COMODULES OVER SEMIPERFECT CORINGS

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Abstract. We discuss when the Rat functor associated to a coring satisfying the left α-condition is exact. We study the category of comodules over a semiperfect coring. We characterize semiperfect corings over artinian rings and over qF-rings.

Introduction

The aim of this note is to generalize properties of semiperfect coalgebras over fields, as discussed in [13], see also [8], to semiperfect corings. We also extend some results given in [4].

Corings were introduced by Sweedler [14]. A coring over a (possibly noncommutative) ring R is a coalgebra (or comonoid) in the category of R-bimodules. Since the beginning of the 21st century, there has been a renewed interest in corings and comodules over a coring, iniated by Brzeziński's paper [3]. The key point is that Hopf modules and most of their generalizations (relative Hopf modules, graded modules, Yetter-Drinfeld modules and many more) are comodules over a certain coring. This observation appeared in MR 2000c 16047 written by Masuoka, who tributed it to Takeuchi, but apparently it was already known by Sweedler, at least in the case of Hopf modules. It has lead to a unified and simplified treatment of the above mentioned modules, and new viewpoints on subjects like descent theory and Galois theory. For an extensive treatment, we refer to [4].

In this paper, we study semiperfect corings. A coring is called right semiperfect if it satisfies the left α-condition, and the (abelian) category of right $\mathcal{C}$-comodules is semiperfect, which means that every simple object has a projective cover. It turns out that this notion is closely related to rationality properties of modules over the dual of the coring (which is a ring). Rationality properties have been studied in [1] and [6]. The Rat functor sends a module over the dual of the coring to its largest rational submodule. It can be described using the category $\sigma[M]$. The category $\sigma[M]$ is discussed briefly in Section 1, and the Rat functor is introduced in Section 2. General facts on the category $\sigma[M]$ show that the exactness of the Rat functor is connected to some topological properties of the base ring R, more precisely the M-adic topology on M. In the case of corings, the $\mathcal{C}$-adic topology on ${}^*\mathcal{C}$ coincides with the finite topology, motivating a general study of the properties of the finite topology. We then give some connections between density properties, direct sum decompositions and the exactness of Rat. We show (see Corollary 2.7) that the Rat functor is exact if the coring $\mathcal{C}$ can be decomposed as a direct sum

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of finitely generated left $\mathcal{C}$-comodules. Under certain conditions, which hold if R is a qF-ring, we can prove the converse, namely if Rat is exact, then there is a direct sum decomposition of $\mathcal{C}$ into finitely generated comodules. This is in fact an application of the duality between left and right finitely generated modules over qF-rings.
In Section 3, we characterize semiperfect corings over artinian rings. The main result is Theorem 3.1, stating that a coring over an artinian ring is right semiperfect if and only if the category of right comodules has enough projectives, if and only if it has a projective generator, if and only if every finitely generated comodule has a finitely generated projective cover.
In Section 4, we discuss some applications and examples. First, we apply our results to the case where R is a qF-ring. We recover a result of [10] telling that a left and right (locally) projective coring over a qF-ring is right semiperfect if and only if the Rat functor is exact. Also two-sided prefectness is equivalent to two-sided semiperfectness for corings over qF-rings.
finally, we give some examples, focussing on the Sweedler coring associated to a ring morphism. In particular, we can describe the Rat functor in this situation, and we can discuss when the assumptions of the results in Section 3 and 4.1 are satisfied.

1. Preliminary results

1.1. **The category** $\sigma[M]$**.** Let R be a ring, and $M \in {}_R\mathcal{M}$. Recall from [15, Sec. 15] that $\sigma[M]$ is the full subcategory of ${}_R\mathcal{M}$ consisting of R-modules that are subgenerated by M, that is, submodules of an epimorphic image of $M^{(I)}$, for some index set I. $\sigma[M]$ is the smallest closed subcategory of ${}_R\mathcal{M}$ containing M. Since epimorphic images of objects of $\sigma[M]$ belong to $\sigma[M]$ (see [15, Prop. 15.1]), we have for any $N \in {}_R\mathcal{M}$ that

$$\mathcal{T}^M(N) = \sum\{f(X) \mid X \in \sigma[M],\ f \in {}_R\mathrm{Hom}(N, X)\} \in \sigma[M].$$

$\mathcal{T}^M :\ {}_R\mathcal{M} \to \sigma[M]$ is called the trace functor, and it is straightforward to show that $\mathcal{T}^M$ is the right adjoint of the inclusion functor $i:\ \sigma[M] \to {}_R\mathcal{M}$. Therefore $\mathcal{T}^M$ is left exact; it is also not difficult to see that

$$\mathcal{T}^M(N) = \sum\{X \mid X \subset \sigma[M],\ X \subset M\}.$$

For $X, Y \in {}_R\mathrm{Hom}(X, Y)$, we consider the finite topology on ${}_R\mathrm{Hom}(X, Y)$. A basis of open sets consists of

$$\mathcal{O}(f, x_1, \cdots, x_n) = \{g \in {}_A\mathrm{Hom}(X, Y) \mid g(x_i) = f(x_i),\ \text{for all } i = 1, \cdots, n\}$$

We have a natural map $r:\ R \to {}_{\mathbb{Z}}\mathrm{Hom}(M, M)$, $r_a(m) = am$. The finite topology on ${}_Z Z\mathrm{Hom}(M, M)$ induces a topology on R, called the M-adic topology.
An ideal T of R is called M-dense in R if it is dense in the M-adic topology. This means that for all $a \in R$ and $m_1, \cdots, m_n \in M$, there exists a $b \in T$ such that $am_i = bm_i$, for all i. A left T-module N is called unital if for every $n \in N$, there exists $t \in T$ such that $tn = n$, or, equivalently, for every finite $\{n_1, \cdots, n_k\} \subset N$, there exists $t \in N$ such that $tn_i = n_i$, for all i.
The proof of Proposition 1.1 is straightforward; we also refer to [4, Sec. 41].

Proposition 1.1. *Let R be a ring, and $M \in {}_R\mathcal{M}$.*
(a) For an ideal T of R, and a faithful R-module M, the following assertions are equivalent.

(i) *T is M-dense in R;*
(ii) *M is a unital T-module (with the induced structure from R);*
(iii) *$TN = N$ for all $N \in \sigma[M]$;*
(iv) *the multiplication map $T \otimes_R N \to N$ is an isomorphism.*

(b) $T = \mathcal{T}^M(A)$ is an ideal of A, and the following assertions are equivalent.

(i) *T is M-dense in A;*
(ii) *M is a T-unital module;*
(iii) *$\mathcal{T}^M$ is exact;*
(iv) *$T^2 = T$ and T is a generator in $\sigma[M]$.*

Let K be an A-submodule of M. Recall (see e.g. [15, 19.1]) that K is called superfluous or small, written $K \ll M$, if for every submodule $L \subset M$, $K + L = M$ implies that $L = M$. An epimorphism $f : M \to N$ is called superfluous if $\operatorname{Ker} f \ll M$. Note that this definition can be extended to abelian categories.

Proposition 1.2. *Assume that $\mathcal{T}^M$ is exact.*

(i) *The class $\sigma[M]$ is closed under small epimorphisms in ${}_A\mathcal{M}$;*
(ii) *the inclusion functor $\sigma[M] \to {}_A\mathcal{M}$ preserves projectives.*

Proof. (i) Take $N \in \sigma[M]$, and let

$$0 \to K \to X \to N$$

be an exact sequence in ${}_A\mathcal{M}$ such that K is small in X. then $Y = X/(K+\mathcal{T}^M(X))$ is a quotient of $X/K = N \in \sigma[M]$, so $Y \in \sigma[M]$, by [15, 15.1], and $\mathcal{T}^M(Y) = Y$. Consider the exact sequence

$$0 \to \mathcal{T}^M(X) \to X \to X/\mathcal{T}^M(X) \to 0.$$

Since $\mathcal{T}^M$ is exact and idempotent, it follows that $\mathcal{T}^M(X/\mathcal{T}^M(X)) = 0$. Now Y is a quotient of $X/\mathcal{T}^M(X)$, and it follows from the exactness of $\mathcal{T}^M$ that $\mathcal{T}^M(Y) = 0$. Thus $Y = 0$, and $K + \mathcal{T}^M(X) = X$. Since $K \ll X$, we have that $\mathcal{T}^M(X) = X$, so $X \in \sigma[M]$, as needed. □

1.2. **Properties of the finite topology.**

Proposition 1.3. *Let R be a ring, and fix a right R-module T. Density will mean density in the finite topology.*

(i) *Let $M = M_1 \oplus M_2$ in $\mathcal{M}_R$, and $X_1 \subset \operatorname{Hom}_R(M_1, T)$, $X_2 \subset \operatorname{Hom}_R(M_2, T)$ If $X_1 \oplus X_2$ is dense in $\operatorname{Hom}_R(M, T) = \operatorname{Hom}_R(M_1, T) \oplus \operatorname{Hom}_R(M_1, T)$, then each X_i is dense in $\operatorname{Hom}_R(M_i, T)$.*
(ii) *Let $(M_i)_{i\in I}$ be a family of R-modules, and $X_i \subset \operatorname{Hom}_R(M_i, T)$ such that each X_i is dense in $\operatorname{Hom}_R(M_i, T)$. Let $M = \bigoplus_{i\in I} M_i$. Then $\bigoplus_{i\in I} X_i$ is dense in $\operatorname{Hom}_R(M, T) = \prod_{i\in I} \operatorname{Hom}_R(M_i, T)$.*

Proof. (i) Take $f \in \operatorname{Hom}_R(M_1, T)$ and F is a finite subset of M_1. Viewing f as the pair $(f, 0) \in \operatorname{Hom}_R(M_1, T) \oplus \operatorname{Hom}_R(M_2, T)$ and $F \subset M_1 \subset M_1 \oplus M_2$, we find a pair $(g, h) \in X_1 \oplus X_2 \subset \operatorname{Hom}_R(M, T) = \operatorname{Hom}_R(M_1, T) \oplus \operatorname{Hom}_R(M_2, T)$ such that $(g, h) = (f, 0)$ on F, so $g = f$ on all $m \in F$, with $g \in X_1 \subset \operatorname{Hom}_R(M_1, T)$.
(ii) Take $(f_i)_{i\in I} \in \operatorname{Hom}_R(M, T) = \prod_{i\in I} \operatorname{Hom}_R(M_i, T)$ and a finite subset $F \subset$

$\bigoplus_{i\in I} M_i$. Then there is a finite subset $J\subset I$ such that $F\subset \bigoplus_{i\in J} M_i$. $F_i = \{m_i \mid m\in F\}$ is finite, and, using the density of X_i in $\mathrm{Hom}_R(M_i,T)$, we find $g_i\in X_i$ such that $g_i=f_i$ on F_i. Now let $g\in \prod_{i\in I}\mathrm{Hom}_R(M_i,T)=\mathrm{Hom}_R(M,T)$ be defined as follows: the i-th component of g is g_i if $i\in J$, and it is zero otherwise. Then $g\in \bigoplus_{i\in I} X_i$ and $g=f$ on all F_i, and a fortiori on F, by linearity. □

Corollary 1.4. *If $(M_i)_{i\in I}$ is a family of R-modules and $X_i\subset \mathrm{Hom}_R(M_i,T)$ then $\bigoplus_{i\in I} X_i$ is dense in $\prod_{i\in I}\mathrm{Hom}_R(M_i,T)=\mathrm{Hom}_R(\bigoplus_{i\in I} M_i,T)$ if and only if all X_i are dense in $\mathrm{Hom}_R(M_i,T)$. Consequently, the direct sum $\bigoplus_{i\in I}\mathrm{Hom}_R(M_i,T)$ is dense in the direct product $\prod_{i\in I}\mathrm{Hom}_R(M_i,T)$.*

Proposition 1.5. *Let $T\in\mathcal{M}_R$ be an injective module, and $u:\ X\to Y$ a monomorphism in $\mathcal{M}_R$. If V is dense in $\mathrm{Hom}_R(Y,T)$, then $\mathrm{Hom}_R(u,T)(V)$ is dense in $\mathrm{Hom}_R(X,T)$.*

Proof. Take $f\in\mathrm{Hom}_R(X,T)$, and a finite subset $F\subset X$. As T is an injective module, we can find $g\in\mathrm{Hom}_R(Y,T)$ such that $g\circ u=f$. As $u(F)$ is a finite subset of Y we can find $h\in V$ such that h equals g on $u(F)$. Now we obviously have that $\mathrm{Hom}_R(u,T)(h)=h\circ u$ equals $g\circ u=f$ on F, hence $\mathrm{Hom}_R(u,T)(V)$ is dense in $\mathrm{Hom}_R(X,T)$. □

2. Corings and the Rat functor

2.1. **Corings.** Let R be a ring. An R-coring is a coalgebra in the monoidal category ${}_R\mathcal{M}_R$. It consists of a triple $\mathcal{C}=(\mathcal{C},\Delta,\varepsilon)$, where $\mathcal{C}$ is an R-bimodule, and $\Delta:\ \mathcal{C}\to\mathcal{C}\otimes_R\mathcal{C}$ and $\varepsilon:\ Cc\to R$ are R-bimodule maps satisfying appropriate coassociativity and counit properties. We refer to [3] and [4] for more detail about corings. We use the Sweedler-Heyneman notation

$$\Delta(c)=c_{(1)}\otimes_R c_{(2)},$$

where the summation is implicitely understood. If $\mathcal{C}$ is an R-coring, then ${}^*\mathcal{C}={}_R\mathrm{Hom}(\mathcal{C},R)$ is a ring with multiplication given by the formula

$$(f\#g)(c)=g(c_{(1)}f(c_{(2)})).$$

The unit of the multiplication is ε. We have a ring morphism

$$\iota:\ R\to{}^*\mathcal{C},\ \iota(r)(c)=\varepsilon(c)r.$$

A right $\mathcal{C}$-comodule consists of a pair (M,ρ^r), where $M\in\mathcal{M}_R$ and $\rho^r:\ M\to M\otimes_R\mathcal{C}$ is a right A-linear map satisfying the conditions

$$(\rho^r\otimes_R\mathcal{C})\circ\rho^r=(M\otimes_R\Delta)\circ\rho^r \text{ and } (M\otimes_R\varepsilon)\circ\rho^r=M.$$

Left $\mathcal{C}$-comodules are defined in a similar way, and the categories of left and right $\mathcal{C}$-comodules are respectively denoted by $\mathcal{M}^{\mathcal{C}}$ and ${}^{\mathcal{C}}\mathcal{M}$. We use the Sweedler-Heyneman notation

$$\rho^r(m)=m_{[0]}\otimes_R m_{[1]} \text{ and } \rho^l(m)=m_{[-1]}\otimes_R m_{[0]}$$

for right and left $\mathcal{C}$-coactions. We have a functor $F:\ \mathcal{M}^{\mathcal{C}}\to\mathcal{M}_{{}^*\mathcal{C}}$, with $F(M)=M$ as an R-module, equipped with the right ${}^*\mathcal{C}$-action $m\cdot f=m_{[0]}f(m_{[1]})$. In particular, $\mathcal{C}$ is a right and left ${}^*\mathcal{C}$-module. If M and N are right $\mathcal{C}$-comodules, then the set of R-linear maps preserving the $\mathcal{C}$-coaction is denoted by $\mathrm{Hom}^{\mathcal{C}}(M,N)$.

2.2. **The α-condition.** $M \in {}_R\mathcal{M}$ satisfies the (left) α-condition if the canonical map

$$\alpha_{N,M}:\ N\otimes_R M \to \mathrm{Hom}_R({}^*M,N),\ \alpha(n\otimes_R m)(f)=nf(m)$$

is injective, for all $N \in \mathcal{M}_R$. Otherwise stated: if $n\otimes_R m \in N\otimes_R M$ is such that $nf(m)=0$ for all $f \in {}^*M$, then $n\otimes m=0$. M satisfies the α-condition if and only if M is locally projective in ${}_R\mathcal{M}$. An R-coring $\mathcal{C}$ satisfies the left α-condition if and only if $\mathcal{M}^{\mathcal{C}}$ is a full subcategory of $\mathcal{M}_{^*\mathcal{C}}$, and the natural functor $\mathcal{M}^{\mathcal{C}} \to \sigma[C_{^*\mathcal{C}}]$ is an isomorphism. In this case, $\mathcal{C}$ is flat as a left R-module, hence $\mathcal{M}^{\mathcal{C}}$ is a Grothendieck category in such a way that the forgetful functor $\mathcal{M}^{\mathcal{C}} \to \mathcal{M}_A$ is exact (see [4, Sec. 19]).

If $\mathcal{C} \in {}_R\mathcal{M}$ is locally projective, then for all $M \in \mathcal{M}^{\mathcal{C}}$, the lattices consisting respectively of all $\mathcal{C}$-subcomodules and of all ${}^*\mathcal{C}$-submodules of M coincide, so it makes sense to talk about the subcomodule generated by a subset of M. ¿From the proof of [4, 19.12], we deduce the following result.

Theorem 2.1. (Finiteness Theorem) *If $\mathcal{C} \in {}_R\mathcal{M}$ is locally projective, then a right $\mathcal{C}$-comodule M is finitely generated as a right $\mathcal{C}$-comodule if and only if it is finitely generated as a right R-module.*

Let $\mathcal{C}$ be locally projective as a left R-module, and M a right ${}^*\mathcal{C}$-module. $\mathrm{Rat}^{\mathcal{C}}(M)$ is by definition the largest ${}^*\mathcal{C}$-submodule N of M, on which there exists a right $\mathcal{C}$-coaction ρ such that $F(N,\rho)=N$. Otherwise stated, $\mathrm{Rat}^{\mathcal{C}}$ is the preradical functor $\mathcal{T}^{\mathcal{C}}$, with $\mathcal{C}$ considered as a right ${}^*\mathcal{C}$-module. We also have that $\mathrm{Rat}^{\mathcal{C}}(M)$ consists of the elements $m \in M$ such that there exists $m_{[0]}\otimes_R m_{[1]} \in M\otimes_R \mathcal{C}$ with $m\cdot f = m_{[0]}f(m_{[1]})$, for all $f \in {}^*\mathcal{C}$. In a similar way, we define the left Rat functor ${}^{\mathcal{C}}\mathrm{Rat}$. The proof of Proposition 2.2 is straightforward, and left to the reader.

Proposition 2.2. *Let $\mathcal{C}$ be an R-coring, and $M \in {}^{\mathcal{C}}\mathcal{M}$.*

- (i) *The R-modules ${}^{\mathcal{C}}\mathrm{Hom}(M,\mathcal{C})$ and ${}^*M = {}_R\mathrm{Hom}(M,R)$ are isomorphic;*
- (ii) *${}^{\mathcal{C}}\mathrm{Hom}(M,\mathcal{C})$ is a right ${}^*\mathcal{C}$-module, via*

$$(\varphi\cdot f)(m) = f(\varphi(m));$$

- (iii) *we have isomorphic functors ${}^{\mathcal{C}}\mathrm{Hom}(-,\mathcal{C})$ and ${}_R\mathrm{Hom}(-,R)$ from ${}^{\mathcal{C}}\mathcal{M}$ to $\mathcal{M}_{^*\mathcal{C}}$; these functors are left exact if $\mathcal{C}$ is locally projective in $\mathcal{M}_R$, and exact if R is injective as a left R-module;*
- (iv) *the isomorphism from (i) defines a ring isomorphism ${}^{\mathcal{C}}\mathrm{End}(\mathcal{C}) \cong {}^*\mathcal{C}$, where the multiplication on ${}^{\mathcal{C}}\mathrm{End}(\mathcal{C})$ is the oppositie composition;*
- (v) *${}^{\mathcal{C}}\mathrm{Hom}(M,\mathcal{C})$ is a right ${}^{\mathcal{C}}\mathrm{End}(\mathcal{C})$-module, via*

$$(\varphi\cdot f)(m) = f(\varphi(m)).$$

Observe that the right coactions defined in (ii) and (v) are the same after we identify ${}^{\mathcal{C}}\mathrm{End}(\mathcal{C})$ and ${}^*\mathcal{C}$ using (iv).
Let ${}^{\mathrm{fg}\mathcal{C}}\mathcal{M}$ be the category of finitely generated left $\mathcal{C}$-comodules. If R is left noetherian, then the kernel of a morphism in ${}^{\mathrm{fg}\mathcal{C}}\mathcal{M}$ is still finitely generated, hence ${}^{\mathrm{fg}\mathcal{C}}\mathcal{M}$ has kernels (and cokernels), and is an abelian category.

Proposition 2.3. *Let R be a left noetherian ring, and $\mathcal{C}$ a locally projective R-coring.*

(i) *For any finitely generated $M \in {}_R\mathcal{M}$, the evaluation map*

$$\psi_M:\ {}_R\mathrm{Hom}(M,R)\otimes\mathcal{C}\to {}_R\mathrm{Hom}(M,\mathcal{C}),\ \psi_M(f\otimes c)(m)=f(m)c$$

is an isomorphism.

(ii) *Let $(M,\rho_M)\in {}^{\mathrm{fg}\mathcal{C}}\mathcal{M}$ and consider the map*

$$\phi_M:\ {}^*M\to {}^*M\otimes_R\mathcal{C},\ \phi_M(f)=\psi_M^{-1}((\mathcal{C}\otimes f)\circ\rho_M)$$

Then $({}^*M,\phi_M)\in\mathcal{M}^{\mathcal{C}}$, and the associated ${}^*\mathcal{C}$-module structure is as defined in Proposition 2.2

Proof. (i) It is straightforward to prove the statement for free modules. Then we can easily show it for finitely presented modules, using the flatness of $\mathcal{C}$ over R. Since R is noetherian, every finitely presented module is finitely generated.
(ii) Take $f\in {}^*M$, and write

$$\phi_M(f)=f_{[0]}\otimes f_{[1]}\in {}^*M\otimes_R\mathcal{C}.$$

Then $m_{[-1]}f(m_{[0]})=f_{[0]}(m)f_{[1]}$, and for very ${}^*c\in{}^*\mathcal{C}$, we find that

$$\begin{aligned}(f\cdot {}^*c)(m)&={}^*c(m_{[-1]}f(m_{[0]}))={}^*c(f_{[0]}(m)f_{[1]})\\&=f_{[0]}(m){}^*c(f_{[1]})=(f_{[0]}\cdot{}^*c(f_{[1]})(m)\end{aligned}$$

This shows that ${}^*M$ is a rational ${}^*\mathcal{C}$-module, and that ϕ_M is a right $\mathcal{C}$-coaction. □

2.3. **The Rat functor.** Assume that $\mathcal{C}$ is a coring satisfying the left α-condition. Then the functor $\mathrm{Rat}^{\mathcal{C}}$ is additive and left exact.

Proposition 2.4. *The following assertions are equivalent.*

(i) $\mathrm{Rat}^{\mathcal{C}}({}^*\mathcal{C})$ *is dense in* ${}^*\mathcal{C}$ *in the* $\mathcal{C}$*-adic topology;*
(ii) $\mathrm{Rat}^{\mathcal{C}}({}^*\mathcal{C})$ *is dense in* ${}^*\mathcal{C}$ *in the finite topology;*
(iii) $\mathrm{Rat}^{\mathcal{C}}$ *is an exact functor.*

Proof. The equivalence of (i) and (iii) follows from Proposition 1.1, invoking the fact that $\mathcal{C}$ is faithful as a right ${}^*\mathcal{C}$-module.
Note that the sets

$$\mathcal{O}_a(F)=\{{}^*c\mid c\cdot{}^*c=0,\ \text{for all } c\in F\},$$

with $F\subset\mathcal{C}$ finite, form a basis of open neighborhoods of $0\in\mathcal{C}$ in the $\mathcal{C}$-adic topology, which is a linear topology. Also

$$\mathcal{O}_f(F)=\{{}^*c\mid {}^*c(c)=0,\ \text{for all } c\in F\},$$

with $F\subset\mathcal{C}$ finite, form a basis of open neighborhoods of 0 for the finite topology, which is also linear.
Let $F\subset\mathcal{C}$ be finite. For each $c\in F$, we fix a tensor representation of $\Delta(c)$, and then consider the finite set F' of all second tensor components. Then we easily see that

$$\mathcal{O}_f(F')\subseteq\mathcal{O}_a(F)\subseteq\mathcal{O}_f(F)$$

and it follows that the two linear topologies on ${}^*\mathcal{C}$ coincide, so it follows that (i) is equivalent to (ii). □

Proposition 2.5. *Suppose we have a decomposition $\mathcal{C}=\bigoplus_{i\in I}C_i$ as left $\mathcal{C}$-comodules. Then $\mathrm{Rat}^{\mathcal{C}}({}^*C_i)$ is dense in ${}^*C_i$ for all $i\in I$ if and only if $\mathrm{Rat}^{\mathcal{C}}({}^*\mathcal{C})$ is dense in ${}^*\mathcal{C}$.*

Proof. Assume that each $\mathrm{Rat}^{\mathcal{C}}({}^*C_i)$ is dense in ${}^*C_i$. It follows from Proposition 1.3 that $\bigoplus_{i\in I}\mathrm{Rat}^{\mathcal{C}}({}^*C_i)$ is dense in ${}^*C$, and then $\mathrm{Rat}^{\mathcal{C}}({}^*C)\supset\bigoplus_{i\in I}\mathrm{Rat}^{\mathcal{C}}({}^*C_i)$ is also dense.
Conversely, let $M=\bigoplus_{j\in I,j\neq i}C_j$, for each $i\in I$. Then $C=C_i\oplus M$ and ${}^*C={}^*C_i\oplus{}^*M$, hence $\mathrm{Rat}^{\mathcal{C}}({}^*\mathcal{C})=\mathrm{Rat}^{\mathcal{C}}({}^*C_i)\oplus Rat^{\mathcal{C}}({}^*M)$ is dense in ${}^*C={}^*C_i\oplus{}^*M$ ($\mathrm{Rat}^{\mathcal{C}}$ is an aditive functor). The result then follows from Proposition 1.3 (ii). □

Lemma 2.6. (i) *Assume that $M\in{}^{\mathcal{C}}\mathcal{M}$ is finitely generated and projective as a left R-module. Then ${}^*M$ is a rational right ${}^*\mathcal{C}$-module.*
(ii) *Suppose that $\mathcal{C}=M\oplus N$ in ${}^{\mathcal{C}}\mathcal{M}$. Then ${}^*M$ is rational if and only if M is finitely generated as a left R-module.*

Proof. (i) We take a finite dual basis $\{(x^i,f_i)\mid i=1,\cdots,n\}$ of $M\in{}_R\mathcal{M}$. For all $h\in{}^*M$ and $\alpha\in{}^*\mathcal{C}$, we have

$$h\cdot\alpha=\sum_i f_i\cdot(h\cdot\alpha)(x^i)=\sum_i f_i\alpha(x^i_{[-1]}h(x^i_{[0]}))$$

This shows that $h_{[0]}\otimes h_{[1]}=f_i\otimes x^i_{[-1]}h(x^i_{[0]})\in{}^*M\otimes\mathcal{C}$ is such that $h\cdot\alpha=h_{[0]}\alpha(h_{[1]})$, and this proves that ${}^*M$ is rational.
(ii) One direction follows from (i). Conversely, assume that ${}^*M$ is rational. Take $e=\varepsilon_{|_M}\in{}^*M$. We can identify ${}^*\mathcal{C}={}^*M\oplus{}^*N$ as right ${}^*\mathcal{C}$ modules. For $h\in{}^*M$ and $c\in\mathcal{C}$, $(e\cdot h)(c)=h(c_{(1)}e(c_{(2)}))=h(c_{(1)}\varepsilon(c_{(2)}))=h(c)$ if $c\in M$ $(c_{(1)}\otimes c_{(2)}\in C\otimes M)$ and $(e\cdot h)(c)=h(c_{(1)}e(c_{(2)}))=0$ if $c\in N$ $(c_{(1)}\otimes c_{(2)}\in C\otimes N)$ showing that $e\cdot h=h$ (the h in the $e\cdot h$ is regarded as belonging to ${}^*\mathcal{C}$). As ${}^*M$ is rational there is $\sum_i f_i\otimes x^i\in{}^*M\otimes\mathcal{C}$ such that $e\cdot\alpha=\sum_i f_i\alpha(x^i)$, for all $\alpha\in{}^*\mathcal{C}$. Then for any $h\in{}^*M$, $h=e\cdot h=\sum_i f_ih(x^i)$, and, for all $m\in M$, we have $h(m)=\sum_i f_i(m)h(x^i)=h(\sum_i f_i(m)x^i)=h(\sum_i f_i(m)m^i)$, where $x^i=m^i+n^i\in M\oplus N$ is the unique representation of x^i in the direct sum $\mathcal{C}=M\oplus N$ and the last equality holds as $h_{|_N}=0$. As this last equality holds for all $h\in{}^*M$, we can easily see that it actually holds for all $\alpha=(h,g)\in{}^*\mathcal{C}={}^*M\oplus{}^*N$ because $m\in M$, and so we now obtain, using the left α-condition on ${}^*\mathcal{C}$, that $m=f_i(m)m^i$, where $m\in M$ is arbitrary and $m^i\in M$ are fixed. Thus M is finitely generated. □

Corollary 2.7. *Assume that $\mathcal{C}=\bigoplus_{i\in I}C_i$ as left $\mathcal{C}$-comodules, and that each C_i is finitely generated. Then $\mathrm{Rat}^{\mathcal{C}}({}^*\mathcal{C})$ is dense in ${}^*\mathcal{C}$, and, equivalently, $\mathrm{Rat}^{\mathcal{C}}$ is an exact functor.*

Proof. This is a direct consequence of Proposition 1.3 (ii) and Lemma 2.6. □

Example 2.8. We now present an example of a coring for which we can explicitly construct the Rat functor. Let G be a group, k a commutative ring, and R a G-graded k-algebra. It is well-known that $\mathcal{C}=R\otimes kG$ is an R-coring. The structure maps are given by the formulas

$$r(s\otimes\sigma)t=\sum_{\rho\in G}rst_\rho\otimes\sigma\rho;$$

$$\Delta_{\mathcal{C}}(s\otimes\sigma)=(s\otimes\sigma)\otimes_R(1\otimes\sigma)\ ;\ \varepsilon(s\otimes\sigma)=s.$$

Here t_ρ is the homogeneous part of degree ρ of t. Clearly $\mathcal{C}=\bigoplus_{\sigma\in G}R\otimes\sigma$ decomposes as the direct sum of finitely generated (free of rank one) left $\mathcal{C}$-comodules, hence it follows from Corollary 2.7 that Rat is exact. We will illustrate this, computing Rat. First observe that

$$^*\mathcal{C} = {}_R\mathrm{Hom}(R \otimes kG, R) \cong \mathrm{Hom}(kG, R) \cong \mathrm{Map}(G, R).$$

The multiplication on $^*\mathcal{C}$ can be transported into a multiplication on $\mathrm{Map}(G, R)$. This multiplication is the following. For $f, g : G \to R$ and $\tau \in G$:

$$(f\#g)(\tau) = \sum_\rho f(\tau)_\rho g(\tau\rho) \tag{1}$$

Let $(kG)^*$ be the dual of the group algebra kG, with free basis $\{v_\sigma \mid \sigma \in G\}$, such that $v_\sigma(\tau) = \delta_{\sigma,\tau}$. then v_σ can also be viewed as a map $G \to R$, and this gives us an algebra embedding $(kG)^* \subset \mathrm{Map}(G, R)$. Indeed, using (1), we easily compute that $v_\sigma \# v_\tau = \delta_{\sigma,\tau} v_\sigma$.
We also have an algebra embedding

$$\iota : R \to \mathrm{Map}(G, R), \ \iota_r(\sigma) = r.$$

Indeed, using (1), we find

$$(\iota_r \# \iota_s)(\tau) = \sum_\rho \iota_r(\tau)_\rho \iota_s(\tau\rho) = \sum_\rho r_\rho s = rs = \iota_{rs}(\tau).$$

Let $r \in R$ be homogeneous of degree ρ, and $f : G \to R$. Using (1), we compute

$$v_\sigma \# \iota_r = \iota_r \# v_{\sigma\rho} \text{ and } v_\sigma \# f = v_\sigma \# \iota_{f(\sigma)}. \tag{2}$$

Now take $M \in \mathcal{M}_{^*\mathcal{C}} \cong \mathcal{M}_{\mathrm{Map}(G,R)}$. By restriction of scalars, M is also a right R-module and a right $(kG)^*$-module. Now put $M_\sigma = M \cdot v_\sigma$.
1) If $\sigma \neq \tau$, then $M_\sigma \cap M_\tau = 0$. Indeed, if $m \cdot v_\sigma = n \cdot v_\tau$, then

$$m \cdot v_\sigma = m \cdot (v_\sigma \# v_\sigma) = (m \cdot v_\sigma) \cdot v_\sigma = (n \cdot v_\tau) \cdot v_\sigma = n \cdot (v_\tau \# v_\sigma) = 0.$$

2) $M_\sigma R_\rho \subset M_{\sigma\rho}$. Take $m \cdot v_\sigma \in M_\sigma$ and $r \in R_\rho$. Using (2), we find

$$(m \cdot v_\sigma) r = m \cdot (v_\sigma \# \iota_r) = m \cdot (\iota_r \# v_{\sigma\rho}) = (mr) \cdot v_{\sigma\rho} \in M_{\sigma\rho}.$$

This shows that $\bigoplus_{\sigma\in G} M_\sigma$ is a G-graded R-module; we will show that it is the rational part of M.
3) $M_\sigma \subset \mathrm{Rat}(M)$. Take $m \cdot v_\sigma \in M_\sigma$ and $f \in \mathrm{Map}(G, R)$. Using (2), we find

$$(m \cdot v_\sigma) \cdot f = m \cdot (v_\sigma \# f) = m \cdot (v_\sigma \# \iota_{f(\sigma)}) = (m \cdot v_\sigma) f(\sigma),$$

so $m \cdot v_\sigma$ is rational.
4) It follows from 3) that $\bigoplus_{\sigma\in G} M_\sigma \subseteq \mathrm{Rat}(M)$.
5) Let $m \in \mathrm{Rat}(M)$. Then there exist $m_1, \cdots, m_n \in M$, $r_1, \cdots, r_n \in R$ and $\sigma_1, \cdots, \sigma_n \in G$ such that, for all $\varphi \in {}^*\mathcal{C}$:

$$m \cdot \varphi = \sum_i m_i \varphi(r_i \otimes \sigma_i).$$

Making the identification $^*\mathcal{C} \cong \mathrm{Map}(G, R)$, we find for all $f : G \to R$:

$$m \cdot f = \sum_i m_i r_i f(\sigma_i).$$

Replacing m_i by $m_i r_i$, it is no restriction to take $r_i = 1$. We can also take the σ_i pairwise different. Taking $f = v_\sigma$, we find that

$$m_\sigma = \sum_i m_i \delta_{\sigma,\sigma_i}$$

so $m_\sigma \neq 0$ for only a finite number of σ, and $m_{\sigma_i} = m_i$. Finally

$$m = m \cdot \iota_1 = \sum_i m_i \iota_1(\sigma_i) = \sum_i m_i = \sum_i m_{\sigma_i} \in \bigoplus_{\sigma \in G} M_\sigma.$$

We conclude that

$$\mathrm{Rat}(M) = \bigoplus_{\sigma \in G} M \cdot v_\sigma,$$

and it is clear that Rat is exact.

In some situations, the converse of Corollary 2.7 also holds. If R is left artinian, then any left comodule contains a simple comodule. The same holds for comodules that are locally artinian, in the sense that any finitely generated submodule is artinian. If this is the case for $\mathcal{C}$, then the left socle of C is essential in $\mathcal{C}$. If moreover $\mathcal{C}$ is injective in ${}^{\mathcal{C}}\mathcal{M}$, then a decomposition $\mathcal{C} = \bigoplus_{i \in I} E(S_i)$ holds with usual arguments, where $\bigoplus_{i \in I} S_i = {}^{\mathcal{C}}s(\mathcal{C})$ is a decomposition of the left socle ${}^{\mathcal{C}}s(\mathcal{C})$ of $\mathcal{C}$ and $E(S_i)$ is the injective hull of S_i contained in $\mathcal{C}$. We will assume that $\mathcal{C}$ is locally projective as a right R-module, which implies that ${}^{\mathcal{C}}\mathcal{M}$ is abelian, so that we have a categorical definition of injective hulls.

Proposition 2.9. *Assume that $\mathcal{C}$ also satisfies the right α-condition, and that the two following conditions hold:*

1. *$\mathcal{C}$ is an injective object of ${}^{\mathcal{C}}\mathcal{M}$;*
2. *R is left artinian or $\mathcal{C}$ is locally artinian in ${}_R\mathcal{M}$ (equivalently in ${}^{\mathcal{C}}\mathcal{M}$).*

Let $\bigoplus_{i \in I} S_i$ be the decomposition of the left socle of $\mathcal{C} \in {}^{\mathcal{C}}\mathcal{M}$ into simple left $\mathcal{C}$-comodules, and $E(S_i)$ an injective envelope of S_i contained in $\mathcal{C}$. Then $\mathrm{Rat}^{\mathcal{C}}$ *is exact if and only if each $E(S_i)$ is finitely generated.*

Proof. We have that $\mathcal{C} = \bigoplus_{i \in I} E(S_i)$, so one direction follows from Corollary 2.7. Conversely, assume that $\mathrm{Rat}^{\mathcal{C}}$ is exact, and let S be a simple subcomodule of C, and $E(S)$ an injective envelope of S contained in C. Then there is a left subcomodule X of $\mathcal{C}$ such that $E(S) \oplus X = \mathcal{C}$ in ${}^{\mathcal{C}}\mathcal{M}$. The functor ${}^{\mathcal{C}}\mathrm{Hom}(-, \mathcal{C})$ is exact since $\mathcal{C} \in \mathcal{M}^{\mathcal{C}}$ is injective, and the composition of ${}^{\mathcal{C}}\mathrm{Hom}(-, \mathcal{C})$ with the natural functor $\mathcal{M}^{\mathcal{C}} \to \mathcal{M}_{^*\mathcal{C}}$ is also exact. Thus we obtain an epimorphism $\pi :\ {}^*E(S) \to {}^*S$, with kernel ${}^\perp S = \{f \in {}^*E(S) \mid f_{|S} = 0\}$.
We will first show that ${}^\perp S \ll {}^*E(S)$. Using the isomorphisms in Proposition 2.2, we can regard π as a left ${}^{\mathcal{C}}\mathrm{End}(\mathcal{C})$-module morphism ${}^{\mathcal{C}}\mathrm{Hom}(E(S), \mathcal{C}) \to {}^{\mathcal{C}}\mathrm{Hom}(S, \mathcal{C})$. Take $f \in {}^{\mathcal{C}}\mathrm{Hom}(E(S), \mathcal{C}) \setminus {}^\perp S$, i.e. $f :\ E(S) \to \mathcal{C}$ such that $f_{|S} \neq 0$. Then $\mathrm{Ker}\, f \cap S = 0$ since S is simple, and therefore $\mathrm{Ker}\, f = 0$, since S is essential in $E(S)$. So $E(S) \cong f(E(S))$, and there exists a left $\mathcal{C}$-subcomodule M of $\mathcal{C}$ such that $\mathcal{C} \cong f(E(S)) \oplus M$. We can extend f to a left $\mathcal{C}$-comodule isomorphism $\overline{f} :\ \mathcal{C} \to \mathcal{C}$, since $X \cong M$. Let h be the inverse of $\overline{f}$. Take an arbitrary $g \in {}^{\mathcal{C}}\mathrm{Hom}(E(S), \mathcal{C})$,and extend g to $\overline{g} :\ \mathcal{C} = E(S) \oplus X \to \mathcal{C}$ by putting $\overline{g}_{|X} = 0$. Then $\overline{g} = \overline{g} \circ h \circ \overline{f}$, which means that ${}^{\mathcal{C}}\mathrm{Hom}(E(S), \mathcal{C})$ is generated by $\overline{f}$ as a left ${}^{\mathcal{C}}\mathrm{End}(\mathcal{C})$-module. Consequently ${}^\perp S \ll {}^*E(S)$.
The Finiteness Theorem 2.1 shows that S is finitely generated and then it follows from Proposition 2.3 (ii) that ${}^*S$ is a rational ${}^*C$-comodule, so $\mathrm{Rat}^{\mathcal{C}}({}^*S) = {}^*S$. $\mathrm{Rat}^{\mathcal{C}}$ is exact, so we have an exact sequence

$$0 \longrightarrow \mathrm{Rat}^{\mathcal{C}}({}^\perp S) \longrightarrow \mathrm{Rat}^{\mathcal{C}}({}^*E(S)) \xrightarrow{\ \pi\ } \mathrm{Rat}^{\mathcal{C}}({}^*S) = {}^*S \longrightarrow 0.$$

We obtain $\pi(\mathrm{Rat}^{\mathcal{C}}({}^*E(S))) = {}^*S$, so ${}^\perp S + \mathrm{Rat}^{\mathcal{C}}({}^*E(S)) = {}^*E(S)$. It then follows that ${}^*E(S)$ is rational. This last part can also be seen as follows. We have an exact sequence

$$0 \longrightarrow {}^\perp S \longrightarrow {}^*E(S) \longrightarrow {}^*S \longrightarrow 0,$$

with ${}^\perp S \ll {}^*E(S)$ and ${}^*S$ rational, so ${}^*E(S)$ is rational by Proposition 1.2(i). Using Lemma 2.6, we find that ${}_RE(S)$ is finitely generated. □

3. Semiperfect corings

Let $\mathcal{C}$ be an abelian category. A projective object $P \in \mathcal{C}$ together with a superfluous epimorphism $P \to M$ is called a projective cover of M. $\mathcal{C}$ is called semiperfect if every simple object has a projective cover. If a coring $\mathcal{C}$ satisfies the left α-condition, then $\mathcal{M}^{\mathcal{C}}$ is an abelian category, and $\mathcal{C}$ is called right semiperfect if $\mathcal{M}^{\mathcal{C}}$ is semiperfect. Semiperfect corings were introduced first in [10].

Theorem 3.1. *Let R be a right artinian ring, and $\mathcal{C}$ an R-coring satisfying the left α-condition. The following statements are equivalent.*

(i) *$\mathcal{C}$ is right semiperfect;*
(ii) *Every finitely generated right comodule has a projective cover;*
(iii) *every finitely generated right comodule has a finitely generated projective cover;*
(iv) *the category $\mathcal{M}^{\mathcal{C}}$ has enough projectives;*
(v) *every simple right comodule has a finitely generated projective cover;*
(vi) *the category $\mathcal{M}^{\mathcal{C}}$ has a progenerator (=projective generator).*

Proof. (i)⇒(ii). First notice that an R-module is finitely generated if and only if it has finite length. Every finitely generated comodule M has a maximal subcomodule, so its Jacobson radical $J(M)$ in $\mathcal{M}^{\mathcal{C}}$ is different from the comodule itself. $J(M) \ll M$, and $M/J(M)$ is a semisimple finitely generated comodule. Every simple component of $M/J(M)$ has a projective cover, and the direct sum of all these projective covers is a projective cover $f : P \to M/J(M)$ of $M/J(M)$. Since P is projective, there exists $g: P \to M$ such that $u \circ g = f$, with $u : M \to M/J(M)$ the canonical projection. Then a usual argument shows that $g : P \to M$ is a projective cover: $u(g(P)) = f(P) = M/J(M)$, hence $u(J(M) + g(P)) = M/J(M)$ and it follows that $J(M) + g(P) = M$. From the fact that $J(M)$ is small in M, it follows that $g(P) = M$ and g is surjective. Finally $\mathrm{Ker}\, g \subset \mathrm{Ker}\, f \ll P$, so $\mathrm{Ker}\, g \ll P$, and $g: P \to M$ is a projective cover of M.

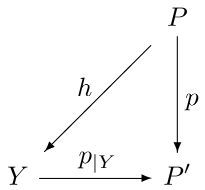
(iv)⇒(iii). Let M be a finitely generated comodule. We know that there exists a projective object $P \in \mathcal{M}^{\mathcal{C}}$ and a $\mathcal{C}$-colinear epimorphism $f : P \to M$. Let $(M_i)_{i\in I}$ be a family of finitely generated comodules such that we have a $\mathcal{C}$-colinear epimorphism $f : \bigoplus_{i\in I} M_i \to P$. As P is projective, we have that $\bigoplus_{i\in I} M_i \cong P \oplus X$ as comodules. Since R is artinian, we can assume that the M_i are indecomposable. As they have finite length in $\mathcal{M}_R$, they also have finite length in $\mathcal{M}^{\mathcal{C}}$ and $\mathcal{M}_{^*\mathcal{C}}$, so their ${}^*\mathcal{C}$-endomorphism rings are local, by the Krull-Schmidt Theorem (see [2, 12.8]). It then follows from the Crawley-Jønsson-Warfield Theorem (see [2, 26.5]) that $P \cong \bigoplus_{i\in J} M_i$, with $J \subset I$. The M_i are finitely generated (rational) ${}^*\mathcal{C}$-modules, and are projective objects of $\mathcal{M}^{\mathcal{C}}$, since they are direct summands of P. Since M is finitely generated, we can find a finite $F \subset J$ and a projection $\bigoplus_{i\in F} M_i \to M$, induced by f. Thus we have found a finitely generated projective object $P \in {}^{\mathcal{C}}\mathcal{M}$ and a $\mathcal{C}$-colinear epimorphism $f : P \to M$. Dualizing the proof of the Eckmann-Schopf Theorem on the existence of the injective envelope of a module, see e.g. [2, 18.10], we can show that M has a projective cover. This works as follows.

• Let $K = \operatorname{Ker} f$, and consider the set V consisting of subcomodules $H \subset K$ such that $K/H \ll P/H$, which is equivalent to

$$H \subset T \subset P, K + T = P \implies T = P$$

$V \neq \emptyset$ since $K \in V$. V contains a minimal element K' since R is artinian.

• Then consider the set W consisting of subcomodules $Y \subset P$ such that $K' + Y = P$. This set is nonempty, since P belongs to it. Then take an element in this set such that $K' \cap Y$ is minimal. Let $p : P \to P' = P/K'$ be the projection. Since P is projective, there exists a comodule morphism $h : P \to Y$ such that $p_{|Y} \circ h = p$, that is, the following diagram commutes:



We will now show that $p_{|Y}$ is an isomorphism.

• h is surjective. Take $y \in Y$. Then

$$p(y - h(y)) = p(y) - p(h(y)) = p(y) - p(y) = 0$$

so $y - h(y) \in K'$ and

$$y = (y - h(y)) + h(y) \in (Y \cap K') + \operatorname{Im} h.$$

It follows that $Y \subset (Y \cap K') + \operatorname{Im} h$. The converse implication is obvious, so

$$Y = (Y \cap K') + \operatorname{Im} h$$

It then follows that

$$P = Y + K' = (Y \cap K') + \operatorname{Im} h + K' = \operatorname{Im} h + K'$$

The minimality condition on Y then yields that $Y = \operatorname{Im} h$, so h is surjective.

• $Y \cap K' \ll Y$. If $H \subset Y$ and $H + (Y \cap K') = Y$, then $H + K' = H + (Y \cap K') + K' = Y + K' = P$. This means that $H \in W$, and the minimality condition on Y gives us that $H \cap K' \supset Y \cap K'$, and $H \cap K' \subset Y \cap K'$ since $H \subset Y$. Then we find that $Y = H + (Y \cap K') = H + (H \cap K') = H$, as needed.

• From the fact that $0 = p(K') = (p \circ h)(K')$, it follows that $h(K') \subset \operatorname{Ker}(p_{|Y}) = Y \cap K'$.

• $\operatorname{Ker} h = K'$. It is clear that $\operatorname{Ker} h \subset K'$. It follows that $K' \subset \operatorname{Ker} h$ if we can show that $\operatorname{Ker} h \in V$, or

$$\operatorname{Ker} h \subset T \subset P, K + T = P \Rightarrow T = P$$

Assume $\operatorname{Ker} h \subset T \subset P$ and $K + T = P$. Since $K' \subset P$, we find that $K + K' + T = P$. Also $K' \subset T + K' \subset P$, so it follows from the fact that $K' \in V$ that $K' + T = P$.Then $h(K') + h(T) = h(P) = Y$, since K is surjective. Since $h(K') \subset Y \cap K'$, this implies that $Y \cap K' + h(T) = Y$, hence $h(T) = Y$, since $Y \cap K' \ll Y$, and finally $T = P$ because $T \subset \operatorname{Ker} h$.

• $p_{|Y}$ is surjective, as $p = p_{|Y} \circ h$ and p is an epimorphism.

• $p_{|Y}$ is injective. Take $y \in Y$ such that $p(y) = 0$. h is surjective, so $y = h(z)$. Then $0 = p(y) = p(h(z)) = p(z)$, so $z \in K' = \ker h$, and $y = h(z) = 0$.
• It now follows that $Y \cap K' = 0$. We know from the definition of Y that $Y + K' = P$. Hence $Y \oplus K' = P$, and $P' \cong Y$ is finitely generated projective, being a direct factor of P. Now look at the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K' & \longrightarrow & P & \xrightarrow{p} & P' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \subset} & & \downarrow{\scriptstyle =} & & & & \\
0 & \longrightarrow & K & \longrightarrow & P & \xrightarrow{f} & M & \longrightarrow & 0
\end{array}$$

It follows that we have an epimorphism $P' \to M$ in $\mathcal{M}^{\mathcal{C}}$, with kernel K/K'. This is a projective cover, since $K/K' \ll P' = P/K'$. Moreover, P' is finitely generated as a quotient of P.

(ii)⇒(vi). Take a family $(M_i)_{i\in I}$ consisting of finitely generated comodules that generate $\mathcal{M}^{\mathcal{C}}$. Let $P_i \to M_i$ be a projective cover of M_i. Then $\bigoplus_{i\in I} P_i$ is a projective generator of $\mathcal{M}^{\mathcal{C}}$.

(vi)⇒(iv), (iii)⇒(ii)⇒(i) and (iii)⇒(v)⇒(i) are obvious. □

Proposition 3.2. *Let R be a right artinian ring, and $\mathcal{C}$ an R-coring satisfying the left α-condition.*

(i) *$\mathcal{M}^{\mathrm{fg}\mathcal{C}}$ is an abelian category;*
(ii) *$Q \in \mathcal{M}^{\mathrm{fg}\mathcal{C}}$ is injective if and only if Q is an injective object in $\mathcal{M}^{\mathcal{C}}$;*
(iii) *$P \in \mathcal{M}^{\mathrm{fg}\mathcal{C}}$ is projective if and only if P is a projective object in $\mathcal{M}^{\mathcal{C}}$.*

Proof. (i) The fact that $\mathcal{M}^{\mathrm{fg}\mathcal{C}}$ has kernels follows from the assumption that R is right artinian and the Finiteness Theorem.

(ii) This is a straighforward adaptation of the corresponding result on comodules over a coalgebra. Let $u:\ N \to M$ be a monomorphism in $\mathcal{M}^{\mathcal{C}}$ and $f:\ N \to Q$. Consider the set

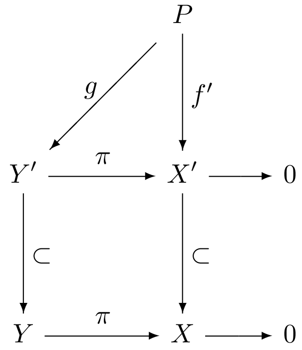
$$X = \{(N', f') \mid N \subset N' \subset M,\ f':\ N' \to Q,\ f'_{|N} = f\}$$

ordered by the relation $(N', f') < (N'', f'')$ if $N' \subset N''$ and $f''_{N'} = f'$. Take a maximal element (N_0, f_0) in X, and assume that $N_0 \neq M$. Take $m \in M \setminus N_0$ and X the subcomodule of M generated by M. By the Finiteness Theorem for comodules, X is finitely generated, so there exists $g:\ X \to Q$ such that the following diagram commutes:

$$\begin{array}{ccccc}
0 & \longrightarrow & N_0 \cap X & \longrightarrow & X \\
 & & \downarrow{\scriptstyle f_{0|N_0\cap X}} & \swarrow{\scriptstyle g} & \\
 & & Q & &
\end{array}$$

Then consider the map $f' :\ N' = N_0 + X \to Q$, defined by $f'(n_0+x) = f_0(n_0)+g(x)$. The usual computation shows that f' is well-defined, and (N', f') is an element in X that is strictly greater than (N_0, f_0), a contradiction.

(3)



(iii) Let $\pi :\ Y \to X$ and $f :\ P \to X$ be morphisms in $\mathcal{M}^{\mathcal{C}}$, with π surjective. Let $\{p_1, \cdots, p_n\}$ be a set of generators of P as an R-module (and a fortiori as a $\mathcal{C}$-comodule). Then $X' = \operatorname{Im} f$ is generated by $\{x_1, \cdots, x_n\}$, with $x_i = f(p_i)$. Take $y_i \in Y_i$ such that $\pi(y_i) = x_i$, and let Y' be the $\mathcal{C}$-submodule (or ${}^*\mathcal{C}$-submodule) of Y generated by $\{y_1, \cdots, y_n\}$. Let $f' :\ P \to X'$ be the corestriction of f. Since X' and Y' are finitely generated and $\pi_{|Y'}$ is still an epimorphism, there exists $g :\ P \to X'$ such that $f' = \pi \circ g$, and the projectivity of P in $\mathcal{M}^{\mathcal{C}}$ follows from the commutativity of the diagram (3). □

4. Applications and examples

4.1. **Application to qF-rings.** In Theorem 3.1, we gave equivalent conditions for the semiperfectness of a left locally projective coring $\mathcal{C}$ over a right artinian ring R. In the case where R is a qF-ring, more characterizations are possible. This has been studied recently by El Kaoutit and Gómex-Torrecillas (see [10, Theorems 3.5, 3.8, 4.2]. Using the results of the previous Sections, we find a different proof of these results.

First recall that a qF ring, or quasi-Frobenius ring, is a ring which is right artinian and injective as a right R-module, or, equivalently, left artinian and injective as a left R-module (in [15], these rings are called noetherian QF rings). In this situation, R is a cogenerator of $\mathcal{M}_R$ and ${}_R\mathcal{M}$, see [15, 48.15]. Since a qF-ring is a left and right perfect ring, local projectivity is equivalent to projectivity. Also recall that flat modules over qF-rings are projective. Let R be a qF-ring, and assume that $\mathcal{C} \in {}_R\mathcal{M}$ is flat (or, equivalently, (locally) projective). Then ${}^{\mathcal{C}}\mathcal{M}$ is a Grothendieck category, and the forgetful functor ${}^{\mathcal{C}}\mathcal{M} \to {}_R\mathcal{M}$ is exact and has a right adjoint $\mathcal{C} \otimes_R -$. Since ${}_R\mathcal{M}$ has enough injectives and the forgetful functor is exact, $\mathcal{C} \otimes_R -$ preserves injectives. Now $R \in {}_R\mathcal{M}$ is injective because R is a qF-ring, so $\mathcal{C} = \mathcal{C} \otimes_R R$ is an injective object of ${}^{\mathcal{C}}\mathcal{M}$, and we can apply Proposition 2.9. We find that $\mathcal{C} = \bigoplus_{i \in I} E(S_i)$, with $\bigoplus_{i \in I} S_i$ the decomposition of the left socle of $\mathcal{C} \in {}^{\mathcal{C}}\mathcal{M}$.

If R is a qF-ring, then the contravariant functors

$$(-)^* = \operatorname{Hom}_R(-, R) :\ \mathcal{M}_R \to {}_R\mathcal{M},\ \ {}^*(-) = {}_R\operatorname{Hom}(-, R) :\ {}_R\mathcal{M} \to \mathcal{M}_R,$$

define an equivalence duality between the categories of finitely generated left R-modules and finitely generated right R-modules. More explicitely, every finitely generated left R-module M is reflexive, that is, the map

$$\Phi_M:\ M \to ({}^*M)^*,\ \Phi_M(m)(f) = f(m)$$

is an isomorphism. This result follows, for example, after we take $U = M = R$ in [15, 47.13(2)].
If M is not finitely generated, then we still have the following result.

Lemma 4.1. *Let R be a qF ring and $M \in {}_R\mathcal{M}$-module. Then* $\mathrm{Im}\,(\Phi_M)$ *is dense in* $({}^*M)^*$ *with respect to the finite topology on* $\mathrm{Hom}_R({}^*M, R)$.

Proof. Take $T \in ({}^*M)^*$ and $F = \{f_1, \ldots, f_n\} \subset {}^*M$. We have to prove that there exists an $m \in M$ such that $T(f_i) = \Phi_M(f_i) = f_i(m)$. Let ${}^\perp F = \bigcap_{i=1}^n \mathrm{Ker}\, f_i \subset M$ and $N = M/{}^\perp F$. Then we have a natural inclusion

$$\frac{M}{\bigcap_{i=\overline{1,n}} \mathrm{Ker}\ f_i} \hookrightarrow \bigoplus_{i=1}^{n} \frac{M}{\mathrm{Ker}\, f_i} \simeq \bigoplus_{i=1}^{n} \mathrm{Im}\ f_i \hookrightarrow R^n$$

and this shows that $N = M/{}^\perp F$ has finite length. Let $\pi:\ M \longrightarrow M/{}^\perp F = N$ be the canonical projection and consider its dual $\pi^*:\ {}^*N \longrightarrow {}^*M$. By the construction of N as a factor module, there are left R-linear maps $\overline{f_i}:\ N \longrightarrow R$ such that $\overline{f_i} \circ \pi = f_i$. Consider $t = T \circ \pi^* \in ({}^*N)^*$. As N is finitely generated, Φ_N is an isomorphism (it gives the above stated duality between ${}_R\mathcal{M}$ and $\mathcal{M}_R$), so there is $n = \hat{m} = \pi(m) \in N$ such that $t = \Phi_N(n)$. Then $T(f_i) = T(\overline{f_i} \circ \pi) = (T \circ \pi^*)(\overline{f_i}) = t(\overline{f_i}) = \Phi_N(n)(\overline{f_i}) = \overline{f_i}(\pi(m)) = f_i(m)$, as needed. □

If $\mathcal{C}$ is a left and right projective R-coring, then the duality is kept after we pass to the categories of finitely generated $\mathcal{C}$-comodules: he functors ${}^*(-) = {}_R\mathrm{Hom}(-, R)$ and $(-)^* = \mathrm{Hom}_R(-, R)$ define an equivalence between the categories ${}^{\mathrm{fg}\mathcal{C}}\mathcal{M}$ and $\mathcal{M}^{\mathrm{fg}\mathcal{C}}$. To prove this, it suffices to show that Φ_M is left $\mathcal{C}$-colinear, or, equivalently, right $\mathcal{C}^*$-linear, for every finitely generated left $\mathcal{C}$-comodule M, and this is a standard computation. From this duality and Proposition 3.2, we obtain the following result.

Corollary 4.2. *Let R be a qF-ring, and $\mathcal{C}$ an R-coring that is projective as a left and right R-module. A finitely generated right $\mathcal{C}$-comodule M is injective (resp. projective) in $\mathcal{M}^{\mathcal{C}}$ if and only if M^* is projective (resp. injective) in ${}^{\mathcal{C}}\mathcal{M}$.*

Theorem 4.3. *Let R be a qF-ring, and $\mathcal{C}$ an R-coring that is (locally) projective as a left and right R-module. The following assertions are equivalent.*

(i) $\mathrm{Rat}^{\mathcal{C}}$ *is exact;*
(ii) $\mathrm{Rat}^{\mathcal{C}}({}^*\mathcal{C})$ *is dense in* ${}^*\mathcal{C}$*;*
(iii) $\mathrm{Rat}^{\mathcal{C}}({}^*M)$ *is dense in* ${}^*M$ *for every left $\mathcal{C}$-comodule M;*
(iv) $\mathrm{Rat}^{\mathcal{C}}({}^*Q)$ *is dense in* ${}^*Q$ *for every left injective $\mathcal{C}$-comodule Q;*
(v) $\mathrm{Rat}^{\mathcal{C}}({}^*Q)$ *is dense in* ${}^*Q$ *for every left injective indecomposable $\mathcal{C}$-comodule Q;*
(vi) ${}^*Q$ *is* ${}^*\mathcal{C}$*-rational for every left injective indecomposable $\mathcal{C}$-comodule Q;*
(vii) $E(S)$ *is finitely generated for every simple left comodule S;*
(viii) *every simple right $\mathcal{C}$-comodule has a finitely generated projective cover;*
(ix) $\mathcal{C}$ *is right semiperfect.*

Proof. (i)$\Longleftrightarrow$(ii) follows from Proposition 2.4.
(ii)$\Longleftrightarrow$(v). As we have seen, $\mathcal{C} = \bigoplus_{i\in I} E(S_i)$, and each injective indecomposable left $\mathcal{C}$-comodule is isomorphic to one of the $E(S_i)$'s, because every comodule

contains a simple comodule. The equivalence of (ii) and (v) then follows from Proposition 2.5.
(v)$\Longrightarrow$(iv). Every left injective comodule Q is a direct sum of injective indecomposable left $\mathcal{C}$-comodules (because its socle is essential), $Q = \bigoplus_{i\in I} Q_i$. Then we have ${}^*Q = \prod_{i\in I} {}^*Q_i$ in $\mathcal{M}_{^*C}$ and $\bigoplus_{i\in I} \mathrm{Rat}^{\mathcal{C}}({}^*Q_i) \subseteq \mathrm{Rat}^{\mathcal{C}}({}^*Q) \subset \prod_{i\in I} {}^*Q_i$ and then it all follows from Proposition 1.3.
(iv)$\Longrightarrow$(iii). Take $M \in {}^{\mathcal{C}}\mathcal{M}$ and an injective envelope $f: M \to Q$ in ${}^{\mathcal{C}}\mathcal{M}$. We know that $\mathrm{Rat}^{\mathcal{C}}({}^*Q)$ is dense in ${}^*Q = {}_R\mathrm{Hom}(Q,R)$. Proposition 1.5 then yields that ${}^*f(\mathrm{Rat}^{\mathcal{C}}({}^*Q))$ is dense in ${}_R\mathrm{Hom}(M,R) = {}^*M$. But ${}^*f(\mathrm{Rat}^{\mathcal{C}}({}^*Q)) \subset \mathrm{Rat}^{\mathcal{C}}({}^*M)$, so $\mathrm{Rat}^{\mathcal{C}}({}^*M)$ is dense in ${}^*M$.
(iii)$\Longrightarrow$(iv)$\Longrightarrow$(v): trivial.
(i)$\Longleftrightarrow$(vii) follows from Proposition 2.9.
(vi)$\Longleftrightarrow$(vii) follows from Lemma 2.6 and the fact that every injective indecomposable is isomorphic to one of the $E(S_i)$'s.
(vii)$\Longleftrightarrow$(viii). Let T be a simple right $\mathcal{C}$-comodule. Then T is finitely generated, and therefore a simple object in $\mathcal{M}^{\mathrm{fg}\mathcal{C}}$. By the duality between ${}^{\mathrm{fg}\mathcal{C}}\mathcal{M}$ and $\mathcal{M}^{\mathrm{fg}\mathcal{C}}$, $T^* \in {}^{\mathrm{fg}\mathcal{C}}\mathcal{M}$ is simple, and $E(T^*)$ is finitely generated by assumption. The monomorphism $T^* \to E(T^*)$ is essential, so, using the duality, the dual map is a superfluous epimorphism ${}^*E(T^*) \to {}^*(T^*) \simeq T$. It follows from Corollary 4.2 that ${}^*E(T^*)$ is projective, and, using again the duality, that it is finitely generated. Hence ${}^*E(T^*)$ is a finitely generated projective cover of T. □

A coring $\mathcal{C}$ is called left (resp. right) perfect if every object in ${}^{\mathcal{C}}\mathcal{M}$ (resp. $\mathcal{M}^{\mathcal{C}}$) has a projective cover. We will now see that, over a qF-ring, perfectness on both sides is equivalent to semiperfectness on both sides. First we need a Lemma.

Lemma 4.4. *Let R be a qF-ring, and $\mathcal{C}$ a right semiperfect coring that is both left and right profective over R. Then every $0 \neq M \in {}^{\mathcal{C}}\mathcal{M}$ contains a maximal subcomodule. Consequently the Jacobson radical $J(M)$ is small in M.*

Proof. ${}^*M \in \mathcal{M}_{^*\mathcal{C}}$, and $\mathrm{Rat}^{\mathcal{C}}({}^*M)$ is dense in ${}^*M$, by Theorem 4.3. Thus, if $\mathrm{Rat}^{\mathcal{C}}({}^*M) = 0$, then ${}^*M = 0$, which is impossible since R is a cogenerator in ${}_R\mathcal{M}$. So $\mathrm{Rat}^{\mathcal{C}}({}^*M) \neq 0$, and we can take a nonzero simple right subcomodule S of $\mathrm{Rat}^{\mathcal{C}}({}^*M)$. Let $u: S \to \mathrm{Rat}^{\mathcal{C}}({}^*M)$ and $v: \mathrm{Rat}^{\mathcal{C}}({}^*M) \to {}^*M$ be the inclusion maps. Then u is right $\mathcal{C}$-colinear, and v is right ${}^*\mathcal{C}$-linear. Now consider the composition $f = u^* \circ v^* \circ \phi$.

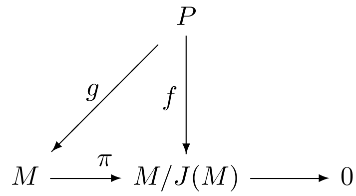
$$M \xrightarrow{\phi} ({}^*M)^* \xrightarrow{v^*} (\mathrm{Rat}^{\mathcal{C}}({}^*M))^* \xrightarrow{u^*} S^*.$$

A straightforward computation shows that $v^* \circ \phi$ is left $\mathcal{C}^*$-linear, and therefore $f = u^* \circ v^* \circ \phi$ is also left $\mathcal{C}^*$-linear. Now $u^* \circ v^*$ is surjective, $\mathrm{Im}\,\phi$ is dense in $({}^*M)^*$, by Lemma 4.1, so $\mathrm{Im}\, f = (u^* \circ v^*)(\mathrm{Im}\,\phi)$ is dense in S^*, by Proposition 1.5. Since S is simple, and therefore finitely generated, the only dense submodule of S^* is S^* itself. So $f: M \to S^*$ is a surjective $\mathcal{C}^*$-linear morphism between the left $\mathcal{C}$-comodules M and S^*, hence it is a left $\mathcal{C}$-colinear surjection. Since S^* is simple in ${}^*\mathcal{M}$, $\mathrm{Ker}\, f$ is a maximal subcomodule of M. □

Proposition 4.5. *Let R be a qF-ring, and $\mathcal{C}$ an R-coring which is left and right (locally) projective over R. Then the following assertions are equivalent.*

(i) *$\mathcal{C}$ is left and right perfect;*
(ii) *$\mathcal{C}$ is left and right semiperfect.*

Proof. The implication (i)⇒(ii) is trivial. Conversely, we will first show that $M/J(M)$ is a semisimple object in $\mathcal{M}^{\mathcal{C}}$, for any $M \in \mathcal{M}^{\mathcal{C}}$. Take $\overline{x} \in M/J(M)$, and let N be the subcomodule of $M/J(M)$ generated by $\overline{x}$. Then $N \subset M/J(M)$, hence $J(N) \subset J(M/J(M)) = 0$. N is finitely generated, and therefore artinian. Let $N_1, \cdots, N_n$ be maximal subcomodules of N such that $\bigcap_{i=1}^n N_i = 0$. Then $N = \bigoplus_{i=1}^n N/N_i$ is semisimple. This shows that every $\overline{x} \in M/J(M)$ belongs to a semisimple subcomodule, so $M/J(M)$ is semisimple.
Since $\mathcal{C}$ is right semiperfect, there exists a projective cover $f: P \to M/J(M)$. Since P is projective, there exists $g \in \mathcal{M}^{\mathcal{C}}$ making the following diagram commutative (π is the canonical projection):



Now $\pi(J(M) + g(P)) = \pi(g(P)) = f(P) = M/J(M)$, so $J(M) + g(P) = M$, since π is surjective. $\mathcal{C}$ is left semiperfect, hence, by Lemma 4.4, $J(M) \ll M$, and we conclude that $g(P) = M$. So g is surjective. $\mathrm{Ker}\, f \ll P$ and $\mathrm{Ker}\, g \subset \mathrm{Ker}\, f$, hence $\mathrm{Ker}\, g \ll P$, and we conclude that $g: P \to M$ is a projective cover of M. □

4.2. **Examples.**

Example 4.6. Let $\mathcal{C}$ be a coring, and assume that $\mathcal{C}$ is finitely generated and projective as a left R-module. Then $\mathcal{M}^{\mathcal{C}}$ is isomorphic to $\mathcal{M}_{^*\mathcal{C}}$, and $\mathrm{Rat}^{\mathcal{C}}$ is an isomorphism of categories. Hence $\mathrm{Rat}^{\mathcal{C}}$ is exact. $\mathcal{M}^{\mathcal{C}}$ has enough projectives, but not necessarily projective covers. As an example, let R be a non-semiperfect ring, and $\mathcal{C} = R$, the trivial R-coring. Then $\mathcal{M}^{\mathcal{C}} = \mathcal{M}_R$ is not semiperfect.

Example 4.7. Let $\mathcal{C}$ be a cosemisimple coring. Then $\mathcal{C}$ is left and right semiperfect, since the categories of left and right $\mathcal{C}$-comodules are semisimple, see [4, 19.14], [9] and [11]. In this case, $\mathcal{C}$ is projective in $_R\mathcal{M}$ and $\mathcal{M}_R$, so $\mathcal{C}$ satisfies the left and right α-condition. $\mathcal{C}$ can then be written as a direct sum of finitely generated left (or right) $\mathcal{C}$-comodules, and the functors $\mathrm{Rat}^{\mathcal{C}}$ and $^{\mathcal{C}}\mathrm{Rat}$ are exact. So all the equivalent statements of Theorem 4.3 hold, without the assumption that the base ring R is a qF-ring.

Example 4.8. To a ring morphism $\iota: R \to S$, we can associate the Sweedler coring $\mathcal{C}$. As an S-bimodule, $\mathcal{C} = S \otimes_R S$, and the comultiplication and counit are given by the formulas

$$\Delta(s \otimes_R s') = (s \otimes_R 1) \otimes_S (1 \otimes_R s') \; ; \; \varepsilon(s \otimes_R s') = ss'$$

The Sweedler coring is important in descent theory: the comodules over $\mathcal{C}$ are exactly the descent data from [12] (in the commutative case) and [7] (in the noncommutative case). If $M \in \mathcal{M}^{\mathcal{C}}$, then M descends to an R-module

$$M^{\mathrm{co}\mathcal{C}} = \{m \in M \mid \rho(m) = m \otimes_R 1\}$$

For a detailed discussion, we refer to [5]. It is also easy to see that we have an isomorphism of R-algebras

$$^*\mathcal{C} = {}_S\mathrm{Hom}(S \otimes_R S, S) \cong {}_R\mathrm{End}(S)$$

(again, ${}_R\mathrm{End}(S)$ is a ring with the opposite composition as multiplication). Also notice that $S \subset {}_R\mathrm{End}(S)$ as algebras, by right multiplication.
If we assume that $S \in {}_R\mathcal{M}$ is locally projective, then $\mathcal{C} \in {}_S\mathcal{M}$ is locally projective, and we can consider the functor

$$\mathrm{Rat}^{\mathcal{C}} :\ \mathcal{M}_{{}_R\mathrm{End}(S)} \to \mathcal{M}^{\mathcal{C}}$$

Let M be a right ${}_R\mathrm{End}(S)$-module, and take $m \in M$. Then $m \in \mathrm{Rat}^{\mathcal{C}}(M)$ if and only if there exists $m_{[0]} \otimes_R m_{[1]} \in M \otimes_R S$ such that $m \cdot f = m_{[0]} f(m_{[1]})$, for all $f \in {}_R\mathrm{End}(S)$. In particular,

$$\begin{aligned}\mathrm{Rat}^{\mathcal{C}}(M)^{\mathrm{co}\mathcal{C}} &= \{m \in \mathrm{Rat}^{\mathcal{C}}(M) \mid \rho(m) = m \otimes_R 1\} \\ &= \{m \in M \mid m \cdot f = mf(1),\ \text{for all } f \in {}_R\mathrm{End}(S)\}\end{aligned}$$

$\mathrm{Rat}^{\mathcal{C}}(M)$ is a right $\mathcal{C}$-comodule, and therefore a right ${}_R\mathrm{End}(S)$-module, and, by restriction of scalars, a right S-module. Therefore

$$\mathrm{Rat}^{\mathcal{C}}(M)^{\mathrm{co}\mathcal{C}} \cdot S \subset \mathrm{Rat}^{\mathcal{C}}(M) \tag{4}$$

If we take $M = {}_R\mathrm{End}(S)$, then we see that

$$\mathrm{Rat}^{\mathcal{C}}(M)^{\mathrm{co}\mathcal{C}} = \{g \in {}_R\mathrm{End}(S) \mid (f \circ g)(s) = g(s)f(1),\ \text{for all } f \in {}_R\mathrm{End}(S)\}$$

Take $h \in {}^*S = {}_R\mathrm{Hom}(S, R)$. Then $\overline{h} = h \circ \iota \in {}_R\mathrm{Hom}(S, S)$, and it follows easily that $\overline{h} \in \mathrm{Rat}^{\mathcal{C}}({}_R\mathrm{End}(S))^{\mathrm{co}\mathcal{C}}$. We will use this to show that $\mathrm{Rat}^{\mathcal{C}}(\mathrm{End}(S))$ is dense in ${}_R\mathrm{End}(S)$.
Take $f \in {}_R\mathrm{End}(S)$, and $F \subset S$ finite. Since S is locally projective, there are $h_1, \cdots, h_n \in {}^*S$ and $x_1, \cdots, x_n \in S$ such that

$$x = \sum_{k=1}^{n} h_k(x) x_k$$

for $x \in F$ and then a simple computation shows that

$$f(x) = \sum_{k=1}^{n} h_k(x) f(x_k) = \Big(\sum_{k=1}^{n} \overline{h}_k \cdot f(x_k)\Big)(x)$$

By (4) and the the fact the above argument, $\sum_{k=1}^{n} \overline{h}_k \cdot f(x_k) \in \mathrm{Rat}^{\mathcal{C}}(\mathrm{End}(S))$. So we have shown that f coincides on F to an element in $\mathrm{Rat}^{\mathcal{C}}(\mathrm{End}(S))$. We conclude that $\mathrm{Rat}^{\mathcal{C}}({}^*\mathcal{C})$ lies dense in ${}^*\mathcal{C}$, and, by Proposition 2.4, $\mathrm{Rat}^{\mathcal{C}}$ is exact.
If S is pure as a left and right R-module, in particular if $S \in {}_R\mathcal{M}$ is faithfully flat, then the categories $\mathcal{M}_R$ and $\mathcal{M}^{\mathcal{C}}$ are equivalent (see [5, 7, 12]). In this case, $\mathcal{M}^{\mathcal{C}}$ has enough projectives.
If $S \in {}_R\mathcal{M}$ is faithfully flat and locally projective, then we have an explicit description of $\mathrm{Rat}^{\mathcal{C}}(M)$, namely

$$\mathrm{Rat}^{\mathcal{C}}(M) = \mathrm{Rat}^{\mathcal{C}}(M)^{\mathrm{co}\mathcal{C}} \otimes_R S,$$

with $\mathrm{Rat}^{\mathcal{C}}(M)^{\mathrm{co}\mathcal{C}}$ given by (4).

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