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A Joint Spectral Characterization of Primeness for C*-Algebras

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Abstract

We prove that a C*-algebra \mathcal{A} is prime if and only if $\sigma_T((L_a, R_b), \mathcal{A}) = \sigma(a) \times \sigma(b)$ for every $a, b \in \mathcal{A}$, where σ_T denotes Taylor spectrum and L_a, R_b are the left and right multiplication operators acting on \mathcal{A} .

Let \mathcal{A} be a unital C*-algebra, let L_a, R_b denote the *left* and *right* multiplication operators induced by $a, b \in \mathcal{A}$ (i.e., $L_a(x) := ax$, $R_b(x) := xb$, $x \in \mathcal{A}$), and set $M_{a,b} := L_a R_b$. We obtain a characterization of primeness for C*-algebras in terms of the spectral theory of (L_a, R_b) . Recall that an ideal \mathcal{I} in \mathcal{A} is said to be *prime* if, whenever $\mathcal{I}_1, \mathcal{I}_2$ are ideals in \mathcal{A} such that $\mathcal{I}_1 \mathcal{I}_2 \subseteq \mathcal{I}$, it follows that $\mathcal{I}_1 \subseteq \mathcal{I}$ or $\mathcal{I}_2 \subseteq \mathcal{I}$; \mathcal{A} is said to be prime if (0) is a prime ideal. In [Ma1], M. Mathieu obtained the following result.

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Theorem 1 ([Ma1]) *Let \mathcal{A} be a unital C^* -algebra. The following statements are equivalent.*

(i) \mathcal{A} is prime.

(ii) $\|M_{a,b}\| = \|a\| \|b\|$ for all $a, b \in \mathcal{A}$.

(iii) $\sigma(M_{a,b}) = \sigma(a)\sigma(b)$ for all $a, b \in \mathcal{A}$.

In this note we prove a joint spectral analogue of Theorem 1. First, we recall the definition of the (joint) Taylor spectrum of (L_a, R_b) . The *Koszul complex* associated with (L_a, R_b) is

$$\mathcal{K}_{a,b} : 0 \longrightarrow \mathcal{A} \xrightarrow{D_{a,b}^0} \mathcal{A} \oplus \mathcal{A} \xrightarrow{D_{a,b}^1} \mathcal{A} \longrightarrow 0,$$

where

$$D_{a,b}^0(x) := ax \oplus xb \quad (x \in \mathcal{A})$$

and

$$D_{a,b}^1(x \oplus y) := -xb + ax \quad (x, y \in \mathcal{A}).$$

We say that (L_a, R_b) is *Taylor invertible* if $\mathcal{K}_{a,b}$ is exact, i.e., if the following three implications hold:

$$\text{Ker} D_{a,b}^0 = 0 : \quad x \in \mathcal{A}, ax = 0 = xb \Rightarrow x = 0;$$

$$\text{Ker} D_{a,b}^1 = \text{Ran} D_{a,b}^0 : \quad x, y \in \mathcal{A}, ay = xb \Rightarrow \exists z \in \mathcal{A} : x = az;$$

and

$$\text{Ran} D_{a,b}^1 = \mathcal{A} : \quad z \in \mathcal{A} \Rightarrow \exists x, y \in \mathcal{A} : -xb + ay = z.$$

The *Taylor spectrum* of (L_a, R_b) is then $\sigma_T((L_a, R_b), \mathcal{A}) := \{(\lambda, \mu) \in \mathbf{C}^2 : (L_a - \lambda, R_b - \mu) \text{ is not Taylor invertible}\}$ ([Ta1], [Ta2], [Cu1]).

Theorem 2 *Let \mathcal{A} be a unital C^* -algebra. The following statements are equivalent.*

(i) \mathcal{A} is prime.

(ii) $\sigma_T((L_a, R_b), \mathcal{A}) = \sigma(a) \times \sigma(b)$ ($a, b \in \mathcal{A}$).

Corollary 3 *Let \mathcal{A} be a unital C^* -algebra, and let $a, b \in \mathcal{A}$. The following statements are equivalent.*

(i) $\sigma_T((L_a, R_b), \mathcal{A}) = \sigma(a) \times \sigma(b)$ ($a, b \in \mathcal{A}$).

(ii) $\sigma(M_{a,b}) = \sigma(a)\sigma(b)$.

Remark 4 (i) *Since the implication (i) \Rightarrow (ii) in Corollary 3 is an easy consequence of the Spectral Mapping Theorem for the Taylor spectrum ([Ta2]), and since (ii) implies the primeness of \mathcal{A} by Theorem 1, the content of Theorem 2 is really the validity of (i) \Rightarrow (ii).*

(ii) *For $\mathcal{A} = \mathcal{L}(\mathcal{H})$, the C^* -algebra of bounded operators on a Hilbert space \mathcal{H} , Theorem 2 is a special case of [CuF, Theorem 3.1].*

(iii) *Theorem 2 gives an affirmative answer to a question raised in [Cu2, Problem 5.3; case $n=1$]; it also sheds new light on the so-called mixed interpolation problem [Har].*

For the proof of Theorem 2 we shall need the following lemma. Recall that the (joint) *Harte* spectrum of (L_a, R_b) is the union of the *left* and *right* spectra; alternatively, $(\lambda, \mu) \notin \sigma_H((L_a, R_b), \mathcal{A})$ if and only if $D_{a,b}^0$ is bounded below and $D_{a,b}^1$ is onto. Clearly $\sigma_H \subseteq \sigma_T$.

Lemma 5 ([Ma2, Theorem 3.12]) *Let \mathcal{A} be a unital prime C^* -algebra, and let $a, b \in \mathcal{A}$. Then*

$$\sigma_H((L_a, R_b), \mathcal{A}) = [\sigma_\ell(a) \times_r (b)] \cup [\sigma_r(a) \times \sigma_\ell(b)].$$

Proof of Theorem 2. Let $\lambda \in \sigma(a)$, $\mu \in \sigma(b)$. We wish to prove that $(\lambda, \mu) \in \sigma_T((L_a, R_b), \mathcal{A})$. Assume not. Without loss of generality, $\lambda = \mu = 0$, and by Lemma 5, we can also assume that either (i) $\lambda \in [\sigma_\ell(a) \setminus \sigma_r(a)]$, $\mu \in [\sigma_\ell(b) \setminus \sigma_r(b)]$ or (ii) $\lambda \in [\sigma_r(a) \setminus \sigma_\ell(a)]$, $\mu \in [\sigma_r(b) \setminus \sigma_\ell(b)]$.

Case (i). Let c be a right inverse for a , i.e., $ac = 1$. Then $x := 1 - ca$ is such that $ax = 0$ and $x \neq 0$. Similarly, $y := 1 - db$ is such that $by = 0$, $y \neq 0$, where $bd = 1$. Since \mathcal{A} is prime, $\|M_{x,y}\| = \|x\| \|y\| > 0$, which implies that

$$xzy \neq 0 \tag{1}$$

for some $z \in \mathcal{A}$. Then $D_{a,b}^1(0 \oplus xz) = axz = 0$. Since $K_{a,b}$ is exact, the kernel of $D_{a,b}^1$ must equal the range of $D_{a,b}^0$, that is, $0 \oplus xz = au \oplus ub$ for some $u \in \mathcal{A}$. Then $au = 0$ and $xz = ub$. It follows that $xzy = uby = 0$, contradicting (1).

Case (ii). Let c', d' be left inverses for a, b , respectively. Then $x' := 1 - ac'$ and $y' := 1 - bd'$ are such that $x'a = 0$, $y'b = 0$, $x' \neq 0$ and $y' \neq 0$. As in the previous case, we can find $z' \in \mathcal{A}$ such that $x'z'y' \neq 0$. Since $D_{a,b}^1(z'y' \oplus 0) = z'y'b = 0$, there exists $u' \in \mathcal{A}$ such that $z'y' \oplus 0 = au' \oplus u'b$. Then $z'y' = au'$, which implies $x'z'y' = x'au' = 0$, a contradiction.

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