

GENERALIZED FROBENIUS ALGEBRAS AND HOPF ALGEBRAS

MIODRAG CRISTIAN IOVANOV

ABSTRACT. "Co-Frobenius" coalgebras were introduced as dualizations of Frobenius algebras. Recently, it was shown in [I] that they admit left-right symmetric characterizations analogue to those of Frobenius algebras: a coalgebra C is co-Frobenius if and only if it is isomorphic to its rational dual. We consider the more general quasi-co-Frobenius (QcF) coalgebras; in the first main result we show that these also admit symmetric characterizations: a coalgebra is QcF if it is weakly isomorphic to its (left, or equivalently right) rational dual $Rat(C^*)$, in the sense that certain coproduct or product powers of these objects are isomorphic. These show that QcF coalgebras can be viewed as generalizations of both co-Frobenius coalgebras and Frobenius algebras. Surprisingly, these turn out to have many applications to fundamental results of Hopf algebras. The equivalent characterizations of Hopf algebras with left (or right) nonzero integrals as left (or right) co-Frobenius, or QcF, or semiperfect or with nonzero rational dual all follow immediately from these results. Also, the celebrated uniqueness of integrals follows at the same time as just another equivalent statement. Moreover, as a by-product of our methods, we observe a short proof for the bijectivity of the antipode of a Hopf algebra with nonzero integral. This gives a purely representation theoretic approach to many of the basic fundamental results in the theory of Hopf algebras. Furthermore, the results on coalgebras allow the introduction of a general concept of Frobenius algebra, which makes sense for infinite dimensional and topological algebras, and specializes to the classical notion in the finite case: this will be a topological algebra A which is isomorphic to its complete topological dual A^\vee . We give many examples of co-Frobenius coalgebras and Hopf algebras connected to category theory, homological algebra and the newer q-homological algebra, topology or graph theory, showing the importance of the concept.

INTRODUCTION

A K algebra A over a field K is called Frobenius if A is isomorphic to A^* as right A -modules. This is equivalent to there being an isomorphism of left A -modules between A and A^* . This is the modern algebra language formulation for an old question posed by Frobenius. Given a finite dimensional algebra with a basis x_1, \dots, x_n , the left multiplication by an element a induces a representation $A \mapsto End_K(A) = M_n(K)$, $a \mapsto (a_{ij})_{i,j}$ ($a_{ij} \in K$), where $a \cdot x_i = \sum_{j=1}^n a_{ij} x_j$. Similarly, the right multiplication produces a matrix a'_{ij} by writing $x_i \cdot a = \sum_{j=1}^n a'_{ji} x_j$, $a'_{ij} \in K$, and this induces another representation $A \ni a \mapsto (a'_{ij})_{i,j}$. Frobenius' problem came as the natural question of when the two representations are equivalent. Frobenius algebras occur in many different fields of mathematics, such as topology (the cohomology ring of a compact oriented manifold with coefficients in a field is a Frobenius algebra by Poincaré duality), topological quantum field theory (there is a one-to-one correspondence between 2-dimensional quantum field theories and

Key words and phrases. coalgebra, Hopf algebra, integral, Frobenius, QcF, co-Frobenius.
2010 *Mathematics Subject Classification.* 16T15, 18G35, 16T05, 20N99, 18D10, 05E10.

commutative Frobenius algebras; see [Ab]), Hopf algebras (a finite dimensional Hopf algebra is a Frobenius algebra), and Frobenius algebras have subsequently developed into a research subfield of algebra.

Co-Frobenius coalgebras were first introduced by Lin in [L] as a dualization of Frobenius algebras. A coalgebra is left (right) co-Frobenius if there is a monomorphism of left (right) C^* -modules $C \subseteq C^*$. However, unlike the algebra case, this concept is not left-right symmetric, as an example in [L] shows. Nevertheless, in the case of Hopf algebras, it was observed that left co-Frobenius implies right co-Frobenius. Also, a left (or right) co-Frobenius coalgebra can be infinite dimensional, while a Frobenius algebra is necessarily finite dimensional. Co-Frobenius coalgebras are coalgebras that are both left and right co-Frobenius. It recently turned out that this notion of co-Frobenius has a nice characterization that is analogue to the characterizations of Frobenius algebras and is also left-right symmetric: a coalgebra C is co-Frobenius if it is isomorphic to its left (or equivalently to its right) rational dual $\text{Rat}_C({}^*C^*)$ (equivalently $C \simeq \text{Rat}(C_{C^*}^*)$; see [I]). This also allowed for a categorical characterization which is again analogue to a characterization of Frobenius algebras: an algebra A is Frobenius iff the functors $\text{Hom}_A(-, A)$ ("the A -dual functor") and $\text{Hom}_K(-, K)$ ("the K -dual functor") are naturally isomorphic. Similarly, a coalgebra is co-Frobenius if the C^* -dual $\text{Hom}_{C^*}(-, C^*)$ and the K -dual $\text{Hom}_K(-, K)$ functors are isomorphic on comodules. If a coalgebra C is finite dimensional then it is co-Frobenius if and only if C is Frobenius, showing that the co-Frobenius coalgebras (or rather their dual) can be seen as the infinite dimensional generalization of Frobenius algebras. One very important example is in again in the topological situation: the homology of a compact oriented manifold M admits a coalgebra structure which is dual to that of the algebra in co-homology, and it becomes a co-Frobenius coalgebra. For any topological space it can actually be described as follows: take an n -simplex $\sigma : [0, 1, \dots, n] \rightarrow M$ in $H_*(M)$ and introduce the comultiplication $\Delta(\sigma) = \sum_{i=0}^n \sigma|_{[0,1,\dots,i]} \otimes \sigma|_{[i,\dots,n]}$, which in fact induces at homology level, that is, the formula can be introduced for cohomology classes. Hence, we can then consider the convolution product $*$ on the dual algebra $((H_*(M))^*, *)$, and then by the definition of the cup product the natural map $(H_*(M), \cup) \rightarrow ((H_*(M))^*, *)$, $c \mapsto (\sigma \mapsto c(\sigma))$ obviously becomes a morphism of rings: $(c*d)(\sigma) = \sum \sum_{\sigma} c(\sigma_1)d(\sigma_2) = \sum_{i=0}^n c(\sigma|_{[0,1,\dots,i]})d(\sigma|_{[i,\dots,n]}) = (c \cup d)(\sigma)$. In the case when M is a compact oriented manifold, this is an isomorphism turning $H_*(M)$ into a co-Frobenius coalgebra.

Quasi-co-Frobenius (QcF) coalgebras were introduced in [GTN] (further investigated in [GMN]), as a natural dualization of quasi-Frobenius algebras (QF algebras), which are algebras that are self-injective, cogenerators and artinian to the left, equivalently, all these conditions to the right. However, in order to allow for infinite dimensional QcF coalgebras (and thus obtain more a general notion), the definition was weakened to the following: a coalgebra is said to be left (right) QcF if it embeds in $\coprod_I C^*$ (a direct coproduct of copies of C^*) as left (right) C^* -modules. These coalgebras were shown to bear many properties that were the dual analogue of the properties of QF algebras. Again, this turned out not to be a left-right symmetric concept, and QcF coalgebras were introduced to be the coalgebras which are both left and right QcF. Our first goal is to note that the results and

techniques of [I] can be extended and applied to obtain a symmetric characterization of these coalgebras. In the first main result we show that a coalgebra is QcF if and only if C is "weakly" isomorphic to $\text{Rat}(C^*C^*)$ as left C^* -modules, in the sense that some (co)product powers of these objects are isomorphic, and this is equivalent to asking that C^* is "weakly" isomorphic to $\text{Rat}(C_{C^*}^*)$ (its right rational dual) as right C^* -modules. In fact, it is enough to have an isomorphism of countable powers of these objects. This also allows for a nice categorical characterization, which states that C is QcF if and only if the above C^* -dual and K -dual functors are (again) "weakly" isomorphic. Besides realizing QcF coalgebras as a left-right symmetric concept which is a generalization of both Frobenius algebras, co-Frobenius co-algebras and co-Frobenius Hopf algebras, we note that this also provides this characterization of finite dimensional quasi-Frobenius algebras: A is QF iff A and A^* are weakly isomorphic in the above sense, equivalently, $\coprod_{\mathbb{N}} A \simeq \coprod_{\mathbb{N}} A^*$.

Thus these results give a nontrivial generalization of Frobenius algebras and of quasi-Frobenius algebras, and the algebras arising as dual of QcF coalgebras are entitled to be called Generalized Frobenius Algebras, or rather Generalized QF Algebras.

These turn out to have a wide range of applications to Hopf algebras. In the theory of Hopf algebras, some of the first fundamental results were concerned with the characterization of Hopf algebras having a nonzero integral. These are in fact generalizations of well known results from the theory of compact groups. Recall that if G is a (locally) compact group, then there is a unique left invariant (Haar) measure and an associated integral \int . Considering the algebra $R_c(G)$ of continuous representative functions on G , i.e. functions $f : G \rightarrow \mathbb{R}$ such that there are $f_i, g_i : G \rightarrow K$ for $i = 1, n$ with $f(xy) = \sum_{i=1}^n f_i(x)g_i(y)$, then this becomes a Hopf algebra with multiplication given by the usual multiplication of functions, comultiplication given by $f \mapsto \sum_{i=1}^n f_i \otimes g_i$ and antipode S given by the composition with the taking of inverses $S(f)(x) = f(x^{-1})$. Then, the integral \int of G restricted to $R_c(G)$ becomes an element of $R_c(G)^*$ that has the following property: $\alpha \cdot \int = \alpha(\mathbf{1}) \int$, with $\mathbf{1}$ being the constant 1 function. Such an element in a general Hopf algebra is called a left integral, and Hopf algebras (quantum groups) having a nonzero left integral can be viewed as ("quantum") generalizations of compact groups (the Hopf Algebra can be thought of as the algebra of continuous representative functions on some abstract quantum space). Among the first the fundamental results in Hopf algebras was (were) the fact(s) that the existence of a left integral is equivalent to the existence of a right integral, and these are equivalent to the (co)representation theoretic properties of the underlying coalgebra of H of being left co-Frobenius, right co-Frobenius, left (or right) QcF, or having nonzero rational dual. These were results obtained in several initiating research papers on Hopf algebras [LS, MTW, R, Su, Sw1]. Then the natural question of whether the integral in a Hopf algebra is unique arose (i.e. the space of left integrals \int_l or that of right integrals \int_r is one dimensional), which would generalize the results from compact groups. The answer to this question turned out positive, as it was proved by Sullivan in [Su]; alternate proofs followed afterwards (see [Ra, St]). Another very important result is that of Radford, who showed that the antipode of a Hopf algebra with nonzero integral is always bijective.

We re-obtain all these results as a byproduct of our co-representation theoretic results and generalizations of Frobenius algebras; they will turn out to be an easy application of

these general results. We also note a very short proof of the bijectivity of the antipode by constructing a certain derived comodule structure on H , obtained by using the antipode and the so called distinguished grouplike element of H , and the properties of the comodule H^H . The only way we need to use the Hopf algebra structure of H is through the classical Fundamental theorem of Hopf modules which gives an isomorphism of H -Hopf modules $\int_l \otimes H \simeq \text{Rat}(H^*H^*)$; however, we will only need to use that this is an isomorphism of comodules. We thus find almost purely representation theoretic proofs of all these classical fundamental results from the theory of Hopf algebras, which become immediate easy applications of the more general results on the "generalized Frobenius algebras". Thus, the methods and results in this paper are also intended to emphasize the potential of these representation theoretic approaches.

The next goal of this paper is to give a new interpretation and provide new understanding of the notion of QcF coalgebra, namely, as the dual of a naturally defined notion of generalized (quasi-)Frobenius algebra. This will show that the duality between finite dimensional (quasi-)Frobenius algebras and finite dimensional (quasi-co-)Frobenius coalgebras can be fully extended to the infinite dimensional situation and can be understood in this new generality. This will be a consequence of the above described symmetric characterizations of QcF coalgebras. More precisely, given a topological algebra A whose topology has a basis of neighborhoods of 0 consisting of two-sided ideals of finite codimension (which we call AT-algebra), it will be called generalized Frobenius if A is isomorphic as a left topological module to a certain continuous dual A^\vee of A . Algebras with this kind of topology occur whenever one is interested in only a certain particular class of finite dimensional representations of an arbitrary algebra. In particular, any algebra can be thought as an AT-algebra by endowing it with the topology in which all cofinite ideals are open. It turns out that these generalized Frobenius algebras are exactly algebras which are the dual of some co-Frobenius coalgebra. Moreover, they also have categorical characterizations which parallel those of finite dimensional Frobenius algebras, and of co-Frobenius coalgebras.

In the last section, we give many examples to show that Frobenius and quasi-co-Frobenius coalgebras appear from many different mathematical situations and form a wide class. Also, some of these provide interesting examples of quantum groups. One very important example comes from homological algebra and from the generalized q-homological algebra ([Kap96]; see [M42a, M42b] for the topological origins): the category of n -chain complexes (that is, representations of the line quiver $\cdots \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots$ but with condition that the n 'th power of the differential is 0) is equivalent to the category of left comodules over a co-Frobenius coalgebra Λ_n , which is in fact also a Hopf algebra. (see [Par81] for the $n = 2$ case; also [B]). This makes the category of these n -chain complexes a monoidal category (and rigid if we restrict to finite dimensional ones), and gives a way to explain, for example, the total complex of the tensor product bicomplex as the internal tensor product in the category, or the total complex of the Hom-complex as the internal Hom in the monoidal category (see [W, 2.7]). Other examples of categories of comodules over a co-Frobenius coalgebra include representations of a cyclic quiver $1 \rightarrow 2 \rightarrow \cdots \rightarrow p \rightarrow 1$, but with the condition that a certain fixed number m of compositions of the morphisms (arrows) in the representation yields 0. Under some conditions (m divides n), this is also a Hopf algebra. In particular, these are a class of quantum groups which generalize Taft algebras. Another example with connections to homological algebra is that of the category of double chain-complexes, or generalizations to m, n -double chain complexes, with $d^n = 0$ on the

horizontal and $d^m = 0$ on the vertical: these categories are equivalent to the category of left comodules over $\Lambda_m \otimes \Lambda_n$. Many examples can be build from (finite or infinite) graphs, where sometimes easy combinatorics can be employed to decide whether the coalgebra is co-Frobenius or QcF. Finally, in many situations (for example, over algebraically closed fields), tensor products of co-Frobenius coalgebras give again examples of co-Frobenius coalgebras.

1. QUASI-CO-FROBENIUS COALGEBRAS

Let C be a coalgebra over a field K . We denote by \mathcal{M}^C (respectively ${}^C\mathcal{M}$) the category of right (left) C -comodules and by ${}_{C^*}\mathcal{M}$ (respectively \mathcal{M}_{C^*}) the category of left (right) C^* -modules. We use the simplified Sweedler's σ -notation for the comultiplication $\rho : M \rightarrow M \otimes C$ of a C -comodule M , $\rho(m) = m_0 \otimes m_1$. We will always use the inclusion of categories $\mathcal{M}^C \hookrightarrow {}_{C^*}\mathcal{M}$, where the left C^* -module structure on M is given by $c^* \cdot m = c^*(m_1)m_0$.

Let \mathcal{S} be a set of representatives for the types of isomorphism of simple left C -comodules and \mathcal{T} be a set of representatives for the simple right comodules. It is well known that we have an isomorphism of left C -comodules (equivalently right C^* -modules) $C \simeq \bigoplus_{S \in \mathcal{S}} E(S)^{n(S)}$, where $E(S)$ is an injective envelope of the left C -comodule S and $n(S)$ are

positive integers. Similarly, $C \simeq \bigoplus_{T \in \mathcal{T}} E(T)^{p(T)}$ in \mathcal{M}^C , with $p(T) \in \mathbb{N}$ (we use the same notation for envelopes of left modules and for those of right modules, as it will always be understood from the context what type of modules we refer to). Also $C^* \simeq \prod_{S \in \mathcal{S}} E(S)^*$ in

${}_{C^*}\mathcal{M}$ and $C^* \simeq \prod_{T \in \mathcal{T}} E(T)^*$ in \mathcal{M}_{C^*} . We refer the reader to [A], [DNR] or [Sw] for these

results and other basic facts of coalgebras. We will use the finite topology on duals of vector spaces: given a vector space V , this is the linear topology on V^* that has a basis of neighbourhoods of 0 formed by the sets $F^\perp = \{f \in V^* \mid f|_F = 0\}$ for finite dimensional subspaces F of V . We also write $W^\perp = \{x \in V \mid f(x) = 0, \forall f \in W\}$ for subsets W of V^* . Any topological reference will be with respect to this topology.

For a module M , we convey to write $M^{(I)}$ for the coproduct (direct sum) of I copies of M and M^I for the product of I copies of M . We recall the following definition from [GTN]

Definition 1.1. *A coalgebra C is called right (left) quasi-co-Frobenius, or shortly right QcF coalgebra, if there is a monomorphism $C \hookrightarrow (C^*)^{(I)}$ of right (left) C^* -modules. C is called QcF coalgebra if it is both a left and right QcF coalgebra.*

Recall that a coalgebra C is called right semiperfect if the category \mathcal{M}^C of right C -comodules is semiperfect, that is, every right C -comodule has a projective cover. This is equivalent to the fact that $E(S)$ is finite dimensional for all $S \in \mathcal{S}$ (see [L]). In fact, this is the definition we will need to use. For convenience, we also recall the following very useful results on injective (projective) comodules, the first one originally given in [D] Proposition 4, p.34 and the second one being Lemma 15 from [L]:

[D, **Proposition 4**] *Let Q be a finite dimensional right C -comodule. Then Q is injective (projective) as a left C^* -module if and only if it is injective (projective) as right C -comodule.*

[L, **Lemma 15**] *Let M be a finite-dimensional right C -comodule. Then M is an injective right C -comodule if and only if M^* is a projective left C -comodule.*

We note the following proposition that will be useful in what follows; (i) \Leftrightarrow (ii) was given in [GTN] and our approach also gives here a different proof, along with the new characterizations.

Proposition 1.2. *Let C be a coalgebra. Then the following assertions are equivalent:*

- (i) C is a right QcF coalgebra.
- (ii) C is a right torsionless module, i.e. there is a monomorphism $C \hookrightarrow (C^*)^I$.
- (iii) There exist a dense morphism $\psi : C^{(I)} \rightarrow C^*$, that is, the image of ψ is dense in C^* .
- (iv) $\forall S \in \mathcal{S}, \exists T \in \mathcal{T}$ such that $E(S) \simeq E(T)^*$.

Proof. (i) \Rightarrow (ii) obvious.

(ii) \Leftrightarrow (iii) Let $\varphi : C \rightarrow (C^*)^I \simeq (C^{(I)})^*$ be a monomorphism of right C^* -modules. Let $\psi : C^{(I)} \rightarrow C^*$ be defined by $\psi(x)(c) = \varphi(c)(x)$. It is straightforward to see that the fact that φ is a morphism of left C^* -modules is equivalent to ψ being a morphism in \mathcal{M}_{C^*} , and that φ injective is equivalent to $(\text{Im } \psi)^\perp = 0$, which is further equivalent to $\text{Im } \psi$ is dense in C^* (for example, by [DNR] Corollary 1.2.9).

(ii),(iii) \Rightarrow (iv) As $\text{Im } \psi \subseteq \text{Rat}_{(C^*C^*)}$, $\text{Rat}_{(C^*C^*)}$ is dense in C^* , so C is right semiperfect by Proposition 3.2.1 [DNR]. Thus $E(S)$ is finite dimensional $\forall S \in \mathcal{S}$. Also by (ii) there is a monomorphism $\iota : E(S) \hookrightarrow \prod_{j \in J} E(T_j)^*$ with $T_j \in \mathcal{T}$, and as $\dim E(S) < \infty$ there is

a monomorphism to a finite direct sum $E(S) \hookrightarrow \prod_{j \in F} E(T_j)^*$ (F finite, $F \subseteq J$). Indeed,

if p_j are the projections of $\prod_{j \in J} E(T_j)^*$, then note that $\bigcap_{j \in J} \ker p_j \circ \iota = 0$, so there must be

$\bigcap_{j \in F} \ker p_j \circ \iota = 0$ for a finite $F \subseteq J$. Then $E(S)$ is injective also as right C^* -modules (see

for example [DNR], Corollary 2.4.19), and so $E(S) \oplus X = \bigoplus_{j \in F} E(T_j)^*$ for some X . By [I,

Lemma 1.4], the $E(T_j)^*$'s are local indecomposable, and as they are also cyclic projective we eventually get $E(S) \simeq E(T_j)^*$ for some $j \in F$. This can be easily seen by noting that there is at least one nonzero morphism $E(S) \hookrightarrow E(S) \oplus X = \bigoplus_{j \in F} E(T_j)^* \rightarrow \bigoplus_{j \in F} T_j^* \rightarrow S_k$

(one looks at Jacobson radicals) and this projection then lifts to a morphism $f : E(S) \rightarrow E(T_k)^*$ as $E(S)$ is obviously projective; this has to be surjective since $E(T_k)^*$ is cyclic local, and then f splits; hence $E(S) \simeq E(T_k)^* \oplus Y$ with $Y = 0$ as $E(S)$ is indecomposable.

(iv) \Rightarrow (i) Any isomorphism $E(S) \simeq E(T)^*$ implies $E(S)$ finite dimensional because then $E(T)^*$ is cyclic rational; therefore $E(T) \simeq E(S)^*$. Thus for each $S \in \mathcal{S}$ there is exactly one $T \in \mathcal{T}$ such that $E(S) \simeq E(T)^*$. If \mathcal{T}' is the set of these T 's, then:

$$\begin{aligned} C &\simeq \bigoplus_{S \in \mathcal{S}} E(S)^{n(S)} \hookrightarrow \bigoplus_{S \in \mathcal{S}} E(S)^{(\mathbb{N})} \simeq \bigoplus_{T \in \mathcal{T}' \subseteq \mathcal{T}} (E(T)^*)^{(\mathbb{N})} \\ &\hookrightarrow \left(\bigoplus_{T \in \mathcal{T}} (E(T)^*)^{p(T)} \right)^{(\mathbb{N})} \subseteq \left(\prod_{T \in \mathcal{T}} (E(T)^*)^{p(T)} \right)^{(\mathbb{N})} = C^{*(\mathbb{N})} \end{aligned}$$

□

From the above proof, we see that when C is right QcF, the $E(S)$'s are finite dimensional projective for $S \in \mathcal{S}$, and we also conclude the following result already known from [GTN] (in fact these conditions are even equivalent); see also [DNR, Theorem 3.3.4].

Corollary 1.3. *If C is right QcF, then C is also right semiperfect and projective as right C^* -module.*

We also immediately conclude the following

Corollary 1.4. *A coalgebra C is QcF if and only if the application*

$$\{E(S) \mid S \in \mathcal{S}\} \ni Q \mapsto Q^* \in \{E(T) \mid T \in \mathcal{T}\}$$

is well defined and bijective.

Definition 1.5. (i) *Let \mathcal{C} be a category having products. We say that $M, N \in \mathcal{C}$ are weakly π -isomorphic if and only if there are some sets I, J such that $M^I \simeq N^J$; we write $M \overset{\pi}{\sim} N$.*
(ii) *Let \mathcal{C} be a category having coproducts. We say that $M, N \in \mathcal{C}$ are weakly σ -isomorphic if and only if there are some sets I, J such that $M^{(I)} \simeq N^{(J)}$; we write $M \overset{\sigma}{\sim} N$.*

The study of objects of a (suitable) category \mathcal{C} up to π (respectively σ)-isomorphism is the study of the localization of \mathcal{C} with respect to the class of all π (or σ)-isomorphisms.

Recall that in the category ${}^C\mathcal{M}$ of left comodules, coproducts are the usual direct sums of (right) C^* -modules and the product \prod^C is given, for a family of comodules $(M_i)_{i \in I}$, by $\prod^C M_i = \text{Rat}(\prod_{i \in I} M_i)$.

For easy future reference, we introduce the following conditions:

(C1) $C \overset{\sigma}{\sim} \text{Rat}(C^*_{C^*})$ in ${}^C\mathcal{M}$ (or equivalently, in \mathcal{M}_{C^*}).

(C2) $C \overset{\pi}{\sim} \text{Rat}(C^*_{C^*})$ in ${}^C\mathcal{M}$.

(C3) $\text{Rat}(C^I) \simeq \text{Rat}(C^{*J})$ for some sets I, J .

(C2') $C \overset{\pi}{\sim} \text{Rat}(C^*_{C^*})$ in \mathcal{M}_{C^*} .

Lemma 1.6. *Either one of the conditions (C1), (C2), (C3), (C2') implies that C is QcF (both left and right).*

Proof. Obviously (C2') \Rightarrow (C2). In all of the above conditions one can find a monomorphism of right C^* -modules $C \hookrightarrow (C^*)^J$, and thus C is right QcF. Then each $E(S)$ for $S \in \mathcal{S}$ is finite dimensional and projective by Corollary 1.3. We first show that C is also left semiperfect, along the same lines as the proofs of [I], Proposition 2.1 and [I] Proposition 2.6. For sake of completeness, we include a short version of these arguments here. Let $T_0 \in \mathcal{T}$ and assume, by contradiction, that $E(T_0)$ is infinite dimensional. We first show that $\text{Rat}(E(T_0)^*) = 0$. Indeed, assume otherwise. Then, since $C^* = \prod_{T \in \mathcal{T}} E(T)^{*p(T)}$

and $C = \bigoplus_{S \in \mathcal{S}} E(S)^{n(S)}$ as right C^* -modules, it is straightforward to see that either one of conditions (C1-C3) implies that $\text{Rat}(E(T_0)^*)$ is injective as left comodule, as a direct summand in an injective comodule. Thus, as $\text{Rat}(E(T_0)^*) \neq 0$, there is a monomorphism $E(S) \hookrightarrow \text{Rat}(E(T_0)^*) \subseteq E(T_0)^*$ for some indecomposable injective $E(S)$ ($S \in \mathcal{S}$). This shows that $E(S)$ is a direct summand in $E(T_0)^*$, since $E(S)$ is injective also as right C^* -module (by the above cited [D, Proposition 4]). But this is a contradiction since $E(S)$ is finite dimensional and $E(T_0)^*$ is indecomposable by [I, Lemma 1.4] and $\dim E(T_0)^* = \infty$. Next, use [I, Proposition 2.3] to get an exact sequence

$$0 \rightarrow H \rightarrow E = \bigoplus_{\alpha \in A} E(S_\alpha)^* \rightarrow E(T) \rightarrow 0$$

with $S_\alpha \in \mathcal{S}$. Since the $E(S_\alpha)^*$'s are injective in ${}_{C^*}\mathcal{M}$ by [L, Lemma 15], we may assume, by [I, Proposition 2.4] that H contains no nonzero injective right comodules. For

some $\beta \in A \neq \emptyset$, put $E' = \bigoplus_{\alpha \in A \setminus \{\beta\}} E(S_\alpha)^*$. Then one sees that $H + E' = E$ (otherwise, since there is an epimorphism $E(T) = \frac{E}{H} \rightarrow \frac{E}{H+E'}$, the finite dimensional rational right C^* -module $\left(\frac{E}{H+E'}\right)^*$ would be a nonzero rational submodule of $E(T)^*$, and this provides an epimorphism $H \rightarrow \frac{H}{H \cap E'} \simeq \frac{H+E'}{E'} \simeq E(S_\beta)^*$. But $E(S_\beta)^*$ is projective, so this epimorphism splits, and this comes in contradiction with the assumption on H (the $E(S_\beta)^*$'s are injective in $C^*\mathcal{M}$). \square

Now, we note that if a coalgebra C is QcF, then all the conditions (C1)-(C3) are fulfilled. Indeed, we have that each $E(S)$ ($S \in \mathcal{S}$) is isomorphic to exactly one $E(T)^*$ ($T \in \mathcal{T}$) and actually all $E(T)^*$'s are isomorphic to some $E(S)$. Then:

(C1)

$$\begin{aligned} C^{(\mathbb{N})} &= \left(\bigoplus_{S \in \mathcal{S}} E(S)^{n(S)}\right)^{(\mathbb{N})} = \bigoplus_{S \in \mathcal{S}} E(S)^{(\mathbb{N})} = \bigoplus_{T \in \mathcal{T}} E(T)^{*(\mathbb{N})} \\ &= \bigoplus_{T \in \mathcal{T}} E(T)^{*(p(T) \times \mathbb{N})} = \bigoplus_{T \in \mathcal{T}} (E(T)^{*p(T)})^{(\mathbb{N})} = (\text{Rat}C^*)^{(\mathbb{N})} \end{aligned}$$

where we use that $\text{Rat}(C^*) = \bigoplus_{T \in \mathcal{T}} E(T)^{*p(T)}$ as right C^* -modules for left and right semiperfect coalgebras (see [DNR, Corollary 3.2.17])

(C2)

$$\begin{aligned} \prod_{\mathbb{N}}^C C &= \text{Rat}(C^{\mathbb{N}}) = \prod_{\mathbb{N}} \bigoplus_{S \in \mathcal{S}} E(S)^{n(S)} \\ &= \prod_{\mathbb{N}} \prod_{S \in \mathcal{S}}^C E(S)^{n(S)} \quad (*) \\ &= \prod_{S \in \mathcal{S}}^C E(S)^{n(S) \times \mathbb{N}} = \prod_{S \in \mathcal{S}}^C E(S)^{\mathbb{N}} = \prod_{T \in \mathcal{T}}^C E(T)^{* \mathbb{N}} \\ &= \prod_{T \in \mathcal{T}}^C E(T)^{* \mathbb{N} \times p(T)} = \prod_{\mathbb{N}} \prod_{T \in \mathcal{T}}^C E(T)^{*p(T)} \\ &= \prod_{\mathbb{N}}^C \text{Rat}\left(\prod_{T \in \mathcal{T}} (E(T)^{p(T)})^*\right) = \prod_{\mathbb{N}}^C \text{Rat}\left(\left(\bigoplus_{T \in \mathcal{T}} E(T)^{p(T)}\right)^*\right) \\ &= \prod_{\mathbb{N}}^C \text{Rat}(C^*) \end{aligned}$$

where for (*) we have used [I, Lemma 2.5] and the fact that $E(T)^*$ are all rational since $E(T)$ are finite dimensional in this case (the product in the category of left comodules is understood whenever \prod is written); also

(C3) holds because $\text{Rat}(C^{\mathbb{N}}) = \prod_{T \in \mathcal{T}}^C E(T)^{* \mathbb{N}}$ by the computations in lines 1 and 3 in the

above equation and because

$$\text{Rat}(C^{*\mathbb{N}}) = \text{Rat}\left(\prod_{\mathbb{N}} \prod_{T \in \mathcal{T}} E(T)^{*p(T)}\right) = \prod_{T \in \mathcal{T}} E(T)^{*p(T) \times \mathbb{N}} = \prod_{T \in \mathcal{T}} E(T)^{*N}$$

Combining all of the above we obtain the following nice symmetric characterization which extends the one of co-Frobenius coalgebras from [I] and those of co-Frobenius Hopf algebras and Frobenius Algebras.

Theorem 1.7. *Let C be a coalgebra. Then the following assertions are equivalent.*

(i) C is a QcF coalgebra.

(ii) $C \overset{\sim}{\simeq} \text{Rat}(C_{C^*}^*)$ or $C \overset{\pi}{\simeq} \text{Rat}(C_{C^*}^*)$ in ${}^C\mathcal{M}$ or $\text{Rat}(C^I) \simeq \text{Rat}(C^{*J})$ in ${}^C\mathcal{M}$ (or \mathcal{M}_{C^*}) for some sets I, J .

(iii) $C^{(\mathbb{N})} \simeq (\text{Rat}(C^*))^{(\mathbb{N})}$ or $\prod_{\mathbb{N}} C \simeq \prod_{\mathbb{N}} \text{Rat}(C^*)$ or $\text{Rat}(C^{\mathbb{N}}) \simeq \text{Rat}(C^{*\mathbb{N}})$ as left C -comodules (right C^* -modules)

(iv) The left hand side version of (i)-(iii).

(v) The association $Q \mapsto Q^*$ determines a duality between the indecomposable injective left comodules and indecomposable injective right comodules.

We note the connection to a recent characterization of quasi-Frobenius (co)algebras from [INV06], and how these results allow a generalization of this. For two objects X, Y of a category \mathcal{C} with coproducts, in [INV06] X is said to divide Y ($X|Y$) if $Y^n \cong X \oplus Z$ for some Z , and X and Y are said to be similar if $X|Y$ and $Y|X$. We say that X weakly divides Y and write if there is some set I and an object Z of \mathcal{C} such that $Y^{(I)} \cong X \oplus Z$ (note that this is equivalent to asking there are sets I, J and an object Z with $Y^{(I)} = X^{(J)} \oplus Z$), and call X and Y weakly similar if $X|_w Y$ and $Y|_w X$. We note that [INV06, Theorem 7.5] applied for coalgebras characterizes quasi-Frobenius coalgebras, which are precisely the finite dimensional quasi-co-Frobenius coalgebras (see (iv), (v)); this is a left and right symmetric concept, equivalent. Then we have the following generalization valid for arbitrary coalgebras:

Theorem 1.8. *The following are equivalent for a coalgebra C :*

(i) C is quasi-co-Frobenius.

(ii) C and $\text{Rat}(C^*C^*)$ are weakly similar.

(iii) C and $\text{Rat}(C_{C^*}^*)$ are weakly similar.

Proof. If C is QcF then $C^{(\mathbb{N})} \cong \text{Rat}(C^*C^*)$ easily implies (ii), and (iii) follows similarly. If C and $\text{Rat}(C^*C^*)$ are weakly similar, then since $\text{Rat}(C^*C^*)$ is a direct summand in some $C^{(I)}$, it follows that $\text{Rat}(C^*C^*)$ is an injective comodule (since coproducts of injective comodules is injective), and so $\text{Rat}(C^*C^*) \cong \bigoplus_{S \in \mathcal{S}} E(S)^{(I_S)}$ (structure of injective left comodules). Also, since C is a direct summand of $\text{Rat}(C^*C^*)$, it follows that $\text{Rat}(C^*C^*)$ contains all types of indecomposable simples, so $I_S \neq \emptyset$ for all $S \in \mathcal{S}$. Hence, if $I := \max\{I_S | S \in \mathcal{S}\} \cup \{\mathbb{N}\}$, then $\text{Rat}(C^*C^*)^{(I)} = \bigoplus_{S \in \mathcal{S}} E(S)^{(I_S) \times I} = \bigoplus_{S \in \mathcal{S}} E(S)^{(I)} \cong \bigoplus_{S \in \mathcal{S}} E(S)^{n(S) \times I} = C^{(I)}$ so C is QcF. \square

1.1. Categorical characterization of QcF coalgebras. We give now a characterization similar to the functorial characterizations of co-Frobenius coalgebras and of Frobenius algebras. For a set I let $\Delta_I : {}^C\mathcal{M} \rightarrow ({}^C\mathcal{M})^I$ be the diagonal functor and let F_I be the

composition functor

$$F_I : {}^C\mathcal{M} \xrightarrow{\Delta_I} ({}^C\mathcal{M})^I \xrightarrow{I} {}^C\mathcal{M}$$

that is $F_I(M) = M^{(I)}$ for any left C -comodule M .

Theorem 1.9. *Let C be a coalgebra. Then the following assertions are equivalent:*

- (i) C is QcF.
- (ii) The functors $\text{Hom}_{C^*}(-, C^*) \circ F_I$ and $\text{Hom}(-, K) \circ F_J$ from ${}^C\mathcal{M} = \text{Rat}(\mathcal{M}_{C^*})$ to ${}_K\mathcal{M}$ are naturally isomorphic for some sets I, J .
- (iii) The functors $\text{Hom}_{C^*}(-, C^*) \circ F_{\mathbb{N}}$ and $\text{Hom}(-, K) \circ F_{\mathbb{N}}$ are naturally isomorphic.
- (iv) The functors $\text{Hom}_{C^*}(-, C^*)$ and $\text{Hom}(-, K)$ from ${}^C\mathcal{M} = \text{Rat}(\mathcal{M}_{C^*})$ to ${}_K\mathcal{M}$ are weakly similar.
- (v) The right hand side version (left-right symmetric) of (ii)-(iv).

Proof. Since for any left comodule M , there is a natural isomorphism of left C^* -modules $\text{Hom}_{C^*}(M, C) \simeq \text{Hom}(M, K)$, then for any sets I, J and any left C -comodule M we have the following natural isomorphisms:

$$\text{Hom}(M^{(I)}, K) \simeq \text{Hom}_{C^*}(M^{(I)}, C) \simeq \text{Hom}_{C^*}(M, C^I) \simeq \text{Hom}_{C^*}(M, \text{Rat}(C^I))$$

$$\text{Hom}_{C^*}(M^{(J)}, C^*) \simeq \text{Hom}_{C^*}(M, (C^*)^J) \simeq \text{Hom}(M, \text{Rat}(C^*)^J)$$

Therefore, by the Yoneda Lemma, the functors of (ii) are naturally isomorphic if and only if $\text{Rat}(C^I) \simeq \text{Rat}(C^*)^J$ as right C^* -modules. Thus, by Theorem 1.7 (ii), these functors are isomorphic if and only if C is QcF. Moreover, in this case, by the same theorem the sets I, J can be chosen countable.

(iv) follows from Theorem 1.8 again by the application of the Yoneda Lemma, and (v) follows by the symmetry of (i). \square

Remark 1.10. The above theorem states that C is QcF if and only if the functors C^* -dual $\text{Hom}(-, C^*)$ and K -dual $\text{Hom}(-, K)$ from ${}^C\mathcal{M}$ to ${}_K\mathcal{M}$ are isomorphic in a "weak" meaning, in the sense that they are isomorphic only on the objects of the form $M^{(\mathbb{N})}$ in a way that is natural in M , i.e. they are isomorphic on the subcategory of ${}^C\mathcal{M}$ consisting of objects $M^{(\mathbb{N})}$ with morphisms $f^{(\mathbb{N})}$ induced by any $f : M \rightarrow N$. If we consider the category \mathcal{C} of functors from ${}^C\mathcal{M}$ to ${}_K\mathcal{M}$ with morphisms the classes (which are not necessarily sets) of natural transformations between functors, then the isomorphism in (ii) can be restated as $(\text{Hom}_{C^*}(-, C^*))^I \simeq (\text{Hom}(-, K))^J$ in \mathcal{C} , i.e. the C^* -dual and the K -dual functors are weakly π -isomorphic objects of \mathcal{C} . This is also the setting of statement (iv) in the above theorem.

2. APPLICATIONS TO HOPF ALGEBRAS

Before giving the main applications to Hopf algebras, we start with two easy propositions that will contain the main ideas of the applications. First, for a QcF coalgebra C , let $\varphi : \mathcal{S} \rightarrow \mathcal{T}$ be the function defined by $\varphi(S) = T$ if and only if $E(T) \simeq E(S)^*$ as left C^* -modules; by the above Corollary 1.4, φ is a bijection.

Proposition 2.1. (i) *Let C be a QcF coalgebra and I, J sets such that $C^{(I)} \simeq (\text{Rat}(C^*))^{(J)}$ as right C^* -modules. If one of I, J is finite then so is the other.*

(ii) *Let C be a coalgebra. Then C is co-Frobenius if and only if $C \simeq \text{Rat}(C^*C^*)$ as left C^* -modules and if and only if $C \simeq \text{Rat}(C^*_{C^*})$ as right C^* -modules.*

Proof. (i) C is left and right semiperfect (Corollary 1.3), so using again [DNR, Corollary 3.2.17] we have $\text{Rat}(C_{C^*}^*) = \bigoplus_{T \in \mathcal{T}} E(T)^{*p(T)} = \bigoplus_{S \in \mathcal{S}} E(S)^{p(\varphi(S))}$ and we get $\bigoplus_{S \in \mathcal{S}} E(S)^{n(S) \times I} \simeq$

$\bigoplus_{S \in \mathcal{S}} E(S)^{p(\varphi(S)) \times J}$. From here, since the $E(S)$'s are indecomposable injective comodules we get an equivalence of sets $n(S) \times I \sim p(\varphi(S)) \times J$ (or directly, by evaluating the socle of these comodules). This finishes the proof, as $n(S), p(\varphi(S))$ are finite.

(ii) If C is co-Frobenius, C is also QcF and a monomorphism $C \hookrightarrow \text{Rat}(C_{C^*}^*)$ of right C^* -modules implies $\bigoplus_{S \in \mathcal{S}} E(S)^{n(S)} \hookrightarrow \bigoplus_{T \in \mathcal{T}} E(T)^{*p(T)} \simeq \bigoplus_{S \in \mathcal{S}} E(S)^{p(\varphi(S))}$ and we get $n(S) \leq p(\varphi(S))$ for all $S \in \mathcal{S}$; similarly, as C is also left co-Frobenius we get $n(S) \geq p(\varphi(S))$ for all $S \in \mathcal{S}$. Hence $n(S) = p(\varphi(S))$ for all $S \in \mathcal{S}$ and this implies $C = \bigoplus_{S \in \mathcal{S}} E(S)^{n(S)} \simeq$

$\bigoplus_{T \in \mathcal{T}} E(T)^{p(T)} = \text{Rat}(C_{C^*}^*)$. Conversely, if $C \simeq \text{Rat}(C_{C^*}^*)$ by the proof of (i), when I and J have one element we get that $n(S) = p(\varphi(S))$ for all $S \in \mathcal{S}$ which implies that we also have $C = \bigoplus_{T \in \mathcal{T}} E(T)^{p(T)} \simeq \bigoplus_{S \in \mathcal{S}} E(S)^{*n(S)} = \text{Rat}(C_{C^*}^*)$ so C is co-Frobenius. \square

The above Proposition 2.1 (ii) shows that the results of this paper are a generalization of the results in [I]. The following two propositions are not necessary in their full generality for the applications to Hopf algebras; however, they show precisely what is the difference between the QcF and co-Frobenius properties. Note that their (dual) version for finite dimensional QF and Frobenius algebras is also true.

Proposition 2.2. *With the above notations, let C be a QcF coalgebra and let $\varphi : \mathcal{S} \rightarrow \mathcal{T}$ be such that $\varphi(S) = T$ if and only if $E(S) \cong E(T)^*$ (this is bijective by Theorem 1.7). Then C is co-Frobenius if and only if $n(\sigma(T)) = p(T)$ for all $T \in \mathcal{T}$.*

Proof. Since $C = \bigoplus_{S \in \mathcal{S}} E(S)^{n(S)}$ and $\text{Rat}(C_{C^*}^*) = \bigoplus_{T \in \mathcal{T}} (E(T)^*)^{p(T)}$ as right C^* -modules (again for example, from [DNR, Corollary 3.2.17]), it follows that they are isomorphic if and only if the indecomposable injective summands have the same multiplicity in both. The multiplicity of $E(S)$ in C is $n(S)$ and the multiplicity of $E(S) = E(\varphi(S))^*$ in $\text{Rat}(C_{C^*}^*)$ is $p(\varphi(S))$. Hence, the conclusion follows. \square

Since T is the socle of $E(T)$ and $\sigma(T)^*$ is the ‘‘cosocle’’ of $E(T)$, the above proposition says that a QcF coalgebra is co-Frobenius if the multiplicities of the socle and the cosocle of any indecomposable injective (left, or equivalently, right) comodule are equal.

Corollary 2.3. *If C is a coalgebra and $C^k \cong (\text{Rat}(C_{C^*}^*))^l$ for some natural numbers k, l , then $k = l = 1$ and C is co-Frobenius.*

Proof. As in the previous propositions, we get that $k \cdot n(S) = l \cdot p(\varphi(S))$ for all S . Note that for any $T \in \mathcal{T}$, $p(T) = n(T^*)$. Indeed, it is enough to consider their multiplicities in C_0 , and then it is enough to consider their multiplicities in the simple subcoalgebra E of C_0 in which they are included (same for both). Since $E^* = M_{p(T)}(\text{End}(T)) = M_{n(T^*)}(\text{End}(T^*)^{op})$, we get the same multiplicity of T and its dual T^* . Thus we have $k \cdot n(S) = l \cdot n(\varphi(S)^*)$, i.e. $\frac{k}{l} \cdot n(S) = n(\lambda(S))$, where $\lambda : \mathcal{S} \rightarrow \mathcal{S}$, $\lambda(S) = \varphi(S)^*$ is bijective. We therefore get $n(\lambda^i(S)) = (\frac{k}{l})^i \cdot n(S)$, for any $i \in \mathbb{Z}$. If $k > l$ we get $1 \leq n(\lambda^i(S)) \rightarrow 0$ for $i \rightarrow -\infty$ and if $k < l$ we get $1 \leq n(\lambda^i(S)) \rightarrow 0$ for $i \rightarrow \infty$. Therefore, we can only have $k = l$, and $n(S) = p(\varphi(S))$. Consequently, by Proposition 2.2 C is co-Frobenius. \square

Hence, for a QcF coalgebra either $C \cong \text{Rat}(C^*)$ or otherwise we need at least countable sets I, J such that $C^{(I)} \cong (\text{Rat}(C^*))^{(J)}$ (and $I = J = \mathbb{N}$ is then always a possible choice). Let H be a Hopf algebra over a base field k . Recall that a left integral for H is an element $\lambda \in H^*$ such that $\alpha \cdot \lambda = \alpha(1)\lambda$, for all $\alpha \in H^*$. The space of left integrals for H is denoted by \int_l . The right integrals and the space of right integrals \int_r are defined by analogy. For basic facts on Hopf algebras we refer to [A], [DNR], [M] and [Sw]. The Hopf algebra structure will come into play only through a basic Theorem of Hopf algebras, the fundamental theorem of Hopf modules which yields the isomorphism of right H -Hopf modules $\int_l \otimes H \simeq \text{Rat}(H^*H^*)$. This isomorphism is given by $t \otimes h \mapsto t \leftarrow h = S(h) \rightarrow t$, where for $x \in H$, $\alpha \in H^*$, $x \rightarrow \alpha$ is defined by $(x \rightarrow \alpha)(y) = \alpha(yx)$ and $\alpha \leftarrow x = S(x) \rightarrow \alpha$. Yet, we will only need that this is an isomorphism of right H -comodules (left H^* -modules). Similarly, $H \otimes \int_r \simeq \text{Rat}(H^*_{H^*})$.

Theorem 2.4. (*Lin, Larson, Sweedler, Sullivan*)

If H is a Hopf algebra, then the following assertions are equivalent.

- (i) H is a right co-Frobenius coalgebra.
- (ii) H is a right QcF coalgebra.
- (iii) H is a right semiperfect coalgebra.
- (iv) $\text{Rat}(H^*H^*) \neq 0$.
- (v) $\int_l \neq 0$.
- (vi) $\dim \int_l = 1$.
- (vii) *The left hand side version of the above.*

Proof. (i) \Rightarrow (ii) \Rightarrow (iii) is clear and (iii) \Rightarrow (iv) is a property of semiperfect coalgebras (see [DNR, Section 3.2]).

(iv) \Rightarrow (v) follows by the isomorphism $\int_l \otimes H \simeq \text{Rat}(H^*H^*)$ and (vi) \Rightarrow (v) is trivial.

(v) \Rightarrow (i), (vi) and (vii). Since $\int_l \otimes H \simeq \text{Rat}(H^*H^*)$ in \mathcal{M}^H , we have $H^{\dim(\int_l)} \simeq \text{Rat}(H^*H^*)$ so by Theorem 1.7 H is QcF (both left and right); it then follows that $\int_r \neq 0$ (by the left hand version of (ii) \Rightarrow (v)) and $H^{\dim(\int_r)} \simeq \text{Rat}(H^*_{H^*})$. We can now apply Propositions 2.1 and 2.3 to first get that $\dim \int_l < \infty$, $\dim \int_r < \infty$ and then that H is co-Frobenius (both left and right) so (i) and (vii) hold. Again by Proposition 2.3 we get that, more precisely, $\dim \int_l = \dim \int_r = 1$. \square

The following corollary was the initial form of the result proved by Sweedler [Sw1]

Corollary 2.5. *The following are equivalent for a Hopf algebra H :*

- (i) H^* contains a finite dimensional left ideal.
- (ii) H contains a left coideal of finite codimension.
- (iii) $\int_l \neq 0$.
- (iv) $\text{Rat}(H^*) \neq 0$.

Proof. (i) \Leftrightarrow (ii) It can be seen by a straightforward computation that there is a bijective correspondence between finite dimensional left ideals I of H^* and coideals K of finite codimension in H , given by $I \mapsto K = I^\perp$. Moreover, it follows that any such finite dimensional ideal I of H^* is of the form $I = K^\perp$ with $\dim(H/K) < \infty$, so $I = K^\perp \simeq (H/K)^*$ is then a rational left H^* -module, thus $I \subseteq \text{Rat}(H^*)$. This shows that (ii) \Rightarrow (iv) also holds, while (iii) \Rightarrow (ii) is trivial. \square

The bijectivity of the antipode. Let t be a nonzero left integral for H . Then it is easy to see that the one dimensional vector space kt is a two sided ideal of H^* . Also, by the definition of integrals, $kt \subseteq \text{Rat}(H^*H^*) = \text{Rat}(H^*_{H^*})$ (since H is semiperfect as a

coalgebra). Thus kt also has a left comultiplication $t \mapsto a \otimes t$, $a \in H$ and then by the coassociativity and counit property for Hkt , a has to be a grouplike element. This element is called the *distinguished grouplike* element of H . In particular $t \cdot \alpha = \alpha(a)t$, $\forall \alpha \in H^*$. See [DNR, Chapter 5] for some more details.

For any right H -comodule M denote aM the left H -comodule structure on M with comultiplication

$$M \ni m \mapsto m_{-1}^a \otimes m_0^a = aS(m_1) \otimes m_0$$

(S denotes the antipode). It is straightforward to see that this defines an H -comodule structure.

Proposition 2.6. *The map $p : {}^aH \rightarrow \text{Rat}(H^*)$, $p(x) = x \rightarrow t$ is a surjective morphism of left H -comodules (right H^* -modules).*

Proof. Since the above isomorphism $H \simeq \int_l \otimes H \simeq \text{Rat}(H^*)$ is given by $h \mapsto t \leftarrow h = S(h) \rightarrow t$, we get the surjectivity of p . We need to show that $p(x)_{-1} \otimes p(x)_0 = x_{-1}^a \otimes p(x)_0^a$ and for this, having the left H -comodule structure of $\text{Rat}(H^*)$ in mind, it is enough to show that for all $\alpha \in H^*$, $p(x)_0 \alpha(p(x)_{-1}) = p(x) \cdot \alpha = \alpha(x_{-1}^a) p(x)_0^a$. Indeed, for $g \in H$ we have:

$$\begin{aligned} ((x \rightarrow t) \cdot \alpha)(g) &= t(g_1x)\alpha(g_2) = t(g_1x_1\varepsilon(x_2))\alpha(g_2) \\ &= t(g_1x_1)\alpha(g_2x_2S(x_3)) = t(g_1x_1)(\alpha \leftarrow x_3)(g_2x_2) \\ &= t((gx_1)_1)(\alpha \leftarrow x_2)((gx_1)_2) = (t \cdot (\alpha \leftarrow x_2))(gx_1) \\ &= (\alpha \leftarrow x_2)(a)t(gx_1) \quad (a \text{ is the distinguished grouplike of } H) \\ &= \alpha(aS(x_2))(x_1 \rightarrow t)(g) \end{aligned}$$

and this ends the proof. \square

Let π be the composition map ${}^aH \xrightarrow{p} \text{Rat}(H_{H^*}^*) \xrightarrow{\sim} H \otimes \int_r \simeq H$, where the isomorphism $H \otimes \int_r \simeq H_{H^*}^*$ is the left analogue of $\int_l \otimes H \simeq \text{Rat}(H^*H)$. Since ${}^H H$ is projective in ${}^H \mathcal{M}$, this surjective map splits by a morphism of left H -comodules $\varphi : H \hookrightarrow {}^aH$, so $\pi\varphi = \text{Id}_H$. Then we can find another proof of:

Theorem 2.7. *The antipode of a co-Frobenius Hopf algebra is bijective.*

Proof. Since the injectivity of S is immediate from the injectivity of the map $H \ni x \mapsto t \leftarrow x \in H^*$, as noticed by Sweedler [Sw1], we only observe the surjectivity. The fact that φ is a morphism of comodules reads $\varphi(x)_{-1}^a \otimes \varphi(x)_0^a = x_1 \otimes \varphi(x_2)$, i.e. $aS(\varphi(x)_2) \otimes \varphi(x)_1 = x_1 \otimes \varphi(x_2)$, and since $a = S(a^{-1}) = S^2(a)$, by applying $\text{Id} \otimes \varepsilon\pi$ we get $S(a^{-1})S(\varphi(x)_2)\varepsilon\pi(\varphi(x)_1) = x_1\varepsilon\pi\varphi(x_2) = x_1\varepsilon(x_2) = x$, so $x = S(\varepsilon\pi(\varphi(x)_1)\varphi(x)_2a^{-1})$. \square

3. GENERALIZED FROBENIUS ALGEBRAS

In this section, we apply these results to K -algebras, and introduce and characterize the notion of Generalized Frobenius Algebra. For this we will first explain the natural setting of the problem.

Let A be a topological algebra. We say that the topology is of “algebraic type” if the topology of A is A -linear and the topology of A has a basis of neighborhoods of 0 consisting of two-sided ideals of finite codimension. Here the field K is considered to have the discrete topology. Let us call such an algebra a topological algebra of algebraic type, or

AT-algebra for short. This is important since it captures the following type of situation: given an arbitrary K -algebra, one might be interested in the study of a certain subcategory \mathcal{C} (closed under finite sums, quotients and subobjects) of the category $\text{Rep}(A)$ of finite dimensional (left) A -modules (representations). Then, we can introduce a topology γ on A generated by a basis of 0 consisting of ideals of A which are annihilators of objects of \mathcal{C} . The category \mathcal{C} can then be viewed as the category of finite dimensional topological A -modules. In this respect, for an AT-algebra it is natural to introduce the category $A - \underline{\text{Mod}}$ consisting of topological left A -modules which have a basis of neighborhoods of 0 consisting of submodules of finite codimension. Call these modules topological modules of algebraic type. Such modules were considered and studied by P.Gabriel in [G]. The pseudocompact modules are also introduced (see [G] and [DNR]): they are modules which are separated, complete and have a basis of neighborhoods of 0 consisting of submodules of finite codimension. Let $A - \text{PSC}$ denote the full subcategory of $A - \underline{\text{Mod}}$ consisting of pseudocompact A -modules. In an analogous way we can define the right topological A -modules of algebraic type $\underline{\text{Mod}} - A$, and the right pseudocompact A -modules $\text{PSC} - A$. We recall some known facts. For an AT-algebra, denote A^0 (also denoted $R_c(A)$) the set of continuous functions $f : A \rightarrow K$, that is $\ker(f)$ contains an open (and cofinite) ideal of A . Such function are precisely the representative functions of A : that is, continuous functions f for which are continuous functions $g_i, h_i : A \rightarrow K$ such that $f(ab) = \sum_i f_i(a)g_i(b)$.

Equivalently, A^0 is spanned by the coefficient functions $(\eta_{ij})_{i,j}$ where $\eta : A \rightarrow \text{End}(V)$ is a continuous finite dimensional representation of A (that is, its annihilator is open) and the coefficient functions are being considered with respect to some basis $(v_i)_i$ of V . Then, it is well known that A^0 is a coalgebra (with comultiplication $\Delta : A^0 \rightarrow A^0 \otimes A^0$, $\delta(f) = \sum_i g_i \otimes h_i$ and counit $\varepsilon(f) = f(1)$) the category of finite dimensional topological left

A -modules is equivalent to that of finite dimensional right A^0 -comodules f.d. \mathcal{M}^{A^0} . Also, note that the category of Pseudocompact left A -modules is dual to the category of right A^0 -comodules. This follows in the same way as for a pseudocompact algebra (see [DNR, Section 2.5]).

Using the same ideas as in [G] or [DNR] we in fact have:

Proposition 3.1. *The forgetful functor $U : A - \text{PSC} \rightarrow A - \underline{\text{Mod}}$ has a left adjoint $P : A - \underline{\text{Mod}} \rightarrow A - \text{PSC}$ defined by $P(M) = \lim_{\leftarrow X \text{ open}} \frac{M}{X}$, where the limit is taken over all*

the open left submodules X of M (automatically, of finite codimension). The basis of open neighborhoods of 0 in $P(M)$ is the set $K_X = \ker(P(M) \rightarrow M/X)_{X \text{ open}}$.

Proof. We need to show that $\text{Hom}_{A - \underline{\text{Mod}}}(P(M), P) \cong \text{Hom}_{A - \underline{\text{Mod}}}(M, U(P))$ naturally in M and P . A continuous morphism $f : P(M) \rightarrow P$ is given by a compatible family $f_Y : P(M) \rightarrow \frac{P}{Y}$ with open kernel, that is, with $\ker(f_Y) = K_{X(Y)}$, for some $X(Y)$ cofinite and open in M . The compatibility condition means that whenever $Y \subseteq Y'$, the diagram

$$\begin{array}{ccc} P(M) & \xrightarrow{f_Y} & \frac{P}{Y} \\ & \searrow f_{Y'} & \downarrow \\ & & \frac{P}{Y'} \end{array}$$

is commutative. This is equivalent to $K_{X(Y)} \subseteq K_{X(Y')}$ whenever $Y \subseteq Y'$. But, it is not difficult to see that $K_X \subseteq K_{X'}$ if and only if $X \subseteq X'$, for open X, X' in M . Thus, the

compatibility condition is further equivalent to $X(Y) \subseteq X(Y')$ whenever $Y \subseteq Y'$. The existence of the family of morphisms $f_Y : P(M) \rightarrow \frac{P}{Y}$ is then equivalent to the existence of a family $\overline{f_Y} : \frac{P(M)}{K_{X(Y)}} \rightarrow \frac{P}{Y}$, which is further equivalent to the existence of a family of morphisms of A -modules $\overline{f'_Y} : \frac{M}{X(Y)} \rightarrow \frac{P}{Y}$, which are compatible: $X(Y) \subseteq X(Y')$ whenever $Y \subseteq Y'$. This is then equivalent to (the existence of) a family of morphisms of A -modules $f'_Y : M \rightarrow \frac{P}{Y}$ which have open kernels and which are compatible in the sense that the following diagram is commutative whenever $Y \subseteq Y'$:

$$\begin{array}{ccc} M & \xrightarrow{f'_Y} & \frac{P}{Y} \\ & \searrow f'_{Y'} & \downarrow \\ & & \frac{P}{Y'} \end{array}$$

This defines equivalently a continuous morphism of A -modules $f' : M \rightarrow P$. The naturality of this bijective correspondence holds as well, and we have proved the proposition. \square

We can now explain the natural setting for a definition of a Generalized Frobenius Algebra (GFA). Let A be a topological AT-algebra. In the spirit of the classical definition of a Frobenius algebra and having the previous section in mind, we need to define a suitable (left, right) dual of A and define the GFA by the property that A is isomorphic to its left dual. We note that A is an object in $A - \underline{\text{Mod}}$, so its dual will have to be an object of this category too. Since A is a topological algebra, A^0 is the natural first step in the construction of this dual, since it consists of the continuous functions on A . A^* is canonically endowed with the finite topology, which is in fact the product topology of K^A or equivalently, the topology of point-wise convergence of functions. In order to view A^* as an object of $A - \underline{\text{Mod}}$, we will consider the largest subtopology of the finite topology on A^* which has a basis of neighborhoods of 0 consisting of left (cofinite) A -modules. Then $A^0 \subseteq A^*$ is regarded as a subobject of A^* in $A - \underline{\text{Mod}}$ by using the trace topology. As usual, it is then natural to look at a completion of this space of continuous functions with respect to this natural topology on it (which makes A^0 an object of the desired category $A - \underline{\text{Mod}}$); using the above notations, this is dual is denoted $P(A^0)$. We can then introduce the more general

Definition 3.2. (i) Given an object $M \in A - \underline{\text{Mod}}$, we consider the usual dual M^* of M as a right A -module of “algebraic type” (so as an object of $\underline{\text{Mod}} - A$), with the subtopology of the finite topology which has a basis of neighborhoods of 0 consisting of open A -submodules. (ii) Denote $\text{Hom}_c(M, K)$ the set of continuous linear functions on M ; this is an A -submodule of M (since if $\ker(f) \supseteq N$, N open cofinite A -submodule then $\ker(a \cdot f) \supseteq N$). (iii) For $M \in A - \underline{\text{Mod}}$ we define its dual $M^\vee \in \underline{\text{Mod}} - A$ by $M^\vee = P(\text{Hom}_c(M, K))$, where on $\text{Hom}_{A - \underline{\text{Mod}}}(M, K)$ we consider the trace topology of that described in (i) for M^* .

Note that the open subspaces of M^* are A -submodules V^\perp where $V^\perp = \{m^* \in M^* | m^*|_V = 0\}$ and V is a finite dimensional subspace of M . In order for this to be a submodule, we must have $(m^* \cdot a)(v) = 0$, for all $a \in A$, $m^* \in V^\perp$, $v \in V$, i.e. $m^*(a \cdot v) = 0$. This means that $a \cdot v \in (V^\perp)^\perp = V$, so V must be a left submodule of M . Then the open subspaces of $\text{Hom}_c(M, K)$ are of the form $V^\perp \cap \text{Hom}_c(M, K)$, V left finite dimensional submodule

of M . Thus the dual M^\vee of M is given by

$$M^\vee = \varprojlim_{V \text{ f.d. } \subseteq_A M} \frac{\text{Hom}_c(M, K)}{V^\perp \cap \text{Hom}_c(M, K)}$$

with the limit taken over all finite dimensional submodules V of M .

We motivate the definition (ii) by noting that the usual dual V^* of a vector space is always complete with respect to the finite topology, and it is common to expect the dual of an object to be a complete object. Also, in our setting, the natural functions on M are not only the linear functions but the linear continuous functions, and the topology is induced on M^* in order to make it an object of the category of right modules of algebraic type.

We can now introduce the Generalized Frobenius Algebras:

Definition 3.3. *An AT-algebra is called Generalized Frobenius if $A \cong A^\vee$ as topological left A -modules (i.e. as objects of $A - \underline{\text{Mod}}$). An AT-algebra is called Generalized Quasi-Frobenius if $A \overset{\pi}{\sim} A^\vee$ as topological left A -modules. Here, A^\vee is the dual of the right module A_A , and A as a left topological AT-module has the topology with a basis neighborhoods of 0 consisting of left open ideals.*

Note that the topology of A as a topological left A -module of algebraic type, which has a basis of neighborhoods of 0 consisting of open left ideals is the same with the topology of A as an AT-algebra. Indeed, any open (cofinite) ideal of A is a left ideal, and if H is a left ideal of A which is open, it must contain an open (and cofinite) two-sided ideal I , since these ideals form a basis of A around 0. Although the following proposition is not needed in its generality, it explicitly describes the dual of an AT-module.

Proposition 3.4. *Let $M \in A - \underline{\text{Mod}}$ be a left AT-module. Let $M_0 = \bigcap_{N \text{ open } \subseteq M} N$ be the intersection of all open (and so, cofinite) submodules of M . Then the dual M^\vee of M is given by*

$$M^\vee = \varprojlim_{V \text{ f.d. } \subseteq_A M} \left(\frac{V}{V \cap M_0} \right)^* = \left(\varprojlim_{V \text{ f.d. } \subseteq_A M} \frac{V}{V \cap M_0} \right)^*$$

Proof. Let $M_0 = \bigcap_{N \text{ open } \subseteq M} N$ be the intersection of all open (and so, cofinite) submodules of M . We have an exact sequence of left M -modules:

$$0 \rightarrow V^\perp \cap \text{Hom}_c(M, K) \rightarrow \text{Hom}_c(M, K) \xrightarrow{r} \left(\frac{V}{V \cap M_0} \right)^* \rightarrow 0$$

where r is the restriction map, $f \mapsto f|_V$. The map is well defined since for any $f \in \text{Hom}_c(M, K)$, there is N open cofinite with $N \subseteq \ker(f)$ so then $M_0 \subseteq \ker(f)$. This shows that $f|_{V \cap M_0} = 0$, so the restriction map r has image contained in the kernel of the morphism $V^* \rightarrow (V \cap M_0)^*$, which is $(V/V \cap M_0)^*$.

We show that r is surjective. Since $V \cap M_0 = \bigcap_{N \text{ open } \subseteq M} V \cap N$ and $V \cap N$ are subspaces of the finite dimensional space V , there is some N such that $V \cap N = V \cap M_0$. Let $f : M \rightarrow K$ be a linear map which cancels on $V \cap M_0 = V \cap N$, thus inducing a map $\bar{f} \in (V/V \cap M_0)^*$.

Then \bar{f} extends to a linear map \bar{g} as in the diagram below:

$$\begin{array}{ccccc}
 V & \xrightarrow{\subseteq} & M & & \\
 \downarrow & & \downarrow & \nearrow g & \\
 \frac{V}{V \cap N} & \xrightarrow{\cong} & \frac{V+N}{N} & \xrightarrow{\subseteq} & \frac{M}{N} \\
 \downarrow \bar{f} & & \downarrow \bar{g} & \nearrow & \\
 K & & K & &
 \end{array}$$

and g from the above picture extends f to M . Moreover, by the diagram, $\ker(g)$ contains N , so $g \in \text{Hom}_c(M, K)$. Furthermore, it obvious that the kernel of r is $V^\perp \cap M$.

Thus we have $M^\vee = \varprojlim_{V \text{ f.d. } \subseteq_A M} \frac{\text{Hom}_c(M, K)}{V^\perp \cap \text{Hom}_c(M, K)} = \varprojlim_{V \text{ f.d. } \subseteq_A M} \left(\frac{V}{V \cap M_0} \right)^* = \left(\varprojlim_{V \text{ f.d. } \subseteq_A M} \frac{V}{V \cap M_0} \right)^*$

□

Theorem 3.5. *Let A be a topological algebra of algebraic type. Then A is a Generalized (quasi)-Frobenius algebra if and only if A is pseudocompact with $A \cong C^*$, where C is a (quasi-)co-Frobenius coalgebra.*

Proof. First, since $A^\Lambda \cong (A^\vee)^\Gamma$ as left topological modules, and A^\vee is pseudocompact, it follows that $(A^\vee)^\Gamma$ is pseudocompact (complete and separated) and then so is A^Λ . Since A can be viewed also as a subspace of A^Λ , it follows that A is separated. Since A^Λ is complete, it easily follows that A is also complete. Therefore, it follows that A is a pseudocompact algebra, and so $A \cong C^*$, where

$$C = A^0 = \varprojlim_{I \text{ open ideal}} \left(\frac{A}{I} \right)^*$$

(see for example [DNR, Section 2.6]).

Now, for the situation when $A = C^*$ is a pseudocompact algebra, we see that the topolog-

ical right dual of A is ${}_{C^*}(C^*)^\vee = \left(\varprojlim_{V \text{ f.d. } \subseteq_{C^*}} V \right)^*$. Here we use the previous proposition

and the fact that the intersection of the open ideals of $A = C^*$ is 0. Hence, we actually get that ${}_{C^*}(C^*)^\vee = (\text{Rat}(C_{C^*}^*))^*$. Therefore, the condition that $A = C^*$ is Generalized quasi-Frobenius means $(C^*)^\Lambda \cong (\text{Rat}(C_{C^*}^*))^{*\Gamma}$ as topological modules, which is equivalent to the fact that $(C^{(\Lambda)})^* \cong (\text{Rat}(C_{C^*}^*)^{(\Gamma)})^*$ as (pseudocompact) topological modules. Since the category of left pseudocompact modules over C^* is dual to that of left comodules over C , the condition translates equivalently to $C^{(\Lambda)} \cong \text{Rat}(C_{C^*}^*)^{(\Gamma)}$ as left comodules. By Theorem 1.7 this is equivalent to C being QcF. When the sets Λ, Γ are singletons, we obtain the characterization of Generalized Frobenius algebras. □

Since any pseudocompact algebra is profinite (=inverse limit of finite dimensional algebras), we get the following nice analogue of the fact that a Frobenius algebra is finite dimensional.

Corollary 3.6. *A generalized (quasi)-Frobenius algebra is profinite.*

3.1. Further remarks. One might introduce a less restrictive ‘‘Frobenius’’ notion which only involves the category of finite dimensional topological modules. Let us call a topological AT algebra weakly (quasi)-Frobenius, or a weak (quasi)-Frobenius algebra, if A^0 is a (quasi)-co-Frobenius coalgebra or, equivalently, its pseudocompact completion $(A^0)^*$ is a (quasi)-Frobenius AT-algebra (topological algebra of algebraic type). Let us call a continuous map $f : X \rightarrow Y$ between two topological spaces a trace map if the topology on X is induced by that on Y through f ; that is, for any $x \in X$ and U open neighborhood of X , there is V open in Y such that $x \in f^{-1}(V) \subseteq U$. Then we have:

Proposition 3.7. *A is a weak (quasi)-Frobenius AT-algebra if and only if there is a dense trace morphism of AT-algebras $\psi : A \rightarrow C^*$ for a (quasi)-co-Frobenius coalgebra C .*

Proof. We note that the canonical morphism $A \rightarrow (A^0)^* = \varinjlim_{I \text{ open} \subseteq A} A/I$ is continuous, dense and trace, and this proves the only if part. Let $\psi : A \rightarrow C^*$ be a dense quotient morphism. For each finite dimensional subcoalgebra $E \subseteq C$, let $I_E = \psi^{-1}(E^\perp)$, which is an open ideal of A . Also, since ψ is a trace, for each I open in A , there is a finite dimensional subcoalgebra $E \subseteq C$ such that $I_E \subseteq I$. Hence, $(I_E)_E$ is a basis around 0 for A . Moreover, by the density, for any $c^* \in C^*$ and any E , there is $a \in A$ such that $\varphi(a)|_E = c^*|_E$, which shows that the induced morphism $A/I_E \rightarrow C^*/E^\perp$ is an isomorphism. Hence, we have an isomorphism

$$(A^0)^* = \varinjlim_{I \text{ open}} A/I \cong \varinjlim_{E \text{ f.d. subcoalgebra} \subseteq C} A/I_E \xrightarrow{\cong} \varinjlim_{E \text{ f.d. subcoalgebra} \subseteq C} C^*/E^\perp = C^*$$

Thus, $(A^0)^* \cong C^*$ is (quasi)-Frobenius, and we are done. \square

We note that for a coalgebra C , being QcF is a categorical property. Indeed, it is not difficult to see that by Theorem 1.7, C is QcF if and only if it is left and right semiperfect and projective finite dimensional comodules coincide with injective finite dimensional comodules. This can thus be rephrased equivalently that the category of finite dimensional right (equivalently, left) comodules $\text{f.d. } \mathcal{M}^C$ has enough injectives and projectives, and injectives and projectives coincide (such a category is called a Frobenius category). C is co-Frobenius if it further satisfies the socle-cosocle multiplicity condition of Proposition 2.2.

Therefore, an AT-algebra A is weak quasi-Frobenius if the category $\text{f.d. } A - \underline{\text{Mod}}$ of finite dimensional topological left A -modules is Frobenius or equivalently, the category $\text{f.d. } \underline{\text{Mod}} - A$ is Frobenius. A is weak Frobenius if either of these categories satisfy the multiplicity condition of 2.2.

Non-topological Algebras. If A is an arbitrary algebra, we can think of it as an AT-algebra if we introduce the topology which has a basis of neighborhoods of 0 consisting of all the ideals of finite codimension. Thus it makes sense to talk about the above Frobenius type notions. Recall that a coalgebra C is coreflexive if $(C^*)^0 = C$ (with the usual topology on C^*). We refer to [HR74] and [R1] for details on coreflexive coalgebras. Because of the special topology, we have the following

Proposition 3.8. *An algebra A is Generalized (quasi)-Frobenius if and only if $A = C^*$ where C is (quasi)-co-Frobenius and C_0 is coreflexive.*

Proof. If A is Generalized (quasi)-Frobenius then $A = C^*$ with $C = A^0$. So $C = A^0 = (C^*)^0$ is coreflexive, and (quasi)-co-Frobenius. Conversely, if C is quasi-co-Frobenius, then C is semiperfect. Then, since C_0 is coreflexive, it follows that C is coreflexive. This follows,

for example, from [CNO, 2.5, 2.12] and [HR74, Remark 3.1.2] combined. Therefore, in C^* any cofinite ideal I is closed (see for example [R1, 2.10]) and so $I = E^\perp$, for E finite dimensional subcoalgebra of C . Hence, I is open and the topology on C^* has all cofinite ideals as open ideals. \square

Note that the extra condition that C_0 is coreflexive is not a restrictive one. Indeed, by the results of [HR74], if K is infinite, C_0 is coreflexive if and only if the coalgebra $K^{(\mathcal{S})}$ is coreflexive. This is true whenever the set of simples \mathcal{S} is non-measurable (that is, every ultrafilter on \mathcal{S} which is closed under countable intersections - an Ulam ultrafilter - is principal). This is a reasonable condition as pointed out in [HR74, Section 3.7] (in fact, no example of a measurable set is known).

4. EXAMPLES IN “NATURE”

In what follows, we give a large class of examples of co-Frobenius and QcF coalgebras, and implicitly, of Generalized (quasi)-Frobenius algebras. This section is aimed to provide examples of such coalgebras appearing in very natural contexts and of importance in various places, such as representation theory, homological algebra and even topology.

First we remark a standard procedure of “simple object multiplicity change”. Recall from [Tak77] that if C is a coalgebra, E is a quasi-finite injective cogenerator of \mathcal{M}^C and $B = \text{coend}^C(E)$ then B is a coalgebra and \mathcal{M}^B is equivalent to \mathcal{M}^C through the cotensor functor $F(-) = -\square_B E : \mathcal{M}^B \rightarrow \mathcal{M}^C$ and let G denote its inverse. Also recall that the cohom functor is defined for two comodules M, N by $\text{cohom}(M, N) = \varinjlim_{N' \text{ f.d. } \subseteq N^C} \text{Hom}(N', M)^*$, where the limit ranges over the finite dimensional subcomodules

N' of N . Hence $\text{coend}(E) = \varinjlim_{E' \text{ f.d. } \subseteq E^C} \text{Hom}(E', E)^*$ and so we see that

$$\begin{aligned} \text{coend}(E)^* &= \left(\varinjlim_{E' \text{ f.d. } \subseteq E^C} \text{Hom}^C(E', E)^* \right)^* = \varinjlim_{E' \text{ f.d. } \subseteq E^C} \text{Hom}^C(E', E)^{**} \\ &= \varinjlim_{E' \text{ f.d. } \subseteq E^C} \text{Hom}^C(E', E) = \text{Hom}^C\left(\varinjlim_{E' \text{ f.d. } \subseteq E^C} E', E \right) = \text{End}^C(E) \end{aligned}$$

Let $E = \bigoplus_{T \in \mathcal{T}} E(T)^{h(T)}$, where $h(T)$ are positive integers representing the multiplicity of the injective indecomposable $E(T)$ in E . Also, $\text{End}^C(E) = \prod_{T \in \mathcal{T}} \text{Hom}^C(E(T), E)^{h(T)}$ as

left $\text{End}^C(E)$ -modules and it is standard to see that $\text{Hom}(E(T), E)$ are indecomposable projective local $\text{End}^C(E)$ -modules, whose maximal simple quotient is $\text{Hom}(T, E)$, with multiplicity $h(T)$ in $\text{End}^C(E)$. Dually, since $B^* = \text{End}^C(E)$ it follows that in B the corresponding simple comodule - which is $G(T)$ - has multiplicity $h(T)$. Thus, the simple objects of C were “replaced” by $G(T)$ in the the new coalgebra B , which is Morita equivalent to the old one.

Hence, if C is QcF, as noticed above, B will be QcF too, and the multiplicities of the simple objects in B - $(h(T))_{T \in \mathcal{T}}$ can be chosen arbitrary. As seen in section 2, if we denote $\lambda : \mathcal{T} \rightarrow \mathcal{T}$ the bijective function given by

$$(1) \quad \lambda(T) = \text{cosocle of } E(T) \text{ (which is well defined)}$$

C is co-Frobenius if $h(T) = h(\lambda(T))$ for all T , i.e. iff h is constant on the orbits of the action of \mathbb{Z} on \mathcal{T} through $n \cdot T \rightarrow \lambda^n(T)$. Hence, each function $h : \mathcal{T}/\mathbb{Z} \rightarrow \mathbb{N}$ defines a

co-Frobenius coalgebra equivalent to C . Obviously, any function $\mathcal{T} \rightarrow \mathbb{Z}$ which is non-constant on these orbits defines a QcF coalgebra which is not co-Frobenius.

4.1. Examples from Representation theory of “sub”-quiver coalgebras. We now construct a concrete class of examples. Let Γ be a graph and $K\Gamma$ the quiver coalgebra. Recall that it has a basis consisting of the paths in Γ and comultiplication for each path $p = [x^{(1)} \dots x^{(n)}]$ starting at some a and ending at some b given by $p = [x^{(1)} \dots x^{(n)}] \xrightarrow{D} \sum_{k=0}^n [x^{(1)} \dots x^{(k)}] \otimes [x^{(k)} \dots x^{(n)}]$ (the term for $k=0$ being $a \otimes p$ and the one for $k=n$ being $p \otimes b$). Write $s(p) = a$ and $t(p) = b$ for the source and target of a path p . The counit is $\varepsilon(p) = \delta_{|p|,0}$. Let $h : V_\Gamma \rightarrow \mathbb{N}$ be an arbitrary function defined on the set V_Γ of vertices of Γ , and let $B = \{x_{ij} | p \text{ path from } a \text{ to } b, i = 1, \dots, h(a); j = 1, \dots, h(b)\}$ (i.e. for each arrow x from a to b we have $h(a) \cdot h(b)$ distinct elements x_{ij}). Now for each path $p = [x^{(1)} \dots x^{(n)}]$ of Γ passing through the vertices a_0, a_1, \dots, a_n we define $h(a_0)h(a_n)$ formally distinct elements of a new set (words in the elements of B) $p_{ij}, i = 1, \dots, h(a_0); j = 1, \dots, h(a_n)$. For a vertex in Γ , we define $h(a)^2$ distinct words “of length 0” $a_{ij}, i = 1, \dots, h(a); j = 1, \dots, h(a)$. Now let $K[\Gamma, h]$ be the vector space with a basis B consisting of all p_{ij} (including those of length 0, i.e. the a_{ij} ’s) and introduce a comultiplication and counit as follows:

$$\Delta([x^{(1)} \dots x^{(n)}]_{ij}) = \sum_{k=0}^n \sum_{s=1}^{t(x^{(k)})} [x^{(1)} \dots x^{(k)}]_{i,s} \otimes [x^{(k)} \dots x^{(n)}]_{s,j}$$

$$\varepsilon(p_{ij}) = \delta_{|p|,0} \delta_{ij}$$

This writes for short $\Delta(p_{ij}) = \sum_p \sum_{s=1}^{t(p_1)=s(p_2)} (p_1)_{i,s} \otimes (p_2)_{s,j}$ where $D(p) = p_1 \otimes p_2$ is the comultiplication D of $K\Gamma$. Using this, it is not difficult to check that $(K[\Gamma, h], \Delta, \varepsilon)$ is a coalgebra. The following proposition uses standard techniques of localization in coalgebras, and it shows the concrete example of this phenomenon; it is also a generalization of the result of [CKQ] describing the injective hulls of simple objects in a path coalgebra (see [Si] for a similar result for incidence coalgebras of partially ordered sets):

Proposition 4.1. *The simple types of right $K[\Gamma, h]$ -comodules are $T_a = K\{a_{1j} | j = 1, \dots, h(a)\}$ and the left simple comodules $S_a = K\{a_{i1} | i = 1, \dots, h(a)\}$, a vertex in Γ . Denoting $T_{a,i} = K\{a_{ij} | j = 1, \dots, h(a)\} \cong T_a$ and $S_{b,j} = K\{b_{ij} | i = 1, \dots, h(b)\} \cong S_b$, we have that the injective envelopes of $T_{a,i}$ and $S_{b,j}$ in C are*

$$E(T_{a,i}) = K\{p_{ik} | s(p) = a, k = 1, \dots, t(p)\}; \quad E(S_{b,j}) = K\{p_{kj} | t(p) = b, k = 1, \dots, s(p)\}$$

Moreover, $K[\Gamma, h]$ is Morita equivalent to $K\Gamma$.

The simple objects are obvious and injective envelopes follow immediately because there is a decomposition of $K[\Gamma, h]$ into right comodules:

$$K[\Gamma, h] = \bigoplus_{a,i} K\{p_{ik} | s(p) = a, k = 1, \dots, t(p)\}$$

and $T_{a,i} \subseteq K\{p_{ik} | s(p) = a, k = 1, \dots, t(p)\}$.

For the last assertion, let e be the idempotent of $K[\Gamma, h]^*$ which is equal to ε on all the elements p_{11} for paths p in Γ and 0 on the other elements of B . For $C = K[\Gamma, h]$ we consider the coalgebra eCe defined in [Rad82]. It has a comultiplication given by $ece \mapsto ec_1e \otimes ec_2e$ and counit $ece \mapsto \varepsilon_C(ece)$. It is then easy to see that $K\Gamma \cong eK[\Gamma, h]e$

as coalgebras by $eK[\Gamma, h]e \ni ep_{ij}e \mapsto p_{11} \in K\Gamma$. Also, in this situation, Ce is seen to be an injective cogenerator of \mathcal{M}^C , and each right indecomposable injective $E(T_a)$ has multiplicity 1 in E . Moreover, $eCe \cong \text{coend}(Ce)$ (see [Rad82] too) so $K[\Gamma, h]$ is Morita equivalent to $K\Gamma \cong eCe$ (eCe is the basic coalgebra of C ; see also [CM]).

Remark 4.2. Now, for a more general construction, note that we may consider any subcoalgebra F of $K\Gamma$ which has a basis B_F of paths (such coalgebras are also called "monomial"), and consider a corresponding coalgebra F_h which will be a subcoalgebra of $K[\Gamma, h]$ which will have a basis $B_{F,h} = \{p_{ij} | p \in B_F; i = 1, \dots, s(p); j = 1, \dots, t(p)\}$. Using the same idempotent as above, we will find that B_F and $B_{F,h}$ are Morita equivalent. If B_F is QcF, then it must be co-Frobenius since the multiplicity of the simples in B_F is 1. Then, choosing a suitable function h , as before, we can get various QcF coalgebras which are not co-Frobenius (provided the function λ from equation (1) is not constant).

4.2. co-Frobenius coalgebras from Homological algebra. We now give several examples of QcF and co-Frobenius coalgebras (and so, implicitly, of Generalized (quasi)-Frobenius algebras) which are connected to situations in category theory, homological algebra and topology. Many of these will be obtained from graphs as above.

Example 4.3. Let Λ be the "line" graph $\dots \xrightarrow{x_{n-1}} a_{n-1} \xrightarrow{x_n} a_n \xrightarrow{x_{n+1}} a_{n+1} \longrightarrow \dots$ and Λ_1 the subcoalgebra of coalgebra $K\Lambda$ which has a basis consisting of the paths of length ≤ 1 : vertices a_n and arrows x_n . This is the first term of the coradical filtration of $K\Lambda$. The comultiplication and counit are given by

$$\Delta : a_n \rightarrow a_n \otimes a_n$$

$$\Delta : x_n \rightarrow a_n \otimes x_n + x_n \otimes a_{n+1}$$

$$\varepsilon(a_n) = 1; \quad \varepsilon(x_n) = 0$$

This is co-Frobenius, since we can easily see that $E(T_{a_n}) \cong E(S_{a_{n+1}})^*$ as right comodules and $E(S_{a_n}) \cong E(T_{a_{n-1}})^*$ as left comodules. This coalgebra is tightly connected to homological algebra: the category of right Λ_1 -comodules is equivalent to the category of chain complexes of K -modules (here, K can be any commutative ring) - see [Par81]. In fact, this category has a monoidal structure, and Λ_1 has a Hopf algebra structure, since it is isomorphic as coalgebras with the Hopf algebra $K \langle s, t, t^{-1} \rangle / (s^2, st + ts)$ with comultiplication $\Delta(t) = t \otimes t$ and $\Delta(s) = t^{-1} \otimes s + s \otimes 1$, counit $\varepsilon(s) = 1$, $\varepsilon(t) = 1$, and antipode $S(t) = t^{-1}$, $S(s) = st = -ts$. The isomorphism at coalgebra level is obviously provided by $a_n \leftrightarrow t^n$, $x_n \leftrightarrow t^n s$.

Example 4.4. Let Λ_p be the subcoalgebra of the quiver algebra of the line graph Λ above, consisting of paths of length $\leq p$. For ease of use, let us denote the path starting at a_n and having length k in Λ (the line graph) by $p_{n,k}$. In this case, with notations as before, we have the injective hulls of right comodules

$$E(T_{a_n}) = K\{a_n, x_{n+1}, [x_{n+1}x_{n+2}], \dots, [x_{n+1} \dots x_{n+p}]\} = K\{p_{n,i} | i \leq p\}$$

and the hulls of left comodules are

$$E(S_{a_n}) = K\{a_n, x_{n-1}, [x_{n-2}x_{n-1}], \dots, [x_{n-p} \dots x_{n-1}]\} = K\{p_{n-i,i} | i \leq p\}.$$

So we get $E(T(a_n)) \cong E(S_{a_{n+p}})^*$ as right comodules and $E(S_{a_n}) \cong E(T_{a_{n-p}})^*$.

Remark 4.5. This coalgebra is furthermore a Hopf algebra. Assume K contains a primitive p 'th root of unity q and let H_p be the Hopf algebra defined by $H_p = K \langle s, t, t^{-1} \rangle / (s^p, st - qts)$ as an algebra and with comultiplication $\Delta(t) = t \otimes t$ and $\Delta(s) = t^{-1} \otimes s + s \otimes 1$, counit $\varepsilon(s) = 1$, $\varepsilon(t) = 1$, and antipode $S(t) = t^{-1}$, $S(s) = -ts = -qst$. In order to see this is a Hopf algebra one only needs to show that Δ can be defined as a morphism of algebras, and so, one needs to show that $K \langle s, t, t^{-1} \rangle \ni s \mapsto t^{-1} \otimes s + s \otimes 1 \in K \langle s, t, t^{-1} \rangle / (s^p, st - qts) \otimes K \langle s, t, t^{-1} \rangle / (s^p, st - qts)$ and $K \langle s, t, t^{-1} \rangle \ni t \mapsto t \otimes t \in K \langle s, t, t^{-1} \rangle / (s^p, st - qts) \otimes K \langle s, t, t^{-1} \rangle / (s^p, st - qts)$ factor through $(s^p, st - qts)$, that is, equivalently $(t^{-1} \otimes s + s \otimes 1)^p = 0$ and $t \otimes t$ is invertible. This follows in precisely the same manner as it does in the case of the classical Taft algebras (for t it is obvious); see [Taf71] or [M]. Similarly, one shows that the antimorphism of algebras S defined this way on generators on $K \langle s, t, t^{-1} \rangle \rightarrow K \langle s, t, t^{-1} \rangle$ factors through the ideal $(s^p, st - qts)$. Now, using the quantum binomial formula we have

$$(t^{-1} \otimes s + s \otimes 1)^k = \sum_{i+j=k} \binom{k}{i}_q t^{-i} s^j \otimes s^i$$

where $\binom{k}{i}_q = \frac{(k)!_q}{(i)!_q (j)!_q}$ are the q -binomial coefficients (e.g. see [Kas]). Dividing by $(k)!_q$ which is non-zero if $k < p$ since q is a primitive p 'th root of unity, it follows that the correspondence on bases $\frac{1}{(k)!_q} t^{n+k} s^k \leftrightarrow p_{n,k}$ gives an isomorphism of coalgebras $\Lambda_{p-1} \cong H_p = K \langle s, t, t^{-1} \rangle / (s^p, st - qts)$ (the counit compatibility is obvious).

We note that there is a result which is analogue to that from [Par81] pointed out in 4.3, which generalizes the result of [Par81] and which intimately connects the coalgebra Λ_p to the p -homological algebra introduced in [Kap96] with roots in topology [M42a, M42b] and investigated later by Kassel and Wamst [KW] and Dubois-Violette [D-V] (see also [B], [ITh]). Let Ch_p denote the category of $p+1$ -chain complexes of K -modules, which are a sequences of morphisms

$$\cdots \longrightarrow M_{n-1} \xrightarrow{d_n} M_n \xrightarrow{d_{n+1}} M_{n+1} \longrightarrow \cdots$$

such that $d^{p+1} = 0$, i.e. $d_n d_{n+1} \dots d_{n+p} = 0$ for all n . The morphisms of between two objects M_* and N_* in this category are collections of the type $f_n : M_n \rightarrow N_n$ making all the appropriate diagrams commutative (i.e. $f_* d_* = d_* f_*$). This category also has a monoidal structure with the tensor product of two complexes (X_*) and (Y_*) being obtained by $(X \otimes Y)_r = \bigoplus_{n+m=r} X_n \otimes Y_m$ and with differential $d(x \otimes y) = q^{\deg(y)} d_X(x) \otimes y + x \otimes d_Y(y)$.

In the case $p = 2$ this coincides to what is usually considered the total complex of a tensor product of complexes and with the one used in [Par81] p.372, and it is similar to the tensor product used in [Kap96]. In fact, our tensor product can be obtained from the tensor product \otimes^{Kap} of [Kap96] by what is in the case $p = 2$ the usual ‘‘sign trick’’: for each $p-1$ -complex X_* we can define the complex $I(X) = (X_*)$ but with differential $I(d_n) = \frac{1}{q^n} d_n$. Then one easily sees that I is an equivalence of monoidal categories from $(Ch_{p-1}, \otimes^{Kap}, \mathbf{1}) \xrightarrow{I} (Ch_{p-1}, \otimes, \mathbf{1})$ where $\mathbf{1} = \cdots \rightarrow 0 \rightarrow (K)_0 \rightarrow 0 \rightarrow \cdots$ is the unit object of these categories. We have the following equivalent statement of the main results in [Par81] and [B]:

Theorem 4.6. *There exists an equivalence of categories $Ch_{p-1} \simeq \Lambda_{p-1}\mathcal{M}$ and one of monoidal categories $G : Ch_{p-1} \rightarrow {}^{H_p}\mathcal{M}$ which commutes with the forgetful functors $U : Ch_{p-1} \rightarrow {}_K\mathcal{M}$, $U(X_*) = \bigoplus_n X_n$ and $V : {}^{H_p}\mathcal{M} \rightarrow {}_K\mathcal{M} : VG \cong U$.*

We remark that these inverse equivalences of categories are defined as follows: for a $p-1$ -complex (X_*, d_{X_*}) define $G(X) = \bigoplus_n X_n$ and for a morphism of complexes $f_* : X_* \rightarrow Y_*$ define $G(f) = \bigoplus_n f_n$. Let us convey to identify the elements $p_{n,k} \in \Lambda_{p-1}$ with $\frac{1}{(k)!_q} t^{n+k} s^k \in H_p$. On $G(X)$ introduce the following left H_p -comodule structure: $\rho_{G(X)} : X \rightarrow H_p \otimes X$ such that for $x_n \in X_n$,

$$(2) \quad \rho_{G(X)}(x_n) = \sum_{i=0}^{p-1} p_{n,i} \otimes d_X^i(x_n) = \sum_{i=0}^{p-1} p_{n,i} \otimes d_{X,n+i-1} \cdots d_{X,n+1} d_{X,n}(x_n)$$

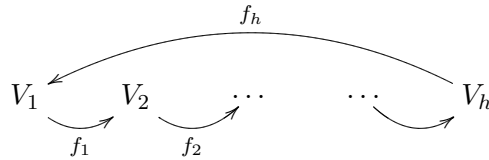
Conversely, define $T : {}^{H_p}\mathcal{M} \rightarrow Ch_{p-1}$ as follows: if $M \in {}^{H_p}\mathcal{M}$ let $M_n = M \cdot p_{n,0}^*$ and $d_{n+1} : M_n \rightarrow M_{n+1}$, $d_{n+1}(x_n) = x_n \cdot p_{n,1}^*$, where the elements $p_{n,k}^* \in H_p^*$ are the dual ‘‘basis’’ for $p_{n,k}$, that is, $p_{n,k}^*(p_{m,l}) = \delta_{n,m} \delta_{k,l}$. Then let $T(M) = ((M_n)_n, (d_n)_n)$. For morphisms $g : M \rightarrow T$ in ${}^{H_p}\mathcal{M}$ we have $g(x \cdot p_{n,0}^*) = g(x) \cdot p_{n,0}^*$ so $g(M_n) \subseteq P_n$, and we can define $g_n : M_n \rightarrow P_n$ by $g_n = g|_{M_n}$.

Example 4.7. *Let Γ be the circle graph*



for a positive integer h and let p be another positive integer. For each vertex a_k denote again $p_{n,k}$ the path of length k in Γ which starts at a_n . Then the coalgebra $\Gamma_{p,h}$ which is the subcoalgebra of $K\Gamma$ having a basis of the paths $\{p_{n,k} | k \leq p\}$ is co-Frobenius (and finite dimensional). Indeed, again we see that the right injective indecomposables are $E(T_{a_n}) = K\{p_{n,k} | k \leq p\}$ and the right injective indecomposables are $E(S_{a_n}) = K\{p_{n-k,k} | k \leq p\}$ (here indices are taken mod h). Then we get that $E(T_{a_n}) \cong E(S_{a_{n+p}})^*$ and we can use again Theorem 1.7.

Remark 4.8. By standard representation theory of quivers, it follows that the category of left $\Gamma_{p,h}$ -comodules is equivalent to the category of the representations of the ‘‘cyclic’’ quiver Γ in the example above, with the condition that the composition of $p+1$ consecutive morphisms is 0; that is the category of diagrams of the form



and such that $f_{i+p} f_{i+p-1} \cdots f_{i+1} f_i = 0$ for all i , where indices are considered mod h . Moreover, let us consider the ideal $I = (t^h - 1)$ of H_p generated by $t^h - 1$. We note that this is a Hopf ideal: $\Delta_{H_p}(t^h - 1) = t^h \otimes t^h - 1 \otimes 1 = (t^h - 1) \otimes t^h + 1 \otimes (t^h - 1)$ and $S_{H_p}(t^h - 1) = t^{-h} - 1 = t^{-h}(1 - t^h)$. Hence, the algebra $H_{p,h} = K \langle s, t, t^{-1} \rangle / (s^p, st - qts, t^h - 1)$ is a Hopf algebra. Now note that in this Hopf algebra $H_{p,h}$, we have $s = st^h = q^h t^h s = q^h s$ and since $s \neq 0$, we must have $q^h = 1$, i.e. $p|h$. If this is the case, applying Bergman’s

Diamond Lemma we easily see that $\{s^i t^j | i = 0, \dots, p-1; j = 0 \dots, h-1\}$ is a K -basis for $H_{p,h}$. Then, by considerations entirely analogue to those in Theorem 4.6 and before, we can see that $K\Gamma_{p-1,h} \cong H_{p,h}$ as coalgebras.

Note also that, denoting by $Ch_{p-1,h}$ the category of such representations of this quiver with condition $f^p = 0$ (defined similarly to Ch_{p-1}), we see that there is an equivalence of tensor categories $Ch_{p-1,h} \simeq^{H_{p,h}} \mathcal{M}$. Note also that this Hopf algebra $H_{p,h}$ generalizes the Taft Hopf algebras of dimension p^2 , only $H_{p,h}$ will have dimension ph , with $p|h$.

Remark 4.9. Moreover, the existence of an antipode S of H_p and $H_{p,h}$ implies that the categories of finite dimensional comodules over H_p and $H_{p,h}$ are tensor categories, i.e. they are rigid (that is, they have left and right duals for objects). This means Ch_{p-1} and $Ch_{p-1,h}$ are also tensor categories. By straightforward computation which uses the equivalence between these categories and those of associated comodules, as well as the antipode of H_p and $H_{p,h}$, we can see that the right dual of a ‘‘complex’’ V in Ch_{p-1} or $Ch_{p-1,h}$ which is bounded and has all V_n 's finite dimensional

$$V : \dots \longrightarrow V_{n-1} \xrightarrow{d_n} V_n \xrightarrow{d_{n+1}} V_{n+1} \longrightarrow \dots$$

is the complex

$$V^* : \dots \longleftarrow (V^*)_{1-n} \xleftarrow{(d^*)_{1-n}} (V^*)_{-n} \xleftarrow{(d^*)_{-n}} (V^*)_{-n-1} \longleftarrow \dots$$

with $(V^*)_n = (V_{-n})^*$ as vector spaces and $(d^*)_{n+1} = -q^{-n-1}(d_{-n})^*$. The left dual of this object is given by

$${}^*V : \dots \longleftarrow ({}^*V)_{1-n} \xleftarrow{({}^*d)_{1-n}} ({}^*V)_{-n} \xleftarrow{({}^*d)_{-n}} ({}^*V)_{-n-1} \longleftarrow \dots$$

with $({}^*V)_n = (V_{-n})^*$ as vector spaces and $({}^*d)_{n+1} = -q^n(d_{-n})^*$ (indices are considered mod h when we talk about $Ch_{p-1,h}$).

An Application

A well known result on the category of (1)-chain (i.e. usual) complexes (X, d) with $d^2 = 0$ is that any such complex is a direct sum of complexes of the type $\dots \rightarrow 0 \rightarrow (K)_n \rightarrow 0 \rightarrow \dots$ and $\dots \rightarrow 0 \rightarrow (K)_n \rightarrow (K)_{n+1} \rightarrow 0 \rightarrow \dots$. We note how the above equivalences and a result on coalgebras imply this result and its generalization to $p-1$ -complexes as a consequence:

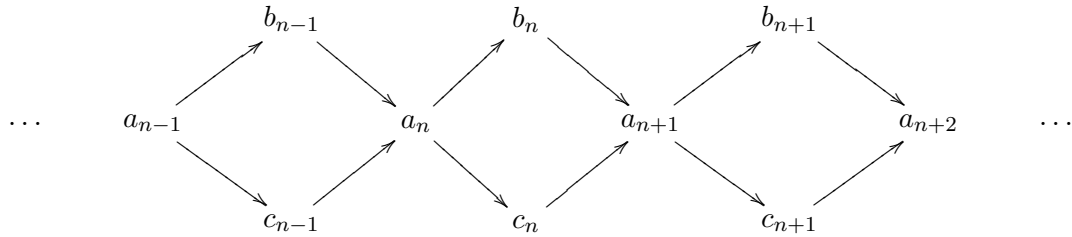
Theorem 4.10. *Any $(p-1)$ complex in Ch_{p-1} or $Ch_{p-1,h}$ is a direct sum of complexes of the type $\dots \rightarrow 0 \rightarrow (K)_{n+1} \xrightarrow{=} (K)_{n+2} \rightarrow \dots \rightarrow (K)_{n+i} \rightarrow 0 \rightarrow \dots$ with $i = 1, 2, \dots, p$ (the indices are always mod h when we are talking about $Ch_{p-1,h}$); here $(K)_i$ means the field K is on position i , and the morphisms are identities.*

Proof. We see that the injective envelopes of simple left comodules for the coalgebras $H_{p,h} \cong \Gamma_{p-1,h}$ and $H_p \cong \Lambda_{p-1}$ are $E(S_{a_n}) = K\{p_{n-k,k} | k \leq p\}$ and they are chain (uniserial) comodules: that is, their lattice of submodules is a chain (see [CT04] or [I09] for properties). This follows easily by noting that the socle of the left comodule $K\{p_{n-k,k} | i \leq k \leq p\} \cong E(S_{a_n})/K\{p_{n-k,k} | k < i\}$ has socle $K\{p_{n-k,k}\}$ which is simple. Thus, we can apply for example [I09, Proposition 3.2] (in fact, it is not difficult to show that $H_{p,h}$ are isomorphic to the $p-1$ 'th term of the coradical filtration of the coalgebra $K_\sigma^h[X]$ of [I09, Example 5.5] for σ a cyclic permutation). Then, since for these coalgebras by [CT04, Proposition 1.13] any left comodule is a direct sum of indecomposable chain

comodules, which must be submodules of the injective envelopes $E(S_{a_n})$, so they are isomorphic to some $E_{n,i} = K\{p_{n-k,k} | 0 \leq k < i\}$ for some $i = 1, 2, \dots, p$. Using the equivalence of categories proved before in Theorem 4.6 and Remark 4.8, we get that each complex is a direct sum of chain complexes corresponding to the $E_{n,i}$'s, which are of the form $\dots \rightarrow 0 \rightarrow (K)_{n+1} \rightarrow (K)_{n+2} \rightarrow \dots \rightarrow (K)_{n+i} \rightarrow 0 \rightarrow \dots$ with $i = 1, 2, \dots, p$. \square

4.3. More examples.

Example 4.11. Consider the graph Θ_∞ obtained by a “string of diamond diagrams”:



for all integers n . We can also obtain a variation of this graph Θ_h if we “close” the string of diamonds into a loop of h such diamonds. Consider the path coalgebra C which is the subcoalgebra of $K\Theta_h$ ($h \in \mathbb{Z} \cup \{\infty\}$) with a basis consisting of the paths of length 0 (vertices) and 1 and also the paths of length 2 ($b_n a_{n+1} b_{n+1}$), ($c_n a_{n+1} c_{n+1}$) and the elements $(a_n b_n a_{n+1}) + (a_n c_n a_{n+1})$ with (uvw) representing the path through the vertices u, v, w . This coalgebra is co-Frobenius: the left injective indecomposables are

- $E(S_{a_n}) = K\{a_{n-1}, (a_{n-1} b_{n-1}), (a_{n-1} c_{n-1}), (a_{n-1} b_{n-1} a_n) + (a_{n-1} c_{n-1} a_n)\}$ (spanned by the paths in C ending at a_n),

- $E(S_{b_n}) = \{b_{n-1}, (b_{n-1} a_n), b_{n-1} a_n b_n\}$ and

- $E(S_{c_n}) = \{c_{n-1}, (c_{n-1} a_n), c_{n-1} a_n c_n\}$

and the right injective indecomposables are

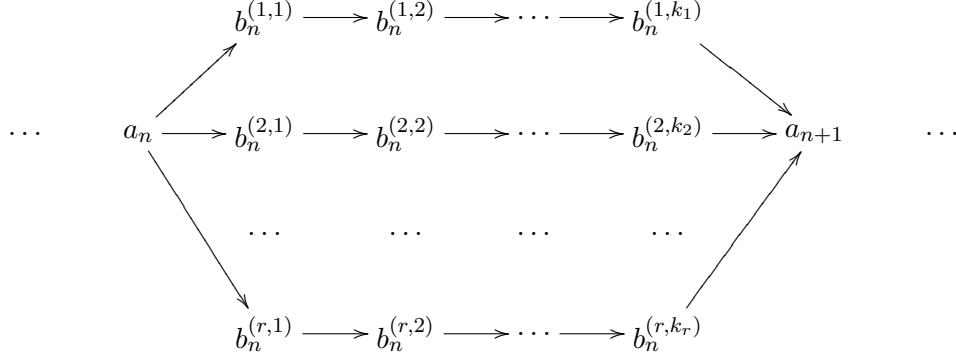
- $E(T_{a_n}) = K\{a_n, (a_n b_n), (a_n c_n), (a_n b_n a_{n+1}) + (a_n c_n a_{n+1})\}$ (spanned by the paths in C starting at a_n),

- $E(T_{b_n}) = \{b_n, (b_n a_{n+1}), b_n a_{n+1} b_{n+1}\}$ and

- $E(T_{c_n}) = \{c_n, (c_n a_{n+1}), c_n a_{n+1} c_{n+1}\}$.

Then we can see that $E(T_{a_n}) \cong E(S_{a_{n+1}})^*$, $E(T_{b_n}) \cong E(S_{b_{n+1}})^*$ and $E(T_{c_n}) \cong E(S_{c_{n+1}})^*$ as right C -comodules which show that C is co-Frobenius. Furthermore, we can extend this example by considering C_p to be the coalgebra with a basis consisting of all paths of length $\leq 2p$ starting at any b_n, c_n , all paths of length $\leq 2p-1$ starting at any a_n , and the elements $(a_n b_n a_{n+1} b_{n+1} \dots a_{n+p-1} b_{n+p-1} a_{n+p}) + (a_n c_n a_{n+1} c_{n+1} \dots a_{n+p-1} c_{n+p-1} a_{n+p})$. We leave out the details which are similar to the one in the previous examples.

Example 4.12. A more general example of the same type is obtained by looking at an infinite string of the following type of diagrams



This quiver $\Delta_\infty(k_1, \dots, k_r)$ consists of such a diagram between a_n and a_{n+1} for each n (r, k_1, \dots, k_r are fixed). We consider the subcoalgebra C' of the full path coalgebra of this quiver, with a basis consisting of the following: the paths of length $\leq k_i + 1$ starting at $b_n^{(i,j)}$ and passing ONLY through points of the type $b_*^{i,t}$ (that is, paths which continuing after a_{n+1} maintain level i), all paths starting at a_n but not reaching a_{n+1} and the elements z_n which equal the sum of all the paths between a_n and a_{n+1} . As before, one can show this is a co-Frobenius coalgebra.

Remark 4.13. We note that in all given examples of co-Frobenius and QcF coalgebras, except Example 4.12, the length of the coradical filtration (Loewy length) of the injective indecomposables are all equal (see coming examples below too); in 4.12, however, we see that they can vary if the k_1, \dots, k_r are different.

It is interesting to note that the above example can be thought as the quotient coalgebra $\bigoplus_{i=1}^r \Lambda_{k_i} / K\{p_{nk_i,0} - p_{nk_j,0} \mid i \neq j\}$, with the space spanned by the elements $\{p_{nk_i,0} - p_{nk_j,0} \mid i \neq j\}$ being a coideal, and a corresponding quotient when we consider the closed “loop” case.

Example 4.14. Let C, D be two QcF coalgebras. Then, by [GMN, Theorem 2.3], $C \otimes D$ is QcF.

Proposition 4.15. Let C and D be co-Frobenius K -coalgebras such that the endomorphism of every C and every D simple left comodule S is 1-dimensional ($\text{End}(S) = K$, in particular, when K is algebraically closed). Then $C \otimes D$ is co-Frobenius.

Proof. If the condition holds, then $C_0 = \bigoplus_S M_{n_S}^c(K)$ and $D_0 = \bigoplus_{S'} M_{m_{S'}}^c(K)$ are sums of comatrix coalgebras. In this case, $(C \otimes D)_0 = C_0 \otimes D_0 = \bigoplus_{S, S'} M_{n_S \cdot m_{S'}}^c(K)$ and the simple left $C \otimes D$ -comodules are $S \otimes S'$ for S, S' simple left C and respectively D -comodules. Also, $C = \bigoplus_S E(S)^{n_S}$ and $D = \bigoplus_{S'} E(S')^{m_{S'}}$ as left comodules easily implies $C \otimes D = (E(S) \otimes E(S'))^{n_S \cdot m_{S'}}$ as left $C \otimes D$ -comodules. Since $S \otimes S' \subseteq E(S) \otimes E(S')$, it follows that $E(S \otimes S') = E(S) \otimes E(S')$. We also have for each simple $C \otimes D$ -comodule $S \otimes S'$ that $E(S) \cong E(T)^*$ and $E(S') \cong E(T')^*$ with T a simple right C -comodule and T' a simple right D -comodule, so $E(S \otimes S') = E(S) \otimes E(S') \cong (E(T) \otimes E(T'))^* = (E(T \otimes T'))^*$. Therefore, if C and D are co-Frobenius, the multiplicities of S and T in C coincide the multiplicities of S' and T' in D coincide too. In this case, this means $\dim(S) = \dim(T)$ and $\dim(S') = \dim(T')$ and it follows that $\dim(S \otimes S') = \dim(T \otimes T')$ i.e. the multiplicity

in $C \otimes D$ of the socle and cosocle of $E(S \otimes S') = E(T \otimes T')^*$ are the same. By Proposition 2.2 and remark thereafter it follows that $C \otimes D$ is co-Frobenius. \square

Example 4.16. Let $BiCh$ be the category of bicomplexes,

$$\begin{array}{ccccccc}
 & & \cdots & & \cdots & & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 \cdots & \longrightarrow & X_{k+1,n-1} & \xrightarrow{d_{k+1,n}} & X_{k+1,n} & \xrightarrow{d_{k+1,n+1}} & X_{k+1,n+1} \longrightarrow \cdots \\
 & & \uparrow d'_{k+1,n-1} & & \uparrow d'_{k+1,n} & & \uparrow d'_{k+1,n+1} \\
 (*) & \cdots & \longrightarrow & X_{k,n-1} & \xrightarrow{d_{k,n}} & X_{k,n} & \xrightarrow{d_{k,n+1}} & X_{k,n+1} \longrightarrow \cdots \\
 & & \uparrow d'_{k,n-1} & & \uparrow d'_{k,n} & & \uparrow d'_{k,n+1} \\
 \cdots & \longrightarrow & X_{k-1,n-1} & \xrightarrow{d_{k-1,n}} & X_{k-1,n} & \xrightarrow{d_{k-1,n+1}} & X_{k-1,n+1} \longrightarrow \cdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \cdots & & \cdots & & \cdots
 \end{array}$$

with the usual conditions that $d'_{k,n}d'_{k-1,n} = 0$, $d_{k,n}d_{k,n-1} = 0$ and the squares commute: $d_{k,n}d'_{k,n-1} = d'_{k,n}d_{k-1,n}$. This is the category of chain complexes in the abelian category $Ch = Ch_1$ of chain complexes of vector spaces. The morphisms are considered as usual, families $f_{k,n} : X_{k,n} \rightarrow Y_{k,n}$ making all the appropriate diagrams commutative. Note that sometimes this category is considered to be such that squares “anticommute”: $d_{k,n}d'_{k-1,n} = -d'_{k-1,n+1}d_{k-1,n}$, but the two are equivalent by a usual sign trick (see, for example [W, 1.2.5]). This category is then equivalent to the category of left comodules over the coalgebra $\Lambda_1 \otimes \Lambda_1$. For this it suffices to see that $\Lambda_1 \otimes \Lambda_1$ has a basis consisting of elements $p_{n,k} \otimes p_{m,l}$ with $0 \leq k, l \leq n$ and to any bicomplex $(X_{k,n})_{k,n}$ we can associate the left $\Lambda_1 \otimes \Lambda_1$ -comodule $\bigoplus_{k,n} X_{k,n}$ with coaction ρ which for $u_{k,n} \in X_{k,n}$ reads $\rho(u_{k,n}) = (a_k \otimes a_n) \otimes u_{k,n} + (a_k \otimes p_{n,1}) \otimes d(u_{k,n}) + (p_{k,1} \otimes a_n) \otimes d'(u_{k,n}) + (p_{k,1} \otimes p_{n,1}) \otimes dd'(u_{k,n})$ (note that $dd' = d'd$, with appropriate indices omitted). It is straightforward to check that this is a comodule structure. Conversely, to each $\Lambda_1 \otimes \Lambda_1$ -comodule Y we associate a double chain complex as follows. Denote, as before, c^* the dual “basis” elements in $(\Lambda_1 \otimes \Lambda_1)^*$ corresponding to $c \in \{p_{n,k} \otimes p_{m,l} | 0 \leq k, l \leq n\}$. We let $Y_{k,n} = Y \cdot (a_k \otimes a_n)^*$, and note that in this situation we have $Y = \bigoplus_{k,n} Y_{k,n}$. Moreover, for $y_{k,n} \in Y_{k,n}$ we let $d(y_{k,n}) =$

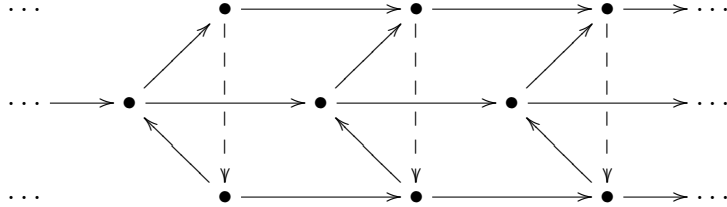
$y_{k,n} \cdot (a_k \otimes p_{n,1})^* = y_{k,n} \cdot (p_{k,0} \otimes p_{n,1})^*$ and $d'(y_{k,n}) = y_{k,n} \cdot (p_{k,1} \otimes p_{n,0})^*$. The fact that $d^2 = 0$ and $(d')^2 = 0$ follows as in the computations in [Par81] and the above examples, and since $(p_{k,0} \otimes p_{n,1})^* \cdot (p_{k,1} \otimes p_{n,0})^* = (p_{k,1} \otimes p_{n,0})^* \cdot (p_{k,0} \otimes p_{n,1})^*$ can be tested by direct application on all the elements of the basis, we get $dd' = d'd$.

Since the computation are similar to those in Theorem 4.6 and in [Par81], we live the details to the reader.

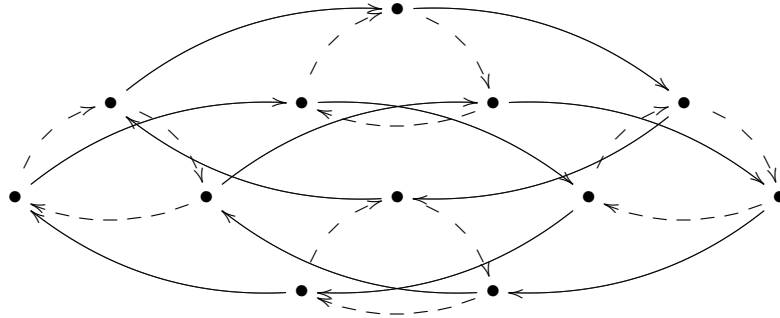
By generalizing the above results, we get the following

Theorem 4.17. Let $Ch_{p-1,r-1}$ be the category of double complexes of the type $(*)$ from Example 4.16 such that $d^p = 0$, $(d')^r = 0$, and the squares commute: $dd' = d'd$. Then $Ch_{p-1,r-1}$ is equivalent to the category of comodules over the Hopf algebra $H_p \otimes H_r$. Moreover, this is an equivalence of monoidal categories.

Note that even more examples can be obtained by combining the Hopf algebras H_p and $H_{p,h}$; denote $H_{p,\infty} = H_p$, so then we have defined $H_{p,h}$ for $h = 1, 2, 3, \dots$, and $h = \infty$ ($H_{p,h}$ is just a coalgebra if $p \nmid h$). Then the category of comodules over $H_{p,h} \otimes H_{r,l}$ can be thought of the category of double chain complexes, with h columns and l lines, where whenever h or l is finite, these are thought of as cycles. In these categories, squares commute, and the relation $d^p = 0$ is needed on the horizontal and $(d')^r = 0$ on the vertical. If h is finite and l is infinite, than these diagrams can be represented by an infinite cylinder with the required properties. When both h, l are finite, these diagrams can be represented by a torus with appropriate commutation relations. For example, for $H_{3,3} \otimes H_{r,\infty}$, the category of comodules is tensor equivalent to the category of diagrams of vector spaces of the following "cylindrical" shape (some arrows are broken only for easier visualization):



and with morphisms being collections of pointwise linear maps (f), making all the appropriate diagrams commutative: $f_*d_* = d_*f_*$. Similarly, for another example, the category of corepresentations (comodules) of the Hopf algebra (or coalgebra) $H_{p,3} \otimes H_{r,4}$ (with $p = 3$ and $r \in \{2, 4\}$ for Hopf algebras) is tensor equivalent to the category of diagrams of the following "torus shape", with appropriate morphisms:



ACKNOWLEDGMENT

This work was supported by the strategic grant POSDRU/89/1.5/S/58852, Project "Post-doctoral programe for training scientific researchers" cofinanced by the European Social Fund within the Sectorial Operational Program Human Resources Development 2007-2013.

REFERENCES

- [A] E. Abe, *Hopf Algebras*, Cambridge Univ. Press, 1977.
- [Ab] L. Abrams, *Two-dimensional topological quantum field theories and Frobenius algebras*, J. Knot Theory Ramifications 5 (1996), no. 5, 569 - 587.
- [AF] D. Anderson, K. Fuller, *Rings and Categories of Modules*, Grad. Texts in Math., Springer, Berlin-Heidelberg-New York, 1974.
- [B] J. Bichon, *N-Complexes et algebres de Hopf*, C. R. Acad. Sci. Paris, Ser. I 337 (2003), 441-444.
- [CKQ] W. Chin, M. Kleiner and D. Quinn, *Almost split sequences for comodules*, J. Algebra 249 (2002), 1-19.
- [CM] W. Chin, S. Montgomery, *Basic coalgebras*, AMS/IP Stud. Adv. Math. 4(1997), 41-47.
- [CNO] J. Cuadra, C. Năstăsescu, F. Van Oystaeyen, *Graded almost noetherian rings and applications to coalgebras*, Journal of Algebra **256** (2002), 97 - 110.
- [CT04] J. Cuadra, B. Torrecillas, *Serial coalgebras*, Journal of Pure and Applied Algebra 189 (2004), 89 - 107.
- [D] Y. Doi, *Homological Coalgebra*, J. Math. Soc. Japan **33**(1981), 31-50.
- [D-V] M. Dubois-Violette, *$dN = 0$: generalized homology*, K-Theory 14 (1998) 371-404.
- [DNR] S. Dăscălescu, C. Năstăsescu, Ş. Raianu, *Hopf Algebras: an introduction*. Vol. 235. Lecture Notes in Pure and Applied Math. Vol.235, Marcel Dekker, New York, 2001.
- [F] C. Faith, *Algebra II: Ring Theory*. Vol 191, Springer-Verlag, Berlin-Heidelberg-New York, 1976.
- [G] P. Gabriel, *Des Catégories Abéliennes*, Bulletin de la S.M.F., tome 90 (1962), p. 323-448.
- [GTN] J. Gomez-Torrecillas, C. Năstăsescu, *Quasi-co-Frobenius coalgebras*, J. Algebra **174** (1995), 909-923.
- [GMN] J. Gomez-Torrecillas, C. Manu, C. Năstăsescu, *Quasi-co-Frobenius coalgebras II*, Comm. Algebra Vol 31, No. 10, pp. 5169-5177, 2003.
- [HR74] R. Heyneman, D. Radford, *Reflexivity and Coalgebras of Finite Type*, J. Algebra **28** (1974), 215-246.
- [INV06] F. Castaño-Iglesias, C. Năstăsescu, J. Vercauteren, *Quasi-Frobenius Functors. Applications*, arXiv:math/0612662.
- [ITh] M. Iovanov, *The Representation Theory of Profinite Algebras - Interactions with Category Theory, Algebraic Topology and Compact Groups*, PhD Thesis, SUNY Buffalo, 2009.
- [I] M.C. Iovanov, *Co-Frobenius Coalgebras*, J. Algebra 303 (2006), no. 1, 146-153; eprint arXiv:math/0604251, <http://xxx.lanl.gov/abs/math.QA/0604251>.
- [IO9] M.C. Iovanov, *When Does the Rational Torsion Split Off for Finitely Generated Modules*, Algebr. Represent. Theory **12** (2009), 287-309. DOI 10.1007/s10468-009-9144-7
- [Kap96] M. Kapranov, *On the q -analogue of Homological algebra*, preprint, ArXiv.9611005v1; http://arxiv.org/PS_cache/q-alg/pdf/9611/9611005v1.pdf
- [Kas] C. Kassel, *Quantum Groups*, Graduate Texts in Mathematics vol 155, Springer-Verlag, 1995.
- [KW] C. Kassel, M. Wambst, *Algebre homologique des N-complexes et homologie de Hochschild aux racines de l'unité*, Publ. Res. Inst. Math. Sci. Kyoto 34 (1998) 91-114.
- [L] B.I-Peng Lin, *Semiperfect coalgebras*, J. Algebra **49** (1977), 357-373.
- [LS] R.G. Larson; M.E. Sweedler, *An associative orthogonal bilinear form for Hopf algebras*, Amer. J. Math. 91 1969 75-94.
- [M42a] W. Mayer, *A new homology theory*, I. Ann. of Math. 43 (1942), 370-380.
- [M42b] W. Mayer, *A new homology theory*, II. Ann. of Math. 43 (1942), 594-605.
- [M] S. Montgomery, *Hopf algebras and their actions on rings*, Amer. Math. Soc., Providence, RI, 1993.
- [MTW] C. Menini, B. Torrecillas Jover, R. Wisbauer, *Strongly rational comodules and semiperfect Hopf algebras over QF rings*, J. Pure Appl. Algebra 155 (2001), no. 2-3, 237-255.
- [Par81] B. Pareigis, *A noncommutative noncocommutative Hopf algebra in "nature"*, J. Algebra **70** (1981), no. 2, 356-374.
- [R] D.E. Radford, *Finiteness conditions for a Hopf algebra with a nonzero integral*, J. Algebra 46 (1977), no. 1, 189-195.
- [R1] D. Radford, *Coreflexive coalgebras*, J. Algebra **26** (1973), 512-535.
- [Rad82] D. E. Radford, *On the structure of pointed coalgebras*, J. Algebra **77** (1982), 1-14.
- [Ra] Ş. Raianu, *An easy proof for the uniqueness of integrals*. Hopf algebras and quantum groups (Brussels, 1998), 237-240, Lecture Notes in Pure and Appl. Math., 209, Dekker, New York, 2000.

- [Si] D. Simson, *Incidence coalgebras of intervally finite posets, their integral quadratic forms and comodule categories*, Colloq. Math. 115 (2009), 259–295.
- [Su] J.B. Sullivan, *The uniqueness of integrals for Hopf algebras and some existence theorems of integrals for commutative Hopf algebras*, J. Algebra 19 1971 426–440.
- [Sw] M.E. Sweedler, *Hopf Algebras*, Benjamin, New York, 1969.
- [Sw1] M.E. Sweedler, *Integrals for Hopf algebras*, Ann. Math 89 (1969), 323–335.
- [St] D. Ştefan, *The uniqueness of integrals (a homological approach)*, Comm. Algebra 23 (1995), no. 5, 1657–1662.
- [Tak] M. Takeuchi, *Topological Coalgebras*, J. Algebra **97** (1985), 505–539.
- [Tak77] M. Takeuchi, *Morita theorems for categories of comodules*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 24 (1977), no. 3, 629–644.
- [Taf71] E.J. Taft, *The Order of the Antipode of Finite-dimensional Hopf Algebra*, Proc. Nat. Acad. Sci. USA, vol. 68, No.11 (Nov 1971), 2631–2633.
- [vD] A. van Daele, *The Haar measure on finite quantum groups*, Proc. Amer. Math. Soc. 125 (1997), no. 12, 3489–3500.
- [vD1] A. Van Daele, *An algebraic framework for group duality*, Adv. Math. 140 (1998), no. 2, 323–366.
- [W] C. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Math., vol. 38, Cambridge Univ. Press, 1994.

MIODRAG C IOVANOV

UNIVERSITY OF SOUTHERN CALIFORNIA

DEPARTMENT OF MATHEMATICS, 3620 SOUTH VERMONT AVE. KAP 108

LOS ANGELES, CALIFORNIA 90089-2532

AND

UNIVERSITY OF BUCHAREST, FACULTY OF MATHEMATICS, STR. ACADEMIEI 14

RO-010014, BUCHAREST, ROMANIA

E-mail address: yovanov@gmail.com; iovanov@usc.edu