SOME EXAMPLES IN MODULES

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Abstract The problem of when the direct product and direct sum of modules are isomorphic is discussed. A series of examples where the product and coproduct of an infinite family of modules are isomorphic is given. One may see that if we require that the isomorphism of $\prod\limits_I$ and $\bigoplus\limits_I$ be a natural (functorial) one, then this can only be done for finite sets I. If this is the case for modules, we show that for comodules over a coalgebra the product and coproduct of a family of comodules can be isomorphic even via the canonical morphism.

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1. INTRODUCTION

Given a family $(M_i)_{i\in I}$ of (left) R modules, we consider the problem of when the direct sum and direct product of this family are isomorphic. It is obvious that if we require that the isomorphism is the canonic isomorphism, then we can easily see that the set must be finite, unless all but a finite part of the modules are 0. Nevertheless we may ask wether the direct product and direct sum of a module can be isomorphic via other isomorphisms. We produce a large class of examples that show that this is possible, so the direct product, by the categorical point of view (classification of modules) is not necessarily very different of the direct sum. We also show that in the case of categories other than categories of modules, namely for comodules over a coalgebra, the direct sum and direct product of a family of objects can be isomorphic even through the canonical morphism.
We can ask the more general question of when the two functors $\prod\limits_{i\in I}$ and $\coprod\limits_{i\in I}$ from the direct product category ${}_R\mathcal{M}^I$ to ${}_R\mathcal{M}$ are isomorphic. It can be shown (not very difficult) that this is only possible for finite sets I.

2. EXAMPLES AND RESULTS

Example 2.1. *Let K be a field and let $(V_i)_{i\in I}$ be vector spaces over K and $V=\prod_{i\in I} V_i$. Then the direct product and direct sum of the family $\{V\}\cup\{V_i \mid i\in I\}$ are isomorphic.*

Proof Denote by A_i a set consisting of a basis for V_i for each $i\in I$ and let A be a basis for V. Then $\bigsqcup_{i\in I} A_i \sqcup A$ is a basis for the coproduct $\bigoplus_{i\in I} V_i \oplus V$ and $A\sqcup A$ is a basis for the direct product $\prod_{i\in I} V_i\times V = V\times V$. But $\mathrm{card}(\bigsqcup_{i\in I} A_i)\leq \mathrm{card}(A)$ because of the natural inclusion of vector spaces $\bigoplus_{i\in I} A_i \hookrightarrow \prod_{i\in I} A_i$ so we have

$$\mathrm{card}(A)\leq \mathrm{card}(\bigsqcup_{i\in I} A_i\sqcup A)\leq \mathrm{card}(A\sqcup A)=\mathrm{card}(A),$$

which shows the desired isomorphism. ∎

Example 2.2. *Let A be a simple Artinian ring, that is, $A\simeq M_n(\Delta)$, with Δ a skewfield. If $(N_i)_{i\in I}$ is a family of A modules and $N=\prod_{i\in I} N_i$, then the direct product and direct sum of the family $(N)\cup(N_i)_{i\in I}$ are isomorphic.*

Proof Let S denote a simple module. As any module is semisimple isomorphic to a direct sum of copies of S (A is semisimple with a single type of simple module), in order for two modules $N\simeq S^{(\alpha)}$ and $M\simeq S^{(\beta)}$ to be isomorphic it is necessary and sufficient for the sets α and β be of the same (infinite) cardinal (by Krull-Remak-Schmidt-Azumaya theorem). Let A_i and A be sets such that $N_i\simeq S^{(A_i)}$ $(\forall i)$ and $N\simeq S^{(A)}$. Then $N\oplus\bigoplus_{i\in I} N_i\simeq S^{(A)}\oplus\bigoplus_{i\in I} S^{(A_i)}\simeq S^{(\alpha)}$ with $\alpha=A\sqcup\bigsqcup_{i\in I} A_i$ and $N\times\prod_{i\in I} N_i\simeq N\times N\simeq N^{(\beta)}$ with $\beta=A\sqcup A$. Using an argument similar to the one in Example 2.1 we obtain that α and β are of the same cardinal and so $N\oplus\bigoplus_{i\in I} N_i\simeq N\times\prod_{i\in I} N_i$. ∎

A ring is said to have finite representation type if there are only finitely many non-isomorphic indecomposable modules. It is known that any module over an Artinian finite representation type ring is a direct sum of indecomposable modules. For algebras the converse is also true; in fact for an Artin algebra (a finite length algebra over a commutative Artinian ring) the following are equivalent:
-every module is a direct sum of finitely generated indecomposable modules;
-there is only a finite number of nonisomorphic finitely generated indecomposable modules;
-every indecomposable module is finitely generated.
Moreover, these statements are left right symetric, that is, the statement for

left modules is equivalent to the one for right modules. We refer to [8], [1], [5], [3] for these facts. For modules over such algebras we can prove a result that gives a large class of examples of isomorphic direct sum and direct product of modules.

Theorem 2.1. *Let A be a (left) Artinian ring with the property that every module decomposes as a direct sum of indecomposable finitely generated modules (for example, A a finite representation type Artin algebra). Then for every family of (left) A modules $(M_n)_{n\in\mathbb{N}}$ there is a module M such that the direct product and direct sum of the family $(M)\cup(M_n)_{n\in\mathbb{N}}$ are isomorphic.*

Proof Let $\{H_j \mid j\in J\}$ be a set of representatives of indecomposable finitely generated A modules (one can see that actually this is a set, not a class!). For every A module M we have a unique decomposition in the sense of Krull-Schmidt decomposition theorem $M=\bigoplus\limits_k M_k$ where all M_k are isomorphic to one of the H_j's (as the generalized Krull-Remak-Schmidt-Azumaya theorem applies, because the endomorphism rings of finitely generated modules over Artinian rings - which are finite length modules - are local). Denote by $\alpha_j(M)$ the 'exponent' of H_j in M, that is a set (cardinal) such that $M\simeq H_j^{\alpha_j(M)}\oplus\bigoplus\limits_{l\in L} M_l$ and M_l not isomorphic to H_j, $\forall\, l\in L$. Then $M\simeq N$ iff $\alpha_j(M)\sim\alpha_j(N)$, $\forall\, j\in J$. By Krull-Schmidt theorem, $\alpha_j(\bigoplus\limits_{l\in L} M_l)\sim\bigsqcup\limits_{l\in L}\alpha_j(M_l)$. For every family $(M_n)_{n\in\mathbb{N}}$ of A modules, let $K=\{j\in J\mid \mathrm{card}(\alpha_j(\bigoplus\limits_{n\in\mathbb{N}} M_n))<\aleph_0\}$ and $M'=\bigoplus\limits_{j\in K} H_j^{(\mathbb{N})}$. Take $M=\prod\limits_{n\in\mathbb{N}} M_n\times M'$. We have $\mathrm{card}(\alpha_j(M_n))\le\mathrm{card}(\alpha_j(M))$ as each M_n is a direct summand in M (by Krull-Schmidt). Notice that $\alpha_j(M)$ is infinite for all j: if $\alpha_j(\bigoplus\limits_{n\in\mathbb{N}} M_n)$ is infinite then $\mathrm{card}(\alpha_j(M_n))$ is nonzero for infinitely many n's, say for all $n\in P$ and so for every finite set $F\subset P$, $\bigoplus\limits_{n\in F} M_n$ is a direct summand in M, showing that $\mathrm{card}(\alpha_j(M))\ge\mathrm{card}(\alpha_j(\bigoplus\limits_{n\in F} M_n))\ge\mathrm{card}(F)$. If $\alpha_j(\bigoplus\limits_{n\in\mathbb{N}} M_n)$ is finite then M contains $H_j^{(\mathbb{N})}$ from M'. Then

$$\mathrm{card}(\alpha_j(M))\le\mathrm{card}(\alpha_j(\bigoplus_{n\in\mathbb{N}} M_n\oplus M))=\mathrm{card}(\bigsqcup_{n\in\mathbb{N}}\alpha_j(M_n)\sqcup\alpha_j(M))$$

$$\le\mathrm{card}(\bigsqcup_{n\in\mathbb{N}}\alpha_j(M)\sqcup\alpha_j(M))=\mathrm{card}(\alpha_j(M)\times\mathbb{N})=\mathrm{card}(\alpha_j(M)).$$

On the other hand we have

$$\mathrm{card}(\alpha_j(M))\le\mathrm{card}(\alpha_j(\prod_{n\in\mathbb{N}} M_n\times M))\le\mathrm{card}(\alpha_j(M\oplus M))$$

$$= \mathrm{card}(\alpha_j(M) \sqcup \alpha_j(M)) = \mathrm{card}(\alpha_j(M)).$$

■

Thus we obtain $\alpha_j(\bigoplus_{n\in\mathbb{N}} M_n \oplus M) \sim \alpha_j(\prod_{n\in\mathbb{N}} M_n \times M)$, so the theorem is proved. Here we have used some well known facts from set theory, such as $a+a=a$ and $a\times\aleph_0=\aleph_0$ for every transfinite cardinal a, which can be found, for example, in [11].

We provide now an example from comodule theory, where the canonical isomorphism from the direct sum to the direct product will be an isomorphism, but with infinite index set. We refer to [6] for basic facts about coalgebras and comodules over coalgebras.

Proposition 2.1. *Let $C = \bigoplus_{i\in I} C_i$ be a cosemisimple coalgebra, with C_i simple coalgebras. Then the canonical morphism from $\bigoplus_{i\in I} C_i$ to $\prod^{C}_{i\in I} C_i$ is an isomorphism (here C_i are right C comodules, and $\prod^{C}$ denotes the direct product in the category $\mathcal{M}^C$ of right comodules).*

Proof It is known (and easy to see) that the direct product of the family $(M_l)_{l\in L}$ of comodules is $\mathrm{Rat}^C(\prod_{l\in L} M_l)$, where $\prod$ represents the direct product of left C^* modules. We also have that $\mathrm{Rat}^C(C^*) = \mathrm{Rat}^C(\prod_{i\in I} C_i^*) = \bigoplus_{i\in I} C_i^*$ (this is true in a more general setting, for a left and right semiperfect coalgebra; see [6], Chapter III). If we denote by S_i the left simple comodule type associated to C_i (that is, a simple left comodule included in C_i) and T_i a right simple C_i module, then we have $T_i \simeq S_i^*$ in $\mathcal{M}^{C_i}$ and then also in $\mathcal{M}^C$ (because there is only a single type of simple left(right) comodule). Also $C_i \simeq S_i^n$ in ${}^{C_i}\mathcal{M}$ and $C_i \simeq T_i^m$ in $\mathcal{M}^{C_i}$ with $m=n$ because $T_i \simeq S_i^*$ implies that S_i and T_i have the same (finite!) dimension. We obtain that $C_i^* \simeq (S_i^*)^n \simeq T_i^n \simeq C_i$ in $\mathcal{M}^{C_i}$ and also in $\mathcal{M}^C$. Therefore we obtain

$$\prod^{C}_{i\in I} C_i = \mathrm{Rat}^C(\prod_{i\in I} C_i) \simeq \mathrm{Rat}^C(\prod_{i\in I} C_i^*) = \bigoplus_{i\in I} C_i^* \ (\mathrm{in}\, \mathcal{M}^C) = \bigoplus_{i\in I} C_i$$

and it is easy to see that the isomorphism is the canonical morphism from the direct sum into the direct product. ■

Remark 2.1. *The above example can be generalized to a more general case, namely for comodules over for (left and right) co-Frobenius coalgebras.*

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