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3	Corresponding Author	Family Name	**Iovanov**
4		Particle	
5		Given Name	**Miodrag-Cristian**
6		Suffix	
7		Organization	University of Bucharest
8		Division	Faculty of Mathematics
9		Address	Str. Academiei 14, Bucharest RO-010014, Romania
10		e-mail	yovanov@walla.com
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The Splitting Problem for Coalgebras: A Direct Approach

Miodrag-Cristian Iovanov

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Abstract In this note we give a new and elementary proof of a result of Năstăsescu and Torrecillas (*J. Algebra*, 281:144–149, 2004) stating that a coalgebra C is finite dimensional if and only if the rational part of any right module M over the dual algebra C^* is a direct summand in M (the splitting problem for coalgebras).

Key words torsion theory · splitting · coalgebra

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Introduction

Let $\mathcal{C}$ be an Abelian category and $\mathcal{A}$ a closed subcategory of $\mathcal{C}$. Then we can define the torsion functor $\mathcal{T} : \mathcal{C} \to \mathcal{A}$ which takes every $X \in \mathcal{C}$ to the subobject $\mathcal{T}(X)$ of X that equals the sum of all subobjects of X that belong to $\mathcal{A}$; we say that $\mathcal{T}(X)$ is the $\mathcal{A}$-torsion part of X. Then the following general question naturally arises: when is the $\mathcal{A}$-torsion part $T(X)$ a direct summand in X for every object X in $\mathcal{C}$ (or in a subclass of $\mathcal{C}$). This is called the splitting problem of $\mathcal{C}$ with respect to $\mathcal{A}$. In the case of the category of modules $\mathcal{C} = {}_R\mathcal{M}$ over a commutative ring R one can consider the splitting problem with respect to the subcategory of all torsion modules; Kaplansky proves that the torsion part of every finitely generated module over a commutative domain R is a direct summand in that module if and only if R is Prüfer (see [5] and [6]) and Rotman [10] proves that if this happens for every R-module then R is a field.

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M.-C. Iovanov (✉)
Faculty of Mathematics, University of Bucharest,
Str. Academiei 14, RO-010014 Bucharest, Romania
e-mail: yovanov@walla.com

Other general results are proved by Teply (see [11–13]). A canonical subcategory of any category $\mathcal{C}$ is the Dickson subcategory, which is defined to be the smallest localizing subcategory of $\mathcal{C}$ that contains all simple subobjects of $\mathcal{C}$. This category coincides with the class of all semiartinian objects of $\mathcal{C}$. Then the splitting problem with respect to the Dickson subcategory of $\mathbb{C}$ is a general question that makes sense for any category $\mathcal{C}$. One can ask whether the splitting with respect to the Dickson subcategory implies that $\mathcal{C}$ actually coincides to this subcategory. In the case of the category of modules over an arbitrary ring R this is a classical open problem.

Let C be a coalgebra over a field k. The category of left (resp. right) C-comodules is a full subcategory of the category of right (resp. left) modules over the dual algebra C^* Năstăsescu and Torrecillas [8] have shown that the rational part of every right C^*-module M is a direct summand in M if and only if C is finite dimensional. In this case, the category of rational right C^*-modules is equal to the category of right C^*-modules, and also to the Dickson subcategory of $\mathcal{M}_{C^*}$.

The aim of this note is to give a new and elementary proof of this result, based on general results on modules and comodules, and an old result of Levitzki, stating that a nil ideal in a right noetherian ring is nilpotent. The proof of Năstăsescu and Torrecillas involve several techniques of general category theory (such as localization), some facts on linearly compact modules and is based on general nontrivial and profound results of Teply regarding the general splitting problem (see [11–13]). We first prove that if C has the splitting property, that is, the rational part of every right C^*-module is a direct summand, then C has only a finite number of isomorphism types of simple (left or right) comodules. We then observe that the injective envelope of every right comodule contains only finite dimensional proper subcomodules. This immediately implies that C^* is right noetherian. Then, using a quite common old idea from Abelian group theory we use the hypothesis for a direct product of modules to obtain that every element of J, the Jacobson radical of C^*, is nilpotent. Using a well known result in noncommutative algebra due to Levitzki, we conclude that J is nilpotent wich combined with the above mentioned key observation immediately yields that C is finite dimensional.

1 The Splitting Problem

We first fix some notations and conventions. Denote by ε the counit of the coalgebra C and by Δ its comultiplication. We use the Sweedler notation convention $\Delta(c) = c_1 \otimes c_2$ for $c \in C$ and the sum sign is omitted. For any $f \in C^*$, denote by $\overline{f} : C \to C$ the right comodule morphism $\overline{f}(x) = f(x_1)x_2$; then $\overline{f}$ is a morphism of right C comodules. As a key technique, we make use of the algebra isomorphism $C^* \simeq \mathrm{Hom}(C^C, C^C)$ given by $f \mapsto \overline{f}$ (with inverse $\alpha \mapsto \varepsilon \circ \alpha$), where $\mathrm{Hom}(C^C, C^C)$ is a ring with multiplication given by opposite composition. Also for a right C-comodule M we have an isomorphism $\mathrm{Hom}^C(M, C) \simeq M^*$, $f \mapsto \varepsilon \circ f$.

For a coalgebra C denote by C_0 the coradical of C. In what follows, we will assume that the coalgebra C has the splitting property for the right C^*-modules, that is, the rational part of every right C^* module is a direct summand in that module.

Lemma 1.1 *If T is a simple right comodule and $E(T)$ is the an injective envelope of T, then $E(T)$ contains only finite dimensional proper subcomodules.*

Proof Let $K \subsetneq E(T)$ be an infinite dimensional subcomodule. Then there is a subcomodule $K \subsetneq F \subset E(T)$ such that F/K is finite dimensional. We have an exact sequence of right C^*-modules:

$$0 \to (F/K)^* \to F^* \to K^* \to 0$$

As F/K is a finite dimensional rational left C^* module, $(F/K)^*$ is a rational right C^*-module; thus $A = \mathrm{Rat} F^* \neq 0$. Denote by $M = T^\perp = \{u \in F^* \mid u|_T = 0\} \subset F^*$. We first show that F^* is generated by any element $u \in F^* \setminus M$. Indeed for $u \in F^* \setminus M$; define $v \in \mathrm{Hom}^C(F, C)$ by $v(x) = u(x_1)x_2$, $\forall x \in F$. Then $v|_T \neq 0$ as $u = \varepsilon \circ v$ and $u \notin M$. We have that v is injective, because T is an essential submodule of $F \subseteq E(T)$ and $T \nsubseteq \mathrm{Ker}(v)$. As C is an injective right C comodule and v is injective, we have a commutative diagram:

$$\begin{array}{ccccc}
C^* & \xrightarrow{\;v^*\;} & F^* & \longrightarrow & 0 \\
\simeq \Big\| & & \simeq \Big\| & & \\
\mathrm{Hom}^C(C, C) & \xrightarrow{\mathrm{Hom}^C(v,C)} & \mathrm{Hom}^C(F, C) & \longrightarrow & 0
\end{array}$$

where the vertical lines are isomorphisms. We see that $\mathrm{Hom}^C(F, C)$ is generated by v as $\mathrm{Hom}^C(C, C) \simeq C^*$ is generated by 1_C, following that F^* is generated by $v^*(\varepsilon) = \varepsilon \circ v = u$. Now if $F^* = A \oplus B$ as right C^*-modules, we see that A is finitely generated as F^* is, so it must be finite dimensional because it is a rational right C^*-module. Thus $A \neq F^*$ by the initial assumption. But now note that $A \subseteq M$, as otherwise if there is $a \in A \setminus M$ and as a generates F^* we would have $A = F^*$. Also $B \neq F^*$ because $A \neq 0$ so by the same argument $B \subseteq M$, and therefore $F^* = A + B \subseteq M$ which is a contradiction ($\varepsilon|_F \notin M$). □

Let $(S_i)_{i \in I}$ be the set of representatives for the simple left comodules. We may assume that each S_i is a subcomodule of C (see for example [3, 2.4.14]). Let $C_i = \sum\limits_{S \subseteq C, S \simeq S_i} S$. Note that C_i is a left subcomodule as well as a right subcomodule (thus a subcoalgebra) of C. Then $C_0 = \sum\limits_{i \in I} C_i$ because the S_i's form a complete set of representatives for the simple left C-comodules and C_0 is essential in C (see, for example, [3, 2.4.12]). Also the sum is direct because the S_i's are pairwise non-isomorphic. Let E_i be an injective envelope of the right comodule C_i; then $C = \bigoplus\limits_{i \in I} E_i$ (see [3, 2.4.16]). We can identify C^* with the direct product $\prod\limits_{i \in I} E_i^*$ where each E_i^* is identified with the set of all elements of C^* that are zero on all E_j's with $j \neq i$. Note that for $c^* = ((c_i^*)_{i \in I}) \in C^*$ and $c_j \in C_j$ we have $c_{j1} \otimes c_{j2} \in C_j \otimes C_j$ and then $c_j \cdot c^* = c^*(c_{j1})c_{j2} = \sum\limits_{i \in I} c_i^*(c_{j1})c_{j2} = c_j^*(c_{j1})c_{j2} = c_j \cdot c_j^*$.

Recall that a coalgebra C is almost connected if C_0 is finite dimensional. As the right comodule C is quasifinite, this is equivalent to the fact that there is only a finite number of types of simple right comodules.

Proposition 1.2 *Let C be a coalgebra such that the rational part of every right C^*-module splits off. Then C is almost connected.*

Proof Consider $M = \prod_{i \in I} S_i$ and take $x = (x_i)_{i \in I} \in M$, such that $x_i \neq 0$, $\forall i \in I$. If $y = (y_i)_{i \in I} \in M$ then for each i we have $S_i = x_i \cdot C^*$ as $x_i \neq 0$ and S_i is simple, so there is $c_i^* \in C^*$ such that $x_i \cdot c_i^* = y_i$. Because $x_i \in C_i$, by the previous considerations we may assume that $c_i^* \in E_i^*$ (that is, it equals zero on all the components of the direct sum decomposition of C except on E_i) and then there is $c^* \in C^*$ with $c^*|_{E_i} = c_i^*|_{E_i}$ ($c^* = (c_i^*)_{i \in I}$). Then one can easily see that $x_i \cdot c^* = x_i \cdot c_i^* = y_i$, thus we may extend this to $x \cdot c* = y$ showing that actually $M = x \cdot C^*$. We have that $Rat(M)$ is a direct summand in M so it must be finitely generated because M is, so $Rat(M)$ must be finite dimensional. But $\bigoplus_{i \in I} S_i \subseteq Rat(\prod_{i \in I} S_i)$, and this shows that I must be finite, equivalently, C is almost connected. □

Corollary 1.3 *C^* is a right noetherian ring.*

Proof Let T be a right simple comodule, $E(T) \subseteq C$ an injective envelope of T and $C = E(T) \oplus X$ as right C comodules. If $0 \neq I$ is a right C^*-submodule of $E(T)^*$, then for $0 \neq f \in I$ put $K = \mathrm{Ker}\,\overline{f}$. We have $K^\perp = \{g \in E(T)^* \mid g|_K = 0\} \supseteq f \cdot C^*$, as $f \in K^\perp$. Conversely, if g is 0 on K, then $K \subseteq \mathrm{Ker}\,\overline{g}$ as K is a right C subcomodule of $E(T)$ and therefore it factors through $\overline{f}$, that is, $\exists \alpha \in \mathrm{Hom}(C^C, C^C)$ such that $\overline{g} = \alpha \overline{f} = \overline{h} \circ \overline{f} = \overline{fh}$ for $h = \varepsilon \circ \alpha$, so $g = f \cdot h \in f \cdot C^*$. This shows that $K^\perp \subseteq f \cdot C^*$, so $K^\perp = f \cdot C^*$. As K is finite dimensional by Lemma 1.1, $K^\perp$ has finite codimension in $E(T)^*$, showing that I has finite codimension ($I \supseteq f \cdot C^* = K^\perp$), which obviously shows that $E(T)^*$ is noetherian. If $C_0 = \bigoplus_{i \in F} T_i$ with T_i simple right comodules then F is finite by Proposition 1.2. Therefore, if for each $i \in F$ $E(T_i)$ is an injective envelope of T_i contained in C, then $C^* = \bigoplus_{i \in F} E(T_i)^*$ as right C^*-modules so $C^*_{C^*}$ is noetherian as each $E(T_i)^*$ is. □

Put $R = C^*$. Note that $J = C_0^\perp = \{f \mid f|_{C_0} = 0\}$ is the Jacobson radical of R and $\bigcap_{n \in \mathbb{N}} J^n = 0$. Also if M is a finite dimensional right R-module, we have $M \cdot J^n = 0$ for some n, because the descending chain of submodules $(MJ^n)_n$ must be stationary and therefore $MJ^n = MJ^{n+1} = MJ^n \cdot J$ implies $MJ^n = 0$ by Nakayama lemma.

Proposition 1.4 *Any element $f \in J$ is nilpotent.*

Proof As C is a finite direct sum of injective envelopes of simple right comodules, it is enough to show that $f^n|_{E(T)} = 0$ for some n for each simple right subcomodule of C and injective envelope $E(T) \subseteq C$. Assume the contrary for some fixed data T, $E(T)$. Let X be a right subcomodule of C such that $C = E(T) \oplus X$ as right C comodules. As $C^* \simeq E(T)^* \oplus X^*$, we identify the any element f of $E(T)^*$ with the element of C^* equal to f on $E(T)$ and 0 on X. Define

$$M = \prod_{n \geq 1} \frac{E(T)^*}{K_n^\perp}$$

where $K_n = \mathrm{Ker}\,\overline{f^n} \cap E(T) \neq E(T)$ (because otherwise $f^n = 0$) and $K_n^\perp = \{g \in E(T)^* \mid g|_{K_n} = 0\}$. For simplicity, if $f \in E(T)^*$ we convey to write f for the element

$f + K_n^{\perp}$, the image of f in $E(T)^*/K_n^{\perp}$. Note that $K_n \subseteq K_{n+1}$. Put $\lambda = (f^{[n/2]})_{n\geq 1} \in M$ where $[x]$ is the smallest integer greater or equal to x. We have:

$$\lambda = (f, f^2, f^2, \dots, f^n, f^n, 0, \dots) + (0, 0, \dots, 0, f^{n+1}, f^{n+1}, f^{n+2}, \dots) = r_n + \mu_n \cdot f^n$$

with $r_n = (f, f^2, f^2, \dots, f^n, f^n, 0, \dots, 0\dots)$ and $\mu_n = (0, 0, \dots, 0, f, f, f^2, \dots)$ (the morphisms are always thought to be 0 on X and they are considered modulo $K_n^{\perp}$). But then $r_n \in \prod\limits_{p\leq n} E(T)^*/K_p^{\perp} \times 0$ which is a rational left C comodule because $E(T)^*/K_p^{\perp} \simeq K_p^*$ and K_p is finite dimensional by Lemma 1.1. Write $M = Rat(M) \oplus \Lambda$ as right R modules and $\mu_n = q_n + \alpha_n$ with $q_n \in Rat(M)$ and $\alpha_n \in \Lambda$. Then if $\lambda = r + \mu$ with $r \in Rat(M)$ and $\mu \in \Lambda$ we have $r + \mu = r_n + \mu_n \cdot f^n = (r_n + q_n \cdot f^n) + \alpha_n \cdot f^n$ which shows that $\mu = \mu_n \cdot f^n$. Then if $\mu = (l_p)_{p\geq 1}$ and $\mu_n = (\mu_{n,p})_{p\geq 1}$ we get that $l_p = \mu_{n,p} \cdot f^n \in (\frac{E(T)^*}{K_p^{\perp}}) \cdot J^n$ for all p. By the previous remark, $(\frac{E(T)^*}{K_p^{\perp}}) \cdot J^n = 0$ for some n (which depends on p) and this shows that $l_p = 0$ for any p and thus $\mu = 0$. Therefore $\lambda \in Rat M$, so $\lambda \cdot R$ is finite dimensional and again we get $\lambda \cdot RJ^n = 0$ for some n. Hence we get $f^{[p/2]+n} = 0$ in $E(T)^*/K_p^{\perp}$ so $f^{[p/2]+n}|_{K_p} = 0$, $\forall p$, equivalently $\overline{f}^{[p/2]+n} = 0$ on K_p (because K_p is a right comodule). For $p = 2n + 1$ we therefore obtain $K_{2n+1} \subseteq K_{2n}$ so $K_m = K_{m+1}$ for $m = 2n$. Then if $I = Im(\overline{f}^m)$, $I \neq 0$ by the assumption ($K_m \neq E(T)$) and there is a simple subcomodule T' of I; then $\overline{f}|_{T'} = 0$ (because $f \in J = C_0^{\perp}$). Take $0 \neq y \in T'$; then $y = \overline{f}^m(x)$, $x \in E(T)$ and $0 = \overline{f}(y) = \overline{f}^{m+1}(x)$ showing that $x \in K_{m+1} = K_m$ and therefore $y = \overline{f}^m(x) = 0$, a contradiction. □

Theorem 1.5 *If the rational part of every right C^*-module splits off, then C is finite dimensional.*

Proof For a right C-comodule M denote by $l_n(M)$ the n-th term in the Loewy series of the comodule M. We first show that $C_n = l_n(C)$ is finite dimensional for all n. We proceed by induction on n; for $n = 0$ this is Proposition 1.2. Assume the statement for $0, 1, \dots, n-1$. Write $C_0 = \bigoplus\limits_{i\in F} T_i$ with T_i simple right C-comodules and $C = \bigoplus\limits_{i\in F} E(T_i)$ with $E(T_i)$ injective envelopes of the T_i's. We know that the set F is finite by Proposition 1.2. We have $C_n = \bigoplus\limits_{i\in F} l_n(E(T_i))$ so it is enough to show that $l_n(E(T_i))$ is finite dimensional for all $i \in F$. If otherwise, we would have $l_n(E(T_i)) = E(T_i)$ by Proposition 1.1. But then one can write $E(T_i)/l_{n-1}(E(T_i)) = l_n(E(T_i))/l_{n-1}(E(T_i)) = T \oplus K$ with T simple finite dimensional, so K must be infinite dimensional because $l_{n-1}(E(T_i))$ is finite dimensional by the induction hypothesis. In this way we can find an infinite dimensional proper subcomodule of $E(T_i)$ corresponding to K which is impossible again by Proposition 1.1.

By Corollary 1.3 C^* is right noetherian and by Proposition 1.4 every element of J is nilpotent. Therefore by Levitzki's Theorem (see for example [4, p. 199, Theorem 1] or [9, Cor. II.4.1.5]) we have that J is nilpotent, so $J^n = 0$ for some n. But $C_{n-1} = (J^n)^{\perp} = \{x \in C \mid h(x) = 0 \ \forall h \in J^n\}$ by [3, Cor. 3.1.10], so $C_{n-1} = C$ and therefore C is finite dimensional. □

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