We actually have very few restrictions on the possible behavior of H's as is asserted by the following fact which is a consequence of a standard existence theorem for ordinary differential equations (see Appendix A.3) and a straight forward construction.

2.9.5 Fact. Suppose $H_0{:}\mathbb{R}^+ \to \mathbb{R}$ is any continuously differentiable decreasing function such that $H_0'(T)$ does not vanish and such that $H_0(T_\star) = 0$. Then, corresponding to each $0 < a < T_\star < b < \infty$, there exist $0 < a_* < a < T_\star < b < b_* < \infty$ and $c_* > 0$ and functions H and F as above (satisfying conditions 2.9.1, 2.9.2, 2.9.3, 2.9.4) such that $H(T) = H_0(T)$ for each $a \le T \le b$.

We henceforth assume that functions H and F satisfying conditions 2.9.1, 2.9.2, 2.9.3, 2.9.4 above are fixed. We also fix the positive constant C_* given in Appendix A.

2.10 Total energy and minimizers. Associated with each crystal $[\![K]\!]$ in $\mathcal{K}$ and each heat distribution Q is the **total energy**

$$\mathcal{E}\big([\![K]\!],\, Q\big) = \Phi\big(\partial[\![K]\!]\big) + \int_\Omega c_{_K} F\left(c_{_K}^{-1} Q\right)\, d\mathcal{L}^n = \Phi\big(\partial[\![K]\!]\big) + \int_\Omega c_{_K} F\,(T)\; d\mathcal{L}^n$$

which is a sum of surface energy and bulk energy; here $T = c_{_K}^{-1} Q$ is the temperature field associated with the crystal $[\![K]\!]$ and the heat distribution Q.

When we say that $[\![K]\!]$, Q **give an $\mathcal{E}$ minimizer for P and Δt** we mean that $[\![K]\!]$ is a crystal in $\mathcal{K}$, P and Q are heat distributions in $\mathcal{Q}$, Δt is a positive time interval, and

$$\mathcal{E}\big([\![K]\!],\, Q\big) + \frac{1}{\Delta t^\alpha}\|Q - P\|_* \le \mathcal{E}\big([\![L]\!],\, R\big) + \frac{1}{\Delta t^\alpha}\|R - P\|_*$$

whenever $[\![L]\!]$ is a crystal in $\mathcal{K}$ and R is a heat distribution in $\mathcal{Q}$. Here α is any fixed small positive number less than $1/24$.

2.11 Construction of approximating evolutions. Suppose that $[\![K_0]\!]$ is a initial crystal in $\mathcal{K}$ and Q_0 is an initial heat distribution satisfying the initial conditions set forth in Section 3.2. For each fixed positive integer j we set $\Delta t = 2^{-j}$ and choose

$$[\![K_j]\!]{:}\mathbb{R}^+ \to \mathcal{K}, \quad Q_j \in \mathbb{L}_{2,loc}\big[\mathbb{R}^+,\, \mathbb{L}_2(\Omega)\big], \quad T_j \in \mathbb{L}_{2,loc}\big[\mathbb{R}^+,\, \mathbb{H}^1(\Omega)\big]$$

by the following requirement and inductive procedure which alternates crystal changing steps with heat flow steps.

2.11.1 $[\![K_j(0)]\!] = [\![K_0]\!]$ and $Q_j(0) = Q_0$.

2.11.2 For each $k = 0, 1, 2, 3, \ldots$ and each $0 \le s < \Delta t$, $[\![K_j\big(k\Delta t + s\big)]\!] = [\![K_j\big(k\Delta t\big)]\!]$.

2.11.3 Suppose $k \in \{0, 1, 2, \ldots\}$ is fixed and that crystal position $K_j\big(k\Delta t\big)$, heat distribution $Q_j\big(x,\, k\Delta t\big)$, and temperature field $T_j\big(x,\, k\Delta t\big)$ have already been constructed. Assuming that functions $Q(x,\, t)$ and $T(x,\, t)$ defined on $\Omega \times [k\Delta t,\, (k+1)\Delta t]$ are a solution to the heat equation (see Theorem 5.1) with fixed crystal $K_j\big(k\Delta t\big)$ and with initial heat distribution $Q_j\big(k\Delta t\big)$. We set

$$Q_j\big(x,\, t\big) = Q\big(x,\, t\big) \quad \text{if } k\Delta t \le t < (k+1)\Delta t,$$

$$T_j\big(x,\, t\big) = T\big(x,\, t\big) \quad \text{if } k\Delta t \le t < (k+1)\Delta t,$$

$$P_j\big(x,\, (k+1)\Delta t\big) = Q\big(x,\, (k+1)\Delta t\big)$$

2.11.4 For each $k = 1,\, 2,\, 3,\, \ldots$ the pair $[\![K_j\big(k\Delta t\big)]\!]$, $Q_j\big(x,\, k\Delta t\big)$ give an $\mathcal{E}$ minimizer for $P_j\big(k\Delta t\big)$ and Δt. Also $T_j\big(\cdot,\, k\Delta t\big) = c^{-1}_{K_j(k\Delta t)} Q_j\big(\cdot,\, k\Delta t\big)$.

As additional notation we set

$$U_j(x,\, t) = \frac{1}{T_j(x,\, t)}$$

for each j, x, and t. The functions U_j are a convenience in controlling the behavior of our evolutions in regions where T_j is small.

We often write $T_j(t) = T_j(\cdot,\, t)$, $\nabla T_j(t) = \nabla T_j(\cdot,\, t)$, $U_j(t) = U_j(\cdot,\, t)$, $\nabla U_j(t) = \nabla U_j(\cdot,\, t)$.

The existence of $\mathcal{E}$ minimizers is established in Chapter 4 and the existence of heat flows is established in Chapter 5. In Chapter 6 we give conditions sufficient to guarantee that crystal positions and temperature fields be close. The main theorems in Chapter 7 assert that, for a subsequence of the j's, the crystals, heat distributions, and temperature fields will converge strongly to limits which constitute a solution to the heat equation with varying crystals. The existence of the Gibbs-Thomson relation is treated in Chapter 8.

2.12 The total energy functions. For each positive integer j we set

$$\mathbb{E}_j\colon \mathbb{R}^+ \to \mathbb{R}^+, \quad \mathbb{E}_j(t) = \mathcal{E}\big([\![K_j(t)]\!],\, Q_j(t)\big) \text{ for each } t.$$

Also, for convenience, we set

$$\mathbb{E}_j(t^-) = \lim_{s\uparrow t} \mathbb{E}_j(s)$$

for each j and each positive t. Since one always has the option of doing nothing at the crystal changing steps $\mathbb{E}_j$ will not increase as a result of crystal changing. We also show in Section 5.2 that $\mathbb{E}_j$ will not increase under any heat flow with fixed crystal. It follows that $\mathbb{E}_j$ is nonincreasing as a function of t. This provides an essential ingredient of our estimates. We set

$$\mathbb{E}_0 = \mathcal{E}\big([\![K_0]\!],\, Q_0\big) = \mathbb{E}_j(0)$$

for any j since this number is independent of j.

3. Initial conditions

In this chapter we discuss some of the phenomena which can arise in our mathematical model of crystal growth and state initial conditions which preclude wildly oscillatory solutions. We show in Theorem 3.15 that there are conditions under which, after a slight delay, a cold region will instantaneously crystallize. Such conditions are illustrated in Figure 3. In particular, crystal evolutions $[\![K_*(t)]\!]$ constructed in the present model need not be continuous.

3.1 Proposition (Some preliminary estimates about our model).
(1) Suppose $Q_\star$ is a prescribed average heat distribution, $0 \le \gamma \le 1$, and $[\![K]\!]$ is a crystal with $\mathcal{L}^n(K) = \gamma\,|\Omega|$. Then the bulk energy

$$\int_\Omega c_{_K} F(T)\, d\mathcal{L}^n$$

is minimized if the temperature field is taken to be constant with value $T(\gamma)$ satisfying the condition

$$\big[\gamma\, c_{_s} + (1-\gamma) c_{_\ell}\big] T(\gamma) = Q_\star.$$

For this value of temperature, the total bulk energy equals

$$\big[\gamma\, c_{_s} + (1-\gamma) c_{_\ell}\big] F\big(T(\gamma)\big)\, |\Omega|.$$

(2) For each $0 \le \gamma \le 1$ there is a crystal $[\![K(\gamma)]\!]$ with $\mathcal{L}^n\big(K(\gamma)\big) = \gamma\,|\Omega|$ such that

$$\Phi\big(\partial[\![K(\gamma)]\!]\big) = \inf\left\{\Phi\big(\partial[\![L]\!]\big)\colon [\![L]\!] \text{ is a crystal with } \mathcal{L}^n(L) = \gamma\,|\Omega|\right\}.$$

For all sufficiently small positive γ, any $K(\gamma)$ will be a Wulff shape for Φ (Wulff shapes in $\mathbb{R}^n$ are defined in [ATW 5.1]*; Wulff shapes in our torus Ω have the obvious definition). Similarly, for all γ sufficiently close to 1, any $K(\gamma)$ will be a Wulff shape for the surface energy function $\Phi^*(v) = \Phi(-v)$.*

PROOF. The uniform convexity of F assumed in 2.9.3 implies that transference of a small amount of heat from a region of higher temperature to a region of lower temperature will always strictly decrease total bulk energy (see the calculations given in the proof of

Proposition A.2). We infer that only constant temperatures can (and indeed do) provide minimal bulk energy for fixed crystal volume. The value of the temperature $T(\gamma)$ in (1) is determined by the prescribed total heat content in accordance with our equation there, and the associated bulk energy in (1) is also determined by our equation there.

The existence of $[\![K(\gamma)]\!]$'s asserted in (2) follows from the compactness theorem for integral currents together with the lower semicontinuity of integrals of convex integrands. As is well known, in $\mathbb{R}^n$ Wulff shapes uniquely minimize surface energy for the volume enclosed. In order to verify our assertions about Wulff shapes, we first translate components of the support of $[\![K(\gamma)]\!]$ in Ω if necessary to make the support of $[\![K(\gamma)]\!]$ connected. We then use the constructions of Section 3.4 of [ATW] (as in the proof of conclusion (1) of Theorem 4.5 below) to verify the existence of a positive lower bound to density ratios

$$\frac{\mathcal{H}^{n-1}\big(\partial K \cap \mathbb{B}^n(p,\, r)\big)}{r^{n-1}}$$

associated with p's in the support of $\partial[\![K]\!]$ and small r's. We now infer from this lower density ratio bound that the diameter of the support of $\partial[\![K]\!]$ will be small if the $n-1$ area of ∂K is small and that this $n-1$ area will be small if the n volume of K is small. We now use the fact that the simply connected covering space of Ω is $\mathbb{R}^n$, etc. ■

3.2 Remarks.

(1) For preassigned heat capacities c_s and c_ℓ and preassigned average heat distribution $Q_\star$, the first formula in 3.1(1) shows $T(\gamma)$ to be an increasing function of γ, i.e. the more of Ω that is frozen, the warmer the the average temperature.

(2) For preassigned heat capacities c_s and c_ℓ, preassigned $0 \le \gamma \le 1$, and preassigned temperature $T(\gamma)$, the first formula in 3.1(1) prescribes the average temperature $Q_\star$. If one preassigns $T(1) = T_\star$, then the lowest energy state occurs only when Ω is completely frozen. Similarly, if one preassigns $T(0) = T_\star$ then the lowest energy state occurs only when Ω is completely melted.

(3) It is not difficult to prescribe F, c_s, c_ℓ, and $Q_\star$ so that

$$c_s F(T(1)) = c_\ell F(T(0)).$$

If Φ is then sufficiently large (when restricted to $\partial\mathbb{B}^n(0,\,1)$) then the total energy function

$$\gamma \mapsto \Phi\big(\partial[\![K(\gamma)]\!]\big) + \big[\gamma\, c_s + (1-\gamma)c_\ell\big]F\big(T(\gamma)\big)\,|\Omega|$$

is minimized only when $\gamma = 0$ or when $\gamma = 1$. Following the terminology of 2.10 we see that $[\![K(0)]\!]$ and $Q_\star$ give an $\mathcal{E}$ minimizer for $Q_\star$ and any value of Δt and also $[\![K(1)]\!]$ and

$Q_\star$ give an $\mathcal{E}$ minimizer for $Q_\star$ and any value of Δt. If one were to construct approximating evolutions by the procedure set forth in 2.11 with initial conditions $[\![K_0]\!] = [\![K(0)]\!]$ and $Q_0 = Q_\star$, then

$$[\![K_j(t)]\!] = \begin{cases} [\![K(0)]\!] \\ \\ [\![K(1)]\!] \end{cases}.$$

are both admissible choices for each j and each t.

3.3 Controls on our evolution process. There are two main controls which we have on our evolution process. The first of these is the *a priori* Hölder estimates we obtain on our heat flow in the Monge-Kantorovich distance; the essential estimate is that set forth in Theorem 5.3, and this estimate depends only on our bound $\mathbb{E}_0$ for our initial total energy (2.12). The second type of control we have control we have is one which says (with the exception noted below) that if crystal positions K and L are substantially different while heat distributions P and Q are close, then one of the associated temperature fields $c_K^{-1}P$ or $c_L^{-1}Q$ or the associated reciprocal temperature fields $c_K P^{-1}$ or $c_L Q^{-1}$ must have large space Dirichlet's integral; this estimate is essentially set forth in Theorem 6.1, Corollary 6.2, and Proposition 6.3. The reason that Dirichlet's integrals are controlled is that the rate of decrease of our Lyapunov function $\mathbb{E}_j(t)$ is bounded from below by the time integral of these space Dirichlet's integrals; this is established as Conclusion (2) of Theorem 5.2.

The exceptional case when different K and L do not necessarily cause large Dirichlet's integrals of temperature or reciprocal temperatures is when K is very small or when K is nearly equal to Ω and L is nearly equal to $\Omega \sim K$. We therefore assume the following:

3.4 Main assumption about initial conditions. We assume that in Section 2.12 our initial crystal $[\![K_0]\!]$ and our initial heat distribution Q_0 are such that:

either the volume of every crystal which can occur in any approximating evolution is uniformly bounded away from zero

or the volume of every crystal which can occur in any approximating evolution is uniformly bounded away from $|\Omega|$.

3.5 Sufficient conditions that our main assumption about initial conditions hold. Two conditions which together imply our main assumption are the following (we use the terminology of 3.1 and 3.2):

(1) There is some number $0 < \gamma_0 < 1$ for which

$$\Phi\big(\partial[\![K(\gamma_0)]\!]\big) + \big[\gamma_0\, c_s + (1-\gamma_0)c_\ell\big]F\big(T(\gamma_0)\big)\,|\Omega| \\ < \sup\big\{c_\ell F\big(T(0)\big)\,|\Omega|,\ c_s F\big(T(1)\big)\,|\Omega|\big\}.$$

(2) Our initial crystal $[\![K_0]\!]$ and initial heat distribution Q_0 are chosen in 2.12 so that

$$\mathbb{E}_0 = \mathcal{E}\big([\![K_0]\!],\, Q_0\big) < \sup\big\{c_\ell F\big(T(0)\big)\,|\Omega|,\ c_s F\big(T(1)\big)\,|\Omega|\big\}.$$

Condition (1) says that it is possible in our model of crystal growth to satisfy condition (2) while condition (2) clearly implies that our main assumption about initial conditions does indeed hold. A typical situation in which conditions (1) and (2) would hold would be that in which (i) the ambient domain $|\Omega|$ is large; (ii) the initial crystal volume $\mathcal{L}^n(\mathbb{K}_0)$ is small; and (iii) the undercooling determined by Q_0 is large enough to permit substantial crystal growth, but is not large enough to permit freezing all of Ω.

We now begin constructing an example in which there is an instantaneous appearance of a large crystal at positive time from a pure liquid environment. This shows, in particular, that there can be no general theorem guaranteeing continuous crystal evolution under the conditions of the present model. First we give a formula for the heat kernel under special conditions and state well known facts about solutions to the heat equation. We do this to be able to justify claims about heat evolution we need in constructing our example.

3.6 Solving the heat equation in Ω with no crystal present. In accordance with our definitions in 2.5 we say that $Q(x,\,t)$ and $T(x,\,t)$ defined on $\Omega \times [0,\,t_0]$ are a solution to the heat equation with initial heat distribution Q_a (and empty crystal $[\![\emptyset]\!]$) provided

$$c_\ell T = Q, \quad \overset{\bullet}{Q} = \operatorname{div}\big(\kappa_\ell \nabla T\big), \quad \text{and} \quad Q(\cdot,\,0) = Q_0.$$

Suppose that our heat capacity c_ℓ equals one and our heat conductivity matrix κ_ℓ equals the identity matrix. Our heat equation then becomes

$$\overset{\bullet}{T} = \Delta T \quad \text{with} \quad T(\cdot,\,0) = T_0 = Q_0. \tag{$*$}$$

If our domain were $\mathbb{R}^n$ instead of Ω we could solve $(*)$ by setting

$$T(x,\,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\xi \in \mathbf{R}^n} T_0(\xi) \exp\left(-\frac{|x-\xi|^2}{4t}\right)\, d\mathcal{L}^n \xi$$

[RZ Problem 10, page 373]. If, additionally, $T_0(\xi+\rho_* z) = T_0(\xi)$ for each ξ and each $z \in \mathbb{Z}^n$ then $(*)$ would become

$$T(x,\,t) = \frac{1}{(4\pi t)^{n/2}} \sum_{z\in\mathbf{Z}^n} \int_{\xi\in[0,\,\rho_*]^n} T_0(\xi) \exp\left(-\frac{|x-(\rho_* z+\xi)|^2}{4t}\right) d\mathcal{L}^n\xi$$

We now fix a standard covering mapping $\Pi{:}\,\mathbb{R}^n \to \Omega$ such that $\Pi(x+\rho_* z) = \Pi(x)$ for each $z \in \mathbb{Z}^n$. We define a heat kernel $Y{:}\,\Omega\times\mathbb{R}^+\times\Omega$ by setting for each $(x,\,t,\,\xi) \in \Omega\times\mathbb{R}^+\times\Omega$,

$$Y(x,\,t,\,\xi) = \frac{1}{(4\pi t)^{n/2}} \sum_{z\in\mathbf{Z^n}} \exp\left(-\frac{|z-(\rho_* z+\xi_*)|^2}{4t}\right);$$

here x_*, ξ_* are any fixed points in $\mathbb{R}^n$ with $\Pi(x_*) = x$, $\Pi(\xi_*) = \xi$. In our original domain Ω we infer that the function

$$T(x,\,t) = \int_{\xi\in\Omega} T_0(\xi)\, Y(x,\,t,\,\xi)\, d\mathcal{L}^n\xi \tag{$**$}$$

solves the heat equation $(*)$ with initial temperature T_0.

We state some well known properties of solutions to our heat equation $(*)$ which follow from $(**)$ and which we will use in constructing our examples.

3.6.1 Maximum and minimum principles. If $-\infty < A \le T_0(x) \le B < \infty$ for each x in Ω, then $-\infty < A \le T(x,\,t) \le B < \infty$ for each x in Ω and each $t > 0$.

3.6.2 Maximum principle for Lipschitz constants. Suppose T_0 is smooth. Then, for each $t > 0$, $\mathrm{Lip}\big(T(\cdot,t)\big) \le \mathrm{Lip}\big(T_0\big)$, since partial derivatives of $T(x,\,t)$ are also solutions to the heat equation.

3.6.3 Uniform rate of dissipation. For each $\epsilon > 0$ and $\tau > 0$ there is $\delta > 0$ such that if

$$\int_\Omega |T_0|\, d\mathcal{L}^n < \delta,$$

then $|T(x,\,t)| < \epsilon$ for each $x \in \Omega$ and each $t > \tau$.

3.7 Proposition (An isoperimetric-intersection estimate). *Corresponding to each $0 < \lambda < 1$ there is $\epsilon > 0$ with the following property. Suppose R and S are positive numbers, $K_0 = \mathbb{B}^n(0,\,R)$, and K is any subset of $\mathbb{R}^n$ of finite perimeter such that*

(a) $\mathcal{L}^n\big(K \triangle K_0\big) < \epsilon R^n$,

(b) $\mathcal{L}^n\big(K\big) = \alpha(n)S^n$ *and* $\mathcal{H}^{n-1}\big(\partial K\big) - n\alpha(n)S^{n-1} < \epsilon S^{n-1}$.

Then, for each $0 < \delta < \epsilon$,

$$\frac{\mathcal{L}^n\Big(K \cap K_0 \cap \{x\colon |x_n| < \delta R\}\Big)}{\mathcal{L}^n\Big(K_0 \cap \{x\colon |x_n| < \delta R\}\Big)} > \lambda.$$

PROOF. It clearly suffices to consider the case $R = 1$. The isoperimetric inequality insures us that in case $\mathcal{H}^{n-1}\big(\partial K\big) = n\alpha(n)S^{n-1}$ then K will be a ball of radius S. We use the terminology of Appendix C.1 and infer from Theorem C.3 that if S is close to 1 and if $\mathcal{H}^{n-1}\big(\partial K\big)$ is only slightly greater than $n\alpha(n)S^{n-1}$ and if we write $\partial[\![K]\!] = \mathbf{t}\big(\partial K,\, 1,\, *\mathbf{n}_K\big)$ then the varifold $\mathbf{v}\big(\partial K,\, 1,\, *\mathbf{n}_K\big)$ is weakly close to the varifold $\mathbf{v}\big(\partial K_0,\, 1,\, *\mathbf{n}_{K_0}\big)$. We set $\xi(x_1,\, \ldots,\, x_n) = (x_1,\, \ldots,\, x_{n-1})$ and infer that for most $z = (x_1,\, \ldots,\, x_{n-1})$ with $r^2 = x_1^2 + \ldots + x_{n-1}^2 < 1$ the slice $\big\langle [\![K]\!],\, \xi,\, z\big\rangle$ equals $[\![(z, h),\, (z,\, -k)]\!]$ for some h and k each close to $\sqrt{1 - r^2}$ [FH 4.3.8, page 444]. For such z's, the set $\{z\} \times [\delta,\, \delta]$ almost surely almost lies in K. Our assertion follows from Fubini's theorem. ■

3.8 Proposition (Isoperimetric inequalities). *There is a positive number* S_0 *and an increasing function* $\delta\colon \mathbb{R}^+ \to \mathbb{R}^+$ *with the following property. Suppose* $0 < S \le S_0\rho_*$ *and* $[\![K]\!]$ *is a crystal in* $\mathcal{K}$ *with* $\mathcal{L}^n(K) = \alpha(n)S^n$. *Then*

(1) $\mathcal{H}^{n-1}(\partial K) \ge n\alpha(n)S^{n-1}$ *with equality if and only if there is* $p \in \Omega$ *such that* $[\![K]\!] = [\![\mathbb{B}^n(p,\, S)]\!]$.

(2) If $\epsilon > 0$ *and* $\mathcal{H}^{n-1}(\partial K) - n\alpha(n)S^{n-1} < \delta(\epsilon)\, S^{n-1}$, *then there is* $p \in \Omega$ *such that* $\mathcal{L}^n\big(K \triangle \mathbb{B}^n(p,\, S)\big) < \epsilon\, S^n$.

PROOF. The first assertion follows from the remarks about Wulff shapes in Proposition 3.1(2); see also [ATW 3.1.7]. The second assertion follows from the first assertion and compactness theorems for integral currents; see, for example, [ATW 3.1.5] ■

3.9 Assumptions and terminology. We specify certain temperatures, bulk energy changes, a radius, and ρ_*. We have in mind regions "A, B, C" (to be specified below) containing liquid at temperatures τ_A, τ_B, τ_C which change to temperatures τ_{AA}, τ_{BB}, τ_{CC} upon crystallization with changes ΔF_A, ΔF_B, ΔF_C in bulk energy densities.

(1) We assume that our liquid heat capacity c_ℓ equals 1 and that our solid heat capacity c_s is a fixed number less than one.

(2) We assume that our liquid heat conductivity κ_{ℓ} is the identity matrix.

(3) We assume that our bulk energy density function F and our equilibrium freezing temperature $T_\star$ are fixed. In accordance with 2.9.2, $F(T_\star) = 0$. In our terminology below, $\tau_{BB} = \tau_C = T_\star$.

(4) We set $\tau_B = c_s T_\star / c_\ell = c_s T_\star < T_\star$ so that liquid at temperature τ_B will change to temperature $\tau_{BB} = T_\star$ on solidification. The negative number $\Delta F_B = c_s F(\tau_{BB}) - c_\ell F(\tau_B) = -F(\tau_B)$ is the change (decrease) in bulk energy density when liquid at temperature τ_B solidifies.

(4) We set $\tau_C = T_\star$ so that liquid at temperature τ_C will change to temperature $\tau_{CC} = c_\ell \tau_C / c_s = \tau_C / c_s$ on solidification. The positive number $\Delta F_C = c_s F(\tau_{CC}) - c_\ell F(T_C) = c_s F(\tau_{CC})$ is the change (increase) in bulk energy density when liquid at temperature τ_C solidifies.

(5) We denote by τ_A a temperature much greater than τ_{CC} and also greater than b_* (see 2.9.4) whose value will be specified later and set $\tau_{AA} = c_\ell \tau_A / c_s = \tau_A / c_s > \tau_A$ so that liquid at temperature τ_A will change to temperature τ_{AA} on solidification. The positive number

$$\Delta F_A = c_s F(\tau_{AA}) - c_\ell F(\tau_A) = c_s \tau_{AA}^2 - c_\ell \tau_A^2 = c_s \left(\frac{\tau_A}{c_s} \right)^2 - \tau_A^2 = \frac{1 - c_s}{c_s} \tau_A^2$$

is the change (increase) in bulk energy density when liquid at temperature τ_A solidifies.

(6) We fix our surface energy Φ to be surface area by requiring $\Phi(v) = |v|$ for each v.

(7) We fix a positive radius R_0 by the requirement that $\Delta F_B \, \alpha(n) \, R_0^n = n \, \alpha(n) \, R_0^{n-1}$. It follows that if $p \in \Omega$ and

$$Q_0 = \left(\tau_B \chi_{\mathbf{B}^n(p,\, R_0)} + \tau_C \chi_{\Omega \sim \mathbf{B}^n(p,\, R_0)} \right)$$

then, for $p \in \Omega$,

$$\mathcal{E}\big([\![\emptyset]\!],\, Q_0 \big) = \mathcal{E}\big([\![\mathbf{B}^n(p,\, R_0)]\!],\, Q_0 \big).$$

(8) We let S_0 have the meaning of Proposition 3.8 and pick $\rho_* = 2R_0/S_0$

The Monge-Kantorovich term in our minimization criterion used in constructing approximating evolutions adds a subtlety to our estimates. The following proposition helps control the influence of this term.

3.10 Proposition (An estimate relating bulk energy change and Monge-Kantorovich distance). *Suppose*

(a) $[\![K]\!]$ *is a crystal in* Ω*.*

(b) P *and* Q *are heat distributions in* $\mathcal{Q}$

(c) $R = P/c_{K}$ *is the temperature field associated with heat distribution* P *with crystal* $[\![K]\!]$ *present.*

(d) $S = Q/c_{K}$ *is the temperature field associated with heat distribution* Q *with crystal* $[\![K]\!]$ *present.*

(e) $T = P/c_{\ell} \leq R$ *is the temperature field associated with heat distribution* P *with no crystal present.*

(f) $\mathrm{Lip}(T) = \mathrm{Lip}(P)/c_\ell < \infty$.

(g) $0 < T_1 = \inf_\Omega P/c_\ell = \inf_\Omega T \leq \sup_\Omega R \leq \sup_\Omega P/c_s = T_2 < \infty$.

(h) $\Gamma(P) = \sup_{T_1 \leq U \leq T_2} F''(U) < \infty$.

Then the savings in total energy

$$\mathcal{E}\big([\![\emptyset]\!],\, P\big) - \mathcal{E}\big([\![K]\!],\, Q\big)$$

from initial empty crystal and heat distribution P *to crystal* $[\![K]\!]$ *and heat distribution* Q *does not exceed*

$$-\Phi\big(\partial[\![K]\!]\big) + \int_K c_\ell F(T) - c_s F(R)\, d\mathcal{L}^n + \Gamma(P)\,\mathrm{Lip}(T)\,\|P-Q\|_* + \Gamma(P)\,\big(T_2 - T_1\big)\int_K P\, d\mathcal{L}^n.$$

PROOF. Since F is convex, $F(S) \geq F(R) + F'(R)\,(S-R)$ so that

$$\begin{aligned} c_K F(R) - c_K F(S) &\leq c_K F'(R)\,(R-S) \\ &= F'(R)(P-Q) \\ &= F'(T)\,(P-Q) + \big(F'(R) - F'(T)\big)\,(P-Q) \\ &= F'(T)\,(P-Q) + F''(U)\,(R-T)\,(P-Q) \end{aligned}$$

for some $T \leq U \leq R$. Hence

$$\begin{aligned} \mathcal{E}\big([\![K]\!],\, P\big) - \mathcal{E}\big([\![K]\!],\, Q\big) &= \int_\Omega c_K F(R) - c_K F(S)\, d\mathcal{L}^n - \\ &\leq \int_\Omega \big(F' \circ T\big)\,(P-Q)\, d\mathcal{L}^n + \Gamma(P)\int_K (R-T)(P-Q)\, d\mathcal{L}^n \\ &\leq \mathrm{Lip}\big(F' \circ T\big)\|P-Q\|_* + \Gamma(P)\,\big(T_2 - T_1\big)\int_K P\, d\mathcal{L}^n. \end{aligned}$$

Hence

$$\begin{aligned}\mathcal{E}\big([\![\emptyset]\!],\,P\big)-\mathcal{E}\big([\![K]\!],\,Q\big)&=\mathcal{E}\big([\![\emptyset]\!],\,P\big)-\mathcal{E}\big([\![K]\!],\,P\big)+\mathcal{E}\big([\![K]\!],\,P\big)-\mathcal{E}\big([\![K]\!],\,Q\big)\\&\le-\Phi\big(\partial[\![K]\!]\big)+\int_\Omega c_\ell F(T)-c_K F(R)\,d\mathcal{L}^n\\&\quad+\Gamma(P)\,\mathrm{Lip}(T)\,\|P-Q\|_*+\Gamma(P)\,\big(T_2-T_1\big)\int_K P\,d\mathcal{L}^n.\end{aligned}$$

■

3.11 Assumptions and terminology.

(1) We assume that our initial crystal $[\![K_0]\!]$ is the zero current corresponding to the empty set. We denote by $T_0 = Q_0$ our initial temperature fields and heat distributions which we assume to be positive smooth functions.

(2) For each j we denote by $\Delta t_j = 2^{-j}$ the time step of the j-th approximating evolution.

(3) For each j and time t we denote by $[\![K_j(t)]\!]$, $Q_j(t)$, $T_j(t)$ the crystals, heat distributions, and temperature fields at time t in a j-th approximating evolution constructed as in 2.11 Additionally for each positive integer k,

$$P_j\big(k\,\Delta t_j\big)=\lim_{t\uparrow k\,\Delta t_j}Q_j(t);$$

compare 2.11.13

3.12 Corollary (The evolution when no crystal is present is pure heat flow). *There is a positive integer J with the following property. Suppose $j \ge J$ and there is a positive time t_0 such that $K_j(t) = \emptyset$ for $0 \le t \le t_0$.*

(1) If k is a positive integer such that $k\,\Delta t_j \le t_0$ then

$$Q_j\big(k\Delta t_j\big)=P_j\big(k\,\Delta t_j\big).$$

(2) The restriction of the function $Q_j(x,\,t) = T_j(x,\,t)$ to the time interval $[0,\,t_0]$ is a solution to the heat equation (with zero crystal) with initial heat distribution Q_0 and final heat distribution $Q_j(t_0)$.

PROOF. Conclusion (1) clearly implies Conclusion (2) since we are solving the ordinary heat equation in Ω. We now verify Conclusion (1). Suppose then $t_1 = k\Delta t_j$. Proposition 3.10 above (with $K = \emptyset$ there) shows that possible decreases in bulk energy from replacing $P_j\big(k\,\Delta t_j\big)$ by $Q_j\big(k\Delta t_j\big)$ are dominated by $\|Q_j\big(k\Delta t_j\big) - P_j\big(k\,\Delta t_j\big)\|_*$ provided one can bound temperatures above and below and also bound the Lipschitz constant of temperatures. Such bounds hold at time $t = 0$ by assumption 3.11(1). In accordance with 3.9(1)(2) together with 3.6.1 and 3.6.2, so long as no crystal is present, the bounds which hold for $Q_j\big((k-1)\,\Delta t_j\big)$ also hold for $P_j\big(k\,\Delta t_j\big)$. In view of the factor $1/\Delta t_j^\alpha$ in the $\mathcal{E}$ minimizing equation given in 2.10, it is always better to take $Q_j\big(k\Delta t_j\big)$ equal to $P_j\big(k\,\Delta t_j\big)$ when Δt_j is small, i.e. j is large. This gives the assertion of the Conclusion (2). ■

3.13 Corollary (A lower bound on the size of first crystallizations). *There are a positive integer J and a positive number η with the following property. Suppose $j \geq J$ and k is a positive integer such that $[\![K_j\big(\ell\,\Delta t_j\big)]\!] = [\![\emptyset]\!]$ for each $\ell < k$ while $[\![K_j\big(k\,\Delta t_j\big)]\!] \neq [\![\emptyset]\!]$. Then $\mathcal{L}^n(K_j) \geq \eta$.*

PROOF. From 2.10 and 2.11 we infer

$$\mathcal{E}\bigg([\![\emptyset]\!],\, P_j(k\,\Delta t_j)\bigg) - \mathcal{E}\bigg([\![K_j(k\,\Delta t_j)]\!],\, Q_j(k\,\Delta t_j)\bigg) - \Delta t_j^{-\alpha}\,\|P_j(k\,\Delta t_j) - Q_j(k\,\Delta t_j)\|_* \geq 0.$$

We infer from assumptions 3.9(1)(2) together with 3.6.1 and 3.6.2 and Corollary 3.12 that upper and lower bounds and Lipschitz bounds which hold for Q_0 also hold for $P_j\big(k\,\Delta t_j\big)$. If j is large then Δt_j will be small so that, in the terminology of Proposition 3.10,

$$\begin{aligned}&\Delta t_j^{-\alpha}\,\|P_j(k\,\Delta t_j) - Q_j(k\,\Delta t_j)\|_* \\ &\qquad > \Gamma\bigg(P_j(k\,\Delta t_j)\bigg)\,\mathrm{Lip}\bigg(P_j(k\,\Delta t_j)\bigg)\|P_j(k\,\Delta t_j) - Q_j(k\,\Delta t_j)\|_*.\end{aligned}$$

In view of Proposition 3.10 and the isoperimetric inequality of Proposition 3.8 (ρ_* was fixed in 3.9(8)), this can happen only if $\mathcal{L}^n\big(K_j(k\,\Delta t_j)\big)$ is large.

3.14 Corollary (A total energy estimate on first crystallizations). *Suppose $j(1) < j(2) < j(3) < \ldots$ and $k(1) < k(2) < k(3) < \ldots$ are positive integers and $t(i) = k(j(i))\,\Delta t_{j(i)}$ for each i such that $[\![K_{j(i)}\big(t\big)]\!] = [\![\emptyset]\!]$ if $t < t(i)$ while $[\![K_{j(i)}\big(t(i)\big)]\!] \neq [\![\emptyset]\!]$. Suppose also $\sup_i t(i) < \infty$. Then there is a subsequence $\ell(1), \ell(2), \ell(3), \ldots$ of $1, 2, 3, \ldots$ together with a time τ, a crystal $[\![K^0]\!]$ in $\mathcal{K}$, and a heat distribution Q^0 in $\mathcal{Q}$ with associated temperature field $T^0 = Q_0/c_\ell = Q^0$ such that*

(1) $\tau = \lim_{\ell\to\infty} t(i(\ell))$.

(2) $\lim_{\ell\to\infty} \mathcal{L}^n\big(K_{j(i(\ell))} \triangle K^0\big) = 0$.

(3) $T_{j(i(\ell))}(t(i(\ell))) \rightharpoondown T^0$ *and* $P_{j(i(\ell))}(t(i(\ell))) \rightharpoondown P^0$ *and* $Q_{j(i(\ell))}(t(i(\ell))) \rightharpoondown P^0$ *in* $\mathbb{L}_2(\Omega)$ *as* $\ell \to \infty$.

(4) If $T(x,\,t)$ *is the solution to the heat equation with initial temperature field* T_0 *given in section 3.6, then* $P^0 = T^0 = T(\cdot,\,\tau)$.

(5) $\mathcal{L}^n(K^0) > 0$.

(6) In terms of total energy, it is not disadvantageous to crystallize $[\![K^0]\!]$ *from pure liquid in the temperature field* $T(\cdot,\,\tau)$ *with no change in the underlying heat distribution, i.e.*

$$\mathcal{E}\big([\![\emptyset]\!],\, P^0\big) - \mathcal{E}\big([\![K^0]\!],\, P^0\big) \geq 0.$$

PROOF. Conclusion (1) is immediate. Conclusion (2) follows from compactness theorems for integral currents ([ATW 3.1.5] for example); we infer further from the lower semicontinuity of integrals of convex integrands under flat convergence that $\Phi\big(\partial[\![K]\!]\big) \leq \liminf_{\ell\to\infty} \Phi\big(\partial[\![K_{j(i(\ell))}(t(i(\ell))]\!]\big)$. Conclusion (3) follows from Appendix B.8. Conclusion (4) follows from Corollary 3.12. Conclusion (5) follows from Corollary 3.13. According to 2.10 and 2.11 we have for each i,

$$\mathcal{E}\Big([\![\emptyset]\!],\, P_{j(i)}(t(i))\Big) - \mathcal{E}\Big([\![K_{j(i)}(t(i))]\!],\, Q_{j(i)}(t(i))\Big) \geq \Delta t_{j(i)}^{-\alpha}\, \|P_{j(i)} - Q_{j(i)}\|_*.$$

We infer from Appendix B.4 that

$$\mathcal{E}\big([\![K^0]\!],\, Q^0\big) \leq \liminf_{\ell\to\infty} \mathcal{E}\big([\![K_{j(i(\ell))}]\!]\big).$$

We then infer from Appendix B.9 that $P^0 = Q^0$. This gives Conclusion (6). ■

3.15 Assumptions and terminology.

(1) $\Psi_1{:}\mathbb{R}^n \to \mathbb{R}^+$ is a radially symmetric smoothing function having support in $\mathbb{B}^n(0,\,1)$ and $\Psi_\epsilon(x) = \epsilon^{-n}\Psi(x/\epsilon)$ for each x and ϵ.

(2) p is a point in Ω and v is a unit vector in $\mathbb{R}^n$.

(4) For $R_0 \leq R < 2R_0$ and $0 < \delta < 1/2$,

$$B(R) = \mathbb{B}^n(p,\,R)$$

is a closed ball,

$$C(R) = \Omega \sim B(R)$$

is the complement of $B(R)$ in Ω, and

$$A(R,\,\delta) = B(R) \cap \{x\colon |(x-p)\bullet v| < \delta R_0\}$$

is a thicked disk in the ball $B(R)$ analogous to that of Proposition 3.7.

(5) Corresponding to radius $R_0 < R < 2R_0$, temperature τ_A, thickness parameter δ, and smoothing radius ϵ we associate the initial heat distribution and temperature field

$$T_0[R,\,\delta,\,\tau_A,\,\epsilon] = Q_0[R,\,\delta,\,\tau_A,\,\epsilon] = \Psi_\epsilon * \Bigl[\tau_A \chi_{A(R,\,\delta)} + \tau_B \chi_{B(R)} + \tau_C \chi_{C(R)}\Bigr].$$

3.16 Theorem (A large crystal which appears instantaneously at positive time). *There exists a choice of radius $R_0 < R < 2R_0$, temperature τ_A, thickness parameter δ, and smoothing radius ϵ such that with the initial heat distribution and temperature field*

$$T_0[R,\,\delta,\,\tau_A,\,\epsilon] = Q_0[R,\,\delta,\,\tau_A,\,\epsilon]$$

there will be positive times $0 < t_* < t^*$ with the following property: For all sufficiently large integers j there will be a time $t_* \le t_j \le t^*$ such that

(1) $[\![K_j(t)]\!] = [\![\emptyset]\!]$ for all $0 \le t < t_j$.

(2) $\mathcal{L}^n\bigl(K_j(t) \bigtriangleup B(R_0)\bigr) \ge \mathcal{L}^n\bigl(B(R_0)\bigr)$ for all $t_j \le t < t^*$.

PROOF. For $i = 1,\,2,\,3,\,\ldots$, we set $R_i = (1 + 2^{-i})R_0$ and

$$Q_i = \Bigl[\tau_B \chi_{B(R_i)} + \tau_C \chi_{C(R_i)}\Bigr].$$

Step 1. For each i there is a crystal $[\![K_i]\!]$ such that

$$\mathcal{E}\bigl([\![\emptyset]\!],\,Q_i\bigr) - \mathcal{E}\bigl([\![K_i]\!],\,Q_i\bigr) > 0$$

Suppose $[\![K_1]\!]$, $[\![K_2]\!]$, $[\![K_3]\!]$, $\ldots$ are crystals for which

$$\mathcal{E}\bigl([\![\emptyset]\!],\,Q_i\bigr) - \mathcal{E}\bigl([\![K_i]\!],\,Q_i\bigr) \ge 0.$$

Then we infer from our construction and Proposition 3.8 that

$$\mathcal{L}^n\bigl(K_i \bigtriangleup B(R_0)\bigr) \to 0 \quad \text{and} \quad \mathcal{H}^{n-1}\bigl(\partial K_i\bigr) - n\,\alpha(n)\,R_0^{n-1} \to 0$$

as $i \to \infty$ and we infer from Proposition 3.7 that

$$\frac{\mathcal{L}^n\big(K_i \cap A(R_0,\, \delta)\big)}{\mathcal{L}^n\big(A(R_0,\, \delta)\big)} \to 1$$

uniformly in δ as $i \to \infty$.

Step 2. There exist positive numbers $\epsilon_1,\, \epsilon_2,\, \epsilon_3, \ldots$ and $\sigma_1,\, \sigma_2,\, \sigma_3, \ldots$ with the following properties. Suppose

$$Q_{i,\epsilon} = \Psi_{\epsilon_i} * Q_i \pm \sigma_i.$$

For each i there is a crystal $[\![K_i]\!]$ such that

$$\mathcal{E}\big([\![\emptyset]\!],\, Q_{i,\epsilon}\big) - \mathcal{E}\big([\![K_i]\!],\, Q_{i,\epsilon}\big) > 0$$

Suppose $[\![K_1]\!]$, $[\![K_2]\!]$, $[\![K_3]\!]$, $\ldots$ are crystals for which

$$\mathcal{E}\big([\![\emptyset]\!],\, Q_{i,\epsilon}\big) - \mathcal{E}\big([\![K_i]\!],\, Q_{i,\epsilon}\big) \geq 0.$$

Then

$$\mathcal{L}^n\big(K_i \bigtriangleup B(R_0)\big) \to 0 \quad \text{and} \quad \mathcal{H}^{n-1}\big(\partial K_i\big) - n\,\alpha(n)\, R_0^{n-1} \to 0$$

as $i \to \infty$ and

$$\frac{\mathcal{L}^n\big(K_i \cap A(R_0,\, \delta)\big)}{\mathcal{L}^n\big(A(R_0,\, \delta)\big)} \to 1$$

uniformly in δ as $i \to \infty$.

Step 3. There exist positive times $0 < t_* < t^*$ together with positive numbers $\epsilon_1,\, \epsilon_2,\, \epsilon_3, \ldots$ and $\sigma_1,\, \sigma_2,\, \sigma_3, \ldots$ and $\delta_1,\, \delta_2,\, \delta_3, \ldots$ nd $\tau_1,\, \tau_2,\, \tau_3, \ldots$ such that with the initial heat distribution and temperature field

$$T_0[R_i,\, \delta_i,\, \tau_{A_i},\, \epsilon_i] = Q_0[R_i,\, \delta_i,\, \tau_{A_i},\, \epsilon_i]$$

there will be positive times $0 < t_* < t^*$ with the following property: For all sufficiently large integers j there will be a time $t_* \leq t_j \leq t^*$ such that

(1) $[\![K_j(t)]\!] = [\![\emptyset]\!]$ for all $0 \leq t < t_j$.

(2) $\mathcal{L}^n\big(K_j(t) \bigtriangleup B(R_0)\big) \geq \mathcal{L}^n|big(B(R_0)\big)$ for all $t_j \leq t < t^*$. The main idea is to choose the δ_i's very small and the τ_{A_i}'s very large in such a way that $\mathcal{L}^n(A(R_i, \delta_i))\tau_{A_i}$ is very small while $\mathcal{L}^n(A(R_i, \delta_i))\tau_{A_i}^2$ is very large. We infer from Step 2 and Corollary 3.12 and 3.6 and

especially Proposition 3.14 that no crystallization will occur until the high temperatures in the $A(R_i,\, \delta_i)$'s will have had time to diffuse away. According to 3.6 there is a positive lower bound to this time provided the δ_i's and τ_{A_i}'s are fixed for large i's. According to 3.6.3 the high temperatures in the $A(R_i,\, \delta_i)$'s will dissipate in uniform time. We infer from Step 2 that crystallization will have occurred by then and that it will have been large. ■

4. Existence and structure of $\mathcal{E}$ minimizers

In this chapter we use existence and regularity theorems from geometric measure theory to show that the variational minimizations required to construct our approximating evolutions can be performed. The resulting temperature fields are Lipschitz continuous, and the Gibbs-Thomson relation between curvature and temperature holds exactly along the crystal interface. In two and three dimensional spaces, the crystal interface is a smooth hypersurface in case Φ is smooth and elliptic.

4.1 Theorem (Existence of $\mathcal{E}$ minimizers). *Corresponding to each heat distribution $P \in \mathcal{Q}$ and each time interval $\Delta t > 0$ there exist a crystal $[\![K]\!] \in \mathcal{K}$ and a heat distribution $Q \in \mathcal{Q}$ which give an $\mathcal{E}$ minimizer for P and Δt.*

PROOF. Suppose $[\![K_1]\!],\, [\![K_2]\!],\, [\![K_3]\!],\, \ldots \in \mathcal{K}$ and $Q_1,\, Q_2,\, Q_3,\, \ldots \in \mathcal{Q}$ such that

$$\lim_i \Big[\mathcal{E}\big([\![K_i]\!],\, Q_i\big)+\Delta t^{-\alpha}\|Q_i-P\|_*\Big] = \inf\Big\{\mathcal{E}\big([\![K]\!],\, Q\big)+\Delta t^{-\alpha}\|Q-P\|_*\colon [\![K]\!] \in \mathcal{K},\, Q \in \mathcal{Q}\Big\}.$$

We set $T_i = c_{K_i}^{-1} Q_i$ for each i as the temperature field associated to Q_i and $[\![K_i]\!]$ so that

$$\mathcal{E}\big([\![K_i]\!],\, Q_i\big) + \Delta t^{-\alpha}\|Q_i - P\|_* = \Phi\big(\partial[\![K_i]\!]\big) + \int_\Omega c_{K_i} F\,(T_i)\; d\mathcal{L}^n + \Delta t^{-\alpha}\|Q_i - P\|_*.$$

The compactness theorem for integral currents guarantees the existence of a subsequence $j(1),\, j(2),\, j(3),\, \ldots$ of $1,\, 2,\, 3,\, \ldots$ and $[\![K]\!] \in \mathcal{K}$ such that $\mathcal{L}^n(K_{j(i)} \triangle K) \to 0$ as $i \to \infty$ and

$$\Phi\big(\partial[\![K]\!]\big) \le \liminf_i \Phi\big(\partial[\![K_{j(i)}]\!]\big).$$

Corollary B.6 guarantees the existence of a further subsequence $k(1),\, k(2),\, k(3),\, \ldots$ of $j(1),\, j(2),\, j(3),\, \ldots$ and $Q \in \mathcal{Q}$ such that

$$\|Q_{j(i)} - Q\|_* \to 0 \quad \text{and} \quad \|Q_{j(i)} - P\|_* \to \|Q - P\|_*$$

as $i \to \infty$. We then apply the convergence criteria given in B.3 together with the lower semicontinuity criterion given in Proposition B.4 (with g_i, f_i, G there replaced by $c_{K_{j(i)}}$, $T_{j(i)}$, F respectively in the present context) to infer the existence of a final subsequence $\ell(1)$, $\ell(2)$, $\ell(3)$, ... of $k(1)$, $k(2)$, $k(3)$, ... and limit temperature field T such that $T_{\ell(i)} \rightharpoonup T$ as $i \to \infty$ (weak convergence in $\mathbb{L}_2(\Omega)$) and

$$\int_\Omega c_K F(T)\, d\mathcal{L}^n \le \liminf_i \int_\Omega c_{K_{\ell(i)}} F(T_{\ell(i)})\, d\mathcal{L}^n.$$

We use the convergence criteria in B.3 to infer that $Q_{\ell(i)} = c_{K_{\ell(i)}} T_{\ell(i)} \rightharpoonup Q = c_K T$ as $i \to \infty$. We do not assert any special convergence of the functions $1/T_{\ell(i)}$ to $1/T$. ■

4.2 Theorem (A priori estimates on temperature fields for $\mathcal{E}$ minimizers). *Suppose crystal $[\![K]\!] \in \mathcal{K}$ and heat distribution $Q \in \mathcal{Q}$ give an $\mathcal{E}$ minimizer for some heat distribution P and time interval Δt. Suppose also $T = c_K^{-1} Q$ is the temperature field associated with Q and $[\![K]\!]$. Then we can redefine T on a set of $\mathcal{L}^n$ measure zero if necessary in order that the following is true:*

(1) For each p in Ω,

$$\Gamma(K, Q)\, \Delta t^{n\alpha/(3n+2)} \le T(p) \le \frac{Q_*|\Omega|}{c_s} + \left(\frac{n^{1/2}\rho_*}{2c_*}\right)\frac{1}{\Delta t^\alpha};$$

here the constant

$$\Gamma(K, Q) = \left(\frac{\alpha(n)}{2^{(3n+2)/3} C_*^n \left[\mathcal{E}\big([\![K]\!], Q\big) + \Omega/a_*^2\right]}\right)^{\frac{1}{3n+2}}$$

is bounded from below once $\mathcal{E}\big([\![K]\!], Q\big)$ is bounded from above.

(2) For each p and q in Ω,

$$\big|F'(T(p)) - F'(T(q))\big| \le \frac{|p-q|}{\Delta t^\alpha},$$

$$\left|\frac{1}{T(p)^3} - \frac{1}{T(q)^3}\right| \le \frac{C_*|p-q|}{\Delta t^\alpha},$$

and

$$\big|T(p) - T(q)\big| \le \frac{|p-q|}{c_* \Delta t^\alpha};$$

in particular, T is Lipschitz continuous.

PROOF. Since we are free to redefine T on a set of measure zero if we like, it is sufficient to verify our Lipschitz relation between points p and q in Ω which are close and which are Lebesgue points of both T and c_K [FH 2.9.9]. Suppose p and q are such points with $T(p) < T(q)$ and set $v = q - p$ (with the obvious meaning). For arbitrary (but fixed) $0 < \epsilon << |v|/3$ we choose a set A in $\mathbb{B}^n(p, \epsilon)$ with $\mathcal{L}^n(A) > 0$ such that for each $x \in A$, the following conditions hold:

$$\begin{aligned} c_K(x) &= c_K(p), & c_K(x+v) &= c_K(q), \\ \big|Q(x) - Q(p)\big| &< \epsilon, & \big|Q(x+v) - Q(q)\big| &< \epsilon, \\ \big|T(x) - T(p)\big| &< \epsilon, & \big|T(x+v) - T(q)\big| &< \epsilon. \end{aligned}$$

For arbitrary (but fixed) $0 < \Delta Q < \epsilon c_{_s}$ we consider a comparison heat distribution $Q^* \in \mathcal{Q}$ given by setting

$$Q^*(x) = \begin{cases} Q(x) & \text{if } x \notin A \cup (\{v\} + A) \\ Q(x) + \Delta Q & \text{if } x \in A \\ Q(x) - \Delta Q & \text{if } x \in \{v\} + A. \end{cases}$$

We also set $\Delta T(p) = c_K(p)^{-1}\Delta Q$ and $\Delta T(q) = c_K(q)^{-1}\Delta Q$. The assumed minimality of $[\![K]\!]$ and Q implies that

$$\mathcal{E}\big([\![K]\!],\, Q\big) + \frac{1}{\Delta t^\alpha}\|Q - P\|_* \le \mathcal{E}\big([\![K]\!],\, Q^*\big) + \frac{1}{\Delta t^\alpha}\|Q^* - P\|_*.$$

We estimate

$$\begin{aligned} \int_\Omega c_K F\big(c_K^{-1} Q\big)\, d\mathcal{L}^n - \int_\Omega c_K F\big(c_K^{-1} Q^*\big)\, d\mathcal{L}^n &\le \frac{1}{\Delta t^\alpha}\|Q^* - P\|_* - \frac{1}{\Delta t^\alpha}\|Q - P\|_* \\ &\le \frac{1}{\Delta t^\alpha}\|Q^* - Q\|_* \\ &\le \frac{1}{\Delta t^\alpha}\mathcal{L}^n(A)\,\Delta Q\,|v|. \end{aligned}$$

The extreme inequality will be used to prove our assertions. We use the Mean Value Theorem in the following sequence of equalities to conclude, for each $x \in A$, the existence of temperatures

$$T(x) < T^*(x) < T(x) + \Delta T(p) < T(x) + \epsilon,$$

$$T(x+v)-\epsilon < T(x+v)-\Delta T(q) < T^*(x+v) < T(x+v)$$

for which the following is true.

$$\begin{aligned}
&\int_\Omega c_K F(c_K^{-1}Q)\,d\mathcal{L}^n - \int_\Omega c_K F\big(c_K^{-1}Q^*\big)\,d\mathcal{L}^n \\
&\qquad\qquad = \int_{x\in A} c_K(p)F\big(T(x)\big) - c_K(p)F\big(T(x)+\Delta T(p)\big)\,d\mathcal{L}^n x \\
&\qquad\qquad\quad + \int_{x\in A} c_K(q)F\big(T(x+v)\big) \\
&\qquad\qquad\qquad\qquad - c_K(q)F\big(T(x+v)-\Delta T(q)\big)\,d\mathcal{L}^n x
\end{aligned}$$

$$\begin{aligned}
&= \int_{x\in A} c_K(p)F'\big(T^*(x)\big)(-\Delta T(p))\,d\mathcal{L}^n x \\
&\quad + \int_{x\in A} c_K(q)F'\big(T^*(x+v)\big)\Delta T(q)\,d\mathcal{L}^n x \\
&= \Delta Q \int_A F'\big(T^*(x+v)\big) - F'\big(T^*(x)\big)\,d\mathcal{L}^n \\
&= \Delta Q \int_A F'\big(T(q)\pm 2\epsilon\big) - F'\big(T(p)\pm 2\epsilon\big)\,d\mathcal{L}^n \\
&= \Delta Q \mathcal{L}^n(A)\Big[F'\big(T(q)\pm 2\epsilon\big) - F'\big(T(p)\pm 2\epsilon\big)\Big].
\end{aligned}$$

Since F' is continuous and ϵ is arbitrary we infer

$$F'\big(T(q)\big) - F'\big(T(p)\big) \le \frac{|v|}{\Delta t^\alpha} = \frac{|p-q|}{\Delta t^\alpha}.$$

The second inequality of conclusion (2) follows from this inequality together with the estimate given in appendix A.1.15. With the continuity of T established the first inequality of conclusion (2) follows. The third inequality of conclusion (2) now follows from the first together with the estimate given in appendix A.1.16.

We use 2.1, 2.2, 2.3 to infer that

$$\begin{aligned}
\sup_\Omega T &\le \frac{\text{Average}(Q)}{c_s} + \text{diam}(\Omega)\text{Lip}(T) \\
&\le \frac{Q_*|\Omega|}{c_s} + \frac{n^{\frac{1}{2}}\rho_*}{2c_*}\frac{1}{\Delta t^\alpha}
\end{aligned}$$

which is the right hand inequality asserted in conclusion (1).

To check the left hand inequality asserted in conclusion (1) we consider p as the center of a ball of radius

$$r = \frac{\Delta t^\alpha}{2C_* T(p)^3}$$

and infer from the second inequality of conclusion (2) that if $|p-q| \le r$ then

$$\frac{1}{T(p)^3} - \frac{1}{T(q)^3} \le \frac{C_* r}{\Delta t^\alpha} = \frac{1}{2T(p)^3}, \qquad \frac{1}{T(q)^3} \ge \frac{1}{2T(p)^3}, \qquad \frac{1}{T(q)^2} \ge \frac{1}{2^{\frac{2}{3}} T(p)^2}.$$

We then use A.1.1 to estimate

$$\begin{aligned} \frac{1}{2^{\frac{2}{3}} T(p)^2} \alpha(n) r^n &\le \int_{\mathbf{B}^n(p,r)} \frac{1}{T^2}\, d\mathcal{L}^n \\ &\le \frac{1}{c_s} \int_{\Omega \cap \{x : T(x), a_*\}} c_K\, F(T)\, d\mathcal{L}^n + \frac{|\Omega|}{a_*^2} \\ &\le \mathcal{E}\big([\![K]\!],\, Q\big) + \frac{|\Omega|}{a_*^2}. \end{aligned}$$

The left hand inequality of conclusion (1) follows. ■

4.3 Theorem (Variations of crystals of $\mathcal{E}$ minimizers). *Suppose crystal $[\![K]\!] \in \mathcal{K}$ and heat distribution $Q \in \mathcal{Q}$ give an $\mathcal{E}$ minimizer for some heat distribution P and time interval Δt. Then there is a decreasing function $\rho{:}\mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{r \downarrow 0} \rho(r) = 0$ depending only on a lower bound for Δt and and upper bound for $\mathcal{E}\big([\![K]\!],\, Q\big)$ for which the following is true Suppose $p \in \Omega$ and $0 < r << 1$ and $[\![L]\!] \in \mathcal{K}$ with $K \triangle L \subset \mathbb{B}^n(p,\, r)$. Then*

$$\Phi\big(\partial[\![L]\!]\big) - \Phi\big(\partial[\![K]\!]\big) - \Big(\mathcal{L}^n(L \sim K) - \mathcal{L}^n(K \sim L)\Big)\, H(T(p)) \pm \rho(r)\mathcal{L}^n(L \triangle K) \ge 0.$$

PROOF The assumed minimality of $[\![K]\!]$ and Q implies that that for every $Q^* \in \mathcal{Q}$,

$$\mathcal{E}\big([\![L]\!],\, Q^*\big) + \Delta t^{-\alpha}\|Q^* - P\|_* - \mathcal{E}\big([\![K]\!],\, Q\big) - \Delta t^{-\alpha}\|Q - P\|_* \ge 0. \qquad (*)$$

Ultimately this inequality $(*)$ will give our assertion.

First we estimate

$$\|Q^* - P\|_* - \|Q - P\|_* \le \|Q^* - Q\|_* \le 2s \int_\Omega |\Delta Q_1|$$

We fix $0 < r < s = 2r << 1$ and set

$$\Delta Q_1(x) = \begin{cases} 0 & \text{if } x \notin K \triangle L \\ \frac{c_s - c_\ell}{c_\ell} Q(x) & \text{if } x \in L \sim K \\ \frac{c_\ell - c_s}{c_s} Q(x) & \text{if } x \in K \sim L \end{cases}$$

and

$$\Delta Q_2(x) = \begin{cases} 0 & \text{if } x \in \mathbb{B}^n(p,\, r) \cup \big(\Omega \sim \mathbb{B}^n(p,\, s)\big) \\ \alpha(n)^{-1}\big(s^n - r^n\big)^{-1} \int_\Omega \Delta Q_1 \, d\mathcal{L}^n & \text{in } x \in \mathbb{B}^n(p,\, s) \sim \mathbb{B}^n(p,\, r) \end{cases}$$

and

$$Q^* = Q + \Delta Q_1 - \Delta Q_2, \qquad T^* = c_L^{-1} Q^*.$$

We check that

$$T^*(x) = \begin{cases} T(x) & \text{if } x \in \mathbb{B}^n(p,\, r) \cup \big(\Omega \sim \mathbb{B}^n(p,\, s)\big) \\ T(x) - c_K^{-1}(x) \Delta Q_2(x) & \text{if } x \in \mathbb{B}^n(p,\, s) \sim \mathbb{B}^n(p,\, r) \end{cases}$$

First we estimate

$$\|Q^* - P\|_* - \|Q - P\|_* \le \|Q^* - Q\|_* \le 4r \int_\Omega |\Delta Q_1| \le 4r c_\ell \sup_\Omega T \, \mathcal{L}^n(K \triangle L)$$

which is an admissible error for our conclusion.

Next we use the Mean Value Theorem to infer the existence of $T^0(x)$ between $T(x) - \Delta Q_2(x)$ and $T(x)$, for $x \in \mathbb{B}^n(p,\, s) \sim \mathbb{B}^n(p,\, r)$, so that we can compute

$$\begin{aligned} &\int_\Omega \big[c_L F(T^*) - c_K F(T)\big] \, d\mathcal{L}^n \\ &\qquad = \big(c_s - c_\ell\big) \int_{L \sim K} F(T) \, d\mathcal{L}^n + \big(c_\ell - c_s\big) \int_{K \sim L} F(T) \, d\mathcal{L}^n \\ &\qquad\qquad + \int_{\mathbf{B}^n(p,\, s) \sim \mathbf{B}^n(p,\, r)} \big[c_K F\left(T - c_K^{-1} \Delta Q_2\right) - c_K F(T)\big] \, d\mathcal{L}^n \\ &\qquad = \big(c_s - c_\ell\big) \int_{L \sim K} F(T) \, d\mathcal{L}^n + \big(c_\ell - c_s\big) \int_{K \sim L} F(T) \, d\mathcal{L}^n \\ &\qquad\qquad - \int_{\mathbf{B}^n(p,\, s) \sim \mathbf{B}^n(p,\, r)} F'\left(T^0\right) \Delta Q_2 \, d\mathcal{L}^n \end{aligned}$$

We can then write

$$(c_s - c_\ell) \int_{L \sim K} F(T)\, d\mathcal{L}^n = (c_s - c_\ell)\mathcal{L}^n(L \sim K)\big[F(T(p)) \pm r \operatorname{Lip}\big(F\big|\operatorname{image}(T)\big) \operatorname{Lip}(T)\big]$$

$$(c_\ell - c_s) \int_{K \sim L} F(T)\, d\mathcal{L}^n = (c_\ell - c_s)\mathcal{L}^n(K \sim L)\big[F(T(p)) \pm r \operatorname{Lip}\big(F\big|\operatorname{image}(T)\big) \operatorname{Lip}(T)\big]$$

$$\begin{aligned}
&\int_{\mathbf{B}^n(p,\,s) \sim \mathbf{B}^n(p,\,r)} F'\left(T^0\right) \Delta Q_2\, d\mathcal{L}^n \\
&\qquad = \left(\int_\Omega \Delta Q_1\, d\mathcal{L}^n\right) \left(\frac{1}{\alpha(n)(s^n - r^n)} \int_{\mathbf{B}^n(p,\,s) \sim \mathbf{B}^n(p,\,r)} F'\left(T^0\right)\, d\mathcal{L}^n\right) \\
&\qquad = \Big[(c_s - c_\ell)\mathcal{L}^n(L \sim K)\big[T(p) \pm 2r \operatorname{Lip}(T)\big)\big] \\
&\qquad\quad + (c_\ell - c_s)\mathcal{L}^n(K \sim L)\big[T(p) \pm 2r \operatorname{Lip}(T)\big]\Big] \\
&\qquad\qquad \Big[F'(T(p)) \pm 2r \operatorname{Lip}\big(F'\big|\operatorname{image}(T)\big)\operatorname{Lip}(T)\Big].
\end{aligned}$$

We now incorporate these equalities into the inequality $(*)$. The theorem follows since the conclusions of Theorem 4.2 give upper and lower bounds for the image of T and an upper bound for the Lipschitz constant of T. ■

4.4 Theorem (The Gibbs-Thomson relation for $\mathcal{E}$ minimizers when Φ is smooth). *Suppose our surface energy integrand Φ is smooth and that $[\![K]\!] \in \mathcal{K}$ and $Q \in \mathcal{Q}$ give an $\mathcal{E}$ minimizer for some heat distribution P and some time interval Δt. Then the Gibbs-Thomson relation holds with Gibbs-Thomson function H. This means (as in 2.8) that for each smooth initial velocity vector field and associated diffeomorphisms $G_t(x) = x + tg(x)$ we have*

$$\frac{d}{dt}\Phi\big(G_{t\sharp}\partial[\![K]\!]\big)\Big|_{t=0} = \int_{x \in \partial K} H\big(T(x)\big)\, \mathbf{n}_K(x) \bullet g(x)\, d\mathcal{H}^{n-1}x.$$

PROOF. Assume that Φ is smooth. As noted in [AW 4.9] the function

$$g \mapsto \frac{d}{dt}\Phi\big(G_{t\sharp}\partial[\![K]\!]\big)\Big|_{t=0}$$

(as above) is well defined and is a linear functional on the vector space of smooth g's. In accordance with [AW 4.1(1)] and [FH 4.5.6(5)] we can compute

$$\frac{d}{dt}\mathcal{L}^n\big(G_t K\big)\Big|_{t=0} = \int_K \operatorname{div}(g)\, d\mathcal{L}^n = \int_{\partial K} \mathbf{n}_K \bullet g\, d\mathcal{H}^{n-1}$$

so that the function

$$g \mapsto \frac{d}{dt}\mathcal{L}^n\big(G_t K\big)\Big|_{t=0}$$

is also well defined as a linear functional on the vector space of smooth g's. We infer by approximation that the function

$$g \mapsto \frac{d}{dt}\mathcal{L}^n\big(G_t(K) \bigtriangleup K\big)\Big|_{t=0}$$

is well defined and equals

$$\int_{\partial K} \big|\mathbf{n}_K \bullet g\big|\, d\mathcal{H}^{n-1}$$

We now fix an initial velocity vector field g. For a given (small) positive radius r we choose nonnegative functions $\varphi_1, \varphi_2, \ldots, \varphi_N$ together with points $p_1, \ldots, p_N$ in Ω such that the φ's constitute a partition of unity for Ω and the support of each φ_i lies in the ball $\mathbb{B}^n(p_i, r)$. For each i we set $g_i = \varphi_i\, g$ and corresponding set $G_t^i(x) = x + t g_i(x)$ for all small t's. For each fixed i we infer from Theorem 4.3 that for all small t's

$$\Phi\big(G^i_{t\sharp}\partial[\![K]\!]\big) - \Phi\big(\partial[\![K]\!]\big) - \Big(\mathcal{L}^n\big(G^i_t(K) \sim K\big) - \mathcal{L}^n\big(K \sim G^i_t(K)\big)\Big) H\big(T(p_i)\big)$$

$$\pm\, \rho(r)\mathcal{L}^n\big(G^i_t(K) \bigtriangleup K\big) \geq 0.$$

In view of the differentiability established above we further infer

$$\frac{d}{dt}\Phi\big(G^i_{t\sharp}\partial[\![K]\!]\big)\Big|_{t=0} - \frac{d}{dt}\mathcal{L}^n\big(G_t K\big)\Big|_{t=0} H\big(T(p_i)\big) + \rho(r)\frac{d}{dt}\mathcal{L}^n\big(G_t(K) \bigtriangleup K\big)\Big|_{t=0} \geq 0$$

so that

$$\frac{d}{dt}\Phi\big(G^i_{t\sharp}\partial[\![K]\!]\big)\Big|_{t=0} - H\big(T(p_i)\big)\int_{\partial K} \mathbf{n}_K \bullet g_i\, d\mathcal{H}^{n-1} \pm \rho(r)\int_{\partial K} \big|\mathbf{n}_K \bullet g_i\big|\, d\mathcal{H}^{n-1} \geq 0.$$

We now sum over i to obtain

$$\frac{d}{dt}\Phi\big(G_{t\sharp}\partial[\![K]\!]\big)\Big|_{t=0} - \sum_{i=1}^{N} H\big(T(p_i)\big)\int_{\partial K} \mathbf{n}_K \bullet g_i\, d\mathcal{H}^{n-1} \pm \sum_{i=1}^{N} \rho(r)\int_{\partial K} \big|\mathbf{n}_K \bullet g_i\big|\, d\mathcal{H}^{n-1} \geq 0$$

so that (since the φ's are nonnegative)

$$\frac{d}{dt}\Phi\big(G_{t\sharp}\partial[\![K]\!]\big)\Big|_{t=0} - \int_{\partial K}\left(\sum_{i=1}^{N}\varphi_i\, H\big(T(p_i)\big)\right)\mathbf{n}_K \bullet g\, d\mathcal{H}^{n-1} \pm \rho(r)\int_{\partial K}\big|\mathbf{n}_K \bullet g\big|\, d\mathcal{H}^{n-1} \geq 0$$

Since our choice of r is arbitrary we use Theorem 4.2 to infer

$$\frac{d}{dt}\Phi\big(G_{t\sharp}\partial[\![K]\!]\big)\Big|_{t=0} - \int_{\partial K} H(T)\,\mathbf{n}_K \bullet g\, d\mathcal{H}^{n-1} \geq 0.$$

Since a similar inequality must hold with g replaced by $-g$ we infer

$$\frac{d}{dt}\Phi\big(G_{t\sharp}\partial[\![K]\!]\big)\Big|_{t=0} - \int_{\partial K} H(T)\,\mathbf{n}_K \bullet g\, d\mathcal{H}^{n-1} = 0$$

which gives the claim of the theorem. ■

4.5 Theorem (Regularity of $\mathcal{E}$ minimizers). *Suppose that $[\![K]\!] \in \mathcal{K}$ and $Q \in \mathcal{Q}$ give an $\mathcal{E}$ minimizer for some heat distribution P and some time interval Δt. Suppose also that T is the temperature field associated with K and Q. Then*

(1) There exist positive numbers δ and μ such that

$$\mathcal{H}^{n-1}\big(\partial K \cap \mathbb{B}^n(p,\, r)\big) \geq \mu\, r^{n-1}$$

for each point p in the support of $\partial[\![K]\!]$ and each radius $0 < r < \delta$.

(2) The support of $\partial[\![K]\!]$ (which equals the closure of ∂K) has finite $\mathcal{H}^{n-1}$ measure and hence zero $\mathcal{L}^n$ measure.

(3) There exist positive numbers δ and C together with a function $\omega(r) = C\, r$ defined for $0 < r < \delta$ such that the current $\partial[\![K]\!]$ is $\big(\Phi,\, \omega,\, \delta\big)$-minimal in the sense of E. Bombieri [BE Definition 1, page 101].

(4) In case Φ is an even integrand there exist positive numbers γ and δ such that the support of $\partial[\![K]\!]$ is $(\gamma,\, \delta)$ restricted with respect to the empty set in the sense of F. Almgren [AF II.1, page 53]

(5) In case Φ is an even integrand there exist positive numbers δ and C together with a function $\epsilon(r) = C\, r$ defined for $0 < r < \delta$ such that the support of $\partial[\![K]\!]$ is $(\Phi,\, \epsilon,\, \delta)$ minimal with respect to the empty set in the sense of F. Almgren [AF III.1, page 75]

Now suppose additionally that our surface energy integrand Φ is smooth and elliptic. Then as additional conclusion we have that

(6) Except possibly for a compact singular set of zero $\mathcal{H}^{n-1}$ measure, the support of $\partial[\![K]\!]$ is a two times Hölder continuously $n-1$ dimensional submanifold of Ω.

(7) If $n = 2$ or $n = 3$ then the support S of $\partial[\![K]\!]$ is a two times Hölder continuously differentiable submanifold of Ω (with no singular set). Furthermore, at every point p of S the weighted mean curvature $wmc(p)$ of S (with respect to the exterior normal of K) exists in the classical sense and equals $H(T(p))$.

PROOF. The proof of the present theorem very closely follows the arguments and constructions of sections 3.5, 3.5, 3.6, 3.8, 3,9, 3,10 of the paper [ATW] mentioned in the introduction in which a similar variation scheme was used by the present authors together with J. Taylor to produce curvature driven motions. Since there are essentially no new ideas required in adapting the arguments there to the present case, we leave verification of the present theorem to the reader. We note, however, that the first conclusion together with [FH 2.10.19(1)] imply the second conclusion. ■

5. Heat flows

In this chapter we study the existence and regularity of heat flows with fixed crystals. Suppose $[\![K]\!]$ and Q give an $\mathcal{E}$ minimizer for some P and Δt in the construction of approximating evolutions in 2.11. According to conclusion (2) of Theorem 4.5 the closure of the boundary ∂K will have zero n dimensional measure. Our first theorem below shows that this condition is sufficient to guarantee the existence of a heat flow in the sense of 2.5. Our second theorem gives integral inequalities for such flows with fixed crystals. In our third theorem we establish the essential Hölder continuity in the Monge-Kantorovich metric of the heat flows which arise in our approximating evolutions. According to conclusion (7) of Theorem 4.5 the boundary of K will be a Hölder continuously differentiable submanifold of Ω in case our ambient manifold Ω is of dimension either two or three and our surface energy function Φ is smooth and elliptic. Under this regularity assumption on ∂K we show in our fourth theorem below that the heat flow within the crystal and without is Hölder continuously differentiable in space and Hölder continuous in time; in particular, there is temperature continuity with simple gradient jumps across the interface..

5.1 Theorem (Existence of heat flows with fixed crystal). *Suppose that $[\![K]\!]$ is a crystal in $\mathcal{K}$ for which the closure of ∂K has zero $\mathcal{L}^n$ measure. Suppose also that $a \in \mathbb{R}$ and that Q_a is a heat distribution in $\mathcal{Q}$ such that the associated temperature field $T_a = c_{K} Q_a$ belongs to $\mathbb{H}^1(\Omega)$. Then there are unique functions $Q(x, t)$ and $T(x, t)$ defined for x in Ω and $a \le t < \infty$ such that for each fixed number $b > a$ the restriction of Q and*

T to $\Omega \times [a,\, b]$ are a solution to the heat equation with fixed crystal $[\![K]\!]$ and with initial heat distribution Q_a and final heat distribution $Q(\cdot,\, b)$ in the sense of 2.5. Furthermore, the functions Q and T are infinitely differentiable on the set

$$\Big(\Omega \sim \mathrm{Closure}(\partial K)\Big) \times (a,\, \infty).$$

Finally, the equation 2.5.7

$$0 = \int_{\Omega\times(a,\,b)} -\overset{\bullet}{\varphi} Q + \kappa_{K}(\nabla\varphi) \bullet \nabla T \, d\mathcal{L}^{n+1} + \int_{\Omega} \varphi(\cdot,\, b)Q_b - \varphi(\cdot,\, a)Q_a \, d\mathcal{L}^n$$

holds whenever

$$\varphi \in \mathbf{C}^0\big([a,\, b], \mathbb{L}_2(\Omega)\big) \cap \mathbb{L}_2\big((a,\, b), \mathbb{H}^1(\Omega)\big) \quad \textit{with} \quad \overset{\bullet}{\varphi} \in \mathbb{L}_2\big([a,\, b], \mathbb{L}_2(\Omega)\big).$$

PROOF. Our existence proof ultimately is based on the theory of analytic semigroups as set forth in [PA]. There are five steps.

Step 1. We denote by $\mathbb{L}_2(\Omega)_K$ the vector space $\mathbb{L}_2(\Omega)$ with inner product $(\cdot,\, \cdot)_K$ and norm $\|\cdot\|_K$ given by setting for $u,\, v \in \mathbb{L}_2(\Omega)$,

$$\big(u,\, v\big)_K = \int_{x\in\Omega} c_K(x)\, u(x)\, v(x)\, d\mathcal{L}^n x, \quad \|u\|_K^2 = \big(u,\, u\big)_K;$$

here c_K is our heat capacity function which equals c_s in K and equals c_ℓ in $\Omega \sim K$.

We define a linear partial differential operator

$$Au = -\frac{1}{c_K}\mathrm{div}\big(\kappa_K \, \nabla u\big)$$

taking values in $\mathbb{L}_2(\Omega)_K$ and having as domain in $\mathbb{L}_2(\Omega)_K$ the vector space,

$$\mathbb{D}(A) = \mathbb{H}^1(\Omega) \cap \big\{u\colon Au \in \mathbb{L}_2(\Omega)_K\big\} \subset \mathbb{L}_2(\Omega)_K.$$

When we write $Au \in \mathbb{L}_2(\Omega)_K$ we mean that there exists a function f in $\mathbb{L}_2(\Omega)_K$, which symbolically is denoted Au, such that

$$\big(Au,\, v\big)_K = \big(f,\, v\big)_K = \int_{x\in\Omega} \big(\kappa_K(x)\nabla u(x)\big) \bullet \nabla v(x)\, d\mathcal{L}^n x$$

for each $v \in \mathbb{H}^1(\Omega)$. Here κ_K is heat conductivity which equals the symmetric matrix $\kappa_{_s}$ in K and equals the symmetric matrix $\kappa_{_\ell}$ (a multiple of the identity) in $\Omega \sim K$.

Additionally we define a Hilbert space

$$\mathbb{H}^1_0(\Omega)_K$$

which as the vector space consists of those functions u in $\mathbb{H}^1(\Omega)$ for which

$$\int_{x\in\Omega} u(x)\, c_K(x)\, d\mathcal{L}^n x = 0$$

and which carries the inner product $\big\langle \cdot\,,\,\cdot \big\rangle_K$ given by setting, for each u and v,

$$\big\langle u,\, v \big\rangle_K = \int_{x\in\Omega} \big(\kappa_K(x)\, \nabla u(x)\big) \bullet \nabla v(x)\, d\mathcal{L}^n x.$$

$\mathbb{H}^1_0(\Omega)_K$ is a Hilbert space since Poincaré's inequality guarantees that

$$\int_\Omega u^2\, c_K\, d\mathcal{L}^n \le \Gamma^2 \int_\Omega \big(\kappa_K\, \nabla u\big) \bullet \nabla v\, d\mathcal{L}^n$$

Step 2. We assert that $\mathbb{D}(A)$ is dense in $\mathbb{L}_2(\Omega)_K$. Since $\mathcal{D}\big(\Omega\big)$ is dense in $\mathbb{L}_2(\Omega)_K$ it is sufficient to show that $\mathcal{D}\big(\Omega\big)$ lies in the closure of $\mathbb{D}(A)$. For small positive ϵ's we pick smooth functions $\varphi_\epsilon\colon \Omega \to [0,\,1]$ such that $\varphi_\epsilon^{-1}\{1\}$ contains a neighborhood of the closure of ∂K and $\int_\Omega \varphi_\epsilon\, d\mathcal{L}^n < \epsilon$. Suppose then $g \in \mathcal{D}\big(\Omega\big)$. Clearly $g\big(1-\varphi_\epsilon\big) \in \mathbb{D}(\Omega)$ and $\lim_{\epsilon\downarrow 0} \|g - g(1-\varphi_\epsilon)\|_K = 0$.

Step 3. We assert that for each function w_0 in $\mathbb{L}_2(\Omega)_K$ with

$$\int_\Omega c_K\, w_0\, d\mathcal{L}^n = 0$$

the equation $Au = w_0$ has a unique solution $u = u_0$ in $\mathbb{D}(A)$.

Proof. The function

$$v \mapsto \big(v,\, w_0\big)_K = \int_\Omega v\, w_0\, c_K\, d\mathcal{L}^n$$

is a continuous linear functional on $\mathbb{H}^1_0(\Omega)_K$ since

$$\left|\big(v,\, w_0\big)_K\right| \le \left(\int_\Omega v^2\, c_K\, d\mathcal{L}^n\right)^{\frac{1}{2}} \left(\int_\Omega w_0^2\, c_K\, d\mathcal{L}^n\right)^{\frac{1}{2}} \le \Big(\Gamma\, \|w_0\|_K\Big)\big\langle u,\, u\big\rangle_K^{\frac{1}{2}}$$

by Poincaré's inequality above. The usual application of Riesz's Representation Theorem guarantees the existence of u_0 in $\mathbb{H}^1_0(\Omega)_K$ such that, for each v in $\mathbb{H}^1_0(\Omega)_K$,

$$\big(v,\, w_0\big)_K = \big\langle v,\, u_0\big\rangle_K.$$

If $v \in \mathbb{H}^1(\Omega)$ and $\gamma = \int_\Omega v\, c_K\, d\mathcal{L}^n / \int_\Omega c_K\, d\mathcal{L}^n$, then $v - \gamma \in \mathbb{H}^1_0(\Omega)_K$. Since

$$\big(\gamma,\, w_0\big)_K = \big\langle \gamma,\, u_0\big\rangle_K = 0$$

whenever γ is constant we infer that $Au_0 = w_0$.

Step 4. We assert that A is a positive operator, i.e. that A is both positive and self-adjoint.

Proof. It is clear from our definition above that A is positive, i.e. $\big(Au,\, u\big)_K \ge 0$ for each u. It is also clear that A is symmetric since

$$\big(Au,\, v\big)_K = \int_\Omega \nabla u\, \kappa_K\, \nabla v\, d\mathcal{L}^n = \big(u,\, Av\big)_K.$$

Suppose then A^* is the adjoint of A and that $\mathbb{D}(A^*)$ is the domain of A^* so that

$$\big(Au,\, v\big)_K = \big(u,\, A^*v\big)_K \tag{$*$}$$

for each $u \in \mathbb{D}(A)$ and each $v \in \mathbb{D}(A^*)$. Since A is symmetric, $\mathbb{D}(A) \subset \mathbb{D}(A^*)$. To show that $A = A^*$ we must show that v_0 in $\mathbb{D}(A^*)$ implies $v_0 \in \mathbb{D}(A)$. Since the constant function $u = 1$ belongs to $\mathbb{D}(A)$ we infer from $(*)$ that

$$0 = \big(A\,1,\, v_0\big)_K = \big(1,\, A^*v_0\big)_K = \int_\Omega A^*v_0\, c_K\, d\mathcal{L}^n$$

We infer from Step 3 that there is a unique u_0 in $\mathbb{H}^1_0(\Omega)_K$ such that

$$Au_0 = A^*v_0; \tag{$*$}$$

clearly $u_0 \in \mathbb{D}(A)$ with $\int_\Omega u c_K \, d\mathcal{L}^n = 0$. For each $u \in \mathbb{D}(A)$ we use $(*)$ to check

$$\big(Au,\, v_0 - u_0\big)_K = \big(Au,\, v_0\big)_K - \big(Au,\, v_0\big)_K = \big(u,\, A^* v_0\big)_K - \big(u,\, Au_0\big)_K = 0 \qquad (**)$$

We now set

$$\gamma = \frac{\int_\Omega v_0 \, c_K \, d\mathcal{L}^n}{\int_\Omega c_K \, d\mathcal{L}^n}$$

and

$$w_0 = v_0 - u_0 - \gamma$$

so that Step 3 again guarantees the existence of a unique z_0 in $\mathbb{H}^1_0(\Omega)_K$ such that $Az_0 = w_0$. Hence we use $(**)$ to infer

$$0 = \big(Az_0,\, v_0 - u_0\big)_K = \big(v_0 - u_0 - \gamma,\, v_0 - u_0\big)_K = \big(v_0 - u_0 - \gamma,\, v_0 - u_0 - \gamma\big)_K$$

which implies that

$$v_0 = u_0 + \gamma$$

so that $v_0 \in \mathbb{D}(A)$ with

$$Av_0 = Au_0 = A^* v_0$$

in accordance with $(*)$.

Step 5. We assert that the operator $-A$ is the generator of a C_0 **semigroup** in the sense of Definition 2.1 on page 4 of [PA]. This is a consequence of Theorem 3.1 (Hille-Yosida) on page 8 of [PA] once we establish that

(i) $-A$ is closed with the closure of $\mathbb{D}(-A)$ equal to $\mathbb{L}_2(\Omega)_K$.

(ii) the resolvent set $\rho(-A)$ contains $\mathbb{R}^+$,

(iii) for every $\lambda > 0$ the mapping norm of $(\lambda I + A)^{-1}$ does not exceed λ^{-1}.

First $-A$ is closed because $A = A^*$ and the adjoint operator is always closed, and we have already established that $\mathbb{D}(A)$ is dense in $\mathbb{L}_2(\Omega)_K$.

We recall that $\rho(-A)$ equals the set of λ's for which the operator $\lambda I + A$ is invertible. Since A is positive, we estimate for x in $\mathbb{D}(A)$,

$$\big((\lambda I + A)x,\, x\big)_K = \lambda\big(x,\, x\big)_K + \big(Ax,\, x\big)_K \geq \lambda\big(x,\, x\big)_K$$

and infer that for all positive λ, $(\lambda I + A)$ is invertible and the inverse satisfies the estimate in (iii).

Step 6. We assert that the operator $-A$ is the generator of an **analytic semigroup** in the sense of Definition 5.1 on page 60 of [PA]. This is a consequence of the condition (b) of Theorem 5.2 on page 61 of [PA]. In a manner similar to our estimate in Step 5 we estimate

$$\left|\Big(((\sigma+\mathbf{i}\tau)I+A)x,\,x\Big)_K\right| = \left|\sigma\big(x,\,x\big)_K + \mathbf{i}\tau\big(x,\,x\big)_K + \big(Ax,\,x\big)_K\right| \geq |\tau|\,\big(x,\,x\big)_K$$

so that the mapping norm of the inverse to

$$\Big(((\sigma+\mathbf{i}\tau)I+A)x,\,x\Big)_K$$

does not exceed $|\tau|$ as required.

Step 7. We assert that the evolution equation

$$c_K\,\overset{\bullet}{T} = \operatorname{div}\big(\kappa_K\,\nabla T\big)$$

$$T(x,\,0) = T_0(x)$$

is uniquely solvable for any initial conditions T_0 belonging to $\mathbb{L}_2(\Omega)_K$. This is a consequence of Corollary 1.5 on page 104 of [PA].

Step 8. We assert that $-A$ is the generator of a **classical semigroup** in the sense of Definition 2.1 on page 104 of [PA]. This is a consequence of Theorem 3.2 on page 111 of [PA].

Step 9 We infer from Step 8 that for $0 < a < b < \infty$,

$$\int_\Omega T^2(\cdot,\,b)c_K\,d\mathcal{L}^n - \int_\Omega T^2(\cdot,\,a)c_K\,d\mathcal{L}^n = -\int_{\Omega\times(a,\,b)} \nabla T\kappa_K\nabla T\,d\mathcal{L}^{n+1}.$$

We use Lebesgue's increasing convergence theorem to infer that our equality also holds for $a = 0$. Hence,

$$\int_{\Omega\times(0,\,b)} \nabla T\kappa_K\nabla T\,d\mathcal{L}^{n+1} < \infty$$

for each b. We assert that for each $0 \leq a < b < \infty$ and each $\varphi \in \mathcal{D}(\Omega\times\mathbb{R})$

$$\int_{x\in\Omega}\varphi(x,\,b)\,c_K(x)\,T(x,\,b)\,d\mathcal{L}^n + \int_{(x,\,t)\in\Omega\times[a,\,b]}\nabla\varphi(x,\,t)\,\kappa_K(x)\,\nabla T(x,\,t)\,d\mathcal{L}^{n+1}(x,\,t)$$

$$= \int_{x\in\Omega} \varphi(x,\, a)\, c_K(x)\, T(x,\, a)\, d\mathcal{L}^n + \int_{(x,\,t)\in\Omega\times[a,\,b]} \overset{\bullet}{\varphi}(x,\, t)\, c_K(x)\, T(x,\, t)\, d\mathcal{L}^{n+1}(x,\, t).$$

For $a > 0$, our assertion is essentially an identity. Clearly it holds for $a = 0$ as well since our semigroup is continuous. The final assertion of our theorem above follows by approximation.

Step 10. The regularity asserted for our solution is a consequence of Theorem 10.1 on page 351 of [LSU]. ■

5.2 Theorem (Inequalities under heat flow). *Suppose that $Q(x,\, t)$ and $T(x,\, t)$ defined on $\Omega \times [a,\, b]$ are a solution to the heat equation with fixed crystal $[\![K]\!]$ and with initial heat distribution Q_a and final heat distribution Q_b as in 2.5. Also set $U = 1/T$.*

(1) For each twice continuously differentiable function convex function $G{:}\,\mathbb{R}^+ \to \mathbb{R}$ we have

$$\int_\Omega c_{_K} G\big(c_{_K}^{-1} Q_a\big)\, d\mathcal{L}^n - \int_\Omega c_{_K} G\big(c_{_K}^{-1} Q_b\big)\, d\mathcal{L}^n = \int_{\Omega\times[a,\,b]} G''(T)\, \big(\nabla T \kappa_{_K} \nabla T\big)\, d\mathcal{L}^{n+1}$$

provided the first term on the left hand side is finite.

(2) For the bulk energy density function F we have

$$\begin{aligned} &\int_\Omega c_{_K} F\big(c_{_K}^{-1} Q_a\big)\, d\mathcal{L}^n - \int_\Omega c_{_K} F\big(c_{_K}^{-1} Q_b\big)\, d\mathcal{L}^n \\ &\qquad \geq \kappa_* \int_{\Omega\times[a,\,b]} \big(c_* + C_*^{-1} T^{-4}\big)\, |\nabla T|^2\, d\mathcal{L}^{n+1} \\ &\qquad = \kappa_* \int_{\Omega\times[a,\,b]} \Big(c_* |\nabla T|^2 + C_*^{-1} |\nabla S|^2\Big)\, d\mathcal{L}^{n+1}. \end{aligned}$$

(3) For each Lipschitz function $\varphi{:}\, \Omega \to \mathbb{R}$,

$$\begin{aligned} \int_\Omega \varphi\big(Q_a - Q_b\big)\, d\mathcal{L}^n &\leq \kappa_0 \mathrm{Lip}(\varphi) \int_{\Omega\times[a,\,b]} |\nabla T|\, d\mathcal{L}^{n+1} \\ &\leq \kappa_0 \mathrm{Lip}(\varphi) |\Omega|^{\frac{1}{2}} (b-a)^{\frac{1}{2}} \left(\int_{\Omega\times[a,\,b]} |\nabla T|^2\, d\mathcal{L}^{n+1} \right)^{\frac{1}{2}}. \end{aligned}$$

(4) As another estimate on changes in temperature under heat flow we have

$$\begin{aligned} &\int_\Omega \Big|\kappa_K^{\frac{1}{2}} \nabla T(\cdot,\, a)\Big|^2\, d\mathcal{L}^n - \int_\Omega \Big|\kappa_K^{\frac{1}{2}} \nabla T(\cdot,\, b)\Big|^2\, d\mathcal{L}^n \\ &\qquad = 2 \int_{(x,\,t)\in\Omega\times[a,\,b]} c_K(x)\, \overset{\bullet}{T}(x,\, t)^2\, d\mathcal{L}^{n+1}(x,\, t). \end{aligned}$$

(5) As a consequence of (4) we have

$$\int_\Omega \big(T(\cdot,\,a) - T(\cdot,\,b)\big)^2\,d\mathcal{L}^n \le \frac{\kappa_0}{2c_{_S}}(b-a)\,\mathrm{DIR}\big(T(\cdot,\,a)\big).$$

PROOF. With regard to assertion (1) we first suppose the second derivatives of G are bounded. In that case we use the final equation of Theorem 5.1 with $\varphi = G'(T)$ together with integration by parts to infer

$$\int_\Omega G(T(\cdot,\,b))\,c_K\,d\mathcal{L}^n + \int_{\Omega\times(a,\,b)} \mathbb{G}''(T)\,\nabla T\,\kappa_K \nabla T\,d\mathcal{L}^{n+1} = \int_\Omega G(T(\cdot,\,a))\,c_K\,d\mathcal{L}^n.$$

The assertion of (1) now follows by approximation. Note also the calculation

$$\begin{aligned}
-\frac{d}{dt}\int_\Omega c_{_K} G\big(c_{_K}^{-1}Q(\cdot,t)\big)\,d\mathcal{L}^n &= -\int_\Omega c_{_K} G'\left(c_{_K}^{-1}Q(\cdot,\,t)\right)c_{_K}^{-1}\,\dot{Q}\,(\cdot,\,t)\,d\mathcal{L}^n \\
&= -\int_\Omega G'\big(T(\cdot,\,t)\big)\mathrm{div}\big(\kappa_{_K}\nabla T(\cdot,\,t)\big)\,d\mathcal{L}^n \\
&= \int_\Omega G''\big(T(\cdot,\,t)\big)\,\big(\nabla T(\cdot,\,t)\kappa_{_K}\nabla T(\cdot,\,t)\big)\,d\mathcal{L}^n.
\end{aligned}$$

Assertion (2) follows from assertion (1) and inequality A.1.10.

With regard to assertion (3) we compute

$$\begin{aligned}
\int_\Omega \varphi\big(Q_a - Q_b\big)\,d\mathcal{L}^n &= -\int_{\Omega\times[a,\,b]} \varphi\big(\,\dot{Q}\,\big)\,d\mathcal{L}^{n+1} \\
&= -\int_{\Omega\times[a,\,b]} \varphi\,\mathrm{div}\big(\kappa_{_K}\nabla T\big)\,d\mathcal{L}^{n+1} \\
&= \int_{\Omega\times[a,\,b]} \nabla\varphi\kappa_{_K}\nabla T\,d\mathcal{L}^{n+1} \\
&\le \kappa_0 \mathrm{Lip}(\varphi)\int_{\Omega\times[a,\,b]} |\nabla T|\,d\mathcal{L}^{n+1}
\end{aligned}$$

and then use Schwarz's inequality.

With regard to assertion (4) we compute

$$\begin{aligned}
\int_\Omega c_{_K}\,\dot{T}^{\,2}\,d\mathcal{L}^n &= \int_\Omega \dot{T}\,\mathrm{div}\big(\kappa_{_L}\nabla T\big)\,d\mathcal{L}^n \\
&= -\int_\Omega \nabla\dot{T}\,\kappa_{_K}\nabla T\,\mathcal{L}^n \\
&= -\frac{1}{2}\int_\Omega \frac{\partial}{\partial t}\left(\big|\kappa_{_K}^{\frac{1}{2}}\nabla T\big|\right)d\mathcal{L}^n.
\end{aligned}$$

With regard to assertion (5) we compute for $\mathcal{L}^n$ almost every x in Ω,

$$|T(x,\,b)-T(x,\,a)| \le \left|\int_{t\in[a,\,b]} \overset{\bullet}{T}(x,\,t)\,d\mathcal{L}^1 t\right| \le (b-a)^{\frac{1}{2}}\left(\int_{t\in[a,\,b]} \overset{\bullet}{T}(x,\,t)^2\,d\mathcal{L}^1 t\right)^{\frac{1}{2}}$$

and use assertion (4) to estimate

$$(b-a)^{-1}\int_{x\in\Omega} |T(x,\,b)-T(x,\,a)|^2\,d\mathcal{L}^n x \le \int_{\Omega\times[a,\,b]} \overset{\bullet}{T}{}^2\,d\mathcal{L}^{n+1} \le \frac{\kappa_0}{2c_s}\int_\Omega |\nabla T(\cdot,\,a)|^2\,d\mathcal{L}^n.$$

■

5.3 Theorem (Hölder estimates for heat flows of approximating evolutions). *Suppose j is a positive integer, $\Delta t = 2^{-j}$, and $0 < s < t < \infty$. Then*

(1) $\mathbb{E}_j(s) \ge \mathbb{E}_j(t)$

and

(2) In our approximating heat evolution $Q_j(x,\,t)$, we have

$$\left\|Q_j(\cdot,\,t)-Q_j(\cdot,\,s)\right\|_* \le \mathbb{E}_0\,\Delta t^\alpha + \frac{\kappa_0|\Omega|^{\frac{1}{2}}}{\kappa_*^{\frac{1}{2}}\,c_*^{\frac{1}{2}}}(t-s)^{\frac{1}{2}}\,\mathbb{E}_0^{\frac{1}{2}}.$$

PROOF. Suppose k is a positive integer and $t_0 = k\Delta t$. In construction of our approximating evolutions we always have the option of taking $[\![K_j(t_0)]\!] = [\![K_j(t_0-\Delta t)]\!]$ and $Q_j(t_0) = P_j(t_0)$. Hence, it will always be true that

$$\mathbb{E}_j\left(t_0^-\right) - \mathbb{E}_j\left(t_0\right) \ge \frac{1}{\Delta t^\alpha}\|Q_j(t_0)-P_j(t_0)\|_* \ge 0.$$

Such inequalities together with conclusion (2) of Theorem 5.2 imply the first assertion above. Furthermore, summation of such inequalities for t_0's between s and t. is dominated by the first term on the right hand side of the inequality asserted in (2) above.

Now suppose $t_0 - \Delta t \le r < t_0$. We use conclusions (3) and (2) of Theorem 5.2 to infer

$$\begin{aligned}\left\|Q_j(\cdot,\,r)-P_j(\cdot,\,t_0)\right\|_* &\le \kappa_0|\Omega|^{\frac{1}{2}}(t_0-r)^{\frac{1}{2}}\left(\int_{\Omega\times[r,\,t_0]} |\nabla T_j|^2\,d\mathcal{L}^{n+1}\right)^{\frac{1}{2}}\\ &\le \frac{\kappa_0|\Omega|^{\frac{1}{2}}}{\kappa_*^{\frac{1}{2}}\,c_*^{\frac{1}{2}}}\left(t_0-r\right)^{\frac{1}{2}}\left(\mathbb{E}_j(r)-\mathbb{E}_j(t_0^-)\right)^{\frac{1}{2}}.\end{aligned}$$

A corresponding inequality holds if $t_0 < r \le t_0 + \Delta t$. The second term on the right hand side of the inequality asserted in (2) above comes from summation of terms associated with intervals $[t_0,\, t_0 + \Delta t]$, etc. for t_0's between s and t and a straight forward application of Schwarz's inequality for sums, viz

$$\sum_i \Delta t_i^{\frac{1}{2}} \Delta E_i^{\frac{1}{2}} \le \left(\sum_i \Delta t_i \right)^{\frac{1}{2}} \left(\sum_i \Delta E_i \right)^{\frac{1}{2}}$$

for positive numbers Δt_i and ΔE_i. ■

The terminology of the following theorem is that common in the theory of linear parabolic partial differential equations.

5.4 Theorem (Regularity of heat flow across an interface). *Suppose*

(a) $\Theta = \mathbf{U}^{n-1}(0,\, 2) \times \mathbf{U}^1(0,\, 2)$

(b) $0 < \alpha < 1$ *and* $\sigma\colon \mathbf{U}^{n-1}(0,\, 2) \to \mathbf{U}^1(0,\, 1/2)$ *is Hölder continuously differentiable with exponent* α. *The means that*

$$\Gamma = \Theta \cap \left\{ x\colon x_n = \sigma\,(x_1,\, \ldots,\, x_{n-1}) \right\}$$

is a Hölder continuously differentiable hypersurface in Θ *and*

$$\Theta^+ = \Theta \cap \left\{ x\colon x_n > \sigma\,(x_1,\, \ldots,\, x_{n-1}) \right\}, \quad \Theta^- = \Theta \cap \left\{ x\colon x_n < \sigma\,(x_1,\, \ldots,\, x_{n-1}) \right\}$$

are regions which share the boundary Γ.

(c) $\gamma^0\colon \Theta \sim \Gamma \to [1,\, \infty)$ *is constant in each component of* $\Theta \sim \Gamma$*; we also write* $\gamma^0(x,\, t) = \gamma^0(x)$.

(d) $\kappa^0\colon \Theta \sim \Gamma \to \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ *is constant in each component of* $\Theta \sim \Gamma$. *For each* x, $\kappa^0(x) = \left\{ \kappa^0_{ij}(x) \right\}$ *is a symmetric* $n \times n$ *matrix with* $v \bullet \left\langle v,\, \kappa^0(x) \right\rangle \ge |v|^2$ *for each* v*; we also write* $\kappa^0(x,\, t) = \kappa^0(x)$.

(e) $u\colon \Theta \times (-2,\, 0] \to \mathbb{R}$ *is a strictly defined function which is a weak solution of the partial differential equation*

$$\gamma^0(x) \frac{\partial u}{\partial t}(x,\, t) = \sum_{j=1}^n \frac{\partial}{\partial x_i} \left(\sum_{i=1}^n \kappa^0_{ij}(x) \, \frac{\partial u}{\partial x_i}(x,\, t) \right).$$

By this we mean that u belongs to $\mathbb{C}^0\big((-2,\,0],\mathbb{H}^1(\Theta)\big)$ and that, for each test function φ in $\mathcal{D}\big(\Theta\times(-2,\,2)\big)$ and each $-2<a<b\le 0$,

$$\begin{aligned}0=\int_{(x,\,t)\in\Theta\times(a,\,b)}\Bigg(&-\gamma^0(x)\,\frac{\partial\varphi}{\partial t}(x,\,t)\;u(x,\,t)\\&+\sum_{j=1}^{n}\frac{\partial\varphi}{\partial x_j}(x,\,t)\sum_{i=1}^{n}\bigg(\kappa^0_{ij}(x)\,\frac{\partial u}{\partial x_i}(x,\,t)\bigg)\Bigg)\,d\mathcal{L}^{n+1}(x,\,t)\\&\qquad+\int_{x\in\Theta}\gamma^0(x)\big(\varphi(x,\,b)\,u(x,\,b)-\varphi(x,\,a)\,u(x,\,a)\big)\,d\mathcal{L}^n x.\end{aligned}$$

Then u is continuous and, corresponding to each compact subset K of $\Theta\times(-2,\,0]$, there is a constant C_K with the following property: Suppose $(x,\,s)$ and $(y,\,s)$ and $(y,\,t)$ belong to K such that either x and y both belong to Θ^+ or they both belong to Θ^-. Then

$$\|\nabla u(x,\,s)-\nabla u(y,\,s)\|\le C_K|x-y|^\alpha\quad\text{and}\quad|u(y,\,s)-u(y,\,t)|\le C_K\,|s-t|^{\frac{1+\alpha}{2}};$$

$$\text{and}\quad\|\nabla u(y,\,s)-\nabla u(y,\,t)\|\le C_K|s-t|^{\frac{\alpha}{2}}.$$

In particular, the restriction of u to $\Theta^+\times(-2,\,0]$ $\big[$resp. to $\Theta^-\times(-2,\,0]$ $\big]$ has an extension to a neighborhood of $\Theta^+\cup\Gamma\times(-2,\,0]$ $\big[$resp. of $\Theta^-\cup\Gamma\times(-2,\,0]$ $\big]$ for which the inequalities above still hold.

PROOF. For $0<r\le 1$ and $(x,\,t)\in\mathbb{R}^n\times\mathbb{R}$ we set

$$Q_r=\mathbb{B}^n(0,\,r)\times(-r^2,\,0]\quad\text{and}\quad Q_r(x,\,t)=Q_r+\{(x,\,t)\}\ \text{(set addition)}.$$

In order to have notation for the average value of a function we set

$$⨍_A h\,d\mathcal{L}^{n+1}=\frac{1}{\mathcal{L}^{n+1}(A)}\int_A h\,d\mathcal{L}^{n+1}$$

for appropriate h and A. Our proof is in several steps. We will use C to denote a general constant which depends only on n and bounds for σ, γ^0 and κ^0 and choices of cutoff functions, etc. whose value may be different in different places, even in the same formula.

Step 1 (Application of Campanato's embedding theorem). In Step 5 below we introduce a Hölder continuously differentiable change of coordinates ψ for our domain Θ and hence for our function u. In these new coordinates our interface Γ becomes part of the hyperplane $x_n=0$, and our function u becomes a new function $U(x,\,t)=u\big(\psi^{-1}(x),\,t\big)$

which satisfies a new perturbed partial differential equation. We denote by Ξ^+ the domain $\psi(\Theta^+) \cap \mathbb{B}^n(0,\,1) \times (-1,\,0]$ and by Ξ^- the domain $\psi(\Theta^-) \cap \mathbb{B}^n(0,\,1) \times (-1,\,0]$, and by Ξ the domain $\psi(\Theta) \cap \mathbb{B}^n(0,\,1) \times (-1,\,0]$; the domains $\Xi^\pm$ are Lipschitz domains. According to S. Campanato's Theorem 2.I in [CS] we know the following:

Suppose $U\colon \Xi \to \mathbb{R}$ is square summable and strictly defined and additionally has square summable first derivatives. Suppose also

$$\fint_{Q_r(x,\,t)\cap\Xi^\pm} \left|\nabla U - \overline{\nabla U}\right|^2 \, d\mathcal{L}^{n+1} \leq C r^{2\alpha} \tag{$*$}$$

for each $(x,\,t)$ in $\Xi^\pm \cup \partial\Xi^\pm$ and each $0 < r \leq 1$. Here and elsewhere we denote by $\overline{\nabla U}$ the average value of ∇U over the domain under consideration.

Then there is a constant C_0 such that that whenever $(x,\,s)$ and $(y,\,s)$ and $(y,\,t)$ belong to $\Xi^\pm$, then

$$\|\nabla U(x,\,s) - \nabla U(y,\,s)\| \leq C_0|x-y|^\alpha \quad \text{and} \quad |U(y,\,s) - U(y,\,t)| \leq C_0\,|s-t|^{\frac{1+\alpha}{2}}.$$

$$\text{and} \quad \|\nabla U(y,\,s) - \nabla U(y,\,t)\| \leq C_0|s-t|^{\frac{\alpha}{2}}.$$

This conclusion together with assertion (2) of Step 3 below imply the assertions of the present theorem. We will show below that the condition $(*)$ is implied by the conditions (a), (b), (c), (d) below for some finite constant C. It is these four conditions which we will then prove. In order to state these conditions we define a parabolic distance function $\delta_\star$ by setting

$$\delta_\star\big((x,\ s)\,,\,(y,\,t)\big) = \begin{cases} \big(|x-y|^2 + |s-t|\big)^{\frac{1}{2}} & \text{if } s \geq t \\ \infty & \text{if } s < t \end{cases}.$$

Our conditions are the following:

(a) Corresponding to each $(x_0,\,t_0) \in \Xi^\pm$ with

$$s = \delta_\star\big((x_0,\,t_0),\,\partial\Xi^\pm\big) = \inf\left\{\delta_\star\big((x_0,\,t_0),\,(x,\,t)\big)\colon (x,\,t) \in \partial\Xi^\pm\right\} < \infty$$

there is a polynomial function $L_0^\pm\colon\mathbb{R}^n \to \mathbb{R}$ of degree one with

$$|L_0^\pm(x_0)| + \|\nabla L_0^\pm(x_0)\|\,s \leq C\left(\fint_{Q_{s/2}(x_0,\,t_0)} U^2\,d\mathcal{L}^{n+1}\right)^{\frac{1}{2}}$$

such that for each $0 < r \le s$

$$\fint_{(x,\,t)\in Q_{r/2}(x_0,\,t_0)} |U(x,\,t) - L_0^{\pm}(x)|^2 \, d\mathcal{L}^{n+1}(x,\,t) \le C \left(\frac{r}{s}\right)^{2(1+\alpha)} \fint_{Q_s(x_0,\,t_0)} U^2 \, d\mathcal{L}^{n+1}.$$

(b) Corresponding to each $(x_1,\, t_1) \in \partial \Xi^{\pm}$ there is a unique polynomial function $L_1^{\pm}\colon \mathbb{R}^n \to \mathbb{R}$ of degree one with

$$|L_1^{\pm}(x_1)| + \|\nabla L_1^{\pm}(x_1)\| \le C$$

such that for each $0 < r \le 4$

$$\fint_{(x,\,t)\in Q_r(x_1,\,t_1)\cap\Xi^{\pm}} |U(x,\,t) - L_1^{\pm}(x)|^2 \, d\mathcal{L}^{n+1}(x,\,t) \le C\, r^{2(1+\alpha)}.$$

(c) If $(x_0,\, t_0) \in \Xi^{\pm}$ and $(x_1,\, t_1) \in \partial\Xi^{\pm}$ with

$$s = \delta_\star\big((x_0,\, t_0),\, (x_1,\, t_1)\big) = \delta_\star\big((x_0,\, t_0),\, \partial\Xi^{\pm}\big) < \infty$$

and if $L_1^{\pm}\colon \mathbb{R}^n \to \mathbb{R}$ is the unique polynomial function of degree one given in (b) associated with $(x_1,\, t_1)$ then the following in true: There is a polynomial function $L_0^{\pm}\colon \mathbb{R}^n \to \mathbb{R}$ of degree one with

$$|L_0^{\pm}(x_0)| + \|\nabla L_0^{\pm}(x_0)\|\, s \le C \left(\fint_{Q_{s/2}(x_0,\,t_0)} \big(U - L_1^{\pm}\big)^2 \, d\mathcal{L}^{n+1} \right)^{\frac{1}{2}} + C\, s^{1+\alpha}$$

such that for each $0 < r \le s$

$$\fint_{(x,\,t)\in Q_{r/2}(x_0,\,t_0)} |U(x,\,t) - L_1^{\pm}(x) - L_0^{\pm}(x)|^2 \, d\mathcal{L}^{n+1}(x,\,t)$$

$$\le C \left(\frac{r}{s}\right)^{2(1+\alpha)} \left(\fint_{Q_s(x_0,\,t_0)} \big(U - L_1\big)^2 \, d\mathcal{L}^{n+1} + C\, s^{2(1+\alpha)} \right).$$

(d) If $L^{\pm}\colon \mathbb{R}^n \to \mathbb{R}$ are any polynomial functions of degree one then

$$r^2 \fint_{Q_r(x,\,t)\cap\Xi^{\pm}} |\nabla U - \nabla L^{\pm}|^2 \, d\mathcal{L}^{n+1}$$

$$\le C \left(\fint_{Q_{2r}(x,\,t)\cap\Xi} |U - L^{\pm}|^2 \, d\mathcal{L}^{n+1} + \|\nabla L^{\pm}(0)\|^2\, r^{2(1+\alpha)} \right)$$

whenever $(x,\, t) \in \Xi^{\pm} \cup \partial\Xi^{\pm}$ and $0 < r \leq 1$. Note that the domain of integration of the right hand integral is $Q_{2r}(x,\, t) \cap \Xi$ not $Q_{2r}(x,\, t) \cap \Xi^{\pm}$. Also, in the right hand integral

$$L^{\pm}(x) = \begin{cases} L^{+}(x) & \text{if } x \in \Xi^{+} \\ L^{-}(x) & \text{if } x \in \Xi^{-} \end{cases}.$$

We now show that our assumptions (a), (b), (c), (d) above imply the condition $(*)$.

In case $(x_1,\, t_1) \in \partial\Xi^{\pm}$ and $0 < r \leq 1$ we infer from (d) and (b) that

$$\begin{aligned} &⨍_{Q_r(x_1,\, t_1)\cap\Xi^{\pm}} \left|\nabla U - \overline{\nabla U}\right|^2 \, d\mathcal{L}^{n+1} \\ &\qquad \leq ⨍_{Q_r(x_1,\, t_1)\cap\Xi^{\pm}} |\nabla U - \nabla L_1^{\pm}|^2 \, d\mathcal{L}^{n+1} \\ &\qquad \leq \frac{C}{r^2}\left(⨍_{Q_{2r}(x_1,\, t_1)\cap\Xi} |U - L_1^{\pm}|^2 \, d\mathcal{L}^{n+1} + \|\nabla L_1^{\pm}(0)\|^2 r^{2(1+\alpha)}\right) \\ &\qquad \leq C\, r^{2\alpha}. \end{aligned}$$

In case $(x_0,\, t_0) \in \Xi^{\pm}$, we pick $(x_1,\, t_1) \in \partial\Xi^{\pm}$ such that

$$s = \delta_\star\big((x_0,\, t_0),\, (x_1,\, t_1)\big) = \delta_\star\big((x_0,\, t_0),\, \partial\Xi^{\pm}\big).$$

Then (b) implies the existence of $L_1^{\pm}$ such that

$$⨍_{Q_r(x_1,\, t_1)\cap\Xi^{\pm}} |U - L_1^{\pm}|^2 \, d\mathcal{L}^{n+1} \leq C\, r^{2(1+\alpha)}$$

for each $0 < r \leq 4$.

Assuming $0 < r \leq 1$ we use (d), (c), (b) and (a) to infer the existence of $L_0^{\pm}$ with $\|L_0\| \leq C\, s^{\alpha}$ such that either $0 < 4r < s \geq 1$ and

$$\fint_{Q_r(x_0,\,t_0)} \left|\nabla U - \overline{\nabla U}\right|^2 \, d\mathcal{L}^{n+1}$$

$$\leq \fint_{Q_r(x_0,\,t_0)} |\nabla U - \nabla L_1^{\pm} - \nabla L_0^{\pm}|^2 \, d\mathcal{L}^{n+1}$$

$$\leq \frac{C}{r^2}\left(\fint_{Q_{2r}(x_0,\,t_0)} |U - L_1^{\pm} - L_0^{\pm}|^2 \, d\mathcal{L}^{n+1} + \|\nabla L_1^{\pm}(0) + \nabla L_0^{\pm}(0)\|^2 \, r^{2(1+\alpha)}\right)$$

$$\leq \frac{C}{r^2}\Bigg[\left(\frac{4r}{s}\right)^{2(1+\alpha)}\left(\fint_{Q_s(x_0,\,t_0)} |U - L_1^{\pm}|^2 \, d\mathcal{L}^{n+1} + C\, s^{2(1+\alpha)}\right)$$

$$+ \|\nabla L_1^{\pm}(0) + \nabla L_0^{\pm}(0)\|^2 \, r^{2(1+\alpha)}\Bigg]$$

$$\leq C\, r^{2\alpha}$$

or $0 < 4r < s < 1$ and

$$\fint_{Q_r(x_0,\,t_0)} \left|\nabla U - \overline{\nabla U}\right|^2 \, d\mathcal{L}^{n+1}$$

$$\leq \fint_{Q_r(x_0,\,t_0)} |\nabla U - \nabla L_1^{\pm} - \nabla L_0^{\pm}|^2 \, d\mathcal{L}^{n+1}$$

$$\leq \frac{C}{r^2}\left(\fint_{Q_{2r}(x_0,\,t_0)} |U - L_1^{\pm} - L_0^{\pm}|^2 \, d\mathcal{L}^{n+1} + \|\nabla L_1^{\pm}(0) + \nabla L_0^{\pm}(0)\|^2 \, r^{2(1+\alpha)}\right)$$

$$\leq \frac{C}{r^2}\Bigg[\left(\frac{4r}{s}\right)^{2(1+\alpha)}\left(\fint_{Q_s(x_0,\,t_0)} |U - L_1^{\pm}|^2 \, d\mathcal{L}^{n+1} + C\, s^{2(1+\alpha)}\right)$$

$$+ \|\nabla L_1^{\pm}(0) + \nabla L_0^{\pm}(0)\|^2 \, r^{2(1+\alpha)}\Bigg]$$

$$\leq \frac{C}{r^2}\Bigg[\left(\frac{4r}{s}\right)^{2(1+\alpha)}\left(\fint_{Q_{2s}(x_1,\,t_1)\cap\Xi^{\pm}} |U - L_1^{\pm}|^2 \, d\mathcal{L}^{n+1} + C\, s^{2(1+\alpha)}\right)$$

$$+ \|\nabla L_1^{\pm}(0) + \nabla L_0^{\pm}(0)\|^2 \, r^{2(1+\alpha)}\Bigg]$$

$$\leq \frac{C}{r^2}\left(\frac{2r}{s}\right)^{2(1+\alpha)}\left(C\,(2s)^{2(1+\alpha)} + C\, s^{2(1+\alpha)}\right) + \|\nabla L_1^{\pm}(0) + \nabla L_0^{\pm}(0)\|^2 \, r^{2(1+\alpha)}$$

$$\leq C\, r^{2\alpha}$$

or $4r \geq s$ and

$$\fint_{Q_r(x_0,\,t_0)\cap\Xi^\pm} \left|\nabla U - \overline{\nabla U}\right|^2 d\mathcal{L}^{n+1} \leq \fint_{Q_r(x_0,\,t_0)\cap\Xi^\pm} |\nabla U - \nabla L_1^\pm - \nabla L_0^\pm|^2 \, d\mathcal{L}^{n+1}$$

$$\leq \frac{C}{r^2}\left(\fint_{Q_{2r}(x_0,\,t_0)\cap\Xi} |U - L^\pm - L_1^\pm|^2 \, d\mathcal{L}^{n+1} + \|\nabla(L_1^\pm) + \nabla(L_0^\pm)\| \, r^{2(1+\alpha)}\right)$$

$$\leq \frac{C}{r^2}\Bigg(2\fint_{Q_{2r}(x_0,\,t_0)\cap\Xi} |U - L_1^\pm|^2 \, d\mathcal{L}^{n+1} + 2\fint_{Q_{2r}(x_0,\,t_0)\cap\Xi} |L_0^\pm|^2 \, d\mathcal{L}^{n+1}$$

$$+ \|\nabla L_1^\pm(0) + \nabla L_0^\pm(0)\|^2 \, r^{2(1+\alpha)}\Bigg)$$

$$\leq \frac{C}{r^2}\Bigg(2\fint_{Q_{4r}(x_1,\,t_1)\cap\Xi} |U - L_1^\pm|^2 \, d\mathcal{L}^{n+1} + C\fint_{Q_{s/2}(x_0,\,t_0)\cap\Xi} |U - L_1^\pm|^2 \, d\mathcal{L}^{n+1}$$

$$+ \|\nabla L_1^\pm(0) + \nabla L_0^\pm(0)\|^2 \, r^{2(1+\alpha)}\Bigg)$$

$$\leq C\, r^{2\alpha} + + \frac{C}{r^2}\fint_{Q_s(x_1,\,t_1)\cap\Xi} |U - L_1^\pm|^2 \, d\mathcal{L}^{n+1} \leq C\, r^{2\alpha}.$$

Step 2 (Local energy estimates). Suppose

(b) Σ is a Hölder continuously differentiable hypersurface in $\mathbb{B}^n(0,\,1)$.

(c) $\gamma\colon \mathbb{B}^n(0,\,1) \sim \Sigma \to [1,\,\infty)$ is continuous and bounded; we also write $\gamma(x,\,t) = \gamma(x)$.

(d) $\kappa\colon \mathbb{B}^n(0,\,1) \sim \Sigma \to \operatorname{Hom}(\mathbb{R}^n,\,\mathbb{R}^n)$ is Hölder continuous with exponent α in each component of its domain such that, for each x, $\kappa(x) = \{\kappa_{ij}(x)\}$ is a symmetric $n \times n$ matrix with $v \bullet \langle v,\,\kappa(x)\rangle \geq |v|^2$ for each v; we also write $\kappa(x,\,t) = \kappa(x)$.

(e) $f\colon \mathbb{B}^n(0,\,1) \sim \Sigma \to \mathbb{R}^n$ is Hölder continuous in each component of its domain; we also write $f(x,\,t) = f(x)$.

(f) $g\colon Q_1 \sim \Sigma \times (-1,\,0] \to \mathbb{R}$ is continuous and bounded; in our applications g will be the zero function.

(g) $w\colon Q_1 \to \mathbb{R}$ is a weak solution of the partial differential equation

$$\gamma(x)\frac{\partial w}{\partial t}(x,\,t) = \sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{i=1}^{n} \kappa_{ij}(x)\,\frac{\partial w}{\partial x_i}(x,\,t) + f^j(x)\right) + g(x,\,t).$$

By this we mean that w belongs to $\mathbb{C}^0\big((-1,\,0],\,\mathbb{H}^1(\mathbb{B}^n(0,\,1))\big)$ and that, for each test function φ in $\mathcal{D}\big(Q_1\big)$ and each $-1 < a < b \leq 0$,

$$0 = \int_{(x,\,t)\in\Theta\times(a,\,b)} \Bigg(-\gamma(x)\frac{\partial\varphi}{\partial t}(x,\,t)\,w(x,\,t) + \sum_{j=1}^{n} \frac{\partial\varphi}{\partial x_j}(x,\,t) \sum_{i=1}^{n} \Big(\kappa_{ij}(x)\frac{\partial w}{\partial x_i}(x,\,t) + f^j(x) \Big)$$

$$- \varphi(x,\,t)\,g(x,\,t) \Bigg)\, d\mathcal{L}^{n+1}(x,\,t) + \int_{x\in\Theta} \gamma(x)\big(\varphi(x,\,b)\,w(x,\,b) - \varphi(x,\,a)\,w(x,\,a)\big)\, d\mathcal{L}^n x.$$

Then we have the inequality

$$\int_{Q_{\frac{1}{2}}} \left(\frac{\partial w}{\partial t}\right)^2 + |\nabla w|^2\,\mathcal{L}^{n+1} \leq C \int_{Q_1} w^2 + |f|^2 + g^2\, d\mathcal{L}^{n+1}.$$

Proof. We denote by $\eta\colon Q_1 \to [0,\,1]$ a smooth cutoff function which equals 1 in $Q_{\frac{1}{2}}$ and vanishes outside $Q_{\frac{3}{4}}$. We first obtain an estimate on ∇w. We compute formally (to be interpreted in the sense of distributions) using integration by parts with respect to both space and time variables, Schwarz's inequality, and the general inequality $2ab \leq \epsilon^2 a^2 + \epsilon^{-2} b^2$.

$$\begin{aligned}
0 &= \int_{Q_1} \eta^2 w \Bigg(\gamma\,\frac{\partial w}{\partial t} - \sum_j \frac{\partial}{\partial x_j} \Big(\sum_i \kappa_{ij}\,\frac{\partial w}{\partial x_i} + f^j \Big) - g \Bigg)\, d\mathcal{L}^{n+1} \\
&= \int_{Q_1} \frac{\partial}{\partial t} \left(\frac{\eta^2\,\gamma\,w^2}{2} \right) d\mathcal{L}^{n+1} - \int_{Q_1} \frac{\partial \eta}{\partial t}\,\eta\,\gamma\,w^2\, d\mathcal{L}^{n+1} \\
&\qquad + \int_{Q_1} \sum_{ij} \frac{\partial(\eta\,w)}{\partial x_j} \kappa_{ij} \frac{\partial(\eta\,w)}{\partial x_i}\, d\mathcal{L}^{n+1} - \int_{Q_1} \sum_{ij} \frac{\partial \eta}{\partial x_j}\, w \kappa_{ij} \frac{\partial \eta}{\partial x_i}\, w\, d\mathcal{L}^{n+1} \\
&\qquad\quad + \int_{Q_1} \sum_j \frac{\partial \eta}{\partial x_j}\, f^j\,\eta\,w + \eta\, f^j\, \frac{\partial(w\,\eta)}{\partial x_j} - \eta^2\, w\, g\; d\mathcal{L}^{n+1} \\
&\geq \int_{x\in\mathbf{B}^n(0,\,3/4)} \frac{\eta^2(x,\,0)\,\gamma(x)\,w^2(x,\,0)}{2}\, d\mathcal{L}^n x - C \int_{Q_1} w^2\, d\mathcal{L}^{n+1} \\
&\qquad + \int_{Q_1} |\nabla(\eta\,w)|^2\,\mathcal{L}^{n+1} - C \int_{Q_1} |f|^2\, d\mathcal{L}^{n+1} \\
&\qquad - \frac{1}{2} \int_{Q_1} |\nabla(\eta\,w)|^2\,\mathcal{L}^{n+1} - C \int_{Q_1} g^2\, d\mathcal{L}^{n+1}
\end{aligned}$$

We infer

$$\int_{Q_{\frac{1}{2}}} |\nabla w|^2 \, d\mathcal{L}^{n+1} \leq \int_{Q_1} |\nabla(\eta \, w)|^2 \, d\mathcal{L}^{n+1} \leq C \int_{Q_1} w^2 + |f|^2 + g^2 \, d\mathcal{L}^{n+1}.$$

In order to control $\frac{\partial w}{\partial t}$ we estimate

$$\begin{aligned}
0 &= \int_{Q_1} \eta^2 \frac{\partial w}{\partial t} \left(\gamma \, \frac{\partial w}{\partial t} - \sum_j \frac{\partial}{\partial x_j} \left(\sum_i \kappa_{ij} \frac{\partial w}{\partial x_i} + f^j \right) - g \right) \mathcal{L}^{n+1} \\
&\geq \int_{Q_1} \eta^2 \, \gamma \, \left(\frac{\partial w}{\partial t} \right)^2 \, d\mathcal{L}^{n+1} \\
&\quad + \int_{Q_1} \left[\frac{1}{2} \frac{\partial}{\partial t} \left(\eta^2 \sum_{ij} \frac{\partial w}{\partial x_j} \kappa_{ij} \frac{\partial w}{\partial x_i} \right) - \frac{\partial \eta}{\partial t} \, \eta \sum_{ij} \frac{\partial w}{\partial x_j} \kappa_{ij} \frac{\partial w}{\partial x_i} \right] d\mathcal{L}^{n+1} \\
&\quad\quad - \frac{1}{4} \int_{Q_1} \eta^2 \, \left(\frac{\partial w}{\partial t} \right)^2 \, d\mathcal{L}^{n+1} - C \int_{Q_1} \eta^2 \, |\nabla w|^2 \, d\mathcal{L}^{n+1} - C \int_{Q_1} \eta^2 \, g^2 \, d\mathcal{L}^{n+1} \\
&\quad\quad\quad - \frac{1}{4} \int_{Q_1} \eta^2 \, \left(\frac{\partial w}{\partial t} \right)^2 \, d\mathcal{L}^{n+1} + \int_{Q_1} \sum_j \frac{\partial}{\partial x_j} \left(\eta^2 \, \frac{\partial w}{\partial t} \right) \, f^j \, d\mathcal{L}^{n+1}
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{2} \int_{Q_1} \eta^2 \, \gamma(x) \, \left(\frac{\partial w}{\partial t} \right)^2 \, d\mathcal{L}^{n+1} \\
&\quad + \frac{1}{2} \int_{x \in \mathbf{B}^n(0,\,3/4)} \eta(x,\,0)^2 \sum_{ij} \frac{\partial w}{\partial x_j}(x,\,0) \, \kappa_{ij}(x) \, \frac{\partial w}{\partial x_i}(x,\,0) \, d\mathcal{L}^n x \\
&\quad\quad - C \int_{Q_1} \eta^2 \, |\nabla w|^2 \, d\mathcal{L}^{n+1} - C \int_{Q_1} g^2 \, d\mathcal{L}^{n+1} + \int_{Q_1} \sum_j \frac{\partial}{\partial x_j} \left(\eta^2 \, \frac{\partial w}{\partial t} \right) \, f^j \, d\mathcal{L}^{n+1} \\
& \hspace{30em} . \\
&\geq \frac{1}{2} \int_{Q_1} \eta^2 \, \gamma(x) \, \frac{\partial w}{\partial t}^2 \, d\mathcal{L}^{n+1} \\
&\quad + \frac{1}{2} \int_{x \in \mathbf{B}^n(0,\,3/4)} \eta(x,\,0)^2 |\nabla w(x,\,0)|^2 \, d\mathcal{L}^n x \\
&\quad\quad - C \int_{Q_1} \eta^2 \, |\nabla w|^2 \, d\mathcal{L}^{n+1} - C \int_{Q_1} g^2 \, d\mathcal{L}^{n+1} + \int_{Q_1} \sum_j \frac{\partial}{\partial x_j} \left(\eta^2 \, \frac{\partial w}{\partial t} \right) \, f^j \, d\mathcal{L}^{n+1}
\end{aligned}$$

We estimate the final term above

$$
\begin{aligned}
&\int_{Q_1} \frac{\partial}{\partial x_j}\left(\eta^2 \frac{\partial w}{\partial t}\right) f^j \, d\mathcal{L}^{n+1} \\
&\quad = \int_{Q_1} 2\eta \frac{\partial \eta}{\partial x_j} \frac{\partial w}{\partial t} f^j \, d\mathcal{L}^{n+1} + \int_{Q_1} \eta^2 \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x_j} f^j\right) d\mathcal{L}^{n+1} \\
&\quad \geq -\frac{1}{4}\int_{Q_1} \eta^2 \left(\frac{\partial w}{\partial t}\right)^2 d\mathcal{L}^{n+1} - C\int_{Q_1} |\nabla \eta|^2 \, |f|^2 \, d\mathcal{L}^n \\
&\qquad + \int_{Q_1} \eta^2 \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x_j} f^j\right) d\mathcal{L}^{n+1} \\
&\quad = -\frac{1}{4}\int_{Q_1} \eta^2 \left(\frac{\partial w}{\partial t}\right)^2 d\mathcal{L}^{n+1} - C\int_{Q_1} |f|^2 \, d\mathcal{L}^{n+1} \\
&\qquad + \int_{x\in \mathbf{B}^n(0,\,3/4)} \eta^2(x,\,0) \frac{\partial w}{\partial x_j}(x,\,0)\, f^j(x)\, d\mathcal{L}^n x - \int_{Q_1} 2\eta \frac{\partial \eta}{\partial t} \frac{\partial w}{\partial x_j} f^j \, d\mathcal{L}^{n+1} \\
&\quad \geq -\frac{1}{4}\int_{Q_1} \eta^2 \frac{\partial w}{\partial t}^2 d\mathcal{L}^{n+1} - C\int_{Q_1} |f|^2 \, d\mathcal{L}^{n+1} \\
&\qquad - \frac{1}{4n^2}\int_{x\in \mathbf{B}^n(0,\,3/4)} \eta^2(x,\,0)\, |\nabla w(x,\,0)|^2 \, d\mathcal{L}^n x \\
&\qquad\quad - C\int_{x\in \mathbf{B}^n(0,\,3/4)} \eta^2(x,\,0)\, |f(x)|^2 \, d\mathcal{L}^n x - \frac{1}{4n^2}\int_{Q_1} \eta^2 \, |\nabla w|^2 \, d\mathcal{L}^{n+1}
\end{aligned}
$$

We note that

$$
\eta^2 |\nabla w|^2 = |\eta\, \nabla w + \nabla \eta\, w - \nabla \eta\, w|^2 \leq 2|\nabla(\eta\, w)|^2 + 2|\nabla \eta|^2 w^2
$$

and that

$$
\int_{x\in \mathbf{B}^n(0,\,3/4)} |f(x)|^2 \, d\mathcal{L}^n x = \int_{(x,\,t)\in \mathbf{B}^n(0,\,3/4)\times(-1,\,0]} |f(x,\,t)|^2 \, d\mathcal{L}^{n+1}(x,\,t) \leq \int_{Q_1} |f|^2 \, d\mathcal{L}^{n+1}
$$

and combine our estimates to obtain

$$
\begin{aligned}
\int_{Q_{\frac{1}{2}}} \left(\frac{\partial w}{\partial t}\right)^2 d\mathcal{L}^{n+1} &\leq \int_{Q_1} \eta^2 \left(\frac{\partial w}{\partial t}\right)^2 d\mathcal{L}^{n+1} \\
&\leq C\int_{Q_1} |\nabla w|^2 + C\int_{Q_1} |f|^2 + g^2 \, d\mathcal{L}^{n+1} \leq C\int_{Q_1} w^2 + |f|^2 + g^2 \, d\mathcal{L}^{n+1}.
\end{aligned}
$$

Step 3 (Regularity in a special case). Suppose that the assumptions of Step 2 above hold except that our hypothesis (b) there is specialized so that

(b*) $\sigma(x) = 0$ for each x so that our hypersurface Γ is part of the vector subspace $x_n = 0$. Then

(1) The restriction of u to each compact subset of $\Theta \times (-2, 0]$ is Lipschitz.

(2) The restriction u^+ of u to $\Theta^+ \cup \Gamma$ is infinitely differentiable as is the the restriction u^- of u to $\Theta^- \cup \Gamma$.

(3) There is a constant C (independent of u) such that

$$|u(0, 0)|^2 + \|\nabla u^+(0, 0)\|^2 + \|\nabla u^-(0, 0)\|^2 \leq C \int_{Q_1} u^2 \, d\mathcal{L}^{n+1}.$$

(4) For each $0 < r < 1$,

$$\frac{1}{r^4} \fint_{Q_r \cap \Theta^+} \left| u - u(0, 0) - \nabla u^+(0, 0) \right|^2 d\mathcal{L}^{n+1} \leq C \int_{Q_1} u^2 \, d\mathcal{L}^{n+1}$$

and

$$\frac{1}{r^4} \fint_{Q_r \cap \Theta^-} \left| u - u(0, 0) - \nabla u^-(0, 0) \right|^2 d\mathcal{L}^{n+1} \leq C \int_{Q_1} u^2 \, d\mathcal{L}^{n+1}.$$

Proof. We infer from our local energy estimate in Step 2 that the restriction of u to the interior Θ_0 of any compact subset of $\Theta \times (-2, 0]$ has square summable first derivatives, i.e. belongs to $\mathbb{H}^1(\Theta_0)$. This implies that the restriction of a strictly defined u to $\Gamma \cap \Theta_0$ belongs to belongs to $\mathbb{H}^{\frac{1}{2}}(\Gamma \cap \Theta_0)$. Since our interface is a subset of the hyperplane $x_n = 0$ and our partial differential equation is linear we infer that any partial derivative of u which does not require differentiation in the x_n variable itself must be a weak solution to our partial differential equation, hence itself must belong to $\mathbb{H}^1(\Theta_0)$ so that restrictions to Γ belong to $\mathbb{H}^{\frac{1}{2}}(\Gamma \cap \Theta_0)$. We infer that the restriction of u to Γ must be infinitely differentiable. The assertions above now follow from the parabolic Schauder estimates as set forth in Theorem 10.1 on page 351 of [LSU].

Step 4 (An approximation estimate). For each $\epsilon > 0$ there is $\delta > 0$ with the following property: Suppose $\gamma^\star$ and $\kappa^\star$ denote what the functions γ^0 and κ^0 respectively of our original hypotheses would be in case $\sigma = 0$ and Γ were part of the vector subspace $x_n = 0$. Suppose also

(b*) $\Sigma = Q_1 \cap \{x : x_n = 0\}$

(c*) $\gamma\colon Q_1 \sim \Sigma \to [1, \infty)$ is continuous and bounded with

$$\|\gamma - \gamma^\star\|_{\mathbf{L}_2} \le \delta.$$

(d*) $\kappa\colon Q_1 \sim \Sigma \to \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ is Hölder continuous with exponent α such that, for each x, $\kappa(x) = \{\kappa_{ij}(x)\}$ is a symmetric $n \times n$ matrix with $v \bullet \langle v, \kappa(x)\rangle \ge |v|^2$ for each v. Also

$$\|\kappa - \kappa^\star\|_{\mathbf{L}_2} \le \delta.$$

(e*) $f\colon Q_1 \sim \Sigma \to \mathbb{R}^n$ is Hölder continuous with

$$\|f\|_{\mathbf{L}_2} \le \delta.$$

(f*) $g\colon Q_1 \sim \Sigma \times (-2, 0] \to \mathbb{R}$ is continuous and bounded with

$$\|g\|_{\mathbf{L}_2} \le \delta.$$

(g*) $w\colon Q_1 \to \mathbb{R}$ is a weak solution of the partial differential equation

$$\gamma(x)\frac{\partial w}{\partial t}(x, t) = \sum_{j=1}^n \frac{\partial}{\partial x_i}\left(\sum_{i=1}^n \kappa_{ij}(x)\,\frac{\partial w}{\partial x_i}(x, t) + f^j(x)\right) + g(x, t).$$

with

$$\int_{Q_1} u^2\, d\mathcal{L}^{n+1} \le 1.$$

By this we mean in particular that w belongs to $\mathbb{C}^0\big((-1, 0], \mathbb{H}^1(\mathbb{B}^n(0, 1))\big)$ and that, for each test function φ in $\mathcal{D}\big(Q_1\big)$ and each $-1 < a < b \le 0$,

$$\begin{aligned} 0 = \int_{(x,\,t)\in\mathbf{B}^n(0,\,1)\times(a,\,b)} \Bigg(&-\gamma(x)\frac{\partial\varphi}{\partial t}(x, t)\, w(x, t) \\ &+ \sum_{j=1}^n \frac{\partial\varphi}{\partial x_j}(x, t)\sum_{i=1}^n\left(\kappa_{ij}(x)\frac{\partial w}{\partial x_i}(x, t) + f^j(x)\right) - \varphi(x, t)\, g(x, t)\Bigg)\, d\mathcal{L}^{n+1}(x, t) \\ &+ \int_{x\in\Theta} \gamma(x)\big(\varphi(x, b)\, w(x, b) - \varphi(x, a)\, w(x, a)\big)\, d\mathcal{L}^n x. \end{aligned}$$

Then there is a function $v\colon Q_{\frac{1}{2}} \to \mathbb{R}$ which is a weak solution of the partial differential equation

$$\gamma^\star(x)\frac{\partial v}{\partial t}(x, t) = \sum_{j=1}^n \frac{\partial}{\partial x_i}\left(\sum_{i=1}^n \kappa^\star_{ij}(x)\,\frac{\partial v}{\partial x_i}(x, t)\right)$$

with

$$\int_{Q_{1/2}} |u - v|^2 \, d\mathcal{L}^{n+1} < \epsilon.$$

Proof. The negation of our assertion implies the existence of $\{\gamma_k\}_k$, $\{\kappa^k\}_k$, $\{f_k\}_k$, $\{g_k\}_k$, $\{w_k\}_k$ for $k = 1, 2, 3, \ldots$ (with the obvious meaning) such that

$$\|\gamma_k - \gamma^0\|_{\mathbf{L}^2(\Theta)} \to 0, \quad \|\kappa^k - \kappa^0\|_{\mathbf{L}^2(\Theta)} \to 0, \quad \|f_k\|_{\mathbf{L}^2(\Theta)} \to 0, \quad \|g_k\|_{\mathbf{L}^2(\Theta)} \to 0$$

as $k \to \infty$ while for each k

$$\int_{Q_1} w_k^2 \, d\mathcal{L}^{n+1} \leq 1$$

and w_k weakly solves the partial differential equation

$$\gamma_k(x) \frac{\partial w_k}{\partial t}(x, t) = \sum_{j=1}^{n} \frac{\partial}{\partial x_i} \left(\sum_{i=1}^{n} \kappa_{ij}^k(x) \, \frac{\partial w_k}{\partial x_i}(x, t) \, + f_k^j(x) \right) + g_k(x, t)$$

and

$$\int_{Q_{1/2}} |w_k - v|^2 \, d\mathcal{L}^{n+1} \geq \epsilon$$

for every function v which is a weak solution of the partial differential equation

$$\gamma^0(x) \frac{\partial v}{\partial t}(x, t) = \sum_{j=1}^{n} \frac{\partial}{\partial x_i} \left(\sum_{i=1}^{n} \kappa_{ij}^0(x) \, \frac{\partial v}{\partial x_i}(x, t) \right).$$

Our interior estimates in Step 2 guarantee that the $\mathbb{H}^1$ norms of the w_k's are uniformly bounded in $Q_{\frac{1}{2}}$. This implies the existence of a function $w_0 \colon Q_{\frac{1}{2}} \to \mathbb{R}$ with square summable first derivatives together with a a subsequence $\ell(1)$, $\ell(2)$, $\ell(3)$, $\ldots$ of 1, 2, 3, $\ldots$ such that (note B.7), as $i \to \infty$, $w_{\ell(i)} \rightharpoonup w_0$ in $\mathbb{H}^1$ (weak convergence) while $w_{\ell(i)} \to w_0$ in $\mathbb{L}_2$ (strong convergence). This convergence is adequate to guarantee that w_0 satisfies the required equation.

Step 5 (How to flatten Γ). We now consider a change of coordinates

$$\psi(x) = \Big(x_1, \, x_2, \, \ldots, x_{n-1}, \, x_n - \sigma(x_1, \, x_2, \, \ldots, x_{n-1}) \Big)$$

which carries our hypersurface Γ to the plane $x_n = 0$. We check that the function

$$U(x, t) = u(\psi^{-1}(x), t)$$

satisfies the partial differential equation

$$\gamma^*(x)\frac{\partial U}{\partial t}(x,\, t) = \sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{i=1}^{n} \kappa^*_{ij}(x)\,\frac{\partial U}{\partial x_i}(x,\, t)\right)$$

where

$$\gamma^* = \gamma^0 \circ \psi^{-1}$$

$$\kappa^*_{k\ell} = \sum_{ij} \left(\kappa^0_{ij} \circ \psi^{-1}\right)\left(\frac{\partial \psi^k}{\partial x_i} \circ \psi^{-1}\right)\left(\frac{\partial \psi^\ell}{\partial x_j} \circ \psi^{-1}\right)$$

We also check that if $L{:}\,\mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree one with $\|\nabla L(0)\| \leq M$ (the size of M will be controlled later) then the function $U^0 = U - L$ satisfies the partial differential equation

$$\begin{aligned}\gamma^*(x)\frac{\partial U^0}{\partial t}(x,\, t) &= \sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{i=1}^{n} \kappa^*_{ij}(x)\left(\frac{\partial U^0}{\partial x_i}(x,\, t) + \frac{\partial L}{\partial x_i}(0)\right)\right)\\ &= \sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{i=1}^{n} \kappa^*_{ij}(x)\frac{\partial U^0}{\partial x_i}(x,\, t) + f^j(x)\right)\end{aligned}$$

where

$$f^j(x) = \sum_{i=1}^{n} \kappa^*_{ij}(x)\,\frac{\partial L}{\partial x_i}(0).$$

We consider a new pointwise Hölder norm which associates to a function $h(x,t)$ at 0 the number

$$[h]_\alpha(0) = \inf_{h_0} \sup_{r \leq 1} \left(\fint_{(x,\, t) \in Q_r} |h(x,\, t) - h_0(x)|^2 \; d\mathcal{L}^{n+1}(x,\, t)\right)^{\frac{1}{2}}$$

where the infimum is taken over all functions $h_0(x) = h_0(x,\, t)$ which are constant when $x_n > 0$ and also when $x_n < 0$. We infer from our original hypothesis (b) and our calculation above (since M is assumed to be fixed) that at 0

$$[\kappa^*_{ij}]_\alpha + [f^j]_\alpha \leq C\,\|\sigma\|_{C^{1,\alpha}}.$$

If we suitably rescale our coordinate system then the new $[\kappa^*_{ij}]_\alpha$ and $[f^j]_\alpha$ can be made arbitrarily small.

Step 6 (A decay estimate). Suppose C_0 is the constant of Step 3 (called C there) and that $0 < r < 1/2$ is fixed with

$$4\, C_0\, r^{2(1-\alpha)} < 1.$$

Pick and fix $\epsilon_0 > 0$ such that

$$4\, \epsilon_0^2 < r^{n+4+2\alpha}.$$

Now choose $\delta_0 > 0$ such that if the hypotheses of Step 4 hold with δ there replaced by δ_0, u there replaced by W, and v there replaced by V, then

$$\int_{Q_r} |V - W|^2 \, d\mathcal{L}^{n+1} \le \epsilon_0^2.$$

We assert the existence of $C < \infty$ with the following property. Suppose U [resp. U^0] is as in Step 5 with

$$\int_{Q_1} U^2 \, d\mathcal{L}^{n+1} \le 1 \quad \left[\text{resp.} \quad \int_{Q_1} (U^0)^2 \, d\mathcal{L}^{n+1} \le 1\right]$$

and

$$[\kappa_{ij}^*]_\alpha \le \frac{1}{n^2}\delta_0 \quad \left[\text{resp.} \quad [\kappa_{ij}^*]_\alpha \le \frac{1}{n^2}\delta_0 \text{ and } [f^j]_\alpha \le \frac{1}{n}\delta_0\right]$$

for each i and j. Then there exist constant functions

$$a_0(x) = 0,\, a_1(x),\, a_2(x),\, \ldots \quad \text{with} \quad \big|a_k(x) - a_{k+1}(x)\big| \le C\, r^{k(1+\alpha)}$$

for each k and x together with piecewise constant vector valued functions

$$A_0(x) = 0,\, A_1(x),\, A_2(x),\, \ldots \quad \text{with} \quad \big|A_k(x) - A_{k+1}(x)\big| \le C\, r^{k\alpha}$$

for each k and x whose values are constant vectors in $\mathbb{R}^n$ when $x_n > 0$ and also constant vectors in $\mathbb{R}^n$ when $x_n < 0$ such that

$$\fint_{(x,\,t)\in Q_{r^k}} \big|U(x,\,t) - a_k(x) - A_k(x) \bullet x\big|^2 \, d\mathcal{L}^{n+1}(x,\,t) \le r^{2k(1+\alpha)}$$

$$\left[\text{resp.} \quad \fint_{(x,\,t)\in Q_{r^k}} \big|U^0(x,\,t) - a_k(x) - A_k(x) \bullet x\big|^2 \, d\mathcal{L}^{n+1}(x,\,t) \le r^{2k(1+\alpha)}\right]$$

for each $k = 0,\, 1,\, 2,\, \ldots$.

Proof. We consider only the case of the function U^0. For $k = 0$ we take $a_0(x) = 0$, $A_0(x) = 0$ as indicated so that our assertion holds by assumption. Assume then that our assertion holds up to including some nonnegative integer k and set

$$W(x,\, t) = \frac{1}{r^{k(1+\alpha)}} \Big(U^0\big(r^k x,\, r^{2k} t\big) - a_k\big(r^k x\big) - A_k\big(r^k x\big) \bullet x \Big).$$

Then

$$\int_{Q_1} |W|^2 \, d\mathcal{L}^{n+1} \le 1$$

and we use Step 5 to infer that W satisfies the partial differential equation

$$\begin{aligned}
&\gamma^\star(x) \frac{\partial W}{\partial t}(x,\, t) \\
&\quad = \sum_{j=1}^n \frac{\partial}{\partial x_j} \left(\sum_{i=1}^n \kappa^\star_{ij}(x) \, \frac{\partial W}{\partial x_i}(x,\, t) + \frac{1}{r^{k\alpha}} \big(\kappa^\star_{ij}(x) - \kappa^\diamond_{ij}(x)\big) \bullet A^i_k(x) + \frac{1}{r^{k\alpha}} f_\star(x) \right)
\end{aligned}$$

where

$$\gamma^\star(x) = \gamma^*\big(r^k x\big) \quad \text{and} \quad \kappa^\star(x) = \kappa^*\big(r^k x\big) \quad \text{and} \quad f^j_\star(x) = f^j(\big(r^k x\big) - \sum_{i=1}^n \kappa^\diamond_{ij}\big(r^k x\big) \, \frac{\partial L}{\partial x_i}(0)$$

and $\kappa^\diamond$ equals κ^0 specialized to the case in which $\sigma = 0$ (and the hyperplane Γ lies in the vector subspace $x_n = 0$). We infer from Step 4 and our choices above the existence of a function $V(x,\, t)$ with

$$\int_{Q_r} |V - W|^2 \, d\mathcal{L}^{n+1} < \epsilon_0^2$$

which is a solution to the partial differential equation

$$\gamma^0(x) \frac{\partial V}{\partial t}(x,\, t) = \sum_{j=1}^n \frac{\partial}{\partial x_i} \left(\sum_{i=1}^n \kappa^\star_{ij}(x) \, \frac{\partial V}{\partial x_i}(x,\, t) \right)$$

associated with the specialization of our original hypothesis (b) to the case $\sigma = 0$. According to Step 3 our function V is smooth for $x_n \ge 0$ and also smooth for $x_n \le 0$. We set $a_*(x) = V(0,\, 0)$ and define $A_*(x)$ by requiring

$$A_*(x) \bullet x = \begin{cases} \big\langle x,\, \nabla^+ V(0,\, 0) \big\rangle & \text{if } x_n > 0 \\[2ex] \big\langle x,\, \nabla^- V(0,\, 0) \big\rangle & \text{if } x_n < 0 \end{cases}$$

and use Step 3 to infer

$$|V(0,\,0)|^2 + \|\nabla V^+(0,\,0)\|^2 + \|\nabla V^-(0,\,0)\|^2 \le C_0 \int_{Q_1} V^2 \, d\mathcal{L}^{n+1}.$$

and

$$\begin{aligned}\int_{(x,\,t)\in Q_r} &\left|W(x,\,t) - a_*(x) - A_*(x) \bullet x\right|^2 d\mathcal{L}^{n+1} \\ &\le 2\epsilon^2 + 2\int_{(x,\,t)\in Q_r} \left|V(x,\,t) - a_*(x) - A - *(x) \bullet x\right|^2 d\mathcal{L}^{n+1} \\ &\le 2\epsilon^2 + 2\,C_0\,r^4 \cdot r^{n+2} \\ &\le r^{2(1+\alpha)} \cdot r^{n+2}\end{aligned}$$

We conclude that

$$\begin{aligned}\fint_{(x,\,t)\in Q_{r^{k+1}}} &\left|U^0(x,\,t) - \big(a_k(x) + r^{k(1+\alpha)} a_*(x)\big) - \big(A_k(x) + r^{k\alpha} A_*(x)\big) \bullet x\right|^2 d\mathcal{L}^{n+1}(x,\,t) \\ &\le r^{2k(1+\alpha)} \cdot r^{2(1+\alpha)} = r^{2(k+1)(1+\alpha)}\end{aligned}$$

as required.

Step 7 (Corollary). Suppose $U(x,\,t)$ is as in Step 5. Then there is a constant function $a_\star(x)$ and a piecewise constant vector valued function $A_\star(x)$ whose value is a constant vector in $\mathbb{R}^n$ when $x_n > 0$ and also a constant vector in $\mathbb{R}^n$ when $x_n < 0$ such that

$$\sup_{0<r<1} \frac{1}{r^{2(1+\alpha)}} \fint_{(x,\,t)\in Q_r} \left|U(x,\,t) - a_\star(x) - A_\star(x) \bullet x\right|^2 d\mathcal{L}^{n+1}(x,\,t) \le C \int_{Q_1} U^2 \, d\mathcal{L}^{n+1}$$

with

$$\left|a_\star(x)\right|^2 + \left|A_\star(x)\right|^2 \le C \int_{Q_1} U^2 \, d\mathcal{L}^{n+1};$$

here the value of the constant C depends only on the value of $[\kappa^*_{ij}]_\alpha$.

Proof. Use parabolic scaling to satisfy the condition

$$[\kappa^*_{ij}]_\alpha \le \frac{1}{n^2}\delta_0 \quad \text{for each } i \text{ and } j$$

of Step 6, and absorb the errors in the constant C.

Step 8 (Corollary). Suppose $U^0(x, t)$ is as in Step 5. Then there is a constant function $a_\star(x)$ and a piecewise constant vector valued function $A_\star(x)$ whose value is a constant vector in $\mathbb{R}^n$ when $x_n > 0$ and also a constant vector in $\mathbb{R}^n$ when $x_n < 0$ such that

$$\sup_{0<r<1} \frac{1}{r^{2(1+\alpha)}} ⨍_{(x,\,t)\in Q_r} \left|U^0(x, t) - a_\star(x) - A_\star(x) \bullet x\right|^2 d\mathcal{L}^{n+1}(x, t) \leq C \left(\int_{Q_1} (U^0)^2 \, d\mathcal{L}^{n+1} + \sum_{j=1}^{n} [f^j]_\alpha^2 \right)$$

with

$$\left|a_\star(x)\right|^2 + \left|A_\star(x)\right|^2 \leq C \left(\int_{Q_1} (U^0)^2 \, d\mathcal{L}^{n+1} + \sum_{j=1}^{n} [f^j]_\alpha^2 \right)$$

here the value of the constant C depends only on the value of $[\kappa^*_{ij}]_\alpha$.

Proof. Dominate M in Step 5 by $\|A_\star(x)\|$ in Step 7. Then apply Step 6 to the function

$$U^1 = \frac{\delta_0 \, U^0(x, t)}{\left(\int_{Q_1} (U^0)^2 \, d\mathcal{L}^{n+1} + \sum_{j=1}^{n} [f^j]_\alpha(0)]^2 \right)^{\frac{1}{2}}}$$

which implies that the "f^j" associated with U^1 has Hölder norm less than δ_0. Use parabolic scaling.

Step 9 (Final estimates). Our object is to show that the asserted regularity of our original function u across the interface Γ. In view of the change of coordinates ψ in Step 5 which flattened Γ, it is sufficient to show that the perturbed function U satisfies the four conditions (a), (b), (c), (d) of Step 1 when Ξ there equals either $\psi(\Theta^+) \cap \mathbb{B}^n(0, 1) \times (-1, 0]$ or $\psi(\Theta^-) \cap \mathbb{B}^n(0, 1) \times (-1, 0]$ so that Campanato's embedding theorem will imply the necessary regularity of U. Away from the interface $x_n = 0$ the function U is Hölder continuously differentiable (since u is infinitely differentiable away from Γ) so that the relevant estimates hold immediately. The main question is about the behavior of U near Γ.

Condition (b) in Step 1 follows from Step 7 applied to the function U. Condition (a) follows from Step 7 applied to the function U since a situation with no interface discontinuities is a special case of Step 7, and we can scale the estimate there to replace Q_1 by

Q_s. Condition (c) follows from Step 8 applied to the function U^0 since a situation with no interface discontinuities is a special case of Step 8, and we can scale the estimate there to replace Q_1 by Q_s. Condition (d) follows from Step 2. ■

6. Sufficient conditions for crystal positions and temperature fields to be close

In the second conclusion of Theorem 5.2 we showed that large space Dirichlet integrals of temperature implied rapid decrease in our total energy functions $\mathbb{E}_j(t)$. In Theorem 5.3 we showed the essential *a priori* Monge-Kantorovich Hölder continuity in time of heat distributions in approximating evolutions. In this chapter we set forth the principal estimates which enable us to control changes in crystal positions and in temperature fields in terms of temperature Dirichlet integrals and heat distribution changes.

6.1 Theorem (Crystal positions, heat distributions, and Dirichlet integrals). *Suppose the following conditions hold*

(a) $[\![K]\!]$ *and* $[\![L]\!]$ *are crystals in* $\mathcal{K}$ *with* $\mathcal{L}^n(K \triangle L) > 0$.

(b) Q *is a single heat distribution in* $\mathcal{Q}$.

(c) $T_K = c_{_K}^{-1}Q$, $T_L = c_{_L}^{-1}Q$ *are associated temperature fields.*

(d) $U_K = 1/T_K$ *and* $U_L = 1/T_L$ *are associated reciprocal temperature fields.*

Then

$$\mathrm{DIR}(U_K) + \mathrm{DIR}(U_L) + \mathrm{DIR}(T_K) + \mathrm{DIR}(T_L) = \infty.$$

PROOF. The proof is in three parts.

Part 1 (Alternatives across detectable interfaces). Suppose the following conditions hold

(a) K and L are sets in $\mathbb{R}^n$.

(b) $Q{:}\mathbb{R}^n \to \mathbb{R}^+$ is a positive function (a single "heat distribution").

(c) $T_K = c_{_K}^{-1}Q$ and $T_L = c_{_L}^{-1}Q$ (are associated temperature fields); here. $c_{_K}$ and $c_{_L}$ are defined by analogy with 2.5.

(d) $U^K = T_K^{-1}$ and $U^L = T_L^{-1}$ (are associated reciprocal temperature fields).

(e) p and q are points in $\mathbb{R}^n$.

Then one of the following two alternatives holds.

First alternative.

EITHER

(1) Based on information at p and q, neither K nor L has a detectable interface between p and q, viz.

$$\Big(\{p,\, q\} \subset K \text{ or } \{p,\, q\} \subset \mathbb{R}^n \sim K\Big) \quad \text{and} \quad \Big(\{p,\, q\} \subset L \text{ or } \{p,\, q\} \subset \mathbb{R}^n \sim L\Big).$$

OR

(2) Based on information at p and q, both K and L detect interfaces between p and q and they are of the same type, viz.

$$\Big(p \in K \cap L \;\text{ and } q \in (\mathbb{R}^n \sim K) \cap (\mathbb{R}^n \sim L)\Big)$$
$$\text{or} \quad \Big(q \in K \cap L \;\text{ and } p \in (\mathbb{R}^n \sim K) \cap (\mathbb{R}^n \sim L)\Big).$$

Second alternative.

$$\begin{aligned} \big|U_K(p) - U_K(q)\big| + \big|U_L(p) - U_L(q)\big| + \big|T_K(p) - T_K(q)\big| + \big|T_L(p) - T_L(q)\big| \\ \geq 2\,(c_s c_\ell)^{\frac{1}{2}} \left(\frac{1}{c_s} - \frac{1}{c_\ell}\right). \end{aligned}$$

Proof of Part 1. The proof of the assertions of Part 1 is in two steps

Step 1. We assert

(1) For any three numbers A, B, C we have the inequality

$$|A - B| + |A - C| \geq |B - C|.$$

Indeed, if $C \leq B$, then

$$|B - C| = B - C = B - A + A - C \leq |B - A| + |A - C|.$$

(2) For any three positive numbers A, B, R we have the inequality

$$AR + B/R \geq 2\sqrt{AB}.$$

Indeed, the function $f(R) = AR + B/R$ has its unique positive minimum when $f'(R) = A - B/R^2 = 0$, $R = \sqrt{B/A}$, at which point the function takes value

$$A\sqrt{B/A} + B/\sqrt{B/A} = 2\sqrt{AB}.$$

Step 2. We will assume the first alternative does not hold and will show that the second alternative must then hold. There is thus a detectable interface between p and q. For the purpose of our proof it is sufficient to treat the case in which $p \in K$, $q \in \mathbb{R}^n \sim K$, which assumption we make. Other cases follow by various interchanges of K with L and/or of p with q. There are then three cases.

Case 1. $p,\, q \in L$.

Case 2. $p,\, q \in \mathbb{R}^n \sim L$.

Case 3. $p \in \mathbb{R}^n \sim L$, $q \in L$.

We abbreviate $a = 1/c_{_s} > b = 1/c_{_\ell}$ and $R = Q(p)$, $S = Q(p)$ and consider our three cases.

Case 1. We use Step 1 to estimate

$$\begin{aligned}
\big|aR - bS\big| + \big|aR - aS\big| + \big|1/aR - 1/bS\big| + \big|1/aR - 1/aS\big| \\
&\geq (a-b)S + (1/b - 1/a)/S \\
&= (a-b)S + (a-b)/abS \\
&\geq 2(a-b)/\sqrt{ab}
\end{aligned}$$

Case 2. We use Step 1 to estimate

$$\begin{aligned}
\big|aR - bS\big| + \big|bR - bS\big| + \big|1/aR - 1/bS\big| + \big|1/bR - 1/bS\big| \\
&\geq (a-b)R + (1/b - 1/a)R \\
&= (a-b)R + (a-b)/abR \\
&\geq 2(a-b)/\sqrt{ab}
\end{aligned}$$

Case 3. We use Step 1 to estimate

$$\begin{aligned}
&\big|aR - bS\big| + \big|bR - aS\big| + \big|1/aR - 1/bS\big| + \big|1/bR - 1/aS\big| \\
&\qquad = (a/b)\big|bR - (b^2/a)S\big| + \big|bR - aS\big| + \big|1/aR - 1/bS\big| + (a/b)\big|1/aR - b/a^2S\big| \\
&\qquad \geq \big|bR - (b^2/a)S\big| + \big|bR - aS\big| + \big|1/aR - 1/bS\big| + \big|1/aR - b/a^2S\big| \\
&\qquad \geq (a - b^2/a)S + (1/b - b/a^2)/S \\
&\qquad = \big((a^2 - b^2)/a\big)S + \big((a^2 - b^2)/ab^2\big)/S \\
&\qquad \geq 2(a^2 - b^2)/ab \\
&\qquad \geq 2(a-b)/\sqrt{ab}
\end{aligned}$$

since $(a+b)/\sqrt{ab} > 1$.

Part 2 (Interfaces and Dirichlet's integrals). Corresponding to each (large) number M there is a (small) positive number δ with with the following property. Suppose the conditions below are satisfied.

(a) K and L are $\mathcal{L}^n$ measurable sets in $\mathbb{R}^n$.

(b) $Q{:}\mathbb{R}^n \to \mathbb{R}$ belongs to $\mathbb{L}_2(\mathbb{R}^n)$ and $1/Q$ also belongs to $\mathbb{L}_2(\mathbb{R}^n)$ (a single "heat distribution").

(c) $T_K = c_{_K}^{-1}, Q\ T_L = c_{_L}^{-1}Q$ (are associated temperature fields); here. $c_{_K}$ and $c_{_L}$ are defined by analogy with 2.5.

(d) $U^K = T_K^{-1}$ and $U^L = T_L^{-1}$ (are associated reciprocal temperature fields).

(e) $A = \mathbb{B}^n(0,\,1) \cap \{x{:}\, x_n > 0\}$.

(f) $B = \mathbb{B}^n(0,\,1) \cap \{x{:}\, x_n < 0\}$.

(g) $A_K = A \cap K$, $A_L = A \cap L$, $B_K = B \cap K$, $B_L = B \cap L$,

(h) $\mathcal{L}^n\big(A \sim A_K\big) + \mathcal{L}^n\big(B_K\big) < \delta$.

(i) One of the following three possibilities holds

(i.1) $\mathcal{L}^n\big(A \sim A_L\big) + \mathcal{L}^n\big(B \sim B_L\big) < \delta$;

(i.2) $\mathcal{L}^n\big(A_L\big) + \mathcal{L}^n\big(B_L\big) < \delta$;

(i.3) $\mathcal{L}^n\big(A_L\big) + \mathcal{L}^n\big(B \sim B_L\big) < \delta$.

Then

$$\begin{aligned}(1)\qquad \mathrm{DIR}\big(U_K;\mathbb{B}^n(0,\,1)\big) + \mathrm{DIR}\big(U_L;\mathbb{B}^n(0,\,1)\big) + \mathrm{DIR}\big(T_K;\mathbb{B}^n(0,\,1)\big)&\\ + \mathrm{DIR}\big(T_L;\mathbb{B}^n(0,\,1)\big) \geq M.&\end{aligned}$$

Furthermore, if $0 < r < 1$ and

$$U_K^r(x) = U_K(x/r),\ U_L^r(x) = U_K(x/r),\ T_K^r(x) = T_K(x/r),\ T_L^r(x) = T_L(x/r),$$

for each x, then

$$\begin{aligned}(2)\qquad \mathrm{DIR}\big(U_K^r;\mathbb{B}^n(0,\,1)\big) + \mathrm{DIR}\big(U_L^r;\mathbb{B}^n(0,\,1)\big) + +\mathrm{DIR}\big(T_K^r;\mathbb{B}^n(0,\,1)\big)&\\ + \mathrm{DIR}\big(T_L^r;\mathbb{B}^n(0,\,1)\big) \geq Mr^{n-2}.&\end{aligned}$$

Proof of Part 2. The proof of Part 2 is in two steps.

Step 1. Suppose $f(t)$ is a function with square summable first derivatives for $|t| \le \epsilon$ and there exist $-\epsilon \le s < t \le \epsilon$ such that

$$|f(s) - f(t)| \ge \frac{1}{2} \left(c_s c_\ell\right)^{\frac{1}{2}} \left(\frac{1}{c_s} - \frac{1}{c_\ell}\right).$$

Then

$$\int_{[s,\,t]} (f')^2 \, d\mathcal{L}^1 \ge \frac{1}{\epsilon} \frac{c_s c_\ell}{8} \left(\frac{1}{c_s} - \frac{1}{c_\ell}\right)^{\frac{1}{2}}.$$

Indeed

$$|f(s) - f(t)| \le \int_{[s,\,t]} |f'|^2 \, d\mathcal{L}^1 \le (t-s)^{\frac{1}{2}} \left(\int_{[s,\,t]} (f')^2 \, d\mathcal{L}^1\right)^{\frac{1}{2}}.$$

Step 2. If any of the Dirichlet integrals in first conclusion are infinite, the first conclusion is immediate. We therefore assume that each of these Dirichlet integrals is finite and further require that U_K, U_L, T_K, T_L be strictly defined at all Lebesgue points. We infer that for $\mathcal{L}^{n-1}$ almost every y in $\mathbb{B}^{n-1}(0,\,1)$ the functions

$$s \mapsto U_K(y,\,s), s \mapsto U_L(y,\,s), s \mapsto T_K(y,\,s), s \mapsto T_L(y,\,s)$$

are absolutely continuous with square summable first s derivatives for $|y|^2 + s^2 < 1$ [MC, Lemma 3.1.1]. According to Fubini's theorem

$$DIR\big(U_K;\mathbb{B}^n(0,\,1)\big) \ge \int_{y \in \mathbf{B}^{n-1}(0,\,1)} \int_{\{s : |y|^2 + s^2 < 1\}} \left(\frac{\partial U_K}{x_n}(y,\,s)\right)^2 d\mathcal{L}^1 s \, d\mathcal{L}^{n-1} y,$$

with corresponding inequalities for U_L, T_K, T_L. Suppose then it is possibility (i.1) above which occurs and $\epsilon > 0$ is very small. If δ is extremely small we infer from hypotheses (h) and (i.1) together with Fubini's theorem that for most y's in $\mathbf{B}^{n-1}(0,\,1)$ there will exist $p = (y,\,s) \in A_K \cap A_L$ and $q = (y,\,t) \in (\mathbb{R}^n \sim K) \cap L$ with $-\epsilon < s < t < \epsilon$. We check that the first alternative of Part 1 does not hold for such p and q and infer from the second alternative that, for such y's,

$$\begin{aligned} \big|U_K(p) - U_K(q)\big| + \big|U_L(p) - U_L(q)\big| + \big|T_K(p) - T_K(q)\big| & \\ + \big|T_L(p) - T_L(q)\big| & \\ \ge 2 \left(c_s c_\ell\right)^{\frac{1}{2}} \left(\frac{1}{c_s} - \frac{1}{c_\ell}\right)&. \end{aligned}$$

Hence, at least one of the four numbers on the left hand side must be at least as large as one fourth of the number on the right hand side. Suppose, say, for a particular y,

$$\left|U_K(p) - U_K(q)\right| \geq \frac{1}{2}\left(c_s c_\ell\right)^{\frac{1}{2}}\left(\frac{1}{c_s} - \frac{1}{c_\ell}\right).$$

Then the estimate of Step 1 implies that

$$\int_{\{s:|y|^2+s^2<1\}} \left(\frac{\partial U_K}{x_n}(y,\, s)\right)^2 d\mathcal{L}^1 s \geq \frac{1}{\epsilon}\frac{c_s c_\ell}{8}\left(\frac{1}{c_s} - \frac{1}{c_\ell}\right)^{\frac{1}{2}}.$$

Corresponding estimates would hold for U_L, T_K, T_L. The same argument would apply if it were possibility (i.2) or (i.3) which held. We infer the first conclusion of Part 2 from Fubini's theorem as indicated above. The second conclusion follows from the first.

Part 3. We now let K, L, T_K, T_L, U_K, U_L be as in the statement of the theorem. The remainder of the proof the the theorem is in two steps.

Step 1. We assert that $\partial[\![K]\!] \neq \partial[\![L]\!]$. Indeed, if $\partial[\![K]\!] = \partial[\![L]\!]$, then the Constancy Theorem [FH 4.1.7] would imply that $[\![K]\!] - [\![L]\!] = z[\![\Omega]\!]$ for some integer z. On the other hand, for $\mathcal{L}^n$ almost every x in Ω,

$$z = \begin{cases} 1 & \text{if } x \in K \sim L \\ -1 & \text{if } x \in L \sim K \\ 0 & \text{if } x \in (K \cap L) \cup (\Omega \sim K) \cap (\Omega \sim L) \end{cases}.$$

It follows that at most one of the three sets on the right hand side can have positive measure. Since $K \triangle L = (K \sim L) \cup (L \sim K)$ has positive measure by assumption, the measure of $(K \cap L) \cup (\Omega \sim K) \cap (\Omega \sim L)$ must vanish so that K and L together $\mathcal{L}^n$ almost partition Ω. We infer from assumption 2.2.1 that either $\mathcal{L}^n(K) > 0$ and $\mathcal{L}^n(L) > 0$ or $\mathcal{L}^n(\Omega \sim K) > 0$ and $\mathcal{L}^n(\Omega \sim L) > 0$. In either case we further infer that $\partial[\![K]\!] = -\partial[\![L]\!] \neq 0$ which implies our assertion.

Step 2. We utilize the structure of sets of finite perimeter set forth in [FH 4.5.6] to infer (interchanging K and L if necessary) that there is a subset J of ∂K of positive $\mathcal{H}^{n-1}$ measure such that, for each p in J, one of the following three alternatives holds: either $p \in L$ with $\Theta^n\left(\mathcal{L}^n \llcorner L,\, p\right) = 1$, or $p \in \Omega \sim L$ with $\Theta^n\left(\mathcal{L}^n \llcorner L,\, p\right) = 0$, or $p \in \partial L$ with

$\mathbf{n}_L(p) = -\mathbf{n}_K(p)$. For each fixed such p in J and (large) number M we then infer from Part 2 the existence of $r_0(p) > 0$ such that whenever $0 < r < r_0(p)$ then

$$\mathrm{DIR}\big(U_K;\mathbb{B}^n(p,\, r)\big) + \mathrm{DIR}\big(U_L;\mathbb{B}^n(p,\, r)\big) + \mathrm{DIR}\big(T_K;\mathbb{B}^n(p,\, r)\big) \\ + \mathrm{DIR}\big(T_L;\mathbb{B}^n(p,\, r)\big) \geq M r^{n-2}.$$

Let $\mathbb{B}^n(p_1,\, r_1), \mathbb{B}^n(p_2,\, r_2), \mathbb{B}^n(p_3,\, r_3), \ldots$ be a covering of J by closed balls in the sense of the Besicovitch-Federer covering theorem [FH 2.8.14] (with $\beta(n)$ equal to the number $2\zeta + 1$ there) with $r_i < r_0(p_i)$ for each i We infer

$$\begin{aligned} \mathrm{DIR}(U_K) + \mathrm{DIR}(U_L) + \mathrm{DIR}(T_K) + \mathrm{DIR}(T_L) &\geq \frac{M}{\beta(n)} \sum_i r_i^{n-2} \\ &\geq \frac{M}{\alpha(n)\beta(n)\sup_i r_i} \alpha(n) \sum_i r_i^{n-1} \\ &\geq \frac{M}{\alpha(n)\beta(n)\sup_i r_i} \mathcal{H}^{n-1}(J). \end{aligned}$$

The conclusion follows since both M can be chosen arbitrarily large and $\sup_i r_i$ can be chosen very small. ■

6.2 Corollary (Sufficient condition for two crystals to be close). *Corresponding to each (small) positive number ϵ and each (large) number M there is a (small) positive number δ with the following property. Suppose*

(a) $[\![K]\!]$ and $[\![L]\!]$ are crystal in $\mathcal{K}$.

(b) P and Q are heat distributions in $\mathcal{Q}$ with $\big\|P - Q\big\| < \delta$.

(c) $S = c_{K}^{-1} P$ and $T = c_{L}^{-1} Q$ are associated temperature fields.

(d) $U = 1/S$ and $V = 1/T$ are associated reciprocal temperature fields.

(e) $\|U\|_{\mathbf{L}_2} + \|V\|_{\mathbf{L}_2} + \|S\|_{\mathbf{L}_2} + \|T\|_{\mathbf{L}_2} \leq M$.

(f) $\mathrm{DIR}(U) + \mathrm{DIR}(V) + \mathrm{DIR}(S) + \mathrm{DIR}(T) \leq M$.

Then

$$\mathcal{L}^n(K \triangle L) < \epsilon.$$

PROOF. The negation of our assertion says that there exist some ϵ and M such that, no matter how small we pick δ, there will be K, L, S, T, U, V satisfying the conditions of the theorem above for which $\mathcal{L}^n(K \triangle L) \geq \epsilon$. For each $i = 1, 2, 3, \ldots$ let K_i, L_i, S_i, T_i, U_i, V_i be such a choice corresponding to $\delta_i = 2^{-i}$. The compactness theorem for integral currents together with Rellich's theorem guarantee the existence of a subsequence $j(1)$, $j(2)$, $j(3)$, $\ldots$ of 1, 2, 3, $\ldots$ and limit (in the mass topology) crystals $[\![K]\!]$, $[\![L]\!]$ with $K \triangle L \geq \epsilon$, limit (in the strong $\mathbb{L}_2(\Omega)$ topology) temperature fields S, T and inverse temperature fields U, V with $\mathrm{DIR}(U)+\mathrm{DIR}(V)+\mathrm{DIR}(S)+\mathrm{DIR}(T) \leq M$, and hence limit (in the strong $\mathbb{L}_2(\Omega)$ topology) heat distributions P, Q. Since $\lim_i \|P_i - Q_i\|_* = 0$ we infer $P = Q$. This is incompatible with Theorem 6.1 which proves the present corollary. ■

6.3 Proposition (Sufficient condition for two temperature fields to be close). *Corresponding to each (small) positive number ϵ and each (large) number M there is a (small) number δ with the following property. Suppose*

(a) $[\![K]\!]$ and $[\![L]\!]$ are crystals in $\mathcal{K}$ with $\mathcal{L}^n(K \triangle L) < \delta$.

(b) P and Q are heat distributions in $\mathcal{Q}$ with $\|P - Q\|_* < \delta$.

(c) $S = c_{_K}^{-1}P$, and $T = c_{_L}^{-1}Q$, are associated temperature fields.

(d) $U = 1/S$, and $V = 1/T$ are associated reciprocal temperature fields.

(e) $\|U\|_{\mathbf{L}_2} + \|V\|_{\mathbf{L}_2} + \|S\|_{\mathbf{L}_2} + \|T\|_{\mathbf{L}_2} \leq M$.

(f) $\mathrm{DIR}(U) + \mathrm{DIR}(V) + \mathrm{DIR}(S) + \mathrm{DIR}(T) \leq M$.

Then

$$\|U - V\|_{\mathbf{L}_2} + \|S - T\|_{\mathbf{L}_2 i} < \epsilon.$$

PROOF. The negation of our assertion says that there exist some ϵ and M such that, no matter how small we pick δ, there will be K, L, P, Q, S, T, U, V satisfying the conditions of the theorem above for which

$$\|U - V\|_{\mathbf{L}_2} + \|S - T\|_{\mathbf{L}_2} \geq \epsilon.$$

For each $i = 1, 2, 3, \ldots$ let K_i, L_i, P_i, Q_i, S_i, T_i, U_i, V_i be such a choice corresponding to $\delta_i = 2^{-i}$. The compactness theorem for integral currents together with the compactness theorem for heat distributions and with Rellich's theorem guarantee the existence of a subsequence $j(1)$, $j(2)$, $j(3)$, $\ldots$ of 1, 2, 3, $\ldots$ and limit (in the mass topology) crystal $[\![K]\!]$, limit (in the $\|\cdot\|_*$ topology) heat distribution P, limit (in the strong $\mathbb{L}_2(\Omega)$ topology) temperature fields S, T and inverse temperature fields U, V with $\|U\|_{\mathbf{L}_2} + \|V\|_{\mathbf{L}_2} + \|S\|_{\mathbf{L}_2} + \|T\|_{\mathbf{L}_2} \le M$ and $\mathrm{DIR}(U) + \mathrm{DIR}(V) + \mathrm{DIR}(S) + \mathrm{DIR}(T) \le M$. In particular,

$$(i) \qquad \lim_i \mathcal{L}^n\big(K_{j(i)} \bigtriangleup K\big) = \lim_i \mathcal{L}^n\big(L_{j(i)} \bigtriangleup K\big) = 0$$

and

$$(ii) \qquad \lim_i \big\|P_i - P\big\|_* = \lim_i \big\|Q_i - P\big\|_* = 0$$

and

$$(iii) \quad \lim_i \|U_{j(i)} - U\|_{\mathbf{L}_2} = \lim_i \|V_{j(i)} - V\|_{\mathbf{L}_2} = \lim_i \|S_{j(i)} - S\|_{\mathbf{L}_2} = \lim_i \|T_{j(i)} - T\|_{\mathbf{L}_2} = 0$$

and

$$(iv) \qquad \|U - V\|_{\mathbf{L}_2} + \|S - T\|_{\mathbf{L}_2} \ge \epsilon.$$

Conditions (i), (ii), and (iii) imply $S = T = c_K^{-1} P$ and $U = V = 1/T$ which is incompatible with (iv). ■

7. Selection of subsequences of approximating evolutions

The convergence of the crystals, heat distributions, and temperature fields in our approximating evolutions to limits which satisfy the heat equation and Gibbs-Thomson relations is fairly complicated. We describe in this chapter how to obtain subsequences of approximating evolutions for which crystals, heat distributions, and temperature fields converge very strongly at almost every time while heat distributions converge at every time to a limit which varies Hölder continuously in the Monge-Kantorovich metric. We verify that the heat equation with varying crystals is satisfied for our limit.

7.1 Subsequences and Cantor's diagonal process. We denote by J_0 the identity function on the positive integers, i.e. $J_0(i) = i$ for each $i = 1, 2, 3, \ldots$. By a **subsequence** of J_0 we mean any increasing function from the positive integers to the positive integers. In case H and J are subsequences of J_0, we say that H is **subsequence** of J if and only of the image of H is a subset of the image of J. If J^1, J^2, J^3, ... are subsequences of J_0 such that J^{i+1} is a subsequence of J^i for each i, then **the result of Cantor's diagonal process** is the subsequence of J_0 having image $\big\{J^1(1), J^2(2), J^3(3), \ldots\big\}$.

7.2 Theorem (General estimates, weakly convergent subsequences, and Monge-Kantorovich Hölder continuity of heat distributions).

(1) For each positive integer j the following is true.

$$\|Q_j(t)\|_{\mathbf{L}_2} + \|1/Q_j(t)\|_{\mathbf{L}_2} + \|T_j(t)\|_{\mathbf{L}_2} + \|U_j(t)\|_{\mathbf{L}_2} \leq C_1 \tag{1.1}$$

for each time $t \in \mathbb{R}^+$; here

$$C_1 = \sup\left\{c_{\ell}^2, c_{s}^{-2}\right\}\left(C_*|\Omega| + c_{s}^{-1}\mathbb{E}_0\right).$$

$$\int_{t\in(0,\,\infty)} \mathrm{DIR}\big(T_j(t)\big) + \mathrm{DIR}\big(U_j(t)\big)\, d\mathcal{L}^1 t \leq C_2; \tag{1.2}$$

here

$$C_2 = \kappa_*^{-1} \sup\left\{c_*^{-1},\, C_*\right\}\mathbb{E}_0.$$

(1.3) For each $0 < M < \infty$,

$$\mathcal{L}^1\left(\mathbb{R}^+ \cap \left\{t\colon \mathrm{DIR}\big(T_j(t)\big) + \mathrm{DIR}\big(U_j(t)\big) > M\right\}\right) < C_1/M.$$

(2) There exists a subsequence J_1 of J_0 and a limit time parametrized family of heat distributions $Q_*(t)$ in $\mathcal{Q}$ such that

$$\left\|Q_{J_1(i)}(t) - Q_*(t)\right\|_* \to 0 \tag{2.1}$$

locally uniformly in t as $i \to \infty$.

$$\|Q_*(s) - Q_*(t)\|_* \leq \rho|s-t|^{\frac{1}{2}} \tag{2.2}$$

whenever $0 \leq s < t < \infty$; here

$$\rho = \frac{\kappa_0|\Omega|^{\frac{1}{2}}}{\kappa_*^{\frac{1}{2}}\, c_*^{\frac{1}{2}}}\mathbb{E}_0.$$

(3) There is a subsequence J_2 of J_1 above for which the following is true.

(3.1) There is a function K_0 in $\mathbb{L}_{1,loc}\big(\Omega\times\mathbb{R}^+,\, [0,\, 1]\big)$ such that

$$\int_{(x,\,t)\in\Omega\times R^+} \varphi(x,\,t)\left(\chi_{K_{J_2(i)}(t)}(x) - K_0(x,\,t)\right)\, d\mathcal{L}^{n+1}(x,\,t) \to 0$$

as $i \to \infty$ for each $\varphi \in \mathcal{D}(\Omega \times \mathbb{R})$; here

$$\chi_{K_{J_2(i)}(t)}(x) = \begin{cases} 1 & \textit{if } x \in K_{J_2(i)}(x) \\ 0 & \textit{otherwise} \end{cases}.$$

(3.2) There are functions Q_* (necessarily the same as in (2) above) and T_* in $\mathbb{L}_{2,loc}(\Omega \times R)$ such that, for each $0 < t < \infty$, $Q_*(t) \in \mathcal{Q}$ and

$$\|Q_*(t)\|_{\mathbf{L}_2} + \|T_*(t)\|_{\mathbf{L}_2} \leq C_1$$

and that

$$\int_{\Omega \times R^+} \varphi \left(Q_{J_2(i)} - Q_*\right)\right) d\mathcal{L}^{n+1} \to 0,$$

$$\int_{\Omega \times R^+} \varphi \left(T_{J_2(i)} - T_*\right)\right) d\mathcal{L}^{n+1} \to 0$$

as $i \to \infty$ for each $\varphi \in \mathcal{D}(\Omega \times \mathbb{R})$. For each fixed time t corresponding to which there is a crystal $[\![K_*(t)]\!]$ in $\mathcal{K}$ such that

$$\mathcal{L}^n\left(K_{J_2(i)} \triangle K_*(t)\right) \to 0 \quad \text{and} \quad \left\|Q_{J_2(i)}(t) - Q_*(t)\right\|_{\mathbf{L}_2} \to 0$$

as $i \to \infty$, then we can (and do) require that

$$c_{K_*(t)} T * (t) = Q_*(t).$$

(2.3) There is a function ∇T_* in $\mathbb{L}_2(\Omega \times \mathbb{R}^+, \mathbb{R}^n)$ such that

$$\int_{\Omega \times (0,\infty)} \|\nabla T_*\|^2 \, d\mathcal{L}^{n+1} \leq C_2;$$

and that

$$\int_{\Omega \times R^+} \varphi \left(\nabla T_{J_2(i)} - \nabla T_*\right) d\mathcal{L}^{n+1} \to 0$$

as $i \to \infty$ and also for each $\varphi \in \mathcal{D}(\Omega \times \mathbb{R})$

$$\int_{\Omega \times (0,\infty)} \psi \nabla T_* - \nabla \psi T_* \, d\mathcal{L}^{n+1} = 0$$

for each $\psi \in \mathcal{D}\left(\Omega \times (0, \infty)\right)$

PROOF. Assertion (1.1) follows from A.1.9 together with 5.3(1). Assertion (1.2) follows from 5.2(2) and 5.3(4). Assertion (1.2) implies assertion (1.3).

We use (1.1) together with the compactness estimate in B.6 applied to the heat distributions $Q_j(t)$ to infer that, for each fixed time t in $\mathbb{R}^+$, there exists a subsequence J of J_0 and a limit heat distribution $Q_*(t)$ such that $\left\|Q_{J(i)} - Q_*(t)\right\|_* \to 0$ as $i \to \infty$. Cantor's diagonal process guarantees the existence of a further subsequence J_1 of J_0 such that such convergence occurs for a countable set of t's which is dense in $\mathbb{R}^+$. The estimate 5.3(2) implies the asserted convergence and Hölder continuity as in Ascoli's theorem.

Conclusion (3) follows from conclusion (1), 2.5.5, and the usual weak compactness of closed and bounded subsets of vector spaces of square summable functions as in B.3. ■

7.3 Terminology. We fix a subsequence J_2 of J_0 for which conclusions (2) and (3) of Theorem 7.2 both hold and let K_0, Q_*, T_*, ∇T_* have the meaning of conclusions (2) and (3) of Theorem 7.2.

7.4 Theorem (Sufficient condition that $[\![K_*(t)]\!]$, $Q_*(x,\, t)$, $T_*(x,\, t)$ be a solution to the heat equation with varying crystals). *If there is a subsequence J of J_2 and a function $[\![K_*]\!]{:}\mathbb{R}^+ \to \mathcal{K}$ such that*

$$\int_{t \in (0,\, c)} \mathcal{L}^n \left(K_*(t) \triangle K_{J(i)}(t)\right) \, d\mathcal{L}^1 t \to 0$$

as $i \to \infty$ for each $0 < c < \infty$, then the crystals $[\![K_*(t)]\!]$ and functions $Q_*(x,\, t)$, and $T_*(x,\, t)$ are a solution to the heat equation with varying crystals with initial head distribution Q_0 and final heat distribution $Q_*(\cdot,\, b)$ for each $0 < b < \infty$.

PROOF. With regard to condition 2.5.13, we let ψ be a smooth function which $\mathbb{L}_2$ closely approximates c_{K_*} and then write

$$\begin{aligned} \varphi\left(c_{K_*} T_* - Q_*\right) = {} & \varphi\left(c_{K_{J(i)}} T_{J(i)} - Q_{J(i)}\right) + \varphi\left(Q_{J(i)} - Q_*\right) \\ & + \varphi\psi\left(T_* - T_{J(i)}\right) + \varphi\left(c_{K_*} - \psi\right)\left(T_* - T_{J(i)}\right) \\ & + \varphi\left(c_{K_*} - c_{K_{J(i)}}\right) T_{J(i)} \end{aligned}$$

.

It follows from 2.5.6 and 5.1 that

$$\int \varphi\left(c_{K_{J(i)}} T_{J(i)} - Q_{J(i)}\right) \, d\mathcal{L}^{n+1} = 0.$$

It follows from 7.2(3.2) that

$$\int \varphi \left(Q_{J(i)} - Q_*\right)\, d\mathcal{L}^{n+1} \to 0$$

and

$$\varphi\psi \left(T_* - T_{J(i)}\right)\, d\mathcal{L}^{n+1} \to 0$$

as $i \to \infty$.

We use Schwarz's inequality and 7.2(1.1)(3.2) to estimate

$$\left|\int \varphi \left(c_{K_*} - \psi\right)\left(T_* - T_{J(i)}\right)\, d\mathcal{L}^{n+1}\right| \leq 2b^{\frac{1}{2}} C_1 \sup|\varphi| \left(\int \left(c_{K_*} - \psi\right)^2\, d\mathcal{L}^{n+1}\right)^{\frac{1}{2}}$$

and

$$\left|\int \varphi \left(c_{K_*} - c_{K_{J(i)}}\right) T_{J(i)}\, d\mathcal{L}^{n+1}\right| \leq 2b^{\frac{1}{2}} \sup|\varphi| c_\ell \left(\int \mathcal{L}^n \left(K_*(t) \triangle K_{J(i)}(t)\right)\, d\mathcal{L}^1 t\right)^{\frac{1}{2}}.$$

The first integral is small if ψ closely approximates c_{K_*} in $\mathbb{L}_2$ while the second integral approaches zero as $i \to \infty$ by assumption.

With regard to 2.5.14 we set $\Delta t = 2^{-j}$ and compare

$$\int_{\Omega\times(0,\,b)} \overset{\bullet}{\varphi} Q_* + \kappa_{K_*}(\nabla\varphi) \bullet \nabla T_*\, d\mathcal{L}^{n+1} + \int_\Omega \varphi(\cdot,\, b) Q_*(\cdot,\, b) - \varphi(\cdot,\, 0) Q_0\, d\mathcal{L}^n$$

with

$$\int \Omega \times (0,\, b) \overset{\bullet}{\varphi} Q_j + \kappa_{K_j}(\nabla\varphi) \bullet \nabla T_j\, d\mathcal{L}^{n+1}$$

$$+ \sum_{k=1}^{b/\Delta t} \int_\Omega \varphi(\cdot,\, k\Delta t) P_j(\cdot,\, k\Delta t) - \varphi(\cdot,\, (k-1)\Delta t) Q_j(\cdot,\, (k-1)\Delta t)\, d\mathcal{L}^n;$$

here for notational simplicity b is assumed to be an integral multiple of Δt. The second expression vanishes in accordance with 2.5.7 since in our approximations the crystal is fixed in each Δt interval. It is a minor extension of the present argument to treat the general case.

An argument similar to that used in checking 2.4.13 shows that

$$\int_{\Omega\times(0,\,b)} \overset{\bullet}{\varphi} Q_{J_1(i)} + \kappa_{K_{J_1(i)}}(\nabla\varphi) \bullet \nabla T_{J_1(i)} j\, d\mathcal{L}^{n+1}$$

$$\to \int_{\Omega\times(0,\,b)} \overset{\bullet}{\varphi} Q_* + \kappa_{K_*}(\nabla\varphi) \bullet \nabla T_*\, d\mathcal{L}^{n+1}$$

as $i \to \infty$.

For the Ω integrals we estimate

$$\begin{aligned}
&\sum_{k=1}^{b/\Delta t} \int_\Omega \varphi(\cdot,\, k\Delta t) P_j(\cdot,\, k\Delta t) - \varphi(\cdot,\, (k-1)\Delta t) Q_j(\cdot,\, (k-1)\Delta t)\, d\mathcal{L}^n \\
&\qquad\qquad - \int_\Omega \varphi(\cdot,\, b) Q_*(\cdot,\, b) - \varphi(\cdot,\, 0) Q_0\, d\mathcal{L}^n \\
&\quad = \sum_{k=1}^{b/\Delta t - 1} \int_\Omega \varphi(\cdot,\, k\Delta t) \left(P_j(\cdot,\, k\Delta t) - Q_j(\cdot,\, k\Delta t)\right) d\mathcal{L}^n \\
&\quad \le \sum_{k=1}^{b/\Delta t - 1} \operatorname{Lip}(\varphi) \left\| P_j(\cdot,\, k\Delta t) - -Q_j(\cdot,\, k\Delta t) \right\|_* \\
&\quad \le \operatorname{Lip}(\varphi) \sum_{k=1}^{b/\Delta t - 1} \Delta t^{-j\alpha} \big(\mathbb{E}_j((k\Delta t)^-) - \mathbb{E}_j(k\Delta t)\big) \\
&\quad \le \Delta t^{-j\alpha} \operatorname{Lip}(\varphi) \mathbb{E}_0.
\end{aligned}$$

Hence we have limit zero as $j \to \infty$. ∎

7.5 Terminology.

(1) We denote by Y the collection of all triples $([\![K]\!],\, Q,\, T)$ for which $[\![K]\!]$ is a crystal in $\mathcal{K}$, Q is a heat distribution in $\mathcal{Q}$, and $T = c_K^{-1} Q$ is the associated temperature field. We equip Y with a metric

$$\mu\colon Y \times Y \to \mathbb{R}^+,$$

$$\begin{aligned}
\mu\Big(\big([\![K]\!],\, P,\, S\big), \big([\![L]\!],\, Q,\, T\big)\Big) &= \mathcal{L}^n(K \triangle L) \\
&\qquad + \|P - Q\|_{\mathbf{L}_2} + \|1/P - 1/Q\|_{\mathbf{L}_2} \\
&\qquad\qquad + \|S - T\|_{\mathbf{L}_2} + \|1/S - 1/T\|_{\mathbf{L}_2}.
\end{aligned}$$

(2) For each $0 < M < \infty$ we denote

$$Y(M) = Y \cap \big\{([\![K]\!],\, Q,\, T)\colon \operatorname{DIR}(T) + \operatorname{DIR}(1/T) \le M\big\}.$$

7.6 Theorem (Compactness of the $Y(M)$'s and the modulus of continuity functions σ_M).

(1) Corresponding to each $0 < M < \infty$ and each sequence $\big([\![K^i]\!],\, Q^i,\, T^i\big) \in Y(M)$, $i = 1, 2, 3, \ldots$ there exists a subsequence J of J_0 together with $\big([\![K]\!],\, Q,\, T\big) \in Y(M)$ such that, as $i \to \infty$,

$$\mu\Big(\big([\![K^{J(i)}]\!],\, Q^{J(i)},\, T^{J(i)}\big),\, \big([\![K]\!],\, Q,\, T\big)\Big) \to 0 \quad \textit{and} \quad \|Q^{J(i)} - Q\|_* \to 0$$

and also

$$\nabla T^{J(i)} \rightharpoonup \nabla T \quad \textit{and} \quad \nabla\left(1/T^{J(i)}\right) \rightharpoonup \nabla\,(1/T) \quad \big(\textit{weak convergence in}\ \mathbb{L}_2\,(\Omega,\, \mathbb{R}^n)\,\big).$$

(2) Corresponding to each $0 < M < \infty$ there is an increasing function $\sigma_M : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{r \downarrow 0} \sigma_M(r) = 0$ such that

$$\mu\Big(\big([\![K]\!],\, P,\, S\big),\, \big([\![L]\!],\, Q,\, T\big)\Big) \leq \sigma_M\big(\|P - Q\|_*\big)$$

whenever $([\![K]\!],\, P,\, S)$, $([\![L]\!],\, Q,\, T)$ belong to $Y(M)$.

PROOF The compactness theorem for integral currents implies the convergence of a subsequence of the $[\![K^i]\!]$'s. Rellich's theorem (B.7) implies the asserted convergences for a further subsequence of the T^i's, $1/T^i$'s, $\nabla T^i)$'s, and $\nabla\left(1/T^i\right)$'s. The convergence of the Q^i's and $1/Q^i$'s follows, which gives the first part of the first assertion. The second part of the first assertion follows from the first. With regard to the second assertion, we see from Corollary 5.5 that $\mathcal{L}^n(K \triangle L)$ is small if $\|P - Q\|_*$ is small, and from Proposition 6.3 that $\|S - T\|_{\mathbf{L}_2} + \|1/S - 1/T\|_{\mathbf{L}_2}$ is small if $\mathcal{L}^n(K \triangle L)$ and $\|P - Q\|_*$ are both small. As above, these conditions imply that $\|P - Q\|_{\mathbf{L}_2} + \|1/P - 1/Q\|_{\mathbf{L}_2}$ is small. The second assertion follows. ■

7.7 Main Convergence Theorem. *There is a subsequence J of J_2 together with varying crystals $[\![K_*(t)]\!]$ in $\mathcal{K}$, varying heat distributions $Q_*(t)$ in $\mathcal{Q}$, and associated varying temperature fields $T_*(t) = c_{K_*(t)}^{-1} Q_*(t)$ for which the following is true.*

(1) For $\mathcal{L}^1$ almost every time t in $\mathbb{R}^+$

$$\mu\Big(\big([\![K_{J(i)}(t)]\!],\, Q_{J(i)}(t),\, T_{J(i)}(t)\big),\, \big([\![K_*(t)]\!],\, Q_*(t),\, T_*(t)\big)\Big) \to 0$$

with

$$\liminf_{i \to \infty} \mathrm{DIR}\big(T_{J(i)}(t)\big) < \infty \quad \textit{and} \quad \liminf_{i \to \infty} \mathrm{DIR}\big(1/T_{J(i)}(t)\big) < \infty.$$

(2) The crystals $[\![K_*(t)]\!]$ and functions $Q_*(x, t)$ and $T(x, t)$ are a solution to the heat equation with varying crystals as in 2.5 with initial crystal $[\![K_0]\!]$ and initial heat distribution Q_0.

(3) The union

$$\bigcup \{K_*(t) : 0 \le t < \infty\}$$

of the crystal sets $K_*(t)$ is $\mathcal{L}^{n+1}$ measurable.

PROOF.

Step 1. We choose and fix a sequence of intervals $I_1, I_2, I_3, \ldots$ in $\mathbb{R}^+$ with diameters converging to zero such that $\bigcup_{\ell=k}^{\infty} I_\ell = \mathbb{R}^+$ for each positive integer k.

Step 2. Suppose M is a positive integer and J is any subsequence of J_0 and S is the family of times t such there are infinitely many j in $\{J(1), J(2), J(3), \ldots\}$ such that

$$\mathrm{DIR}\big(T_j(t)\big) + \mathrm{DIR}\big(U_j(t)\big) \le M.$$

Then $\mathcal{L}^1\left(\mathbb{R}^+ \sim S\right) \le C_1/M$ where C_1 is the constant of 7.2(1.3).

Proof. Let $X(k)$ denote the set of times t such that whenever $i \ge k$ then

$$\mathrm{DIR}\big(T_{J(i)}(t)\big) + \mathrm{DIR}\big(U_{J(i)}(t)\big) > M$$

so that $X(1) \subset X(2) \subset X(3) \subset \ldots$. Since $X(k)$ is a subset of the t's for which

$$\mathrm{DIR}\big(T_{J(k)}(t)\big) + \mathrm{DIR}\big(U_{J(k)}(t)\big) > M$$

we infer from 7.2(1.3) that $\mathcal{L}^1(X(k)) \le C_1/M$. The assertions follows since

$$\mathbb{R}^+ \sim S = \cup\big\{X(k) : k = 1, 2, 3, \ldots\big\}.$$

Step 3. Our proof requires that we construct for all positive integers M and k the following:

(a) Subsequences J^M, J^M_*, $M^{M,k}$ of J_0 subject to the following conditions:

(a.1) $J^{1,1} = J_2$ as in 7.5.

(a.2) J^M result of Cantor's diagonal process applied to the sequences $J^{M,1}$, $J^{M,2}$, $J^{M,3}$, ...

(a.3) J^M_* is a subsequence of J^M selected in accordance with Step 4 below.

(a.4) $J^{M+1,1} = J_*^M$.

Our final sequence J is the result of Cantor's diagonal process applied to the sequences $J_*^M,\ J_*^{M+1},\ J_*^{M+2},\ \ldots$.

(b) Sets of times $S(M)$, $S_*(M)$, $S\left(M,\ J^{M,k}\right)$ subject to the following conditions:

(b.1) $S\left(M,\ J^{M,k}\right)$ is the set of times t in $\mathbb{R}^+$ for which there are infinitely many j's in $\left\{J^{M,k}(1),\ J^{M,k}(2),\ J^{M,k}(3),\ \ldots\right\}$ for which

$$\mathrm{DIR}\big(T_j(t)\big) + \mathrm{DIR}\big(U_j(t)\big) \leq M.$$

We infer that $S\left(M,\ J^{M,k+1}\right) \subset S\left(M,\ J^{M,k}\right)$. We further infer from Step 2 that

$$\mathcal{L}^1\big(\mathbb{R}^+ \sim S\left(M,\ J^{M,k}\right)\big) \leq C_1/M.$$

(b.2) $S(M) = \bigcap\left\{S\left(M,\ J^{M,\ell}\right) : \ell = 1,\ 2,\ 3,\ \ldots\right\}$. Since

$$\mathbb{R}^+ \sim S(M) = \bigcup\left\{\mathbb{R}^+ \sim S\left(M,\ J^{M,\ell}\right) : \ell = 1,\ 2,\ 3,\ \ldots\right\},$$

we infer $\mathcal{L}^1\left(\mathbb{R}^+ \sim S(M)\right) \leq C_1/M$.

(b.3) $S_*(M) \subset \mathbb{R}^+$ with $\mathcal{L}^1\left(S(M) \sim S_*(M)\right) = 0$ is constructed below.

(c) Countable families $W(M) = \left\{t^{M,1},\ t^{M,2},\ t^{M,3},\ \ldots\right\}$ of times subject to the following constraints:

(c.1) $t^{M,k} \in S\left(M,\ J^{M,k}\right)$.

(c.2) If $I_k \cap S\left(M,\ J^{M,k}\right) \neq \emptyset$, then $t^{M,k}$ also lies in the interval I_k,

(c.3) If $M \geq 2$ and it is consistent with the requirements (c.1) and (c.2) above then $t^{M,k} = t^{M-1,k}$.

It follows from our construction and constraint (b.1) and the choice of I_k's in Step 1 that

(c.4) $W(M) \subset S(M)$, and $W(M)$ is dense in $S(M)$.

Our construction procedure is the following:

(1) Assuming $J^{M,k}$ has been constructed we pick $t^{M,k} \in S\left(M,\ J^{M,k}\right)$ in accordance with (c) above.

(2) Assuming $J^{M,k}$ and $t^{M,k}$ have been fixed, we use conclusion (1) of Theorem 7.6 to choose a subsequence $J^{M,k+1}$ of J^M together with a limit

$$\big([\![K_*(t^{M,k})]\!],\ Q_*(t^{M,k}),\ T_*(t^{M,k})\big) \in Y(M)$$

such that, abbreviating $j(i) = J^{M,k+1}(i)$, we have

$$\mu\Big(\big(\llbracket K_{j(i)}(t^{M,k})\rrbracket,\, Q_{j(i)}(t^{M,k}),\, T_{j(i)}(t^{M,k})\big),\, \big(\llbracket K_*(t^{M,k})\rrbracket,\, Q_*(t^{M,k}),\, T_*(t^{M,k})\big)\Big) \to 0$$

and also

$$\nabla T_{j(i)}(t^{M,k}) \rightharpoonup \nabla T_*(t^{M,k}) \quad \text{and} \quad \nabla \left(1/T_{J(j)}(t^{M,k})\right) \rightharpoonup \nabla \left(1/T_*(t^{M,k})\right)$$

(weak convergence in $\mathbb{L}_2(\Omega, \mathbb{R}^n)$) as $i \to \infty$.

(3) It follows by construction that $Q_*(t^{M,k})$ and $T_*(t^{M,k})$ are the heat distribution and temperature field specified in 7.3; note 7.2(2)(3).

(4) It further follows from 7.2(2.2) and 7.6(2) that whenever $s, t \in W(M)$ then

$$\mu\Big(\big(\llbracket K_*(s)\rrbracket,\, Q_*(s),\, T_*(s)\big),\, \big(\llbracket K_*(t)\rrbracket,\, Q_*(t),\, T_*(t)\big)\Big)$$

$$\le \sigma_M\left(\|Q(s) - Q(t)\|_*\right) \le \sigma_M\left(\rho|s-t|^{\frac{1}{2}}\right).$$

Since $Y(M)$ is complete there is a unique continuous extension of the function

$$\big(\llbracket K_*\rrbracket,\, Q_*,\, T_*\big)\colon W(M) \to Y(M)$$

to a function

$$\big(\llbracket K\rrbracket,\, Q,\, T\big)\colon \mathrm{Closure}\big[W(M)\big] \to Y(M).$$

We do not claim convergence of the

$$\big(\llbracket K_{J^M(i)}(t)\rrbracket,\, Q_{J^M(i)}(t),\, T_{J^M(i)}(t)\big)\text{'s}$$

for all t's in $\mathrm{Closure}\big[W(M)\big]$. The construction of $J_*(M)$ and $S_*(M)$ for which there is convergence is given in Step 4 below.

Step 4. We assert convergence in measure with respect to the μ metric of the $\big(\llbracket K_{J^M(i)}(t)\rrbracket,\, Q_{J^M(i)}(t),\, T_{J^M(i)}(t)\big)$'s to the $\big(\llbracket K(t)\rrbracket,\, Q(t),\, T(t)\big)$'s as $i \to \infty$ for t's in $\mathrm{Closure}\big[W(M)\big]$. As a consequence of this convergence in measure there is a subsequence J_*^M of J^M (as in Step 3(a.4) above) and an $\mathcal{L}^1$ measurable set $S_*(M) \subset \mathrm{Closure} W(M)$ (as in Step 3(b.3) above) with

$$\mathcal{L}^1\Big(\mathrm{Closure}\big[W(M)\big] \sim S_*(M)\Big) = \mathcal{L}^1\big(S(M) \sim S_*(M)\big) = 0$$

such that, for each $t \in S_*(M)$,

$$\mu\bigg(\big([\![K_{J^M(i)}(t)]\!],\, Q_{J^M(i)}(t),\, T_{J^M(i)}(t) \big),\, \big([\![K(t)]\!],\, Q(t),\, T(t) \big) \bigg) \to 0$$

as $i \to \infty$. For such t's we infer $Q(t) = Q_*(t)$, $T(t) = T_*(t)$, and we set $[\![K_*(t)]\!] = [\![K(t)]\!]$.

Proof. We will verify the asserted convergence in measure. Suppose then $\epsilon > 0$ and $0 < b < \infty$. We will show that, for all sufficiently large j's in $\{J^M(1),\, J^M(2),\, J^M(3),\, \ldots\}$, the one dimensional Lesbegue measure of the set

$$[0,\, b] \cap \text{Closure}\big[W(M)\big] \cap \bigg\{ t \colon \mu\bigg(\big([\![K_j(t)]\!],\, Q_j(t),\, T_j(t) \big),\, \big([\![K(t)]\!],\, Q(t),\, T(t) \big) \bigg) > \epsilon \bigg\}$$

is less than ϵ. There are five steps in the demonstration.

(1) First we use 7.2(1.3) to pick and fix an integer $N \geq M$ sufficiently large so that, for each $j \in \{J^M(1),\, J^M(2),\, J^M(3),\, \ldots\}$, if

$$W_j = \bigg\{ t \colon \text{DIR}\big(T_j(t)\big) + \text{DIR}\big(U_j(t)\big) \leq N \bigg\}$$

then $\mathcal{L}^1\big([0,\, b] \sim W_j\big) < \epsilon$.

(2) We now use 7.6(2) to pick and fix $\Delta Q > 0$ sufficiently small so that $\sigma_N(4\Delta Q) < \epsilon$.

(3) We fix a finite set of times $t_1,\, t_2,\, \ldots,\, t_m$ in W(M) such that, for each t belonging to $[0,\, b] \cap \text{Closure} W(M)$, there is $\alpha \in \{1,\, 2,\, \ldots,\, m\}$ such that

$$\|Q(t) - Q(t_\alpha)\|_* < \Delta Q \quad \text{and} \quad \rho\,|t - t_\alpha| < \Delta Q.$$

(4) We infer from 5.3(2), 7.2(2.1), Step 3(a.1), and (3) above that for all sufficiently large j in $\{J^M(1),\, J^M(2),\, J^M(3),\, \ldots\}$ it will be true for each t belonging to $[0,\, b] \cap \text{Closure}\big[W(M)\big]$, that there is $\alpha \in \{1,\, 2,\, \ldots,\, m\}$ such that

$$\|Q_j(t) - Q_j(t_\alpha)\|_* < 2\Delta Q \quad , \quad \|Q_j(t_\alpha) - Q(t_\alpha)\|_* < \Delta Q, \quad \text{and} \quad \|Q(t) - Q(t_\alpha)\|_* < \Delta Q$$

so that

$$\|Q_j(t) - Q(t)\|_* < 4\Delta Q$$

(5) For each fixed sufficiently large j in $\{J^M(1),\, J^M(2),\, J^M(3),\, \ldots\}$ we use (1), (2), (3), (4) to estimate

$$\mathcal{L}^1\bigg([0,\, b] \cap \text{Closure}\big[W(M)\big] \sim W_j \bigg) < \epsilon$$

while, for each $t \in \Big([0, b] \cap \mathrm{Closure}\big[W(M)\big] \sim W_j\Big)$,

$$\big([\![K_j(t)]\!], Q_j(t), T_j(t)\big) \quad \text{and} \quad \big([\![K(t)]\!], Q(t), T(t)\big)$$

both belong to $Y(M)$ with

$$\sigma_N\big(\|Q_j(t) - Q_(t)\|_*\big) < \sigma_N(4\Delta Q) < \epsilon.$$

Step 5. Conclusion (1) of our theorem now follows from the construction in view of Step 2(b.2) and Steps 4 and 5. Our subsequence of subsequence construction guarantees that convergence is never lost for any time for which it is established. Conclusion (2) of our theorem follows from conclusion (1) and Theorem 7.4. By construction in 2.11, the characteristic function $\chi_{J(i)}$ of the set $\bigcup\big\{K_{J(i)}(t): 0 \le t < \infty\big\}$ is $\mathcal{L}^{n+1}$ measurable for each i. We infer from (b.2), (b.3), (c.4) and Step 4 above that the $\chi_{J(i)}$'s converge in measure to the characteristic function χ_* of the set $\bigcup\big\{K_*(t): 0 \le t < \infty\big\}$. We infer that χ_* is $\mathcal{L}^{n+1}$ measurable so that $\bigcup\big\{K_*(t): 0 \le t < \infty\big\}$ is also $\mathcal{L}^{n+1}$ measurable. This is Conclusion (3) of our theorem. ■

8. The Gibbs-Thomson condition holds for elliptic Φ's

According to Theorem 4.4 the Gibbs-Thomson relation between first variation distributions and temperature holds exactly for the $\mathcal{E}$ minimizers constructed in our approximating evolutions in case our surface energy function Φ is smooth. In the present chapter we show that in case Φ is elliptic then the Gibbs-Thomson relation also holds in the final evolution which is a limit of the approximating ones. The sequence of approximations which leads this equality is the following. First we fix a time t which is "good" according to the criteria of Theorem 8.1 below (almost every time is good). We then fix an initial velocity vector field g_* and associated deformation $G_s(x) = x + s\, g_*(x)$ Associated with each large integer i is the time $t_0 = t_0(i)$ at the the beginning of the $\Delta t = 2^{-J(i)}$ time interval which contains t in construction of the $J(i)$-th approximating evolution. For fixed i we have:

$$\frac{d}{ds}\Phi\Big(G_{s\sharp}\partial[\![K_*(t)]\!]\Big)\bigg|_{s=0}$$
$$\approx \frac{d}{ds}\Phi\Big(G_{s\sharp}\partial[\![K_{J(i)}(t)]\!]\Big)\bigg|_{s=0}$$

by Theorem 8.6(3) below

$$= \int_{x \in \partial K_{J(i)}(t_0)} H\left(T_{J(i)}(x,\, t_0)\right)\, g_* \bullet \mathbf{n}_{K_{J(i)}(t_0)}\, d\mathcal{H}^{n-1}x$$

by Theorem 4.4

$$\approx \int_{x \in \partial K_{J(i)}(t)} H\left(T_{J(i)}(x,\, t)\right)\, g_* \bullet \mathbf{n}_{K_{J(i)}(t)}\, d\mathcal{H}^{n-1}x$$

by Theorem 8.3 below

$$= \int_{x \in K_{J(i)}(t)} \operatorname{div}\left(H\left(T_{J(i)}(x,\, t)\right)\, g_*\right)\, d\mathcal{L}^n x$$

by the Gauss-Green theorem [FH 4.5.6(5)]

$$\approx \int_{x \in K_*(t)} \operatorname{div}\left(H\left(T_*(x,\, t)\right)\, g_*\right)\, d\mathcal{L}^n x$$

by Theorem 8.1 below

$$= \lim_{\epsilon \downarrow 0} \int_{x \in \partial K_*(t)} H_\epsilon\big(T_*(x,\, t)\big)\, \mathbf{n}_{K_*(t)}(x) \bullet g_*(x)\, d\mathcal{H}^{n-1}x$$

by Theorem 8.7 below

Equalities result in the limit as $i \to \infty$. In Theorem 8.6 we show that $\partial[\![K_*(t)]\!]$ itself as a solution to a variational problem. This enables us in Theorem 8.8 to conclude $\partial K_*(t)$ is a smooth one dimensional submanifold of Ω without boundary in case $n = 2$ and is a homeomorphic embedding of a compact continuous two dimensional manifold without boundary in case $n = 3$.

8.1 Theorem (Some integral convergences for sequences of crystals). *Suppose, for some fixed time t there is a subsequence (just denoted by the index j) of the integers for which the following is true as $j \to \infty$.*

(a) $\mathcal{L}^n\big(K_j(t) \triangle K(t)\big) \to 0$;

(b) $T_j(t) \to T(t)$;

(c) $U_j(t) \to U(t)$;

(d) $\nabla T_j(t) \rightharpoonup \nabla T(t)$ with $\sup_j \mathrm{DIR}(T_j(t)) < \infty$;

(e) $\nabla U_j(t) \rightharpoonup \nabla U(t)$ with $\sup_j \mathrm{DIR}(U_j(t)) < \infty$.

Then, for each smooth function $g\colon \Omega \to \mathbb{R}$ and for each $i = 1, 2, \ldots, n$,

$$\int_{x \in K_j(t)} \frac{\partial}{\partial x_i}\Big(H\big(T_j(x,\, t)\big)\, g(x)\Big)\, d\mathcal{L}^n x \to \int_{x \in K(t)} \frac{\partial}{\partial x_i}\Big(H\big(T(x,\, t)\big)\, g(x)\Big)\, d\mathcal{L}^n x$$

as $j \to \infty$.

PROOF. It clearly suffices to consider only the case $i = n$. Also, in view of [MC Lemma 3.1.1] we can assume without loss of generality that the functions

$$x_n \mapsto T_j\big(x_1, \ldots, x_{n-1}, x_n, t\big), \quad x_n \mapsto T\big(x_1, \ldots, x_{n-1}, x_n, t\big)$$

are simultaneously absolutely continuous for each j for $\mathcal{L}^{n-1}$ almost all $\big(x_1, \ldots, x_{n-1}\big)$ (with the obvious meaning in Ω). We can therefore evaluate

$$\frac{\partial}{\partial x_i}\Big(H\big(T_j(x, t)\big)\, g(x)\Big) = H'\big(T_j(x, t)\big)\frac{\partial T_j}{\partial x_i}(x, t)\, g(x) + H\big(T_j(x, t)\big)\big)\frac{\partial g}{\partial x_i}(x).$$

For notational simplicity we will drop references to the variables x and the variable t (since it is fixed) and will abbreviate by χ_j the characteristic function of K_j for each j and similarly will abbreviate by χ the characteristic function of K.

We infer from the equations A.1.6 and A.1.6 the existence of bounded Lipschitz functions J, L:$\mathbb{R}^+ \to \mathbb{R}$ such that

$$H(T_*) = (c_\ell - c_s)\left(3U_*^2 - T_*^2 + J(T_*)\right), \quad H'(T_*) = (c_\ell - c_s)\left(-6U_*^3 - 2T_* + L(T_*)\right)$$

for each $0 < T_* < \infty$ with $U_* = 1/T_*$. We note

$$\frac{\partial U_j}{\partial x_n} = -U_j^2\frac{\partial T_j}{x_n}, \quad \frac{\partial U}{\partial x_n} = -U^2\frac{\partial T}{x_n}$$

for each j and estimate

$$\chi_j \frac{\partial}{\partial x_n}\Big(H\left(T_j\right) g\Big) - \chi \frac{\partial}{\partial x_n}\Big(H\left(T\right)\, g\Big)$$

$$\begin{aligned}
&= \chi_j H'\left(T_j\right) \frac{\partial T_j}{\partial x_n}\, g - \chi H'\left(T\right) \frac{\partial T}{\partial x_n}\, g \\
&\qquad\qquad + \chi_j H\left(T_j\right) \frac{\partial g}{\partial x_n} - \chi H\left(T\right) \frac{\partial g}{\partial x_n} \\
&= g\left(c_{\ell} - c_{s}\right) \Bigg(\chi_j \big[-6U_j^3 - 2T_j + L(T_j)\big] \frac{\partial T_j}{\partial x_n} \\
&\qquad\qquad - \chi\big[-6U^3 - 2T + L(T)\big] \frac{\partial T}{\partial x_n}\Bigg) \\
&\quad + \frac{\partial g}{\partial x_n}\left(c_{\ell} - c_{s}\right) \Bigg(\chi_j \big[3U_j^2 - 2T_j^2 + J(T_j)\big] - \chi\big[3U^2 - 2T^2 + J(T)\big]\Bigg) \\
&= g\left(c_{\ell} - c_{s}\right) \Bigg(6\left[\chi_j U_j \frac{\partial U_j}{\partial x_n} - \chi U \frac{\partial U}{\partial x_n}\right] - 2\left[\chi_j T_j \frac{\partial T_j}{\partial x_n} - \chi T \frac{\partial T}{\partial x_n}\right] \\
&\qquad\qquad + \left[\chi_j L\left(T_j\right) \frac{\partial T_j}{\partial x_n} - \chi L\left(T\right) \frac{\partial T}{\partial x_n}\right]\Bigg) \\
&\quad + \frac{\partial g}{\partial x_n}\left(c_{\ell} - c_{s}\right) \Bigg(3\left(\chi_j - \chi\right) U^2 + 3\chi_j \left(U_j - U\right)\left(U_j + U\right) \\
&\qquad\qquad + \left(\chi_j - \chi\right) T^2 + \chi_j \left(T_j - T\right)\left(T_j + T\right) \\
&\qquad\qquad + \left(\chi_j - \chi\right) J(T) + \chi_j \left(J\left(T_j\right) - J\left(T\right)\right)\Bigg)
\end{aligned}$$

We write

$$\chi_j U_j - \chi U = \left(\chi_j - \chi\right) U + \chi_j \left(U_j - U\right), \quad \chi_j T_j - \chi T = \left(\chi_j - \chi\right) T + \chi_j \left(T_j - T\right),$$

$$\chi_j L\left(T_j\right) - \chi L(T) = \left(\chi_j - \chi\right) L(T) + \chi_j \left(L(T_j) - L(T)\right)$$

and infer

$$g\left(c_{\ell} - c_{s}\right) 6\chi_j U_j \to g\left(c_{\ell} - c_{s}\right) 6\chi U,$$

$$g\left(c_{\ell} - c_{s}\right)(-2)\chi_j T_j \to g\left(c_{\ell} - c_{s}\right)(-2)\chi T,$$

$$g\left(c_{\ell} - c_{s}\right) \chi_j L\left(T_j\right) \to g\left(c_{\ell} - c_{s}\right) \chi L(T)$$

as $j \to \infty$ (strong convergence in $\mathbb{L}_2(\Omega)$) while

$$\frac{\partial U_j}{\partial x_n} \rightharpoonup \frac{\partial U}{\partial x_n}, \quad \frac{\partial T_j}{\partial x_n} \rightharpoonup \frac{\partial T}{\partial x_n}$$

as $j \to \infty$ (weak convergence in $\mathbb{L}_2(\Omega)$). We can then invoke the convergence properties set forth in Appendix B.3 to conclude that the $\mathcal{L}^n \llcorner \Omega$ integrals of functions appearing in the first summand of the final expression in our sequence of equalities above converge to zero as $j \to \infty$. We invoke the absolute continuity of integrals of summable functions set forth in [FH 2.4.11] to conclude that the $\mathcal{L}^n \llcorner \Omega$ integrals of the first, third, and fifth functions appearing in the second summand in the same final expression also converge to zero as $j \to \infty$. Finally we use Schwarz's inequality in checking that the $\mathcal{L}^n \llcorner \Omega$ integrals of the second, fourth, and sixth functions appearing in the same second summand of the same final expression also converge to zero as $j \to \infty$. This completes the proof. ■

8.2 Theorem (Some oriented integral estimates over boundaries of crystals). *Suppose K is a crystal in $\mathcal{K}$ and $f\colon \Omega \to \mathbb{R}$ is a Lipschitz function having support in an open ball $\mathbb{U}^n\big(p,\, \rho_*/2\big)$ in Ω. Then, for each positive integer M, the oriented surface integral*

$$\left| \int_{p \in \partial K} f(p)\, \mathbf{n}_K(p) \bullet \mathbf{e}_n \, d\mathcal{H}^{n-1} p \right|$$

does not exceed the sum

$$2^{\frac{3}{2}} M \left(n\alpha(n) \left(\frac{\rho_*}{2}\right)^{n-1} \right)^{\frac{1}{2}} \left(\int_\Omega f^2 \, d\mathcal{L}^n \right)^{\frac{1}{4}} \left(\int_\Omega |\nabla f|^2 \, d\mathcal{L}^n \right)^{\frac{1}{4}}$$

$$+ \left(\frac{\rho_* \big(\mathcal{H}^{n-1}(\partial K) + n\alpha(n) \left(\frac{\rho_*}{2}\right)^{n-1}\big)}{2M} \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla f|^2 \, d\mathcal{L}^n \right)^{\frac{1}{2}}.$$

If we replace $2^{\frac{3}{2}}$ by $2^{\frac{3}{2}} + 1$ in the above sum, then the inequality holds for all $M \geq 2$ (not just integers).

PROOF.

Step 1. If $f\colon\mathbb{R} \to \mathbb{R}^+$ is a Lipschitz function having compact support with

$$h = \sup f \quad \text{and} \quad 2r = \mathcal{L}^1\{s \colon f(s) \geq h/2\},$$

then

$$r\,h^2 \le 2\int_{\mathbf{R}} f^2\,d\mathcal{L}^1 \quad \text{and} \quad \frac{h^2}{r} \le 2\int_{\mathbf{R}} \left(\frac{\partial f}{\partial r}\right)^2 d\mathcal{L}^1.$$

Proof. If $g{:}\mathbb{R} \to \mathbb{R}^+$ is the symmetric decreasing rearrangement [AL] of f then

$$\sup g = \sup f = h, \quad \int_{\mathbf{R}} g^2 = \int_{\mathbf{R}} f^2, \quad \text{and} \quad \int_{\mathbf{R}} \left(\frac{\partial g}{\partial r}\right)^2 d\mathcal{L}^1 \le \int_{\mathbf{R}} \left(\frac{\partial f}{\partial r}\right)^2 d\mathcal{L}^1.$$

Hence it is sufficient to prove our assertion in case $g = f$ which is left to the reader.

Step 2. Suppose $f{:}\mathbb{R}^n \to \mathbb{R}^+$ is a Lipschitz function having compact support and functions h, $r{:}\mathbb{R}^{n-1} \to \mathbb{R}^+$ are defined by setting

$$h(x) = \sup\{f(x,\, s){:}\, s \in \mathbb{R}\}, \quad 2r(x) = \mathcal{L}^1\{s{:}\, f(x,\, s) \ge h(x)/2\}$$

for each $x \in \mathbb{R}^{n-1}$. Then, for each measurable subset A of $\mathbb{R}^{n-1}$,

$$\int_A h\,d\mathcal{L}^{n-1} \le 2^{\frac{1}{2}}\mathcal{L}^{n-1}(A)^{\frac{1}{2}} \left(\int_{\mathbf{R}^n} f^2\,d\mathcal{L}^n\right)^{\frac{1}{4}} \left(\int_{\mathbf{R}^n} |\nabla f|^2\,d\mathcal{L}^n\right)^{\frac{1}{4}}.$$

Proof. According to Schwarz's inequality we have

$$\int_A h\,d\mathcal{L}^{n-1} \le \mathcal{L}^{n-1}(A)^{\frac{1}{2}} \left(\int_A h^2\,d\mathcal{L}^{n-1}\right)^{\frac{1}{2}},$$

and we can use Schwarz's inequality together with the estimate from Step 1 to estimate

$$\begin{aligned}
\int_A h^2\,d\mathcal{L}^{n-1} &\le \int_{x\in\mathbf{R}^{n-1}} h^2(x)\,d\mathcal{L}^{n-1}x \\
&\le \left(\int_{x\in\mathbf{R}^{n-1}} h^2(x)\,r(x)\,d\mathcal{L}^{n-1}x\right)^{\frac{1}{2}} \left(\int_{x\in\mathbf{R}^{n-1}} \frac{h^2(x)}{r(x)}\,d\mathcal{L}^{n-1}x\right)^{\frac{1}{2}} \\
&\le \left(2\int_{x\in\mathbf{R}^{n-1}}\int_{s\in\mathbf{R}} f^2(x,s)\,d\mathcal{L}^1 s\,d\mathcal{L}^{n-1}x\right)^{\frac{1}{2}} \\
&\qquad \left(2\int_{x\in\mathbf{R}^{n-1}}\int_{s\in\mathbf{R}} \left(\frac{\partial f}{\partial r}(x,\,s)\right)^2 d\mathcal{L}^1 s\,d\mathcal{L}^{n-1}x\right)^{\frac{1}{2}}
\end{aligned}$$

from which two inequalities together with Fubini's theorem our assertion follows.

Step 3. Suppose $f:\mathbb{R}^n \to \mathbb{R}^+$ is a Lipschitz function having compact support and $K \subset \mathbb{R}^{n-1} \times [R, S] \subset \mathbb{R}^n$ is a set of finite perimeter. Then, for each positive integer M, the oriented surface integral

$$\left| \int_{p\in\partial K} f(p)\,\mathbf{n}_K(p) \bullet \mathbf{e}_n \, d\mathcal{H}^{n-1}p \right|$$

does not exceed the sum

$$2^{\frac{3}{2}} M\, \mathcal{L}^{n-1}\big(\pi\partial K\big)^{\frac{1}{2}} \left(\int_{\mathbf{R}^n} f^2 \, d\mathcal{L}^n\right)^{\frac{1}{4}} \left(\int_{\mathbf{R}^n} |\nabla f|^2 \, d\mathcal{L}^n\right)^{\frac{1}{4}}$$

$$+\left(\frac{(S-R)\mathcal{H}^{n-1}(\partial K)}{2M}\right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^n} |\nabla f|^2 \, d\mathcal{L}^n\right)^{\frac{1}{2}}$$

Proof. We let $\pi:\mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^{n-1}$ denote projection on the first factor. We infer from the theory of slicing set forth in [FH 4.3.8] that, for $\mathcal{L}^{n-1}$ almost every x in $\mathbb{R}^{n-1}$, there will be a nonnegative integer $N(x)$ together with numbers

$$R \le r_1(x) < s_1(x) < r_2(x) < s_2(x) < \ldots < r_{N(x)} < s_{N(x)} \le S$$

such that the slice $\big\langle \partial[\![K]\!],\, \pi,\, x\big\rangle$ can be written

$$\big\langle \partial[\![K]\!],\, \pi,\, x\big\rangle = \sum_{i=1}^{N(x)} [\![(x,\, s_i(x))]\!] - [\![(x,\, r_i(x))]\!]$$

so that

$$\int_{p\in\partial K} f(p)\,\mathbf{n}_K(p) \bullet \mathbf{e}_n \, d\mathcal{H}^{n-1}p = \int_{x\in\mathbf{R}^{n-1}} \sum_{i=1}^{N(x)} f\big(x,\, s_i(x)\big) - f\big(x,\, r_i(x)\big)\, d\mathcal{L}x.$$

For each positive integer M we set

$$A_M = \mathbb{R}^{n-1} \cap \{x\colon 1 \le N(x) \le M\} \quad \text{and} \quad B_M = \mathbb{R}^{n-1} \cap \{x\colon N(x) > M\}$$

and we estimate

$$\begin{aligned}
&\left| \int_{p\in\partial K} f(p)\, \mathbf{n}_K(p) \bullet \mathbf{e}_n \, d\mathcal{H}^{n-1}p \right| \\
&\qquad \le 2M \int_{x\in A_M} \sup\{|f(x,\, r)| \colon R \le r \le S\}\, d\mathcal{L}^{n-1}x \\
&\qquad\qquad + \int_{x\in B_M} \int_{t\in[R,\, S]} \left| \frac{\partial f}{\partial r}(x,\, t) \right| d\mathcal{L}^1\, d\mathcal{L}^{n-1}x \\
&\qquad \le 2^{\frac{3}{2}} M\, \mathcal{L}^{n-1}(\pi\partial K)^{\frac{1}{2}} \left(\int_{\mathbf{R}^n} f^2\, d\mathcal{L}^n \right)^{\frac{1}{4}} \left(\int_{\mathbf{R}^n} |\nabla f|^2\, d\mathcal{L}^n \right)^{\frac{1}{4}} \\
&\qquad\qquad + \mathcal{L}^n\big(B_M \times [R,\, S]\big)^{\frac{1}{2}} \left(\int_{\mathbf{R}^n} |\nabla f|^2\, d\mathcal{L}^n \right)^{\frac{1}{2}} \\
&\qquad \le 2^{\frac{3}{2}} M\, \mathcal{L}^{n-1}(\pi\partial K)^{\frac{1}{2}} \left(\int_{\mathbf{R}^n} f^2\, d\mathcal{L}^n \right)^{\frac{1}{4}} \left(\int_{\mathbf{R}^n} |\nabla f|^2\, d\mathcal{L}^n \right)^{\frac{1}{4}} \\
&\qquad\qquad + \left(\frac{(S-R)\mathcal{H}^{n-1}\big(\partial K\big)}{2M} \right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^n} |\nabla f|^2\, d\mathcal{L}^n \right)^{\frac{1}{2}};
\end{aligned}$$

In making the estimates above we have used Step 2 applied to the function $|f|$, the Fundamental Theorem of Calculus, the Hausdorff area formula [FH 3.2.20], Schwarz's inequality, and Fubini's theorem.

Step 4. Let K and f be as in the hypotheses of the present theorem. Since the support of f lies in the open ball $\mathbb{U}^n\big(p,\, \rho_*/2\big)$ it will lie in some closed ball $\mathbb{B}^n\big(p,\, \rho_*/2 - 2\epsilon\big)$ for some ϵ. The set $L = K \cap \mathbb{U}^n\big(p,\, \rho_*/2 - \epsilon\big)$ belongs to $\mathcal{K}$ with

$$\mathcal{L}^{n-1}\big(\pi\, \partial L\big) < \alpha(n) \left(\frac{\rho_*}{2}\right)^{n-1} \text{ and } \mathcal{H}^{n-1}(\partial L) < \mathcal{H}^{n-1}(\partial K) + n\alpha(n) \left(\frac{\rho_*}{2}\right)^{n-1}.$$

Identification of Ω with the quotient of $\mathbb{R}^n$ by ρ_* times the integer lattice enables us to reduce our problem to a problem in $\mathbb{R}^n$ as in Step 3 in which K there (a single inverse image of L) lies in the cube $\big[0,\, \rho_*\big]^n$ and f there has its support in the same cube. The assertion of our theorem now follows from the estimate in Step 3. ■

8.3 Theorem (Bounds on the change in surface integrals of $H(T)$ under heat flow). *There is a constant C with the following property.*
Suppose

(a) $[\![K_0]\!] \in \mathcal{K}$ *and* $Q_0 \in \mathcal{Q}$ *give an* $\mathcal{E}$ *minimizer for some heat distribution* P *and some time interval* $\Delta t \le 1$;

(b) $0 < t_1 < \Delta t$ *and* $Q(x, t)$ *and* $T(x, t)$ *are functions defined on* $\Omega \times [0, t_1]$ *which are a solution to the heat equation with fixed crystal* $[\![K_0]\!]$ *and initial heat distribution* Q_0 *and final heat distribution* Q_1 *as in 2.5, 5.1, 5.2;*

(e) $T_0 = c_{K_0}^{-1} Q_0$ *is the temperature field associated with* $[\![K_0]\!]$ *and* Q_0 *satisfying the a priori estimates of 4.2, and* $T_1 = c_{K_0}^{-1} Q_1$ *is the temperature field associated with* $[\![K_0]\!]$ *and* Q_1;

(d) $g\colon \Omega \to \mathbb{R}$ *is a smooth function having support in an open ball* $\mathbb{U}^n\big(p, \rho_*/2\big)$.

Then, for each $i = 1, 2, \ldots, n$,

$$\left| \int_{x \in \partial K_0} H(T_0(x))\, g(x)\, \mathbf{n}_K(x) \bullet \mathbf{e}_i \, d\mathcal{L}^n x - \int_{x \in \partial K_0} H(T_1(x))\, g(x)\, \mathbf{n}_K(x) \bullet \mathbf{e}_i \, d\mathcal{L}^n x \right|$$

$$\le C \Delta t^{1/12 - 2\alpha} \sup \left\{ \sup_\Omega |g|, \sup_\Omega |Dg| \right\}.$$

PROOF. It clearly suffices to treat only the case in which $i = n$ which assumption we make. The basic calculations in our proof are in terms of our time interval Δt raised to various powers. In order to simplify the exposition we will keep accurate accounting of the exponents of Δt but will denote by C a general (large) constant which will possibly have different values, even in the same expression. We will prove our theorem by invoking Theorem 8.2 with function

$$f(x) = \Big[H\big(T_0(x)\big) - H\big(T_1(x)\big) \Big]\, g(x)$$

and with constant

$$M = \Delta t^{-\frac{1}{6}}.$$

From Theorem 4.3 we know that, for each x,

$$(i) \qquad C^{-1} \Delta t^{\frac{n\alpha}{3n+2}} \le T_0(x) \le C + C\Delta t^{-\alpha} \le C \Delta t^{-\alpha}.$$

The maximum principle for solutions to the heat equation together with the corresponding minimum principle imply that

$$(ii) \qquad C^{-1} \Delta t^{\frac{n\alpha}{3n+2}} \le T_1(x) \le C \Delta t^{-\alpha}.$$

We infer from Theorem 4.3 that T_0 satisfies a Lipschitz condition with constant $C\Delta t^{-\alpha}$ so that

(iii) $\mathrm{DIR}(T_0) \le C\Delta t^{-2\alpha}$.

We use conclusion (4) of Theorem 5.2 to infer that

(iv) $\mathrm{DIR}(T_1) \le C\Delta t^{-2\alpha}$.

We use conclusion (5) of Theorem 5.2 to infer

$$\int_\Omega \big|T_0 - T_1\big|^2 \, d\mathcal{L}^n \le C\Delta t^{1-2\alpha}. \tag{v}$$

We use estimate A.1.14 of Appendix A.1 together with (i) above to infer, for each x,

$$\begin{aligned}
&\Big|H\big(T_0(x)\big) - H\big(T_1(x)\big)\Big| \\
&\qquad \le C\big|T_0(x) - T_1(x)\big| \left(\frac{1}{T_0(x)T_1(x)^2} + \frac{1}{T_0(x)^2T_1(x)} + T_0(x) + T_1(x)\right) \\
&\qquad \le C\big|T_0(x) - T_1(x)\big| \left(\frac{1}{\Delta t^{\frac{3n\alpha}{3n+2}}} + \frac{1}{\Delta t^{\alpha}}\right) \\
&\qquad \le C\Delta t^{-\alpha}\big|T_0(x) - T_1(x)\big|.
\end{aligned} \tag{vi}$$

Similarly we use estimate A.1.7 of Appendix A.1 together with (i) above to infer, for each x,

$$\begin{aligned}
&\Big|H'\big(T_0(x)\big)\nabla T_0(x) - H'\big(T_1(x)\big)\nabla T_1(x)\Big| \\
&\qquad \le C\Big(\big|\nabla T_0(x)\big| + \big|\nabla T_1(x)\big|\Big)\left(\frac{1}{\Delta t^{\frac{3n\alpha}{3n+2}}} + \frac{1}{\Delta t^{\alpha}}\right) \\
&\qquad \le C\Big(\big|\nabla T_0(x)\big| + \big|\nabla T_1(x)\big|\Big)\Delta t^{-\alpha}.
\end{aligned} \tag{vii}$$

We combine (v) and (vi) above to estimate

$$\int_\Omega f^2 \, d\mathcal{L}^n \le \sup_\Omega g^2 \, \Delta t^{1-4\alpha} \tag{$viii$}$$

and combine (i), (iii), (iv), (v), and (vii) above together with the chain rule to estimate

(ix)

$$\int_\Omega |\nabla f|^2 \, d\mathcal{L}^n \le C \sup_\Omega g^2 \, \Delta t^{-4\alpha} + C\sup_\Omega |\nabla g|^2 \, \Delta t^{1-4\alpha} \le C\sup\big\{\sup_\Omega g^2, \sup_\Omega |\nabla g|^2\big\}\Delta t^{-4\alpha}$$

We now use 8.2 to estimate

$$\left| \int_{\Omega} f\, \mathbf{n}_{K_0} \bullet \mathbf{e}_n \, d\mathcal{L}^n \right| \leq C \Delta t^{-\frac{1}{6}} \, \Delta t^{\frac{1-4\alpha}{4}} \, \Delta t^{-\frac{4\alpha}{4}} + C \Delta t^{\frac{1}{12}} \, \Delta t^{-\frac{4\alpha}{2}} = C \Delta t^{\frac{1}{12} - 2\alpha}.$$

The theorem follows. ■

8.4 Terminology.

(1) For each $\epsilon > 0$ we set

$$H_\epsilon(T) = \begin{cases} H(\epsilon) & \text{if } 0 < T < \epsilon \\ H(T) & \text{if } \epsilon \leq T \leq \frac{1}{\epsilon} \\ H\left(\frac{1}{\epsilon}\right) & \text{if } \frac{1}{\epsilon} < T < \infty. \end{cases}$$

(2) We say that a function $T \colon \Omega \to \mathbb{R}$ is **strictly defined** if and only if

$$T(p) = \lim_{r \downarrow 0} \alpha(n)^{-1} r^{-n} \int_{\mathbf{B}^n(p,\, r)} T \, d\mathcal{L}^n$$

at each point p at which the limit above exists.

8.5 Theorem (An approximate Stokes theorem for the wmc **integral).** *Suppose that t is one of the almost all positive times for which crystals, heat distributions, and temperature fields converge as in conclusion (1) of Theorem 7.7. Suppose also that $T_*(t)$ is strictly defined. Then*

$$\lim_{\epsilon \downarrow 0} \int_{x \in \partial K_*(t)} H_\epsilon\big(T_*(x,\, t)\big)\, \mathbf{n}_{K_*(t)}(x) \bullet g_*(x) \, d\mathcal{H}^{n-1} x$$

$$= \int_{x \in K_*(t)} \operatorname{div}\left(H\big(T_*(x,\, t)\big)\, g_*(x) \right) \, d\mathcal{H}^{n-1} x$$

for each smooth vector field $g_* \colon \Omega \to \mathbb{R}^n$.

PROOF. It clearly suffices to consider only the case $g_* = g\, \mathbf{e}_n$ for some smooth function $g \colon \Omega \to \mathbb{R}$. In view of B.7 we infer from our hypotheses that $\mathrm{DIR}(T_*(t)) < \infty$. We infer from [MC Lemma 3.1.1] that the functions

$$x_n \mapsto T_*\big(x_1,\, \ldots,\, x_{n-1},\, x_n,\, t\big)$$

are absolutely continuous for $\mathcal{L}^{n-1}$ almost all $\big(x_1,\,\ldots,\,x_{n-1}\big)$ (with the obvious meaning in Ω). One uses a representation of $\partial[\![K_*(t)]\!]$ in terms of slices as in the proof of Step 3 in the proof of Theorem 8.2 (note Step 4 there) together with the fundamental theorem of calculus to infer

$$\int_{x\in\partial K_*(t)} H_\epsilon\big(T_*(x,\,t)\big)\,g(x)\,\mathbf{e}_n\bullet\mathbf{n}_{K_*(t)}(x)\,d\mathcal{H}^{n-1}x$$

$$=\int_{x\in K_*(t)}\frac{\partial}{\partial x_n}\left(H_\epsilon\big(T_*(x,\,t)\big)\,g(x)\right)\,d\mathcal{H}^{n-1}x.$$

We check that the the integral

$$\int_{x\in K_*(t)}\operatorname{div}\left(H\big(T_*(x,\,t)\big)\,g_*(x)\right)\,d\mathcal{H}^{n-1}x$$

exists and is finite and use Lebesgue's bounded convergence theorem to conclude that

$$\lim_{\epsilon\downarrow 0}\int_{x\in K_*(t)}\operatorname{div}\left(H_\epsilon\big(T_*(x,\,t)\big)\,g_*(x)\right)\,d\mathcal{H}^{n-1}x=\int_{x\in K_*(t)}\operatorname{div}\left(H\big(T_*(x,\,t)\big)\,g_*(x)\right)\,d\mathcal{H}^{n-1}x.$$

The theorem follows. ■

8.6 Theorem (Minimizing properties of $[\![K_*(t)]\!]$'s). *Suppose that t is one of the almost all positive times for which the crystals, heat distributions, and temperature fields converge as in conclusion (1) of Theorem 7.7. Then*

(1) $$\mathcal{E}\big([\![K_*(t)]\!],\,Q_*(t)\big)=\inf\Big\{\mathcal{E}\big([\![K]\!],\,Q_*(t)\big)\colon[\![K]\!]\in\mathcal{K}\Big\};$$

(2) $$\Phi\big(\partial[\![K_*(t)]\!]\big)=\lim_{i\to\infty}\Phi\big(\partial[\![K_{J(i)}(t)]\!]\big);$$

If we assume additionally that Φ is smooth and elliptic the we further conclude

(3) $$\mathbf{M}\big(\partial[\![K_*(t)]\!]\big)=\lim_{i\to\infty}\mathbf{M}\big(\partial[\![K_{J(i)}(t)]\!]\big);$$

(4). $$\frac{d}{ds}\Phi\Big(G_{s\sharp}\partial[\![K_*(t)]\!]\Big)\bigg|_{s=0}=\lim_{i\to\infty}\frac{d}{ds}\Phi\Big(G_{s\sharp}\partial[\![K_{J(i)}(t)]\!]\Big)\bigg|_{s=0}$$

for each (fixed) smooth vector field $g\colon \Omega \to \mathbb{R}^n$ and associated deformation of Ω given by setting $G_t(x) = x + s\,g(x)$ for small times s.

PROOF. For each large i we set $\Delta t_i = 2^{-J(i)}$ (so that Δt_i is the time step used in constructing the $J(i)$-th approximating evolution) and denote by $k(i)$ the unique positive integer for which

$$k(i)\Delta t_i \le t < (k(i)+1)\Delta t_i$$

$k(i)\Delta t_i$ is thus the beginning of the interval in which the time t appears in constructing the $J(i)$-th approximating evolution. The idea in proving (1) and (2) is to assume the negation and show that this will violate the property

$$\mathcal{E}\big([\![K_{J(i)}(k(i)\Delta t_i)]\!],\, Q_{J(i)}(k(i)\Delta t_i)\big) = \inf\Big\{\mathcal{E}\big([\![K]\!],\, Q_{J(i)}(k(i)\Delta t_i)\big)\colon [\![K]\!] \in \mathcal{K}\Big\}$$

which holds by construction (see 2.11.4). There is a lot of terminology involved here, and it seems useful to make abbreviations in the proof.

For large i we further set

$$K_i = K_{J(i)}(t) = K_{J(i)}(k(i)\Delta t_i) \quad K_* = K_*(t)$$

and denote by $[\![K]\!]$ a general (but fixed) crystal. Additionally we set

$$c_i = c_{K_i}, \quad c_* = c_{K_*}, \quad c = c_K$$

$$P_i = Q_{J(i)}(k(i)\Delta t_i), \quad Q_i = Q_{J(i)}(t), \quad Q_* = Q_*(t)$$

$$S_i = T_{J(i)}(k(i)\Delta t_i), \quad T_i = T_{J(i)}(t), \quad T_* = T_*(t)$$

(our use of “P” here differs from the usage in 2.11.3). We will denote by C a general (large) constant (which is independent of i and K) which will possibly have different values in the same expression. The remainder of the proof occurs in six steps.

Step 1. According the conclusion (1) of Theorem 4.3 we know

$$(i) \qquad C^{-1}\Delta t_i^{\frac{n\alpha}{3n+2}} \le S_i \le C\Delta t_i^{-\alpha}.$$

The maximum principle for solutions to the heat equation together with the corresponding minimal principle imply that also

$$(ii) \qquad C^{-1}\Delta t_i^{\frac{n\alpha}{3n+2}} \le T_i \le C\Delta t_i^{-\alpha}.$$

We infer from conclusion (2) of Theorem 4.3 that each S_i satisfies a Lipschitz condition with constant $C\Delta t_i^{-\alpha}$ so that

(iii) $\mathrm{DIR}(S_i) \le C\Delta t_i^{-2\alpha}$.

We use conclusion (4) of Theorem 5.2 to infer that

(iv) $\mathrm{DIR}(T_i) \le C\Delta t_i^{-2\alpha}$.

We use conclusion (5) of Theorem 5.2 to infer

$$\int_\Omega \big|S_i - T_i\big|^2 \, d\mathcal{L}^n \le C\Delta t_i^{1-2\alpha}. \tag{v}$$

Step 2. For each i,

$$\left| \int_\Omega cF\big(c^{-1}P_i\big)\, d\mathcal{L}^n - \int_\Omega cF\big(c^{-1}Q_i\big)\, d\mathcal{L}^n \right| \le C\, \Delta t_i^{\frac{1}{2}-2\alpha}.$$

Proof. We use the extreme inequality of A.1.13 to infer that the difference of the integrals on the left above does not exceed

$$C\int_\Omega \left|c^{-1}P_i - c^{-1}Q_i\right| \left(\frac{c^3}{P_i^2 Q_i} + \frac{c^3}{P_i Q_i^2} + \frac{P_i}{c} + \frac{Q_i}{c} \right) d\mathcal{L}^n$$

which in turn does not exceed

$$C\int_\Omega \left|S_i - T_i\right| \left(\frac{1}{S_i^2 T_i} + \frac{1}{S_i T_i^2} + S_i + T_i \right) d\mathcal{L}^n.$$

We use estimates (i) and (ii) in Step 1 above to dominate the right hand factor in the integrand above. We then use Schwarz's inequality together with estimate (v) in Step 1 above to dominate the contribution of the left hand side. The assertion of Step 2 follows.

Step 3. We assert

$$\lim_{i\to\infty} \int_\Omega c_i F\big(T_i\big)\, d\mathcal{L}^n = \int_\Omega c_* F\big(T_*\big)\, d\mathcal{L}^n.$$

Proof. We write

$$\int_\Omega c_i F\big(T_i\big) - c_* F\big(T_*\big)\, d\mathcal{L}^n = \int_\Omega (c_i - c_*) F\big(T_*\big)\, d\mathcal{L}^n + \int_\Omega c_i \Big(F\big(T_i\big) - F\big(T_*\big) \Big)\, d\mathcal{L}^n.$$

The left hand integral on right side of the equality immediately above converges to zero as $i \to \infty$ as a consequence the convergence of the crystals $[\![K_i]\!]$ to $[\![K_*]\!]$ and the absolute continuity of integrals of summable functions. The right hand integral there above converges to zero as a consequence of the left hand inequality of A.1.13 together with the strong convergence of temperature fields T_i to T_* and of reciprocal temperature fields $1/T_i$ to $1/T_*$.

Step 4. We assert

$$\lim_{i\to\infty} \int_\Omega cF\big(c^{-1}Q_i\big)\, d\mathcal{L}^n = \int_\Omega cF\big(c^{-1}Q_*\big)\, d\mathcal{L}^n.$$

Proof. We use the left hand inequality of A.1.13 to estimate

$$\begin{aligned}
\Big|cF\big(c^{-1}Q_i\big) - cF\big(c^{-1}Q_*\big)\Big| &\le C\bigg(\Big|\frac{1}{Q_i^2} - \frac{1}{Q_*^2}\Big| + |Q_i - Q_*|\bigg)\\
&\le C\bigg(\Big|\frac{c_i^2}{Q_i^2} - \frac{c_*^2}{Q_*^2}\Big| + \Big|\Big(\frac{c_*^2}{c_i^2} - 1\Big)\frac{c_i^2}{Q_i}\Big|\\
&\qquad + \Big|\frac{Q_i}{c_i} - \frac{Q_*}{c_*}\Big| + \Big|\Big(\frac{c_i}{c_*} - 1\Big)\frac{Q_i}{c_i}\Big|\bigg).
\end{aligned}$$

The integrals of the first and third summands immediately above converge to zero as $i \to \infty$ as a consequence of the strong convergence of temperature fields T_i to T_* and of reciprocal temperature fields $1/T_i$ to $1/T_*$.

Step 5. The negation of the first assertion of our theorem would imply the existence of a crystal $[\![K]\!]$ such that

$$\Phi\big(\partial[\![K]\!]\big) + \int_\Omega cF\big(c^{-1}Q_*\big) < \Phi\big(\partial[\![K_*]\!]\big) + \int_\Omega c_*F\big(c_*^{-1}Q_*\big).$$

The reason this does not happen is that, for large i's, $[\![K]\!]$ would then also be a better choice than $[\![K_i]\!]$ in the minimization at the i-th step which occurred at time $k(i)\Delta t_i$, i.e. we would have

$$\Phi\big(\partial[\![K]\!]\big) + \int_\Omega cF\big(c^{-1}P_i\big) < \Phi\big(\partial[\![K_i]\!]\big) + \int_\Omega cF\big(c_i^{-1}P_i\big).$$

This last inequality holds for large i as a consequence of the first inequality by virtue of the bulk integral estimate in Step 2 together with the convergence of bulk integrals asserted in Steps 3 and 4 and the lower semicontinuity of interface integrals, i.e. the fact that $\Phi\left(\llbracket K_*\rrbracket\right) \le \liminf_i \Phi\left(\llbracket K_i\rrbracket\right)$. The negation of the second assertion of our theorem would imply $\Phi\left(\llbracket K_*\rrbracket\right) < \limsup_i \Phi\left(\llbracket K_i\rrbracket\right)$. An argument similar to that above would show that, for arbitrarily large i's, $\llbracket K_*\rrbracket$ would then also be a better choice than $\llbracket K_i\rrbracket$ for the minimization at the i-th step which occurred at time $k(i)\Delta t_i$, i.e.

$$\Phi\big(\partial\llbracket K_*\rrbracket\big) + \int_\Omega c_* F\big(c_*^{-1} P_i\big) < \Phi\big(\partial\llbracket K_i\rrbracket\big) + \int_\Omega c_i F\big(c_i^{-1} P_i\big).$$

Step 6. We now consider assertions (3) and (4) of our theorem above. Under the assumption that Φ is smooth and elliptic we infer from Conclusion (2) of the present theorem together with Theorem C.3 in Appendix C that there is weak varifold convergence

$$\mathbf{v}\bigg(\partial K_{J(i)}(t)\,,\,1\,,\,\mathbf{n}_{K_{J(i)}(t)}\bigg) \rightharpoonup \mathbf{v}\bigg(\partial K_*(t)\,,\,1\,,\,\mathbf{n}_{K_*(t)}\bigg)$$

as $i \to \infty$ in the sense of the terminology set forth in C.1(6). This varifold convergence immediately implies Conclusion (3) of the present theorem. The varifold convergence together with Theorem C.2 imply Conclusion (4) of the present theorem. ■

8.7 Theorem (The Gibbs-Thomson condition holds in the limit in the sense of first variation distributions when Φ is smooth and elliptic). *Suppose that Φ is smooth and elliptic and that t is one of the almost all positive times for which the crystals, heat distributions, and temperature fields converge as in conclusion (1) of Theorem 7.7. Then*

$$\begin{aligned}\frac{d}{ds}\Phi\bigg(G_{s\sharp}\partial\llbracket K_*(t)\rrbracket\bigg)\bigg|_{s=0} &= \lim_{\epsilon\downarrow 0}\int_{x\in\partial K_*(t)} H_\epsilon\big(T_*(x,\,t)\big)\,\mathbf{n}_{K_*(t)}(x)\bullet g_*(x)\,d\mathcal{H}^{n-1}x \\ &= \int_{x\in K_*(t)} \operatorname{div}\big(H\big(T_*(x,\,t)\big)\,g_*(x)\big)\;d\mathcal{L}^n x\end{aligned}$$

for each smooth initial velocity vector field $g_*\colon\Omega\to\mathbb{R}^n$ and associated deformation $G_s(x) = x + s\,g_*(x)$ for small s.

PROOF. Since both sides of the asserted equation are linear in g_* [AW 4.9], it is clearly sufficient to prove the assertion in case $g_* = g\,\mathbf{e}_n$ for some smooth function g having small support in Ω. We fix such a g. We also fix a subsequence of the $J(i)$'s, which we will simply index by the subscript j, for which $\nabla T_j(t) \rightharpoonup \nabla T(t)$ and $\nabla U_j(t) \rightharpoonup \nabla U(t)$ as $j \to \infty$ with $\sup_j \mathrm{DIR}\big(T_j(t)\big) < \infty$ and $\sup_j \mathrm{DIR}\big(U_j(t)\big) < \infty$. This is possible by 7.7(1) and B.7. We set $\Delta t_j = 2^{-J(i)}$ as the time step used in constructing the $J(i)$-th approximating evolution associated with the present subscript j. We further denote by $k(j)$ the unique positive integer for which

$$k(j)\,\Delta t_j \le t < \big(k(j)+1\big)\,\Delta t_j$$

so that $k(j)\Delta t_j$ is the beginning of the time interval in which t appears. According to Theorem 4.4,

$$\frac{d}{ds}\Phi\bigg(G_{s\sharp}\partial[\![K_j(k(j)\Delta t_j)]\!]\bigg)\bigg|_{s=0}$$

equals

$$\int_{x\in\partial K_j(k(j)\Delta t_j)} H\big(T_j(x,\,k(j)\Delta t_j)\big)\,g(x)\,\mathbf{n}_{K_j(k(j)\Delta t_j)}(x)\bullet\mathbf{e}_n\,d\mathcal{H}^{n-1}x.$$

According to conclusion (4) of Theorem 8.4,

$$\frac{d}{ds}\Phi\bigg(G_{s\sharp}\partial[\![K_j(k(j)\Delta t_j)]\!]\bigg)\bigg|_{s=0} = \frac{d}{ds}\Phi\bigg(G_{s\sharp}\partial[\![K_j(t)]\!]\bigg)\bigg|_{s=0} \to \frac{d}{ds}\Phi\bigg(G_{s\sharp}\partial[\![K_*(t)]\!]\bigg)\bigg|_{s=0}$$

as $j\to\infty$. According to Theorem 8.3 the difference between

$$\int_{x\in\partial K_j(k(j)\Delta t_j)} H\big(T_j(x,\,k(j)\Delta t_j)\big)\,g(x)\,\mathbf{n}_{K_j(k(j)\Delta t_j)}(x)\bullet\mathbf{e}_n\,d\mathcal{H}^{n-1}x.$$

and

$$\int_{x\in\partial K_j(t)} H\big(T_j(x,\,t)\big)\,g(x)\,\mathbf{n}_{K_j(t)}(x)\bullet\mathbf{e}_n\,d\mathcal{H}^{n-1}x$$

converges to zero as $j\to\infty$ since $\Delta t_j \to 0$ and $\alpha < 1/24$, in accordance with 2.10. According to the general Gauss-Green Theorem [FH 4.5.6(5)]

$$\int_{x\in\partial K_j(t)} H\big(T_j(x,\,t)\big)\,g(x)\,\mathbf{n}_{K_j(t)}(x)\bullet\mathbf{e}_n\,d\mathcal{H}^{n-1}x$$

equals

$$\int_{x\in K_j(t)} \frac{\partial}{\partial x_n}\bigg(H\big(T_j(x,\,t)\big)\,g(x)\bigg)\,d\mathcal{H}^{n-1}x$$

and we know from Theorem 8.5 that

$$\lim_{\epsilon\downarrow 0}\int_{x\in\partial K*(t)} H_\epsilon\big(T_*(x,\,t)\big)\,g(x)\,\mathbf{n}_{K_*(t)}(x)\bullet\mathbf{e}_n\,d\mathcal{H}^{n-1}x$$

equals

$$\int_{x\in K_*(t)}\frac{\partial}{\partial x_n}\bigg(H\big(T_*(x,\,t)\big)\,g(x)\bigg)\,d\mathcal{H}^{n-1}x$$

According to Theorem 8.1

$$\int_{x\in K_j(t)}\frac{\partial}{\partial x_n}\bigg(H\big(T_j(x,\,t)\big)\,g(x)\bigg)\,d\mathcal{H}^{n-1}x$$

converges to

$$\int_{x\in K_*(t)}\frac{\partial}{\partial x_n}\bigg(H\big(T_*(x,\,t)\big)\,g(x)\bigg)\,d\mathcal{H}^{n-1}x$$

as $j\to\infty$. The theorem follows. ■

8.8 Theorem (Minimality and regularity of the limit crystals in low dimension when Φ is smooth and elliptic). *Suppose that our surface energy integrand Φ is smooth and elliptic and that t is one of the almost all positive times for which the crystals, heat distributions, temperature fields, and reciprocal temperature fields converge to limits $[\![K_*(t)]\!]$, $Q_*(t)$, $T_*(t)$, $U_*(t)$ as in conclusion (1) of Theorem 7.7 or as in Theorem 8.6. Then*

(1) If the ambient dimension n equals 2 or 3 then there is a function $\omega\colon(0,\,1)\to\mathbb{R}^+$ with $\lim_{s\downarrow 0}\omega(s)=0$ together with a positive number δ such that $\partial[\![K(t)]\!]$ is $\big(\Phi,\,\omega,\,\delta\big)$-minimal in the sense of Bombieri [BE Definition 1, page 101]. This minimality means means that whenever W is a compact subset of Ω with small diameter and Q is an n dimensional integral current in Ω having support in W with $r=\mathrm{diam}(W)\le\delta$ then

$$\Phi\big(\partial[\![K_*(t)]\!]\llcorner W\big)\le\Phi\big(\partial[\![K_*(t)]\!]\llcorner W+\partial Q\big)+\omega(r)\mathbb{M}\big(\partial[\![K_*(t)]\!]\llcorner W+Q\big).$$

(2) In case $n=2$ then

$$\int_{(0,\,1)}\frac{\omega(s)^{\frac{1}{2}}}{s}\,d\mathcal{L}^1<\infty$$

from which it follows that $\partial K_*(t)$ is a one dimensional differentiable submanifold of Ω without boundary and

$$\frac{d}{ds}\Phi\bigg(G_{s\sharp}\partial[\![K_*(t)]\!]\bigg)\bigg|_{s=0}=\int_{x\in\partial K_*(t)}H\big(T_*(x,\,t)\big)\,\mathbf{n}_{K_*(t)}(x)\bullet g_*(x)\,d\mathcal{H}^{n-1}x$$

for each smooth initial velocity vector field $g_*: \Omega \to \mathbb{R}^n$ and associated deformation $G_s(x) = x + s\, g_*(x)$ for small s.

(3) In case $n = 3$ then $\partial K_*(t)$ is the homeomorphic image in Ω of a compact two dimensional manifold without boundary.

PROOF. Our proof is based on the regularity theorem of almost minimal currents [BE Main Theorem, page 102] together with Reifenberg's topological disk theorem [MC Theorem 10.5.1, page 439] with use of the regularity estimate of [ASS]. We abbreviate $K = K_*(t)$, $Q = Q_*(t)$, $T = T_*(t)$, $U = U_*(t)$. In the following we denote by C a general constant whose value may be different at different times (even in the same equation) which depends only on n, ρ and upper bounds for the integral

$$\int_\Omega U^2 + |\nabla U|^2 + T^2 + |\nabla T|^2 \, d\mathcal{L}^n.$$

Step 1. Some preliminary estimates. As in [ATW 3.1.4] we can write

$$[\![K]\!] + Q = \mathbf{t}\big(A,\, \theta,\, \sigma\big) = \sum_{i=1}^{\infty} [\![P_i]\!] - [\![N_i]\!]$$

where

$$P_i = \big\{x: \sigma(x) = \mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_n \text{ and } \theta(i) \ge i\big\}, \quad N_i = \big\{x: \sigma(x) = -\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_n \text{ and } \theta(i) \ge i\big\}$$

so that $P_1 \cap N_1 = \emptyset$ and $P_1 \supset P_2 \supset P_3 \supset \ldots$ and $N_1 \supset N_2 \supset N_3 \supset \ldots$ and

$$\Phi\Big(\partial\big([\![K]\!] + Q\big)\Big) = \sum_i \Phi\big(\partial[\![P_i]\!]\big) = \sum_i \Phi\big(-\partial[\![N_i]\!]\big)$$

and

$$\mathbb{M}\Big(\partial\big([\![K]\!] + Q\big)\Big) = \sum_i \mathbb{M}\big(\partial[\![P_i]\!]\big) + \sum_i \mathbb{M}\big(\partial[\![N_i]\!]\big).$$

If we set $L = P_1$ then

$$\Phi\big(\partial[\![L]\!]\big) \le \Phi\Big(\partial\big([\![K]\!] + Q\big)\Big)$$

and

$$\mathbb{M}\big(\partial[\![L]\!]\big) \le \mathbb{M}\Big(\partial\big([\![K]\!] + Q\big)\Big).$$

We abbreviate $|K \triangle L| = \mathcal{L}^n(K \triangle L)$ check that $|K \triangle L| = \mathbb{M}([\![K]\!] - [\![L]\!])$ and that $\partial([\![K]\!] - [\![L]\!]) = \partial[\![K]\!] \llcorner W - \partial[\![L]\!] \llcorner W$ so that, by the isoperimetric inequality [ATW 3.1.7],

$$\begin{aligned}|K \triangle L| &\leq \frac{1}{n^{\frac{n}{n-1}} \alpha(n)^{\frac{1}{n-1}}} \mathbb{M}\Big(\partial([\![K]\!] \llcorner W - \partial[\![L]\!] \llcorner W)\Big)^{\frac{n}{n-1}} \\ &\leq \frac{1}{n^{\frac{n}{n-1}} \alpha(n)^{\frac{1}{n-1}}} \Big(\mathbb{M}(\partial[\![K]\!] \llcorner W) + \mathbb{M}(\partial[\![L]\!] \llcorner W)\Big)^{\frac{n}{n-1}}.\end{aligned}$$

Step 2. More preliminary estimates. We set

$$\sigma = \left(|K \triangle L|^{1/n} + |K \triangle L|^{\frac{3-n}{3n}} \left(\int_{K \triangle L} U^6 + T^6 \, d\mathcal{L}^n\right)^{\frac{1}{3}}\right)$$

In case $n = 2$ we use Proposition B.10 in Appendix B to infer

$$\sigma \leq C \, |K \triangle L|^{\frac{1}{6}} \leq C r^{\frac{1}{3}}.$$

In case $n = 3$ we use B.10 again to infer

$$\sigma \leq |K \triangle L|^{\frac{1}{3}} + C \int_{K \triangle L} U^2 + |\nabla U|^2 + T^2 + |\nabla T|^2 \, d\mathcal{L}^n$$

and then use the absolute continuity of integrals [FH 2.4.11, page 85] to infer that σ converges to 0 as the magnitude of $|K \triangle L|$ goes to 0. In particular σ converges to 0 as r converges to 0. We now use inequality A.1.8 in Appendix A together with Hölder's inequality to help estimate

$$\begin{aligned}&\int_{K \triangle L} c_L F(Q/c_L) \, d\mathcal{L}^n \\ &\qquad \leq C \int_{K \triangle L} 1 + U^2 + T^2 \, d\mathcal{L}^n \\ &\qquad \leq C\left(|K \triangle L| + \left(\int_{K \triangle L} U^6 + T^6 \, d\mathcal{L}^n\right)^{\frac{1}{3}} |K \triangle L|^{\frac{2}{3}}\right) \\ &\qquad \leq C \, |K \triangle L|^{\frac{n-1}{n}} \left(|K \triangle L|^{1/n} + |K \triangle L|^{\frac{3-n}{3n}} \left(\int_{K \triangle L} U^6 + T^6 \, d\mathcal{L}^n\right)^{\frac{1}{3}}\right) \\ &\qquad = C \, |K \triangle L|^{\frac{n-1}{n}} \sigma \\ &\qquad \leq C \, \sigma \Big(\mathbb{M}(\partial[\![K]\!] \llcorner W) + \mathbb{M}(\partial[\![L]\!] \llcorner W)\Big) \\ &\qquad \leq C \, \sigma \Big(\Phi(\partial[\![K]\!] \llcorner W) + \Phi(\partial[\![L]\!] \llcorner W)\Big)\end{aligned}$$

Our $\mathcal{E}$ minimality implies

$$\Phi\big(\partial[\![K]\!]\big) + \int_\Omega c_K\big(Q/c_K\big)\, d\mathcal{L}^n \leq \Phi\big(\partial[\![L]\!]\big) + \int_\Omega c_L\big(Q/c_L\big)\, d\mathcal{L}^n$$

from which we infer

$$\begin{aligned}
\Phi\big(\partial[\![K]\!]\llcorner W\big) &\leq \Phi\big(\partial[\![L]\!]\llcorner W\big) + \int_\Omega c_L F\big(Q/c_L\big) - c_K F\big(Q/c_K\big)\, d\mathcal{L}^n \\
&\leq \Phi\big(\partial[\![L]\!]\llcorner W\big) + \int_{K\triangle L} c_L F\big(Q/c_L\big)\, d\mathcal{L}^n \\
&\leq \Phi\big(\partial[\![L]\!]\llcorner W\big) + C\,\sigma\,\Big(\Phi\big(\partial[\![K]\!]\llcorner W\big) + \Phi\big(\partial[\![L]\!]\llcorner W\big)\Big)
\end{aligned}$$

Hence

$$(1 - C\sigma)\,\Phi\big(\partial[\![K]\!]\llcorner W\big) \leq (1 + C\sigma)\,\Phi\big(\partial[\![L]\!]\llcorner W\big)$$

so that, in case r and hence $|K \triangle L|$ is sufficiently small

$$\begin{aligned}
\Phi\big(\partial[\![K]\!]\llcorner W\big) &\leq (1 + 3C\sigma)\,\Phi\big(\partial[\![L]\!]\llcorner W\big) \\
&\leq \Phi\big(\partial[\![L]\!]\llcorner W\big) + C\sigma\mathbf{M}\big(\partial[\![L]\!]\llcorner W\big) \\
&\leq \Phi\big(\partial[\![K]\!]\llcorner W + \partial Q\big) + C\sigma\mathbf{M}\big(\partial[\![K]\!]\llcorner W + \partial Q\big).
\end{aligned}$$

We now choose $\omega(r)$ so that $\omega(r) \geq C\sigma$ while $\lim_{s\downarrow 0}\omega(s) = 0$ and the integral inequality of conclusion (2) above holds in case $n = 2$. This gives the asserted minimality of conclusion (1). Conclusion (2) is a consequence of our estimate on σ above the the Main Theorem in [BE]. The first variation formula in conclusion 2 is a consequence of Theorem 8.7 together with the estimates of [SE definition, page 135; Proposition 1, page 122; further results 4.2 and 4.3, page 192, 193] which guarantee the $\mathcal{H}^{n-1}$ summability of the restriction of $H(T_*(t))$ to $\partial\ K_*(t)$. The argument to obtain the tangent cone criterion of [BE Remark, page 125] there closely parallels the tangent cone argument given in [ATW Theorems 3.9 and 3.10] based on the 1 dimensional regularity of [FH 5.3.10, page 617]; details are left to the reader. Conclusion (3) is a consequence of Reifenberg's theorem noted above. The argument to show there that the closure of ∂K satisfies the $(\epsilon,\, R_1)$ condition there closely parallels the tangent cone argument given in [ATW Theorems 3.9 and 3.10] based on the 2 dimensional regularity of [ASS Corollary 3.2, page 255]; details are left to the reader.

8.9 Remark. The C^0 regularity asserted in Theorem 8.8 for crystal boundaries when $n = 3$ is weaker than the C^1 regularity asserted in [LS Section 3, page 106] when Φ is the 2 dimensional area integrand.

Appendix A. The bulk energy density function F and the Gibbs-Thomson function H

A.1 Proposition (Facts about F and H). *There is a positive constant C_* with the following property. Suppose T, T_1, T_2 are positive numbers with $T_1 < T_2$ and $U = 1/T$, $U_1 = 1/T_1$, $U_2 = 1/T_2$.*

A.1.1
$$F(T) = \begin{cases} U^2 & \text{if } 0 < T \le a_* \\ T^2 & \text{if } b_* \le T < \infty. \end{cases}$$

A.1.2
$$F'(T) = \begin{cases} -2U^3 & \text{if } 0 < T \le a_* \\ 2T & \text{if } b_* \le T < \infty. \end{cases}$$

A.1.3
$$F''(T) = \begin{cases} 6U^4 & \text{if } 0 < T \le a_* \\ 2 & \text{if } b_* \le T < \infty. \end{cases}$$

A.1.4
$$H(T) = (c_\ell - c_s)\big(F(T) - TF'(T)\big)$$

A.1.5
$$H'(T) = -(c_\ell - c_s)\, T\, F''(T)$$

A.1.6
$$H(T) = (c_\ell - c_s) \begin{cases} 3U^2 & \text{if } 0 < T \le a_* \\ -T^2 & \text{if } b_* \le T < \infty. \end{cases}$$

A.1.7
$$H'(T) = (c_\ell - c_s) \begin{cases} -6U^3 & \text{if } 0 < T \le a_* \\ -2T & \text{if } b_* \le T < \infty. \end{cases}$$

A.1.8
$$F(T) \le C_*\big(U^2 + T^2\big).$$

A.1.9 $$U^2 + T^2 \le C_* + F(T).$$

A.1.10 $$F''(T) \ge c_* + C_*^{-1} U^4.$$

A.1.11 $$|H(T)| \le C_* \big(U^2 + T^2\big).$$

A.1.12 $$|H'(T)| \le C_* \big(U^3 + T\big).$$

A.1.13
$$\big|F(T_1) - F(T_2)\big| \le C_* \big|U_1^2 - U_2^2\big| + C_* \big|T_1 - T_2\big| = C_* \big|T_1 - T_2\big| \Big(U_1 U_2^2 + U_1^2 U_2 + T_1 + T_2\Big).$$

A.1.14
$$\big|H(T_1) - H(T_2)\big| \le C_* \big|U_1^2 - U_2^2\big| + C_* \big|T_1 - T_2\big| = C_* \big|T_1 - T_2\big| \Big(U_1 U_2^2 + U_1^2 U_2 + T_1 + T_2\Big).$$

A.1.15. $$T_2 - T_1 \le c_*^{-1} \Big(F'\big(T_2\big) - F'\big(T_1\big)\Big)$$

A.1.16. $$U_1^3 - U_2^3 \le C_* \Big(F'\big(T_2\big) - F'\big(T_1\big)\Big)$$

PROOF. The first twelve assertions above are either definitions from Section 2.9 or follow immediately from these definitions. The rest of the proof is in three steps.

Step 1. Suppose $h{:}\mathbb{R}^+ \to \mathbb{R}$ is a continuously differentiable decreasing function such that

$$h(T) = \begin{cases} AT^{-2} & \text{if } 0 < T \le a_* \\ -BT^2 & \text{if } b_* \le T < \infty. \end{cases}$$

for positive constants A and B. Then there is a number c (depending only on h) such that

$$\big|h(T_1)-h(T_2)\big| \le c\big|T_1^{-2}-T_2^{-2}\big|+c\big|T_1-T_2\big| = c\big|T_1-T_2\big|\left(T_1^{-1}T_2^{-2}+T_1^{-2}T_2^{-1}+T_1+T_2\right).$$

Proof. According to the Mean Value Theorem there is a number $T_1 < S < T_2$ such that

$$\big|h(T_1)-h(T_2)\big| = \big|T_1-T_2\big|\, h'(S)$$

We consider several cases.

Case 1. Suppose $b_* \le S$. Then $\big|h'(S)\big| = 2BS \le 2BT_2$ so that any c not less than $2B$ will do.

Case 2. Suppose $a_*/2 \le S \le b_*$. If D is a bound for $|h'(s)|$ for $a_*/2 \le s \le b_*$ then $h'(S) \le (2D/a_*)T_2$ so that any c not less than $(2D/a_*)$ will do.

Case 3. Suppose $S \le a_*$ and $T_2 \le a_*$. Then $h(T_1)-h(T_2) = A\big(T_1^{-2}-T_2^{-2}\big)$ so that an c not less than A will do.

Case 4. Suppose $S \le a_*$ and $T_2 \ge a_*$. Then

$$h(T_1) = \frac{A}{T_1^2} = \frac{4A}{3}\left(\frac{1}{T_1^2}-\frac{1}{(2T_1^2)}\right) \ge \frac{4A}{3}\left(\frac{1}{T_1^2}-\frac{1}{(T_2^2)}\right)$$

and, since $T_2 \ge a_*$ there is some constant c_1 such that $h(T_2) \le c_1T_2^2 \le (4c_1/3)\big(T_2^2-T_1^2\big)$. Hence any c not less than both $4A/3$ and $4c_1/3$ will do.

Assertion A.1.14 now follows immediately from the estimate by setting $h(T) = H(T)$. To verify assertion A.1.13 we set

$$h(T) = \begin{cases} F(T) & \text{if } 0 < T \le T_\star \\ -F(T) & \text{if } T_\star \le T < \infty \end{cases}$$

and check that $\big|F(T_1)-F(T_2)\big| \le \big|h(T_1)-h(T_2)\big|$.

Step 2. We use the Mean Value Theorem together with A.1.10 to infer the existence of $T_1 < T^* < T_2$ with

$$F'\big(T_2\big)-F'\big(T_1\big) = F''\big(T^*\big)\big(T_2-T_1\big) \ge c_*\big(T_2-T_1\big)$$

which gives A.1.15

Step 3 In order to prove assertion A.1.16 we define $a - 2^{-1/3}a_*$ so that

$$\frac{1}{a^3} - \frac{1}{a_*^3} = \frac{1}{a_*^3}$$

and consider three cases.

Case 1. If $0 < T_1 < T_2 \le a_*$ then

$$\frac{1}{T_1^3} - \frac{1}{T_2^3} = \frac{1}{2}\Big(F'(T_2) - F'(T_1)\Big)$$

from which A.1.16 follows in case $C_* \ge 1/2$.

Case 2. If $0 < a \le T_1 < T_2 < \infty$ then

$$\frac{1}{T_1^3} - \frac{1}{T_2^3} = \big(T_2 - T_1\big)\frac{\big(T_2^2 + T_2T_1 + T_1^2\big)}{T_1^3T_2^3} < 3\frac{\big(T_2 - T_1\big)}{T_1^3T_2} < \frac{3}{a^4}\big(T_2 - T_1\big)$$

from which A.1.16 follows if $C_* \ge 3/a^4$.

Case 3. If $0 < T_1 < a < a_* < T_2 < \infty$ then

$$\begin{aligned} \frac{1}{T_1^3} - \frac{1}{T_2^3} &= \frac{1}{T_1^3} - \frac{1}{a_*^3} + \frac{1}{a_*^3} - \frac{1}{T_2^3} \\ &< \frac{1}{T_1^3} - \frac{1}{a_*^3} + \frac{1}{a_*^3} \\ &\le 2\left(\frac{1}{T_1^3} - \frac{1}{a_*^3}\right) \\ &= F'\big(a_*\big) - F'\big(T_1\big) \\ &< F'\big(T_2\big) - F'\big(T_1\big) \end{aligned}$$

from which A.1.16 follows in case $C_* \ge 1$. ■

A.2 Proposition (Bulk energy functions whose integrals decrease when heat flows from hot to cold). *Suppose G_s, G_ℓ:$\mathbb{R}^+ \to \mathbb{R}^+$ are continuously differentiable functions and $[\![K]\!]$ is a crystal in $\mathcal{K}$. Then the following two conditions are equivalent.*

(1) $$\int_K G_s(T + \Delta T)\,d\mathcal{L}^n + \int_{\Omega\sim K} G_\ell(T + \Delta T)\,d\mathcal{L}^n \le \int_K G_s(T)\,d\mathcal{L}^n + \int_{\Omega\sim K} G_\ell(T)\,d\mathcal{L}^n$$

whenever T and $T + \Delta T$ are temperature fields with

$$\sup\big\{T(x) + \Delta T(x) \colon \Delta T(x) > 0\big\} < \inf\big\{T(x) + \Delta T(x) \colon \Delta T(x) < 0\big\}.$$

In saying the T and $T+\Delta T$ are temperature fields we mean that $c_K T$ and $c_K(T+\Delta T)$ are heat distributions in $\mathcal{Q}$.

(2) There is a continuously differentiable convex function $F_*{:}\mathbb{R}^+ \to \mathbb{R}^+$ and a constant B such that, for each T,

$$G_s(T) = c_s F_*(T) \quad \text{and} \quad G_\ell(T) = c_\ell F_*(T) + B.$$

PROOF. Assume the first condition holds. We infer that

$$dG_s(S) - dG_s(T) \le 0,\ dG_\ell(S) - dG_\ell(T) \le 0,\ dG_s(S) - dG_\ell(T) \le 0,\ dG_\ell(S) - dG_s(T) \le 0$$

whenever $0 < S < T < \infty$. Since $T = c_K^{-1} Q$ we have

$$dG_s(T) = G_s'(T)\frac{dT}{dQ}dQ = \frac{1}{c_s}G_s'(T)dQ, \quad \text{and} \quad dG_\ell(T) = G_\ell'(T)\frac{dT}{dQ}dQ = \frac{1}{c_\ell}G_\ell'(T)dQ.$$

from which we infer that

$$\frac{1}{c_s}G_s'(S) \le \frac{1}{c_s}G_s'(T),\ \frac{1}{c_\ell}G_\ell'(S) \le \frac{1}{c_\ell}G_\ell'(T),\ \frac{1}{c_s}G_s'(S) \le \frac{1}{c_\ell}G_\ell'(T),\ \frac{1}{c_\ell}G_\ell'(S) \le \frac{1}{c_s}G_s'(T).$$

It follows that $\frac{1}{c_s}G_s'$ and $\frac{1}{c_\ell}G_\ell'$ are convex functions whose difference is constant. We infer that the first condition implies the second. The other implication is left to the reader. ■

A.3 Proposition (The Gibbs-Thomson function H effectively determines the bulk energy function F). *Suppose $H_*{:}\mathbb{R}^+ \to \mathbb{R}$ is continuous and*

$$0 < c_0 < \infty, \quad 0 < T_0 < \infty, \quad -\infty < F_0 < \infty.$$

Then there is a unique continuously differentiable function $F_*{:}\mathbb{R}^+ \to \mathbb{R}$ with $F_*(T_0) = F_0$ such that

$$H_*(T) = c\big(F_*(T) - TF_*'(T)\big).$$

If H_* is continuously differentiable and nonincreasing, then F_* is convex.

PROOF. Our differential equation above can be rewritten

$$F_*'(T) - \frac{1}{T}F_*(T) = \frac{-c}{T}H_*(T)$$

which is a standard first order ordinary differential equation for which existence and uniqueness of solutions is well known for arbitrary initial conditions as set forth on page 11 of [BR]. If H_* is continuously differentiable we compute

$$\begin{aligned} F_*''(T) &= \frac{-1}{T^2}F_*(T) + \frac{1}{T}F_*'(T) + \frac{1}{T^2}H_*(T) - \frac{1}{T}H_*'(T)T \\ &= \frac{-1}{T^2}F_*(T) + \frac{1}{T}\Big(\frac{1}{T}F_*(T) + \frac{-c}{T}H_*(T)\Big) + \frac{1}{T^2}H_*(T) - \frac{1}{T}H_*'(T)T \\ &= \frac{-1}{T}H_*'(T) \end{aligned}$$

so that $F_*''(T)$ is nonnegative in case H_* is nonincreasing. ■

Appendix B. Measure and function spaces

B.1 A metric space of Radon measures. We denote by $\mathcal{M}$ the vector space of all signed Radon measures (Daniell integrals [F 2.5]) μ on Ω such that

$$\mu(\Omega) = 0.$$

Associated with such μ's are its **mass norm**

$$|\mu| = \sup\left\{\int_\Omega \varphi\, d\mu \colon \varphi \text{ is smooth with } \sup|\varphi| \le 1\right\}$$

and its **Monge-Kantorovich norm**

$$\|\mu\|_* = \sup\left\{\int_\Omega \varphi\, d\mu \colon \mathrm{Lip}(\varphi) \le 1\right\}.$$

For each $M < \infty$ the set

$$\mathcal{M} \cap \{\mu \colon |\mu| \le M\}$$

is compact in the $\|\cdot\|_*$ topology.

B.2 The vector space $\mathbb{L}_1(\Omega)$. We denote by $\mathbb{L}_1(\Omega)$ the real vector space of all $\mathcal{L}^n$ summable functions $f\colon \Omega \to \mathbb{R}$. The $\mathbb{L}_1$ norm of such an f is the number

$$\|f\|_{\mathbb{L}_1} = \int_\Omega |f|\, d\mathcal{L}^n.$$

If f belongs to $\mathbb{L}_1(\Omega)$ with $\int_\Omega f \, d\mathcal{L}^n = 0$ then the measure $\mathcal{L}^n \wedge f$ belongs to $\mathcal{M}$ and one abbreviates.

$$\|f\|_* = \|\mathcal{L}^n \wedge f\|_* .$$

If g and h belong to $\mathbb{L}_1(\Omega)$ with $\int_\Omega g \, d\mathcal{L}^n = \int_\Omega h \, d\mathcal{L}^n$ then the number $\|g - h\|_*$ gives a the **Monge-Kantorovich distance** between g and h.

B.3 The vector spaces $\mathbb{L}_p(\Omega)$. For $p > 1$ we denote by $\mathbb{L}_p(\Omega)$ the vector subspace of $\mathbb{L}_1(\Omega)$ consisting of f's whose absolute values are p-th power summable. The $\mathbb{L}_p$ norm of such an f is the number

$$\|f\|_{\mathbf{L}_p} = \left(\int_\Omega |f|^p \, d\mathcal{L}^n \right)^{\frac{1}{p}} .$$

Equivalently we can set $q = p/(p-1)$ (so that $1/p + 1/q = 1$) and realize $\|f\|_{\mathbf{L}_p}$ as the supremum of the numbers

$$\int_\Omega \varphi f \, d\mathcal{L}^n$$

corresponding to all φ's in $\mathbb{L}_q(\Omega)$ with $\|\varphi\|_{\mathbf{L}_q} = 1$ [FH2.4.16]. With this norm $\mathbb{L}_p(\Omega)$ is a Banach space.

Suppose $f, f_1, f_2, f_3, \ldots \in \mathbb{L}_p(\Omega)$. We say that the f_i's **converge strongly to** f, written $f_i \to f$, as $i \to \infty$ if and only if

$$\|f_i - f\|_{\mathbf{L}_p} \to 0 \quad \text{as } i \to \infty.$$

We say that the f_i's **converge weakly to** f, written $f_i \rightharpoonup f$, as $i \to \infty$ if and only if

$$\int_\Omega \varphi \big(f_i - f \big) \, d\mathcal{L}^n \to 0 \quad \text{as } i \to \infty$$

for each (fixed) function φ in $\mathbb{L}_q(\Omega)$; it follows that

$$\|f\|_{\mathbf{L}_p} \le \liminf_i \|f_i\|_{\mathbf{L}_p} .$$

Suppose $f, f_1, f_2, f_3, \ldots \in \mathbb{L}_p(\Omega)$ with $\sup_i \|f_i\|_{\mathbf{L}_p} < \infty$ and that $f_i \rightharpoonup f$ as $i \to \infty$ (weak convergence in $\mathbb{L}_p(\Omega)$). Suppose also that $\varphi, \varphi_1, \varphi_2, \varphi_3, \ldots \in \mathbb{L}_q(\Omega)$ and that $\varphi_i \to \varphi$ as $i \to \infty$ (strong convergence in $\mathbb{L}_q(\Omega)$). Then one uses Hölder's inequality [FH 2.4.14] to check

$$\int_\Omega \varphi_i \, f_i \, d\mathcal{L}^n \to \int_\Omega \varphi \, f \, d\mathcal{L}^n \quad \text{as } i \to \infty.$$

One can then combine this estimate with Ascoli's theorem to infer additionally that

$$\|f_i - f\|_* \to 0 \quad \text{as } i \to \infty.$$

Whenever $f_1, f_2, f_3, \ldots \in \mathbb{L}_p(\Omega)$ with $\sup_i \|f_i\|_{\mathbf{L}_p} < \infty$ then there will always be some subsequence $i(1), i(2), i(3), \ldots$ of $1, 2, 3, \ldots$ and $f \in \mathbb{L}_p(\Omega)$ such that $f_{j(i)}$'s converge to f weakly in $\mathbb{L}_p(\Omega)$ as $i \to \infty$ [GT Theorem 5.12][YK][MC Theorem 3.2.4(e)].

B.4 Proposition (Lower semicontinuity of convex integrals). *Suppose*

(a) $p, q > 1$ with $1/p + 1/q = 1$ and $0 < M < \infty$, and

(b) $G\colon (0, \infty) \to \mathbb{R}^+$ is a convex function with $\lim_{t\downarrow 0} G(t) = \lim_{t\to\infty} G(t) = \infty$.

The convexity of G implies the existence of nondecreasing right hand derivates $H\colon (0, \infty) \to \mathbb{R}$ with $G(y) - G(z) \geq H(z)(y - z)$ whenever $y, z \in (0, \infty)$. Now suppose additionally

(c) $H(z) \leq M z^{p-1}$ for each $z > M$;

(d) $f, f_1, f_2, f_3, \ldots \in \mathbb{L}_p(\Omega)$ such that, for each i;

$$\|f_i\|_{\mathbf{L}_p} \leq M \quad \text{and} \quad \int_\Omega G\left(f_i\right)\, d\mathcal{L}^n \leq M,$$

and $f_i \rightharpoonup f$ as $i \to \infty$ in $\mathbb{L}_p(\Omega)$;

(g) $g, g_1, g_2, g_3, \ldots \in \mathbb{L}_1(\Omega)$ such that, for each i and each x, $0 \leq g_i(x) \leq M$, and $\|g_i - g\|_{\mathbf{L}_1} \to 0$ as $i \to \infty$.

Then

$$\int_\Omega g\, G\left(f\right)\, d\mathcal{L}^n \leq \liminf_i \int_\Omega g_i\, G\left(f_i\right)\, d\mathcal{L}^n.$$

PROOF. By contradiction. The negation of our assertion implies the existence of $\epsilon > 0$ and a subsequence of $1, 2, 3, \ldots$ (which, for notational simplicity, we assume is our original sequence $1, 2, 3, \ldots$) such that

$$\int_\Omega g\, G\left(f\right)\, d\mathcal{L}^n \geq \int_\Omega g_i\, G\left(f_i\right)\, d\mathcal{L}^n - 2\epsilon$$

for each i. The convexity of G together with its divergence to ∞ at 0 and ∞ imply that G assumes its minimum value at some point $0 < z < \infty$. We fix such z and for $t \geq 0$ and $0 < x < \infty$ we set

$$h(t, x) = z + (1 - t)(f(x) - z) = tz + (1 - t)f(x),$$

$$h_i(t,\,x) = z + (1-t)(f_i(x) - z) = tz + (1-t)f_i(x).$$

Since we have used only translations and rescalings in the range in constructing our $h_i(t,\,\cdot)$'s and $h(t,\,\cdot)$ we infer, for each $t \geq 0$ that $h_i(t,\,\cdot) \rightharpoonup h(t,\,\cdot)$ in $\mathbb{L}_p(\Omega)$ as $i \to \infty$. We check also, for each $0 < x < \infty$, that $G(h(t,\,x)) \uparrow G(h(0,\,x))$ as $t \downarrow 0$. We then use Lebesgue's increasing convergence theorem [FH 2.4.7] to conclude the existence of $t > 0$ such that

$$(*)\qquad \begin{aligned} \int_{x\in\Omega} g(x)\, G\left(h(t,\,x)\right)\, d\mathcal{L}^n x &\geq \int_{x\in\Omega} g_i(x)\, G\left(f_i(x)\right)\, d\mathcal{L}^n x - \epsilon \\ &\geq \int_{x\in\Omega} g(x_i)\, G\left(h_i(t,\,x)\right)\, d\mathcal{L}^n x - \epsilon \end{aligned}$$

for each i. We now estimate

$$(**)\qquad \begin{aligned} &\int_{x\in\Omega} g_i(x)\, G\left(h_i(t,\,x)\right)\, d\mathcal{L}^n x - \int_{x\in\Omega} g(x)\, G\left(h(t,\,x)\right)\, d\mathcal{L}^n x \\ &\qquad = \int_{x\in\Omega} g_i(x)\, \big(G\left(h_i(t,\,x)\right) - G\left(h(t,\,x)\right)\big)\, d\mathcal{L}^n x \\ &\qquad\quad + \int_{x\in\Omega} \big(g_i(x) - g(x)\big) G\left(h(t,\,x)\right)\, d\mathcal{L}^n x \\ &\qquad \geq \int_{x\in\Omega} g_i(x)\, H(h(t,\,x))\big(h_i(t,\,x) - h(t,\,x)\big)\, d\mathcal{L}^n x. \\ &\qquad\quad + \int_{x\in\Omega} \big(g_i(x) - g(x)\big) G\left(h(t,\,x)\right)\, d\mathcal{L}^n x \end{aligned}$$

We combine assumption (c) with the fact that $h(t,\,x) \geq tz$ for each x to conclude that the functions $x \mapsto g_i(x)H(h(t,\,x))$ belong to $\mathbb{L}_q(\Omega)$ with uniformly bounded $\mathbb{L}_q$ norms and that they converge strongly in $\mathbb{L}_q(\Omega)$ to $x \mapsto g(x)H(h(t,\,x))$. The weak convergence of the $h_i(t,\,\cdot)$'s to $h(t,\,\cdot)$ then implies that the first integral in the final expression in $(**)$ converges to zero as $i \to \infty$ as in B.3. We infer further from assumption (c) and the fact that $h(t,\,t) \geq tz$ for each x that the function $G(h)$ is $\mathcal{L}^n$ summable. We infer from Lebesgue's bounded convergence theorem that the second integral in the final expression in $(**)$ also converges to zero as $i \to \infty$. These two estimates are incompatible with $(*)$. This completes our proof by contradiction of our assertion. ■

B.5 Remark. Results analogous to that of Proposition B.4 hold in case the domain of the convex function G is a general interval $(a,\,b)$ which also contains the images of the f_i's. Additionally the assumption analogous to (c) can be relaxed provided that at endpoints at which G does not diverge to ∞ one extends G continuously to include that endpoint in its domain.

B.6 Corollary (A compactness estimate). *Suppose L and M are positive numbers and $\mathcal{P}(L, M)$ is the space of all nonnegative functions f for which*
(a) $f \in \mathbb{L}_2(\Omega)$ with $\int_\Omega f\, d\mathcal{L}^n = L$ and $\|f\|_{\mathbb{L}_2} \le M$.
(b) $1/f \in \mathbb{L}_2(\Omega)$ with $\|1/f\|_{\mathbb{L}_2} \le M$.
Then $\mathcal{P}(L, M)$ is compact in the topology of the Monge-Kantorovich distance $\|\cdot\|_*$.

B.7 The vector space $\mathbb{H}^1(\Omega)$. We denote by $\mathbb{H}^1(\Omega)$ the vector subspace of $\mathbb{L}_2(\Omega)$ of all functions f which have square summable partial derivatives in the sense of distributions, i.e. for which there exist functions $g_1, \ldots, g_n$ in $\mathbb{L}_2(\Omega)$ such that

$$\int_\Omega f \frac{\partial \varphi}{\partial x_i}\, d\mathcal{L}^n = \int_\Omega -g_i \varphi\, d\mathcal{L}^n$$

for each i and for each smooth function $\varphi\colon \Omega \to \mathbb{R}$. For such f and $g_1, \ldots, g_n$ we abbreviate $\nabla f = (g_1, g_2, \ldots, g_n) : \Omega \to \mathbb{R}^n$, and for each $\mathcal{L}^n$ measurable subset A of Ω we denote

$$\mathrm{DIR}(f;\, A) = \int_A \left(g_1^2 + \ldots + g_n^2\right)\, d\mathcal{L}^n = \int_A |\nabla f|^2\, d\mathcal{L}^n$$

as the usual Dirichlet's integral of f. We abbreviate

$$\mathrm{DIR}(f) = \mathrm{DIR}(f;\, \Omega).$$

If function f belongs to $\mathbb{H}^1(\Omega)$ then the usual Poincaré's inequality for the cube $\left[0,\, \rho_*\right]^n$ guarantees that

$$\int_\Omega f^2\, d\mathcal{L}^n \le \frac{1}{\rho_*^n} \left(\int_\Omega f\, d\mathcal{L}^n\right)^2 + \rho_*^2 \mathrm{DIR}(f).$$

Suppose $f_1, f_2, f_3, \ldots \in \mathbb{H}^1(\Omega)$ with $\infty > \sup_i \|f_i\|_{\mathbb{L}_2} + \sup_i \mathrm{DIR}\big(f_i\big)$. Then there is a subsequence $j(1), j(2), j(3), \ldots$ of $1, 2, 3, \ldots$ and $f \in \mathbb{H}^1(\Omega)$ such that

$$\lim_{i \to \infty} \|f - f_{j(i)}\|_{\mathbb{L}_2} = 0,$$

$$\nabla f_{j(i)} \rightharpoonup \nabla f \text{ as } i \to \infty \quad \left(\text{weak convergence in } \mathbb{L}_2\left(\Omega, \mathbb{R}^n\right)\right),$$

and hence

$$\mathrm{DIR}(f) \le \liminf_i \mathrm{DIR}\left(f_{j(i)}\right).$$

[MC Theorem 3.4.4, Rellich's theorem].

B.8 Proposition (Functions in $\mathbb{H}^1(\Omega)$ have small integrals over small sets). *Suppose A is an $\mathcal{L}^n$ measurable subset of Ω and f in $\mathbb{H}^1(\Omega)$ has average value*

$$AV(f) = \frac{1}{|\Omega|}\int_\Omega f\, d\mathcal{L}^n.$$

(1) If $n \geq 3$ then

$$\int_A f^2\, d\mathcal{L}^n \leq C\,\mathcal{L}^n(A)^{\frac{2}{n}}\,\mathrm{DIR}(f) + 2\mathcal{L}^n(A)\,AV(f)^2;$$

here C is a constant depending only on n and Ω.

(2) If $n = 2$ and $0 < \epsilon < 1$ then

$$\int_A f^2\, d\mathcal{L}^n \leq C_\epsilon\,\mathcal{L}^n(A)^{1-\epsilon}\,\mathrm{DIR}(f) + 2\mathcal{L}^n(A)\,AV(f)^2;$$

here C_ϵ is a constant depending only on ϵ and Ω.

PROOF. We will combine various Hölder's inequalities with the generalized Poincaré type inequality given in [GT Theorem 7.10, page 148] which says, in particular, for suitable functions $u{:}\mathbb{R}^n \to \mathbb{R}$ having compact support and numbers $1 < p < n$ that there is a constant C such that

$$\|u\|_{\mathbf{L}_{np/(n-2)}} \leq C\|Du\|_{\mathbf{L}_p}.$$

Here and below we denote by "C" a general finite constant which need not have the same value in different equations.

Step 1. We set $g = f - AV(f)$ so that

$$\int_A f^2\, d\mathcal{L}^n \leq 2\int_A g^2\, d\mathcal{L}^n + 2\mathcal{L}^n(A)\,AV(f)^2,$$

and, by Poincaré's inequality,

$$\int_\Omega g^2\, d\mathcal{L}^n \leq \rho_*^2\,\mathrm{DIR}(g) = \rho_*^2\,\mathrm{DIR}(f).$$

Step 2. We denote by $\pi{:}\mathbb{R}^n \to \Omega$ the simply connected locally isometric covering mapping of our torus Ω and define $h = g \circ \pi{:}\mathbb{R}^n \to \mathbb{R}$ and $B = \pi^{-1}(A) \cap [0,\, \rho_*]^n$. We fix a Lipschitz function $\varphi{:}\mathbb{R}^n \to [0,\, 1]$ with $\mathrm{Lip}(\varphi) = 1/\rho_*$ and

$$\varphi(x) = \begin{cases} 1 & \text{in } x \in [0,\, \rho_*]^n \\ 0 & \text{if } \operatorname{dist}(x, [0,\, \rho_*]^n) \geq \rho \end{cases}$$

and set $k = \varphi h$ so that $|k| \leq |h|$ and $\nabla k = \varphi \nabla h + h \nabla \varphi$ and hence

$$\int_{\mathbf{R}^n} k^2 \, d\mathcal{L}^n \leq \int_{[-\rho_*,\, 2\rho_*]^n} h^2 \, d\mathcal{L}^n = 3^n \int_\Omega g^2 \, d\mathcal{L}^n,$$

$$\begin{aligned} \int_{\mathbf{R}^n} |\nabla h|^2 \, d\mathcal{L}^n &\leq 2 \int_{[\rho_*,\, 2\rho_*]^n} \left(|\nabla h|^2 + \frac{1}{\rho_*^2} |h|^2 \right) d\mathcal{L}^n \\ &= 2 \cdot 3^n \left[\mathrm{DIR}(f) + \frac{1}{\rho_*^2} \int_\Omega g^2 \, d\mathcal{L}^n \right] \\ &\leq 4 \cdot 3^n \mathrm{DIR}(f) \end{aligned}$$

by Step 1.

Step 3. If $n \geq 3$ we set

$$p = 2, \qquad P = \frac{np}{n-p} = \frac{2n}{n-2}, \qquad R = \frac{P}{2} = \frac{n}{n-2}, \qquad Q = \frac{n}{2}$$

and use Hölder's inequality and [GT Theorem 7.10] to estimate

$$\begin{aligned} \int_A g^2 d\mathcal{L}^n &= \int_B h^2 \, d\mathcal{L}^n \\ &\leq \mathcal{L}^n(B)^{\frac{1}{Q}} \, \|h^2\|_{\mathbf{L}_R} \\ &= \mathcal{L}^n(A)^{\frac{2}{n}} \, \|h\|^2_{\mathbf{L}_P} \\ &\leq C \, \mathcal{L}^n(A)^{\frac{2}{n}} \, \|\nabla h\|^2_{\mathbf{L}_2} \\ &\leq C \, \mathcal{L}^n(A)^{\frac{2}{n}} \, 4 \cdot 3^n \, \mathrm{DIR}(f) \end{aligned}$$

by Step 2. Our first conclusion follows.

Step 4. If $n = 2$ and $0 < \delta < 1$ we set

$$p = 2 - \delta, \quad P = \frac{np}{n-p} = \frac{2(2-\delta)}{\delta}, \quad R = \frac{P}{2} = \frac{2-\delta}{\delta}, \quad Q = \frac{2-\delta}{2-2\delta}$$

and use Hölder's inequality twice and [GT Theorem 7.10] to estimate

$$\begin{aligned}\int_A g^2 d\mathcal{L}^n &= \int_B h^2\, d\mathcal{L}^n \\ &\le \mathcal{L}^n(B)^{\frac{1}{Q}}\, \|h^2\|_{\mathbf{L}_R} \\ &= \mathcal{L}^n(A)^{\frac{2-2\delta}{2-\delta}}\, \|h\|_{\mathbf{L}_P}^2 \\ &\le C\mathcal{L}^n(A)^{\frac{2-2\delta}{2-\delta}}\, \|\nabla h\|_{\mathbf{L}_{2-\epsilon}}^2 \\ &\le C\mathcal{L}^n(A)^{1-\epsilon}\, 4\cdot 3^n\, \mathrm{DIR}(f)\end{aligned}$$

for $1-\epsilon = (2-2\delta)/(2-\delta)$ by Step 2. Our second conclusion follows.

B.9 Proposition (A domination estimate). *Corresponding to each $\epsilon > 0$ there exists $M < \infty$ such that*

$$\|f\|_{\mathbf{L}_2} \le M\|f\|_* + \epsilon \mathrm{DIR}(f)$$

whenever f belongs to $\mathbb{H}^1(\Omega)$ with $\int_\Omega f\, d\mathcal{L}^n = 0$.

PROOF. By contradiction. The negation of the assertion implies the existence of some $\epsilon > 0$ (which we fix) such that for each M there will exists some f with $\|f\|_{\mathbf{L}_2} > M\|f\|_* + \epsilon\mathrm{DIR}(f)$ or, equivalently, $(1+\epsilon)\|f\|_{\mathbf{L}_2} > M\|f\|_* + \epsilon\left(\|f\|_{\mathbf{L}_2} + \mathrm{DIR}(f)\right)$. Without loss of generality we assume $\|f\|_{\mathbf{L}_2} + \mathrm{DIR}(f) = 1$; this implies $\|f\|_{\mathbf{L}_2} \ge \epsilon/(1+\epsilon)$. We fix f_i to be such an f chosen corresponding to $M = i$; this implies $\lim_i \|f_i\|_* = 0$. We use B.1 and B.8 to infer the existence of a limit g for which $\|g\|_{\mathbf{L}_2} \ge \epsilon/(1+\epsilon)$ while $\|g\|_* = 0$. This is not possible. ■

B.10 Proposition (Sobolev estimates in dimensions 2 and 3).
(1) If the ambient dimension n equals 2 there are constants C_p (depending on $1 < p < \infty$ and on ρ_*) such that, for each f in $\mathbb{H}^1(\Omega)$,

$$\left(\int_\Omega f^p\, d\mathcal{L}^2\right)^{\frac{1}{p}} \le C_p \left(\int_\Omega f^2\, d\mathcal{L}^2\right)^{\frac{1}{2}} + C_p \left(\int_\Omega |\nabla f|^2\, d\mathcal{L}^2\right)^{\frac{1}{2}}.$$

(2) If the ambient dimension n equals 3 there is a constant C (depending only on ρ_*) such that, for each f in $\mathbb{H}^1(\Omega)$,

$$\left(\int_\Omega f^6\, d\mathcal{L}^3\right)^{\frac{1}{6}} \le C \left(\int_\Omega f^2\, d\mathcal{L}^3\right)^{\frac{1}{2}} + C \left(\int_\Omega |\nabla f|^2\, d\mathcal{L}^3\right)^{\frac{1}{2}}.$$

PROOF. Our torus Ω is naturally identified with the quotient of R^n by the lattice generated by $\rho_* \mathbf{e}_i$ $(i = 1, \ldots, n)$. Set $J = [-\rho_*, 2\rho_*]^n \subset \mathbb{R}^n$ and let $g: J \to \mathbb{R}$ denote the pullback of f to J. Also set $h(x) = \sup\{1, \rho_*^{-1}\operatorname{dist}(x, \partial J)\}$ for x in Ω. We estimate

$$\int_J |\nabla(gh)|^2 \, d\mathcal{L}^n \le C_0 \int_\Omega f^2 \, d\mathcal{L}^n + \le C_0 \int_\Omega |\nabla f|^2 \, d\mathcal{L}^n$$

for a suitable constant C_0 depending on n and ρ_*. We then apply the Sobolev estimates given in [GT Theorem 7.10, page 148] to the function gh and then by restriction (from J to $[0, \rho_*]^n$) to f. ■

Appendix C. Currents and varifolds and convergence

In this appendix we review some terminology and basic facts about currents and varifolds. We assume the reader will consult the discussion in Chapter 3 of the paper [ATW] as necessary.

C.1 Terminology for variations of integrals over currents and varifolds. Suppose $1 \le k \le n$ are fixed integers.

(1) We identify the Grassmann manifold $\mathbb{G}_0(n, k)$ of all oriented k dimensional vector subspaces of $\mathbb{R}^n$ with the subset of the Grassmann vector space $\wedge_k \mathbb{R}^n$ consisting of all simple unit k vectors in $\mathbb{R}^n$.

(2) By a *parametric integrand* we mean a continuous function $\Psi: \mathbb{R}^n \times \mathbb{G}_0(n, g) \to \mathbb{R}$. For our purposes here we call a parametric integrand Ψ *elliptic* provided that for each x there is a convex function $F: \wedge_k \mathbb{R}^n \to \mathbb{R}^+$ satisfying the following conditions:

(i) F is positively homogeneous of degree 1 and is positive and twice continuously differentiable on $\wedge_k R^n \sim \{0\}$;

(ii) F has positive second derivatives when restricted to each straight line not containing the origin.

(3) Associated with each smooth vector field $g: \mathbb{R}^n \to \mathbb{R}^n$ having compact support are the *deformations* $G_t: \mathbb{R}^n \to \mathbb{R}^n$, $t \in \mathbb{R}$, given by setting

$$G_t(x) = x + t\, g(x)$$

for each t and x. The initial velocity vector field of G_t is g.

(4) Associated with each smooth parametric integrand $\Psi: \mathbb{R}^n \times \mathbb{G}_0(n, k) \to \mathbb{R}$. and with each smooth initial velocity vector field g is the function $H(\Psi, g): \mathbb{R}^n \times \mathbb{G}_0(n, k) \to \mathbb{R}$ given by setting

$$H(\Psi, g)(x, \sigma) = \frac{d}{dt}\Big[|\sigma(t)|\Psi\left(|\sigma(t)|^{-1}\sigma(t)\right)\Big]\Bigg|_{t=0} \quad \text{where} \quad \sigma(t) = \big\langle \sigma, \wedge_k DG_t(x) \big\rangle.$$

(5) For our purposes here, a *k current T in* $\mathbb{R}^n$ is a continuous linear functional on the real vector space of infinitely differentiable differential k forms ω in $\mathbb{R}^n$ having compact support which current also has compact support. In accordance with [FH 4.1.7, page 357] such a k current T is *representable by integration* when there is a Radon measure $\|T\|$ over $\mathbb{R}^n$ having compact support and a $\|T\|$ measurable $\wedge_k \mathbb{R}^n$ valued function $\vec{T}$ of unit length such that

$$T(\omega) = \big(\|T\| \wedge \vec{T}\big)(\omega) = \int_{\mathbf{R}^n} \left\langle \vec{T}, \omega \right\rangle \, d\|T\|.$$

A k current T is called (real) *rectifiable* when there is a k rectifiable set S, a $\mathcal{H}^k$ summable positive density function $\theta: S \to \mathbb{R}^+$ and an associated oriented Borel tangent plane valued function $\sigma: S \to \mathbb{G}_0(n, k)$ such that

$$T(\omega) = \mathbf{t}\big(S, \theta, \sigma\big)(\omega) = \int_S \langle \sigma, \omega \rangle \, \theta \, d\mathcal{H}^k,$$

in particular, $\|T\| = \mathcal{H}^k \llcorner S \wedge \theta$ and $\vec{T} = \sigma$.

In accordance with [FH 4.1.7, page 359], each smooth mapping $f: \mathbb{R}^n \to \mathbb{R}^n$ naturally maps a k currents T in $\mathbb{R}^n$ to a k currents $f_\sharp T$ in $\mathbb{R}^n$.

Associated with each (real) rectifiable k current $T = \mathbf{t}(S, \theta, \sigma)$ and each parametric integrand $\Psi: \mathbb{R}^n \times \mathbb{G}_0(n, g) \to \mathbb{R}$ is the *integral* $\Psi(T)$ *of* Ψ *over* T given by setting

$$\Psi(T) = \int_{x \in S} \Psi(x, \sigma(x))\, \theta(x) \, d\mathcal{H}^k x.$$

(6) By an *oriented k varifold in* $\mathbb{R}^n$ we mean a Radon measure over $\mathbb{R}^n \times \mathbb{G}_0(n, k)$ having compact support. Suppose $V, V_1, V_2, V_3, \ldots$ are k varifolds in $\mathbb{R}^n$. We say that V_j *converges to V weakly* as $j \to \infty$ and write $V_j \rightharpoondown V$ as $j \to \infty$ if and only if

$$\int_{\mathbf{R}^n \times \mathbf{G}_0(n, k)} \varphi \, dV_j \to \int_{\mathbf{R}^n \times \mathbf{G}_0(n, k)} \varphi \, dV$$

as $j \to \infty$ for each continuous function $\varphi : \mathbb{R}^n \times \mathbb{G}_0(n, k) \to \mathbb{R}$ having compact support. Associated with each rectifiable k current $\mathbf{t}(S, \theta, \sigma)$ as above is the *rectifiable oriented k varifold* $\mathbf{v}(S, \theta, \sigma)$ defined by requiring for each continuous function $\varphi : \mathbb{R}^n \times \mathbb{G}_0(n, k) \to \mathbb{R}$ having compact support that

$$\int_{\mathbf{R}^n \times \mathbf{G}_0(n, k)} \varphi \, d\mathbf{v}(S, \theta, \sigma) = \int_{x \in S} \varphi\big(x, \sigma(x)\big)\, \theta(x)\, d\mathcal{H}^k x.$$

By a direct extension of the formula given in [AW 3.2, page 430] each smooth mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ naturally maps a k varifold V in $\mathbb{R}^n$ to a k varifold $f_\sharp T$ in $\mathbb{R}^n$.

Associated with each k varifold V and each parametric integrand $\Psi : \mathbb{R}^n \times \mathbb{G}_0(n, g) \to \mathbb{R}$ is the *integral* $\Psi(V)$ *of* Ψ *over* V given by setting

$$\Psi(V) = \int_{\mathbf{R}^n \times \mathbf{G}_0(n, k)} \Psi d\mathcal{H}^k x.$$

C.2 Theorem (Computing first variations). *Suppose $\Psi : \mathbb{R}^n \times \mathbb{G}_0(n, g) \to \mathbb{R}$ is a smooth parametric integrand and that $g : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth initial velocity vector field in $\mathbb{R}^n$ having compact support and G_t gives the associated deformations as in C.1(3) above. Let $H(\Psi, g) : \mathbb{R}^n \times \mathbb{G}_0(n, k) \to \mathbb{R}$ be the function given in C.1(4) above. Then*

(1) For each k current $T = \mathbf{t}(S, \theta, \sigma)$ as above

$$\frac{d}{dt}\Big(\Psi\big(G_{t\sharp}(T)\big)\Big)\Big|_{t=0} = \int_{x \in S} H(\Psi, g)\big(x, \sigma(x)\big)\, \theta(x)\, d\mathcal{H}^k x.$$

(2) For each k varifold V as above

$$\frac{d}{dt}\Big(\Psi\big(G_{t\sharp}(V)\big)\Big)\Big|_{t=0} = \int_{(x, \sigma) \in \mathbf{R}^n \times \mathbf{G}_0(n, k)} H(\Psi, g)\big(x, \sigma\big)\, dV(x, \sigma).$$

(3) Suppose

$$T = \mathbf{t}(S, \theta, \sigma),\ T_1 = \mathbf{t}(S_1, \theta_1, \sigma_1),\ T_2 = \mathbf{t}(S_2, \theta_2, \sigma_2),\ T_3 = \mathbf{t}(S_3, \theta_3, \sigma_3),\ \ldots$$

are k currents in $\mathbb{R}^n$ associated with the varifolds

$$V = \mathbf{v}(S, \theta, \sigma),\ V_1 = \mathbf{v}(S_1, \theta_1, \sigma_1),\ V_2 = \mathbf{v}(S_2, \theta_2, \sigma_2),\ V_3 = \mathbf{v}(S_3, \theta_3, \sigma_3),\ \ldots.$$

In order that

$$\lim_{j\to\infty}\frac{d}{dt}\left(\Psi\big(G_{t\sharp}(T_j)\big)\right)\Big|_{t=0}\to\frac{d}{dt}\left(\Psi\big(G_{t\sharp}(T)\big)\right)\Big|_{t=0}$$

as $j\to\infty$ it is sufficient that $V_j \rightharpoondown V$ as $j\to\infty$.

PROOF. The proof is left to the reader.

C.3 Theorem (Current and varifold convergence). *Suppose*

$$T=\mathbf{t}(S,\,\theta,\,\sigma),\,T_1=\mathbf{t}(S_1,\,\theta_1,\,\sigma_1),\,T_2=\mathbf{t}(S_2,\,\theta_2,\,\sigma_2),\,T_3=\mathbf{t}(S_3,\,\theta_3,\,\sigma_3),\,\ldots$$

are k currents in $\mathbb{R}^n$ associated with the varifolds

$$V=\mathbf{v}(S,\,\theta,\,\sigma),\,V_1=\mathbf{v}(S_1,\,\theta_1,\,\sigma_1),\,V_2=\mathbf{v}(S_2,\,\theta_2,\,\sigma_2),\,V_3=\mathbf{v}(S_3,\,\theta_3,\,\sigma_3),\,\ldots.$$

In order that that $V_j \rightharpoondown V$ as $j\to\infty$, it is sufficient that the following conditions hold:

(a) the union of the supports of the T_j's be bounded;

(b) the masses $\|T_j\|\mathbb{R}^n$ be bounded;

(c) $T_j(\omega)\to T(\omega)$ as $j\to\infty$ for each smooth differential k form;

(d) there exists an elliptic parametric integrand $\Psi:\mathbb{R}^n\times\mathbb{G}_0(n,\,g)\to\mathbb{R}$ such that $\Psi(T_j)\to\Psi(T)$ as $j\to\infty$.

PROOF. The proof depends on the two propositions and corollary following.

C.4 Proposition (A variant of Jensen's inequality). *Suppose $F,\,G:\mathbb{R}^n\to\mathbb{R}^+$ are convex functions which are twice continuously differentiable on $\mathbb{R}^n\sim\{0\}$. Suppose also that F, G are positively homogeneous of degree 1 and have positive second derivatives when restricted to each straight line not containing the origin. In case $n=1$ we assume additionally that neither F nor G is everywhere equal to zero. Then there is a number $0<\lambda<\infty$ (depending only on F and G) with the following property.*

Suppose φ measures X with $0<\varphi(X)<\infty$ and $h\colon X\to\mathbb{R}^n$ is φ summable. Then the two numbers

$$\int_X (F\circ h)\,d\varphi - F\left(\int_X h\,d\varphi\right)\quad\text{and}\quad\int_X (G\circ h)\,d\varphi - G\left(\int_X h\,d\varphi\right)$$

are within factor λ of each other above and below.

PROOF. **Step 1.** We assert there is at most one point p in $\partial\mathbb{B}^n(0, 1)$ with $F(p) = 0$ [resp. with $G(p) = 0$]. The is clear if $n = 1$; we therefore assume $n \geq 2$ and suppose $p \neq q \in \partial\mathbb{B}^n(0, 1)$ with $F(p) = F(q) = 0$. If $q \neq -p$ then the convexity and positivity of F imply that F vanishes on the line segment $\overline{pq}$ contrary to our assumption about second derivatives. If $q = -p$ we choose $z \in \partial\mathbb{B}^n(0, 1)$ with $z \bullet p = 0$ and estimate

$$F\left(\frac{p+z}{2}\right) < \frac{F(z)}{2} \quad \text{and} \quad F\left(\frac{q+z}{2}\right) < \frac{F(z)}{2}$$

(by Taylor's formula and our assumption about second derivatives applied to the line segments $\overline{pz}$ and $\overline{qz}$) and

$$F\left(\frac{z}{2}\right) = F\left(\frac{\frac{p+z}{2} + \frac{q+z}{2}}{2}\right) < \frac{\frac{F(z)}{2} + \frac{F(z)}{2}}{2} = \frac{F(z)}{2} = F\left(\frac{z}{2}\right)$$

(by the convexity of F applied to the midpoint $\frac{z}{2}$ between $\frac{p+z}{2}$ and $\frac{q+z}{2}$ together with the homogeneity of F). Since this is false, our assertion is proved.

Step 2. Suppose $p \in \partial\mathbb{B}^n(0, 1)$ with $F(p) = G(p) = 0$. Then there is a number $1 < \lambda < \infty$ such that

$$F(q) \leq \lambda G(q) \leq \lambda^2 F(q)$$

for each $q \in \mathbb{R}^n$. Outside a small neighborhood of p we infer from Step 1 and the continuity of F and G that the restrictions of both F and G to $\partial\mathbb{B}^n(0, 1)$ are uniformly bounded away from zero and infinity so that choice of such a λ is possible. We also infer from Taylor's theorem and our assumptions about continuity and positivity of second derivatives that the ratios of $F(q)$ and $G(q)$ are bounded above and below for q's which are near p with $q \bullet p = 0$. Our assertion now follows from the assumed homogeneity of F and G.

Step 3. There is a number $1 < \lambda < \infty$ such that

$$\big(F - DF(p)\big)(q) \leq \lambda\big(G - DG(p)\big)(q) \leq \lambda^2\big(F - DF(p)\big)(q)$$

whenever $0 \neq p,\ q \in \mathbb{R}^n$ Indeed, for each $p \neq 0$ the functions $\big(F - DF(p)\big)$ and $\big(G - DG(p)\big)$ satisfy the same general assumptions as do F and G but also vanish at p. For each fixed p we infer the inequality of Step 3 from Step 2. The arguments of Step 2 in fact give uniform estimates so that such an inequality holds simultaneously for all $p \in \partial\mathbb{B}^n(0, 1)$. The positive homogeneity assumed implies that such p's are sufficient for the general assertion.

Step 4. Suppose $L:\mathbb{R}^n \to \mathbb{R}$ is linear. Then

$$\int_X (L \circ h)\, d\varphi = L\left(\int_X h\, d\varphi\right)$$

for X and h as above. To see this one approximates h by functions with finite images.

Step 5. Suppose X and h are as above and $p = \int_X h\, d\varphi$. If $p = 0$ replace p by any other vector. Set $F_* = F - DF(p)$ and $G_* = G - DG(p)$ so that $F_*(p) = G_*(p) = 0$. We infer from Step 3 that the bounded ratios asserted in the Proposition hold if we replace F by F_* and G by G_*. We infer from Step 4 that this is adequate to prove the original bounded ratios asserted. ■

C.5 Proposition (A criterion for vector functions to be nearly constant). *Suppose*

(a) φ *measures* X *with* $\varphi(X) < \infty$.

(b) $f: X \to \mathbb{R}^n$ *is* φ *summable so that* $W = \int_X f\, d\varphi \in \mathbb{R}^n$.

$W = \int_X f\, d\varphi \in \mathbb{R}^n$. *(c)* $0 < \epsilon < 1$ *and*

$$\int_X |f|\, d\varphi - \left|\int_X f\, d\varphi\right| \le \epsilon^2 \left|\int_X f\, d\varphi\right|.$$

(d) $Y = X \cap \{x: f(x) \bullet W \le (1-\epsilon)|f(x)|\,|W|\}$.

Then

$$\int_Y |f|\, d\varphi \le \epsilon \left|\int_X f\, d\varphi\right|.$$

PROOF. The case $W = 0$ is left to the reader. For $W \ne 0$ we set $u = W/|W|$ and, for $x \in X$, set

$$g(x) = f(x) \bullet u, \quad h(x) = \left(|f(x)|^2 - g(x)^2\right)^{\frac{1}{2}}.$$

Since

$$\left|\int_X f\, d\varphi\right| = u \bullet \int_X f\, d\varphi = \int_X u \bullet f\, d\varphi = \int_X g\, d\varphi,$$

supposition (c) becomes

$$\int_X \left(g^2 + h^2\right)^{\frac{1}{2}} d\varphi \le \left(1 + \epsilon^2\right) \int_X g\, d\varphi$$

and our desired conclusion is

$$\int_Y \left(g^2+h^2\right)^{\frac{1}{2}}\,d\varphi \le \epsilon \int_X g\,d\varphi.$$

We compute

$$\begin{aligned}\int_X g\,d\varphi &= \int_Y g\,d\varphi + \int_{X\sim Y} g\,d\varphi \\ &\le (1-\epsilon)\int_Y \left(g^2+h^2\right)^{\frac{1}{2}}\,d\varphi + \int_{X\sim Y}\left(g^2+h^2\right)^{\frac{1}{2}}\,d\varphi \\ &\le \left(1+\epsilon^2\right)\int_X g\,d\varphi - \epsilon\int_Y \left(g^2+h^2\right)^{\frac{1}{2}}\,d\varphi\end{aligned}$$

so that

$$\epsilon\int_Y \left(g^2+h^2\right)^{\frac{1}{2}}\,d\varphi \le \epsilon^2\int_X g\,d\varphi$$

which implies our conclusion. ■

C.6 Corollary (Another criterion for vector functions to be nearly constant). *Suppose $F\colon\mathbb{R}^n \to \mathbb{R}^+$ is a convex function which is twice continuously differentiable on $\mathbb{R}^n \sim \{0\}$. Suppose also that F is positively homogeneous of degree 1 and has positive second derivatives when restricted to each straight line not containing the origin. In case $n=1$ we assume additionally that F is not everywhere equal to zero. Then there is $\lambda > 0$ with the following property: Suppose*

(a) φ measures X with $\varphi(X) < \infty$.

(b) $f\colon X \to \mathbb{R}^n$ is φ summable so that $p = \int_X f\,d\varphi \in \mathbb{R}^n$.

(c) $0 < \epsilon < 1$ and

$$\int_X F\circ f\,d\varphi - F\left(\int_X f\,d\varphi\right) \le \lambda\epsilon^2\left|\int_X f\,d\varphi\right|.$$

(d) $Y = X \cap \big\{x\colon f(x)\bullet p \le (1-\epsilon)|f(x)|\,|p|\big\}$.

Then

$$\int_Y |f|\,d\varphi \le \epsilon\left|\int_X f\,d\varphi\right|.$$

PROOF. Set $G(v) = |v|$ in Proposition C.4 and use Proposition C.5. ■

PROOF OF THEOREM C.3 We will treat only the case in which Ψ does not depend on its $\mathbb{R}^n$ variables and leave the general case to the reader. Initially we consider the case in which T and the T_j's are general k currents representable by integration (not necessarily rectifiable). Suppose then that we set $N = \binom{n}{k}$ and identify $\wedge_k\mathbb{R}^n$ with $\mathbb{R}^N$. We use our ellipticity assumption to infer the existence of a convex function

$$F{:}\mathbb{R}^N \to \mathbb{R}^+ \quad \text{with} \quad F|\mathbb{G}_0(n,\,k) = \Psi$$

which is positive and twice continuously differentiable on $\mathbb{R}^N \sim \{0\}$ and which is positively homogeneous of degree 1 and has positive second derivatives when restricted to each straight line not containing the origin. By the integral of F over T we mean the number

$$\mathbb{F}(T) = \int_{\mathbf{R}^n} F\big(\vec{T}\big)\, d\|T\|.$$

we leave it to the reader to use the convexity of F to verify that

$$\mathbb{F}(T) = \sup\Big\{T(\omega){:}\, \omega(x) \le F \text{ for each } x\Big\}.$$

We infer from this estimate that

$$F(T) \le \liminf_j F(T_j).$$

Associated with T and each smooth $\varphi{:}\mathbb{R}^n \to \mathbb{R}^+$ having compact support is the current $T \llcorner \varphi$, given by setting

$$(T \llcorner \varphi)(\omega) = T(\varphi \wedge \omega)$$

for each ω. Clearly

$$T \llcorner \varphi = (\|T\| \llcorner \varphi) \wedge \vec{T}.$$

Corresponding to each $\epsilon > 0$ we can apply Besicovitch's covering theorem [FH 2.8.18] to balls centered in the Lebesgue set of $\|T\|$ [FH 2.9.9] to infer the existence of $\|T\|$ Lebesgue points points $p_1,\, p_2,\, \ldots,\, p_M$ for $\vec{T}$ and radii $r_1,\, r_2,\, \ldots,\, r_M$ and smooth functions $\varphi_1,\, \varphi_2,\, \ldots,\, \varphi_M$ and

$$\varphi_0 = 1 - \sum_{j-1}^{M} \varphi_j.$$

having the following properties:

(i) $|p_i - p_j| > r_i + r_j$ for each $i \neq j$;

(ii) each φ_i is nearly the characteristic function of $\mathbb{B}^n(p_i, r_i)$;

(iii) for each $i = 1, \ldots, M$,

$$\int_{x \in \mathbf{R}^n} \varphi_i(x)\big(\vec{T}(x) - \vec{T}(p_i)\big)\, d\|T\|x < \epsilon \int_{\mathbf{R}^n} \varphi_i \, d\|T\|.$$

(iv)
$$\int_{\mathbf{R}^n} \varphi_0 \, d\|T\| < \epsilon;$$

We then have

(1) For each $i = 0, \ldots, M$ the weak convergence of our currents implies

$$\int_{\mathbf{R}^n} \vec{T}_j \, d\|T_j \llcorner \varphi_i\| \to \int_{\mathbf{R}^n} \vec{T} \, d\|T \llcorner \varphi_i\| \quad \text{as } j \to \infty;$$

(2) If ϵ is very small then, for all sufficiently large j's and for each $i = 1, \ldots, M$,

$$\begin{aligned}
\int_{\mathbf{R}^n} \vec{T}_j \, d\|T_j \llcorner \varphi_i\| &\approx \int_{\mathbf{R}^n} \vec{T} \, d\|T \llcorner \varphi_i\| \\
&\approx \int_{\mathbf{R}^n} \vec{T}(p_i) \, d\|T \llcorner \varphi_i\| \\
&= \vec{T}(p_i)\|T \llcorner \varphi_i\| \mathbb{R}^n.
\end{aligned}$$

Now suppose additionally that $\mathbb{F}(T_j) \to \mathbb{F}(T)$ as $J \to \infty$. Then

(3) For each $i = 0, \ldots, M$, the convergence of our $\mathbb{F}$ integrals implies

$$\int_{\mathbf{R}^n} F(\vec{T}_j) \, d\|T_j \llcorner \varphi_i\| \to \int_{\mathbf{R}^n} F(\vec{T}) \, d\|T \llcorner \varphi_i\| \quad \text{as } j \to \infty;$$

(4) If ϵ is very small then, for all sufficiently large j's and for each $i = 1, \ldots, M$,

$$\begin{aligned}
\int_{\mathbf{R}^n} F(\vec{T}_j) \, d\|T_j \llcorner \varphi_i\| &\approx \int_{\mathbf{R}^n} F(\vec{T}) \, d\|T \llcorner \varphi_i\| \\
&\approx \int_{\mathbf{R}^n} F(\vec{T}(p_i)) \, d\|T \llcorner \varphi_i\| \\
&= \vec{T}(p_i)\|T \llcorner \varphi_i\| \mathbb{R}^n.
\end{aligned}$$

(5) If ϵ is very small we combine (2) and (4) above to conclude for large j's and each $i = 1, \ldots, M$,

$$\int_{\mathbf{R}^n} \vec{T}_j \, d\|T_j \llcorner \varphi_i\| \approx F\left(\int_{\mathbf{R}^n} F(\vec{T}_j) \, d\|T_j \llcorner \varphi_i\|\right)$$

so that

$$\int_{\mathbf{R}^n} \vec{T}_j \, d\|T_j \llcorner \varphi_i\| - F\left(\int_{\mathbf{R}^n} F(\vec{T}_j) \, d\|T_j \llcorner \varphi_i\|\right)$$

is very small.

(6) For large j and each $i = 1, \ldots, M$ we set

$$W_{ij} = \int_{\mathbf{R}^n} \vec{T}_j \, d\|T_j \llcorner \varphi_i\|, \quad w_{ij} = \frac{W_{ij}}{|W_{ij}|}$$

and, in case ϵ is very small infer from (5) Corollary C.6 that, except for very small measure, the k vectors $\vec{T}_j(x)$, $x \in \mathbb{B}^n(p_i, \ r_i)$ are very close to w_{ij}. We then use (2) again to infer that $\|T_j \llcorner \varphi_i\|\mathbb{R}^n$ is very close to $\|T \llcorner \varphi_i\|\mathbb{R}^n$.

(7) In case ϵ is very small, we infer from (6) that for each $i = 1, \ldots, M$ the varifolds

$$\mathbf{v}\big(S_j, \, \theta_j \, \varphi_i, \, \sigma_j\big) \quad \text{and} \quad \mathbf{v}\big(S, \, \theta \, \varphi_i, \, \sigma\big)$$

are weakly close for large j. Again, if ϵ is very small we infer from (3) and the fact that the restriction of F to $\mathbb{G}_0(n, \, k)$ is bounded above and below that the varifolds

$$\mathbf{v}\big(S_j, \, \theta_j \, \varphi_i, \, \sigma_j\big) \quad \text{and} \quad \mathbf{v}\big(S, \, \theta \, \varphi_i, \, \sigma\big)$$

are weakly close to 0 for large j. Since our choice of ϵ was arbitrary, we infer that

$$\mathbf{v}\big(S_j, \, \theta_j, \, \sigma_j\big) \rightharpoonup \mathbf{v}\big(S, \, \theta, \, \sigma_j\big) \quad \text{as } j \to \infty.$$

This is the assertion of Theorem C.3 ■

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