

## SUBNORMALITY AND $k$ -HYPONORMALITY OF TOEPLITZ OPERATORS: A BRIEF SURVEY AND OPEN QUESTIONS

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The present note concerns subnormality and  $k$ -hyponormality of Toeplitz operators. We begin with a brief survey of research related to P.R. Halmos's Problem 5 (cf. [Ha1],[Ha2]):

(Prob 5) Is every subnormal Toeplitz operator either normal or analytic ?

As we know, (Prob 5) was answered in the negative by C. Cowen and J. Long [CoL]; directly connected with it is the following problem:

(0.1) Which Toeplitz operators are subnormal ?

Let  $\mathcal{H}$  and  $\mathcal{K}$  be complex Hilbert spaces, let  $\mathcal{L}(\mathcal{H}, \mathcal{K})$  be the set of bounded linear operators from  $\mathcal{H}$  to  $\mathcal{K}$  and write  $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$ . An operator  $T \in \mathcal{L}(\mathcal{H})$  is said to be normal if  $T^*T = TT^*$ , hyponormal if  $T^*T \geq TT^*$ , and subnormal if  $T = N|_{\mathcal{H}}$ , where  $N$  is normal

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1991 *Mathematics Subject Classification*: Primary 47B20, 47B35, 47A63; Secondary 47B37, 47B38, 42A05, 30D50.

The paper is in final form and no version of it will be published elsewhere.

*Key words and phrases*. 2-hyponormality, subnormality, Toeplitz operator, Blaschke product, trigonometric polynomial, invariant subspaces.

The work of the first-named author was partially supported by NSF research grant DMS-9800931.

The work of the second-named author was partially supported by the Brain Korea 21 Project from the Korean Ministry of Education.

on some Hilbert space  $\mathcal{K} \supseteq \mathcal{H}$ . If  $T$  is subnormal then  $T$  is also hyponormal. Recall that the Hilbert space  $L^2(\mathbf{T})$  has a canonical orthonormal basis given by the trigonometric functions  $e_n(z) = z^n$ , for all  $n \in \mathbf{Z}$ , and that the Hardy space  $H^2(\mathbf{T})$  is the closed linear span of  $\{e_n : n = 0, 1, \dots\}$ . An element  $f \in L^2(\mathbf{T})$  is said to be analytic if  $f \in H^2(\mathbf{T})$ , and co-analytic if  $f \in L^2(\mathbf{T}) \ominus H^2(\mathbf{T})$ . If  $P$  denotes the orthogonal projection from  $L^2(\mathbf{T})$  to  $H^2(\mathbf{T})$ , then for every  $\varphi \in L^\infty(\mathbf{T})$  the operators  $T_\varphi$  and  $H_\varphi$  on  $H^2(\mathbf{T})$  defined by

$$T_\varphi g := P(\varphi g) \quad \text{and} \quad H_\varphi(g) := (I - P)(\varphi g) \quad (g \in H^2(\mathbf{T}))$$

are called the *Toeplitz operator* and the *Hankel operator*, respectively, with symbol  $\varphi$ .

(Prob 5) has been answered in the affirmative for *trigonometric* Toeplitz operators [ItW], and for *quasinormal* Toeplitz operators [AIW]. In 1976, M. Abrahamse [Abr] gave a general sufficient condition for the answer to (Prob 5) to be affirmative.

THEOREM 1 ([Abr]). *If*

- (i)  $T_\varphi$  is hyponormal;
- (ii)  $\varphi$  or  $\bar{\varphi}$  is of bounded type (i.e.,  $\varphi$  or  $\bar{\varphi}$  is a quotient of two analytic functions);
- (iii)  $\ker[T_\varphi^*, T_\varphi]$  is invariant for  $T_\varphi$ ,

then  $T_\varphi$  is normal or analytic.

Since  $\ker[T^*, T]$  is invariant for every subnormal operator  $T$ , Theorem 1 answers (Prob 5) affirmatively when  $\varphi$  or  $\bar{\varphi}$  is of bounded type. Also, in [Abr], Abrahamse proposed the following question, as a strategy to answer (Prob 5):

(Abr) Is the Bergman shift unitarily equivalent to a Toeplitz operator ?

To study this question, recall that given a bounded sequence of positive numbers  $\alpha : \alpha_0, \alpha_1, \dots$  (called *weights*), the (*unilateral*) *weighted shift*  $W_\alpha$  associated with  $\alpha$  is the operator on  $\ell^2(\mathbf{Z}_+)$  defined by  $W_\alpha e_n := \alpha_n e_{n+1}$  for all  $n \geq 0$ , where  $\{e_n\}_{n=0}^\infty$  is the canonical orthonormal basis for  $\ell^2$ . It is straightforward to check that  $W_\alpha$  can never be *normal*, and that  $W_\alpha$  is *hyponormal* if and only if  $\alpha_n \leq \alpha_{n+1}$  for all  $n \geq 0$ . The Bergman shift is a weighted shift  $W_\alpha$  with weights  $\alpha := \left\{ \sqrt{\frac{n}{n+1}} \right\}_{n=1}^\infty$ ; it is well known that the Bergman shift is subnormal. In 1983, S. Sun [Sun] showed that if a Toeplitz operator  $T_\varphi$  is unitarily equivalent to a hyponormal weighted shift  $W_\alpha$  with weight sequence  $\alpha$ , then  $\alpha$  must be of the form

$$(1.1) \quad \alpha = \left\{ (1 - \beta^{2n+2})^{\frac{1}{2}} \|T_\varphi\| \right\}_{n=0}^\infty$$

for some  $\beta$  ( $0 < \beta < 1$ ), thus answering (Abr) in the negative. Cowen and Long [CoL] showed that a unilateral weighted shift with weight sequence of the form (1.1) must be subnormal (see also [Fa2]). Consequently, we have:

THEOREM 2 ([Sun], [Cow2]). *Every hyponormal Toeplitz operator which is unitarily equivalent to a weighted shift must be subnormal.*

Finally, in 1984 Cowen and Long [CoL] constructed a symbol  $\varphi$  for which  $T_\varphi$  is unitarily equivalent to a weighted shift with weight sequence (1.1). This helped answer (Prob 5) in the negative.

THEOREM 3 ([CoL],[Cow2]). *Let  $0 < \alpha < 1$  and let  $\psi$  be a conformal map of the unit disk onto the interior of the ellipse with vertices  $\pm(1 + \alpha)i$  and passing through  $\pm(1 - \alpha)$ . If  $\varphi = (1 - \alpha^2)^{-1}(\psi + \alpha\bar{\psi})$ , then  $T_\varphi$  is a weighted shift with weight sequence  $\alpha_n = (1 - \alpha^{2n+2})^{-\frac{1}{2}}$ . Therefore,  $T_\varphi$  is subnormal but neither normal nor analytic.*

In view of Theorem 3, it is worth turning one's attention to hyponormality of Toeplitz operators, which has been studied by M. Abrahamse [Abr], C. Cowen [Cow1],[Cow2], P. Fan [Fa1], C. Gu [Gu], T. Ito and T. Wong [ItW], T. Nakazi and K. Takahashi [NaT], D. Yu [Yu], K. Zhu [Zhu], D. Farenick, the authors, and their collaborators (cf. [FaL1],[FaL2],[CuL1],[HKL],[KiL]). An elegant theorem of C. Cowen [Cow3] characterizes the hyponormality of a Toeplitz operator  $T_\varphi$  on  $H^2(\mathbf{T})$  by properties of the symbol  $\varphi \in L^\infty(\mathbf{T})$ . K. Zhu [Zhu] reformulated Cowen's criterion and then showed that the hyponormality of  $T_\varphi$  with polynomial symbols  $\varphi$  can be decided by a method based on the classical interpolation theorem of I. Schur [Sch]. Here, we shall use a variant of Cowen's theorem [Cow3] that was first proposed by Nakazi and Takahashi [NaT].

COWEN'S THEOREM . *Suppose  $\varphi \in L^\infty(\mathbf{T})$  is arbitrary and write*

$$\mathcal{E}(\varphi) = \{k \in H^\infty(\mathbf{T}) : \|k\|_\infty \leq 1 \text{ and } \varphi - k\bar{\varphi} \in H^\infty(\mathbf{T})\}.$$

*Then  $T_\varphi$  is hyponormal if and only if the set  $\mathcal{E}(\varphi)$  is nonempty.*

On the other hand, the Bram–Halmos criterion for subnormality states that an operator  $T$  is subnormal if and only if

$$\sum_{i,j} (T^i x_j, T^j x_i) \geq 0$$

for all finite collections  $x_0, x_1, \dots, x_k \in \mathcal{H}$  ([Bra],[Con, II.1.9]). It is easy to see that this is equivalent to the following positivity test:

$$(3.1) \quad \begin{pmatrix} I & T^* & \dots & T^{*k} \\ T & T^*T & \dots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \dots & T^{*k}T^k \end{pmatrix} \geq 0 \quad (\text{all } k \geq 1).$$

Condition (3.1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (3.1) for  $k = 1$  is equivalent to the hyponormality of  $T$ , while subnormality requires the validity of (3.1) for all  $k$ . If we denote by  $[A, B] := AB - BA$  the commutator of two operators  $A$  and  $B$ , and if we define  $T$  to be  $k$ -hyponormal whenever the  $k \times k$  operator matrix

$$(3.2) \quad M_k(T) := ([T^{*j}, T^i]_{i,j=1}^k)$$

is positive, or equivalently, the  $(k + 1) \times (k + 1)$  operator matrix in (3.1) is positive (via the operator version of Choleski's Algorithm), then the Bram–Halmos criterion can be rephrased as saying that  $T$  is subnormal if and only if  $T$  is  $k$ -hyponormal for every  $k \geq 1$  ([CMX]).

Recall now ([Ath],[Cu2],[CMX]) that  $T \in \mathcal{L}(\mathcal{H})$  is said to be *weakly  $k$ -hyponormal* if

$$LS(T, T^2, \dots, T^k) := \left\{ \sum_{j=1}^k \alpha_j T^j : \alpha = (\alpha_1, \dots, \alpha_k) \in \mathbf{C}^k \right\}$$

consists entirely of hyponormal operators, or equivalently,  $M_k(T)$  is *weakly positive*, i.e.,

$$(M_k(T) \begin{pmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{pmatrix}, \begin{pmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{pmatrix}) \geq 0 \quad \text{for } x \in \mathcal{H} \text{ and } \lambda_0, \dots, \lambda_k \in \mathbf{C} \quad ([\text{CMX}]).$$

If  $k = 2$  then  $T$  is said to be *quadratically hyponormal*. Similarly,  $T$  is said to be *polynomially hyponormal* if  $p(T)$  is hyponormal for every polynomial  $p \in \mathbf{C}[z]$ . It is known that  $k$ -hyponormal  $\Rightarrow$  weakly  $k$ -hyponormal, but the converse is not true in general.

It is now natural to try to understand the gap between  $k$ -hyponormality and subnormality for Toeplitz operators. As a first inquiry in this line of thought we pose the following ([CuL1]):

QUESTION A. *Is every 2-hyponormal Toeplitz operator subnormal ?*

In [CuL1], the following was shown:

THEOREM 4 ([CuL1]). *Every trigonometric Toeplitz operator whose square is hyponormal must be normal or analytic. Hence, in particular, every 2-hyponormal trigonometric Toeplitz operator is subnormal.*

It is well known ([Cu1],[Cu2]) that, for weighted shifts, there are gaps between hyponormality and quadratic hyponormality, and between quadratic hyponormality and 2-hyponormality. Note that Theorem 4 says more: every *quadratically* hyponormal trigonometric Toeplitz operator is subnormal. Thus Theorem 4 shows that there is a big gap between hyponormality and quadratic hyponormality for Toeplitz operators. For example, if

$$\varphi(z) \equiv \sum_{n=-m}^N a_n z^n \quad (m < N)$$

is such that  $T_\varphi$  is hyponormal, then by Theorem 4  $T_\varphi$  is never quadratically hyponormal, since  $T_\varphi$  is neither analytic nor normal. (Recall that if such a  $T_\varphi$  is normal then  $m = N$  (cf. [FaL1]).)

In view of Theorem 4, the following question arises naturally:

QUESTION B. *Is every quadratically hyponormal Toeplitz operator 2-hyponormal ?*

An affirmative answer to Question B would show that there exists no gap between quadratic hyponormality and 2-hyponormality for Toeplitz operators. A negative answer would give rise to a challenging problem: *Characterize non-2-hyponormal quadratically hyponormal Toeplitz operators; more generally, characterize non- $k$ -hyponormal weakly  $k$ -hyponormal Toeplitz operators.*

We can extend Theorem 4. First we observe:

PROPOSITION 5 ([CuL2]). *If  $T \in \mathcal{L}(\mathcal{H})$  is 2-hyponormal then*

$$(5.1) \quad T(\ker[T^*, T]) \subseteq \ker[T^*, T].$$

PROOF. Suppose that  $[T^*, T]f = 0$ . Since  $T$  is 2-hyponormal, it follows from (3.2) that (cf. [CMX, Lemma 1.4])

$$|([T^{*2}, T]g, f)|^2 \leq ([T^*, T]f, f)([T^{*2}, T^2]g, g) \quad \text{for all } g \in \mathcal{H}.$$

By assumption, we have that for all  $g \in \mathcal{H}$ ,  $0 = ([T^{*2}, T]g, f) = (g, [T^{*2}, T]^*f)$ , so that  $[T^{*2}, T]^*f = 0$ , i.e.,  $T^*T^2f = T^2T^*f$ . Therefore,

$$[T^*, T]Tf = (T^*T^2 - TT^*T)f = (T^2T^* - TT^*T)f = T[T^*, T]f = 0,$$

which proves (5.1). ■

COROLLARY 6. *If  $T_\varphi$  is 2-hyponormal and if  $\varphi$  or  $\bar{\varphi}$  is of bounded type then  $T_\varphi$  is normal or analytic, so that  $T_\varphi$  is subnormal.*

PROOF. This follows at once from Theorem 1 and Proposition 5. ■

COROLLARY 7. *If  $T_\varphi$  is a 2-hyponormal operator such that  $\mathcal{E}(\varphi)$  contains at least two elements then  $T_\varphi$  is normal or analytic, so that  $T_\varphi$  is subnormal.*

PROOF. This follows from Corollary 6 and the fact ([NaT, Proposition 8]) that if  $\mathcal{E}(\varphi)$  contains at least two elements then  $\varphi$  is of bounded type. ■

From Corollaries 6 and 7, we can see that if  $T_\varphi$  is 2-hyponormal but not subnormal then  $\varphi$  is not of bounded type and  $\mathcal{E}(\varphi)$  consists of exactly one element.

From Corollary 6 we can see that if  $T_\varphi$  is a 2-hyponormal operator such that  $\varphi$  or  $\bar{\varphi}$  is of bounded type then  $T_\varphi$  has a nontrivial invariant subspace. The following question is naturally raised:

QUESTION C. *Does every 2-hyponormal Toeplitz operator have a nontrivial invariant subspace? More generally, does every 2-hyponormal operator have a nontrivial invariant subspace?*

It is well known ([Bro]) that if  $T$  is a hyponormal operator such that  $R(\sigma(T)) \neq C(\sigma(T))$  then  $T$  has a nontrivial invariant subspace. But it remains still open whether every hyponormal operator with  $R(\sigma(T)) = C(\sigma(T))$  (i.e., with a *thin* spectrum) has a nontrivial invariant subspace. Recall that  $T \in \mathcal{L}(\mathcal{H})$  is called a *von Neumann operator* if  $\sigma(T)$  is a spectral set for  $T$ ; as shown by J. Agler [Agl], every von Neumann operator has a nontrivial invariant subspace. Recently, B. Prunaru [Pru] established that polynomially hyponormal operators also possess the same property. The following is a sub-question of Question C.

QUESTION D. *Is every 2-hyponormal operator with thin spectrum a von Neumann operator?*

Recall that  $\varphi \in L^\infty(\mathbf{T})$  is called *almost analytic* if  $z^n \varphi$  is analytic for some positive  $n$  and is called *almost coanalytic* if  $\bar{\varphi}$  is almost analytic. Observe that if  $\varphi$  is real-valued and almost analytic, then  $\varphi$  is constant. It is easy to check that if  $\varphi$  is almost coanalytic and  $T_\varphi$  is hyponormal then  $\varphi$  must be a trigonometric polynomial. But this is not the case for almost analytic functions  $\varphi$ . To see this, we reformulate Cowen's Theorem.

Suppose  $\varphi \in L^\infty(\mathbf{T})$  is of the form  $\varphi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$  and  $k(z) = \sum_{n=0}^{\infty} c_n z^n$  is in  $H^2(\mathbf{T})$ . Then  $\varphi - k\bar{\varphi} \in H^\infty$  has a solution  $k \in H^\infty$  if and only if

$$(7.1) \quad \begin{pmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 & \cdots & \bar{a}_n & \cdots \\ \bar{a}_2 & \bar{a}_3 & \cdots & \bar{a}_n & \cdots & \\ \bar{a}_3 & \cdots & \cdots & \cdots & \cdots & \\ \vdots & \bar{a}_n & \cdots & \cdots & \cdots & \\ \bar{a}_n & \cdots & \cdots & \cdots & \cdots & \\ \vdots & \cdots & \cdots & \cdots & \cdots & \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} a_{-1} \\ a_{-2} \\ a_{-3} \\ \vdots \\ \vdots \\ \vdots \end{pmatrix},$$

that is,  $H_{\bar{\varphi}}k = H_{\varphi}e_0$ , where  $e_0 = (1, 0, 0, \dots)$ . Thus, by Cowen's Theorem,  $T_{\varphi}$  is hyponormal if and only if there exists a solution  $k \in H^\infty(\mathbf{T})$  of the equation (7.1) such that  $\|k\|_\infty \leq 1$ .

Now suppose  $\varphi \in L^\infty(\mathbf{T})$  is a function of the form

$$\varphi(z) = \frac{1}{6}z^{-1} + \sum_{n=2}^{\infty} \frac{1}{2^{n-1}}z^n.$$

Then  $k(z) = \sum_{n=0}^{\infty} c_n z^n$  satisfies  $\varphi - k\bar{\varphi} \in H^\infty$  if and only if

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots & \\ \frac{1}{4} & \frac{1}{8} & \cdots & \cdots & \\ \frac{1}{8} & \cdots & \cdots & \cdots & \\ \vdots & \cdots & \cdots & \cdots & \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} \frac{1}{6} \\ 0 \\ 0 \\ \vdots \\ \vdots \end{pmatrix}.$$

A straightforward calculation shows that

$$k(z) = -\frac{1}{6} + \sum_{n=1}^{\infty} \frac{1}{2^{n+1}}z^n$$

satisfies (7.1). Also, it is easy to see that  $k(z) = \frac{1}{3} \frac{z - \frac{1}{2}}{1 - \frac{1}{2}z}$ , so  $\|k\|_\infty = \frac{1}{3}$ . Therefore  $T_{\varphi}$  is hyponormal (cf. [CuL1, Example 2.3]).

However we have:

**THEOREM 8.** *If  $T_{\varphi}$  is 2-hyponormal with non-analytic almost analytic symbol  $\varphi$  then  $\varphi$  must be a trigonometric polynomial.*

**PROOF.** Since almost analytic functions are of bounded type it follows from Corollary 6 that if  $T_{\varphi}$  is 2-hyponormal with non-analytic almost analytic symbol  $\varphi$  then  $T_{\varphi}$  must be normal. Since by the Brown–Halmos Theorem [BrH], every normal Toeplitz operator is a rotation and a translation of a hermitian Toeplitz operator, it follows that  $\varphi$  must be a trigonometric polynomial. ■

Although the existence of a non-subnormal polynomially hyponormal weighted shift was established in [CuP1] and [CuP2], it is still an open question whether the implication “polynomially hyponormal  $\Rightarrow$  subnormal” can be disproved with a Toeplitz operator.

QUESTION E. *Does there exist a Toeplitz operator which is polynomially hyponormal but not subnormal ?*

It is well known that  $T$  is a von Neumann operator if and only if  $q(T)$  is normaloid (i.e., norm equals spectral radius) for every rational function  $q$  with poles outside  $\sigma(T)$ . Thus if  $T$  is *rationally* hyponormal, i.e.,  $q(T)$  is hyponormal for every rational function  $q$  with poles outside  $\sigma(T)$ , then  $T$  is a von Neumann operator. Thus the following question arises naturally:

QUESTION F. *Does there exist a polynomially hyponormal operator which is not a von Neumann operator ? And within the class of Toeplitz operators ?*

An affirmative answer to Question F guarantees the existence of polynomially hyponormal operators which are not rationally hyponormal (and hence not subnormal). If  $T_\varphi$  is a trigonometric Toeplitz operator then evidently, by Theorem 4, if  $T_\varphi$  is polynomially hyponormal then it is a von Neumann operator.

In [CuL2] it was shown that every pure 2-hyponormal operator with rank-one self-commutator is a linear function of the unilateral shift. On the other hand, J.E. McCarthy and L. Yang [McCY] have classified all rationally cyclic subnormal operators with finite rank self-commutators. However it is still open which are the pure subnormal operators with finite rank self-commutator. Related to this, we formulate the following:

QUESTION G. *If  $T_\varphi$  is a 2-hyponormal Toeplitz operator with nonzero finite rank self-commutator, does it follow that  $T_\varphi$  is analytic ? If the answer is affirmative, is  $\varphi$  a linear function of a finite Blaschke product ?*

We shall give a partial positive answer to Question G. To do this we recall Theorem 15 in [NaT] which states that if  $T_\varphi$  is subnormal and  $\varphi = q\bar{\varphi}$ , where  $q$  is a finite Blaschke product, then  $T_\varphi$  is normal or analytic. A careful examination of the proof of that theorem reveals that it uses the subnormality assumption only for the fact that  $\ker [T_\varphi^*, T_\varphi]$  is invariant under  $T_\varphi$ . Thus in view of Proposition 5, the theorem is still valid for “2-hyponormal” in place of “subnormal”. We thus have:

LEMMA 9. *If  $T_\varphi$  is 2-hyponormal and  $\varphi = q\bar{\varphi}$ , where  $q$  is a finite Blaschke product, then  $T_\varphi$  is normal or analytic.*

We now give a partial answer to Question G.

THEOREM 10. *Suppose  $\log |\varphi|$  is not integrable. If  $T_\varphi$  is a 2-hyponormal operator with nonzero finite rank self-commutator then  $T_\varphi$  is analytic.*

PROOF. If  $T_\varphi$  is hyponormal such that  $\log |\varphi|$  is not integrable then by an argument of [NaT, Theorem 4],  $\varphi = q\bar{\varphi}$  for some inner function  $q$ . Also if  $T_\varphi$  has a finite rank self-commutator then, by [NaT, Theorem 10], there exists a finite Blaschke product  $b \in \mathcal{E}(\varphi)$ . If  $q \neq b$ , so that  $\mathcal{E}(\varphi)$  contains at least two elements, then by Corollary 7,  $T_\varphi$  is normal or analytic. If instead  $q = b$  then by Lemma 9,  $T_\varphi$  is also normal or analytic. ■

Theorem 10 reduces Question G to the class of Toeplitz operators such that  $\log |\varphi|$  is integrable. If  $\log |\varphi|$  is integrable then there exists an outer function  $e$  such that  $|\varphi| = |e|$ . Thus we may write  $\varphi = ue$ , where  $u$  is a unimodular function. Since by the Douglas–Rudin

Theorem (cf. [Gar, p.192]), every unimodular function can be approximated by quotients of inner functions, it follows that if  $\log |\varphi|$  is integrable then  $\varphi$  can be approximated by functions of bounded type. Therefore if we could obtain a sequence  $\psi_n$  converging to  $\varphi$  such that  $T_{\psi_n}$  is 2-hyponormal with finite rank self-commutator for each  $n$ , then we would answer Question G affirmatively. On the other hand, if  $T_\varphi$  attains its norm then by a result of Brown and Douglas [BrD],  $\varphi$  is of the form  $\varphi = \lambda \frac{\psi}{\theta}$  with  $\lambda > 0$ ,  $\psi$  and  $\theta$  inner. Thus  $\varphi$  is of bounded type. Therefore, by Corollary 7, if  $T_\varphi$  is 2-hyponormal and attains its norm then  $T_\varphi$  is normal or analytic. However we have not been able to decide that if  $T_\varphi$  is a 2-hyponormal operator with finite rank self-commutator then  $T_\varphi$  attains its norm.

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