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# SUMMARY

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# NONHYPERBOLIC SINGULARITIES IN A CHEMICAL MODEL

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***Abstract:*** *A two-dimensional system with four positive parameters is investigated from dynamical bifurcation point of view. The manifolds where a triple and a double equilibrium exists, and the manifold defined by the presence of an equilibrium with a double-zero eigenvalue are defined by parametric expressions. Then on the basis on these parametric expressions, we prove in the system the existence of a saddle-node and cusp bifurcation.*

***Key words:*** *dynamical system, saddle-node, cusp.*

## 1. Introduction

The Gray-Scott model for a cubic autocatalytic reaction in a continuous flow stirred tank reactor (CSTR) was introduced in 1983 and it describes a group of three chemical reactions for two reactants, $A$ and $B$ [7]. In the last years there was an increasing interest in the Gray-Scott model. Studies concerning it, especially as a pair of coupled reaction-diffusion equations were done [6], [10], [12]. Various models obtained as a generalization of the original Gray-Scott model in CSTR were also presented [1].

Here we analyze the case of homogeneous compositions of reactants in the presence of uncatalyzed conversion. Therefore, we deal with a two-dimensional ODE system depending on four positive parameters, $\dot{u} = a(1-u) - uv^2 - bu$, $\dot{v} = a(c-v) + uv^2 + bu - dv$, or equivalently, by the change of variable $w = 1-u$:

$$\begin{cases} \dot{w} = -(a+b)w + v^2 - wv^2 + b, \\ \dot{v} = -bw - (a+d)v + v^2 - wv^2 + b + ac. \end{cases} \tag{1}$$

The dimensionless parameters $a$, $b$, $c$ and $d$ correspond to the chemical parameters: the residence time ($1/a$), the uncatalyzed rate constant in the conversion of $A$ to $B$ ($b$), the catalyst inflow ($c$) and the decay rate constant in the reaction $B \to C$ ($d$).

By a singular perturbation technique, static bifurcation diagrams (as isolas, mushrooms and simple hystereses) were drawn for (1) [9], [3]. Nevetheless all others previous theoretical studies we are aware of were done in the hypothesis of $b = 0$, that is only with three parameters [8], [11].

The goal of our paper is to analyze the system (1) in the presence of uncatalyzed conversion ($b$ positive), from dynamical bifurcation point of view. Thus in the parameter space we determine some manifolds that play an important role in the dynamical bifurcation diagram and then we prove the existence of the saddle-node and cusp

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bifurcation points. The manifolds found here allow us to investigate the Takens-Bogdanov bifurcation of (1), but this study will be published elsewhere.

The equilibria of the system (1) are the points situated at the intersections of nullclines, *i.e.* the points ($w_e$, $v_e$) satisfying $\dot{w}=0$ and $\dot{v}=0$. Their eigenvalues play a key role in determining the stability properties of the system. They are the roots of the characteristic equation $\lambda^2 - t\lambda + \det = 0$ where $t$ is the trace, and "det" is the determinant of the Jacoby:

$$\det = 3(a+d)v_e^2 - 2a(c+1)v_e + (a+d)(a+b),$$

matrix $J$:

$$t = -\frac{[(3a+2d)v_e^2 - 2a(c+1)v_e + a(2a+b+d)]}{a}.$$

## 2. Preliminary Results

Previous studies on the mathematical model (1) investigated various manifolds in the parameter space, and characterized them by parametric expressions [2], [5]. Those are either associated with the existence of a triple and a double equilibrium, or defined by the presence of an equilibrium point with a double-zero eigenvalue. On the basis of these parametric expressions we are now able to prove further results: we will prove the existence of both saddle-node and cusp bifurcations in the dynamical system (1). In order to do this let us first summarize the main results we need for later proofs.

**Proposition 1.**

*i*) The Gray-Scott system (1) has a triple equilibrium point iff in the parameter space $\mu = (a, b, c, d)$ belongs to the surface:

$$\mathcal{T}:\ a = \frac{8(c+1)^3 p^2}{27},\quad b = \frac{(1-8c)(c+1)^2 p^2}{27},\quad c = c,\quad d = \frac{8(c+1)^3 p(1-p)}{27}, \tag{2}$$

where $c \in \left(0, \frac{1}{8}\right)$ and $p \in (0,1)$. Then the triple equilibrium point has the coordinates $w_{\mathcal{T}} = \frac{1-2c}{3}$, $v_{\mathcal{T}} = \frac{(c+1)p}{3}$.

*ii*) If $\mu \in \mathcal{T}$, then the eigenvalues of the triple point equilibrium ($w_{\mathcal{T}}$, $v_{\mathcal{T}}$) satisfy: $\lambda_1 = 0$, $\lambda_2 > 0$ if $0 < b < b^*$; $\lambda_1 = \lambda_2 = 0$ at $b = b^*$ and $\lambda_1 = 0$, $\lambda_2 < 0$ if $b^* < b < b_f$ where $b^*$ and $b_f$ are defined by:

$$b^* = \frac{(1-2c)^2(1-8c)(c+1)^2}{243},\quad b_f = \frac{(1-8c)(c+1)^2}{27}. \tag{3}$$

**Remark 1.** For reasons to be seen later, let us denote by $\mu_{\mathcal{T}^*}$ the point $\mu_{\mathcal{T}}$ in the parameter space that corresponds to $b = b^*$ and denote its projection on the plane ($a$, $d$) by $\mathcal{T}^*$. Therefore $\mu_{\mathcal{T}^*}$ belongs to the curve:

$$\mathcal{T}^*: a^* = \frac{8(1-2c)^2(c+1)^3}{243},\ b^* = \frac{(1-2c)^2(1-8c)(c+1)^2}{243},\ c^* = c,$$
$$d^* = \frac{16(1-2c)(c+1)^4}{243}, \tag{4}$$

defined for $c \in \left(0, \frac{1}{8}\right)$. The current point $\mu_{\mathcal{T}}$ describes the surface associated with the bands $0 < b < b^*$ and $b^* < b < b_f$ that form $\mathcal{T} \setminus \mathcal{T}^*$:

In view of Proposition 1 (*ii*), $\mu_{\mathcal{T}} \in \mathcal{T} \setminus \mathcal{T}^*$ is presumably a saddle-node or a cusp and $\mu_{\mathcal{T}^*}$ is a candidate for a Takens-Bogdanov bifurcation point. Subsequent proofs show that $\mu_{\mathcal{T}}$ is actually a cusp (Sec. 3) and $\mu_{\mathcal{T}^*}$ is a ***degenerate*** Takens-Bogdanov bifurcation point because of a zero coefficient (for $x^2$) in the normal form.

Let us consider now the following equations:

$$a = 2p^2(w+c)(1-w)^2,\quad b = p^2(w+c)(-2w^2+w-c),\quad c = c,$$
$$d = 2p(1-p)(w+c)(1-w)^2. \tag{5}$$

**Proposition 2.**

*i*) The Gray-Scott system (1) has exactly two equilibrium points iff in the parameter space, the point $\mu = (a, b, c, d)$ belongs to the three dimensional manifold $\mathcal{S} = [\mathcal{S}_1 \cup \mathcal{S}_2] \setminus \mathcal{T}$ where $\mathcal{T}$ is defined by Eq. (2), and:

$$\begin{cases} \mathcal{S}_1: (5) \text{ with } c \in \left(0, \frac{1}{8}\right),\ w \in \left(\frac{1-\sqrt{1-8c}}{4}, \frac{1-2c}{3}\right],\ p \in (0,1), \\ \mathcal{S}_2: (5) \text{ with } c \in \left(0, \frac{1}{8}\right),\ w \in \left[\frac{1-2c}{3}, \frac{1+\sqrt{1-8c}}{4}\right),\ p \in (0,1). \end{cases}$$

The coordinates of the double point equilibrium are $w_{SN} = w$, $v_{SN} = p(w + c)$; the coordinates of the simple point equilibrium are $w_{SS} = 1 - 2c - 2w$, $v_{SS} = p(1 - c - 2w)$.

*ii*) If $\mu \in \mathcal{S}$, then the eigenvalues of the double point equilibrium $(w_{SN}, v_{SN})$ are $\lambda_1 = 0$ and $\lambda_2 = 2p(1 - w)(w + c)(w - p)$. Therefore we have $\lambda_1 = 0$, $\lambda_2 > 0$ for $p \in (0, w)$; $\lambda_1 = \lambda_2 = 0$ at $p = w$ and $\lambda_1 = 0$, $\lambda_2 < 0$ for $p \in (w, 1)$.

**Remark 2.** Obviously, if in Proposition 2 we take $w = \frac{1-2c}{3}$ we get (2).

That is: $\mathcal{S}_1 \cap \mathcal{S}_2 = \mathcal{T}$. Moreover, the system (1) has an equilibrium with a double-zero eigenvalue iff the parameter $\mu$ belongs to some surface, say $\mathcal{DZ}$, on the manifold $\mathcal{S}_1 \cup \mathcal{S}_2$, obtained by choosing $p = w$ as in Proposition 2. Thus, Eqs. (5) become:

$$\mathcal{DZ}: \begin{cases} a = 2w^2(1-w)^2(w+c),\quad b = w^2(w+c)(-2w^2+w-c), \\ c = c,\quad d = 2w(1-w)^3(w+c), \end{cases} \tag{6}$$

with $c \in \left(0, \frac{1}{8}\right)$ and $w \in \left(\frac{1-\sqrt{1-8c}}{4}, \frac{1+\sqrt{1-8c}}{4}\right)$. Then the set $\mathcal{DZ}$ can be decomposed into three disjoint sets such as $\mathcal{DZ} = \mathcal{BT}_1 \cup \mathcal{T}^* \cup \mathcal{BT}_2$, with branches $\mathcal{BT}_1$ and $\mathcal{BT}_2$ laying on $S_1$ and $S_2$ respectively,

$$\mathcal{BT}_1 = \left\{(a,b,c,d) \in \mathbb{R}^4 \mid \text{Eqs. (6) with } c \in \left(0, \frac{1}{8}\right),\ w \in \left(\frac{1-\sqrt{1-8c}}{4}, \frac{1-2c}{3}\right)\right\},$$

$$\mathcal{BT}_2 = \left\{(a,b,c,d) \in \mathbb{R}^4 \mid \text{Eqs. (6) with } c \in \left(0, \frac{1}{8}\right),\ w \in \left(\frac{1-2c}{3}, \frac{1+\sqrt{1-8c}}{4}\right)\right\}.$$

**Proposition 3.** An equilibrium point with a double-zero eigenvalue exists in the Gray-Scott system (1) iff $\mu = (a, b, c, d) \in \mathcal{DZ}$, where $\mathcal{DZ} = \mathcal{BT}_1 \cup \mathcal{T}^* \cup \mathcal{BT}_2$, is defined by Eqs. (6).

*i*) If $\mu \in \mathcal{T}^*$, then the system (1) has a triple equilibrium point $(w_{\mathcal{T}^*}, v_{\mathcal{T}^*})$ with a double-zero eigenvalue. It has the coordinates $w_{\mathcal{T}^*} = \frac{1-2c}{3}$, $v_{\mathcal{T}^*} = \frac{(1-2c)(c+1)}{9}$.

*ii*) If $\mu \in \mathcal{BT}_1 \cup \mathcal{BT}_2$, then the system (1) has two equilibrium points: $(w_{SN}, v_{SN})$ with a double-zero eigenvalue and $(w_{SS}, v_{SS})$ with no zero eigenvalue. They have the coordinates $w_{SN} = w$, $v_{SN} = w(w + c)$ and $w_{SS} = 1 - 2c - 2w$, $v_{SS} = w(1 - c - 2w)$. Moreover $\mathrm{Re}(\lambda_1^{SS}) > 0$, $\mathrm{Re}(\lambda_2^{SS}) > 0$ if $\mu \in \mathcal{BT}_1$, *i.e.* $(w_{SS}, v_{SS})$ is unstable (repulsive), and $\mathrm{Re}(\lambda_1^{SS}) < 0$, $\mathrm{Re}(\lambda_2^{SS}) < 0$ if $\mu \in \mathcal{BT}_2$, *i.e.* $(w_{SS}, v_{SS})$ is asymptotically stable (attractive).

## 3. Cups and Saddle-Node Bifurcations

In this section we investigate (1) from dynamical bifurcation point of view and characterize its saddle-node and cusp bifurcation points. We base our proofs on a more general criterion, obtaining in the end that $\mu \in S \setminus \mathcal{DZ} = [\, S_1 \cup S_2 \,] \setminus \mathcal{DZ}$ corresponds to saddle-node bifurcation points and $\mu \in \mathcal{T} \setminus \mathcal{T}^*$ to cusp bifurcation points for (1). This criterion was part of our previous work [4] on the dynamical system (1), and states as follows. Let:

$$\dot{X} = A_0 X + \alpha_1(A_1 X + V_1) + \alpha_2(A_2 X + V_2) + B(X,X) + C(X,X,X), \tag{7}$$

be a $n$-dimensional ODE system ($n \geq 2$) depending on two parameters $\alpha_1$ and $\alpha_2$. Here $A_0$, $A_1$ and $A_2$ are $n$ by $n$ constant matrices, $V_1$ and $V_2$ are constant column matrices, and $B$ and $C$ are column matrices that contain the second and third order terms with constant coefficients in $X$ respectively. In addition we assume that the matrix $A_0$ has exactly one zero eigenvalue, $\lambda = 0$, all others satisfying the relation $\mathrm{Re}(\lambda_j) \neq 0$.

In these hypotheses the system (7) has at $\alpha_1 = \alpha_2 = 0$ a nonhyperbolic equilibrium $X \equiv 0$ with a one-dimensional central manifold. The dimension of the (generalized) eigenspaces $E^c$ and $E^{su}$ ($\mathbb{R}^n = E^c \oplus E^{su}$) associated with the eigenvalues $\lambda = 0$ and $\lambda_j$ with $\mathrm{Re}(\lambda_j) \neq 0$

are dim $E^c = 1$ and dim $E^{su} = n - 1$. Therefore it makes sense to consider in $\mathbb{R}^n$ two eigenvectors $q$ and $p$ of the matrices $A_0$ and $A_0^T$ such that $A_0q = 0$ and $A_0^T p = 0, p \cdot q = 1$.

**Proposition 4.** In the above hypotheses, consider the $n$-dimensional ODE system (7) depending on two real parameters $\alpha_1$, $\alpha_2$ and choose two vectors $q$ and $p$ in $\mathbb{R}^n$ such that $A_0q = 0$ and $A_0^T p = 0, p \cdot q = 1$.

*i*) (***Saddle-node bifurcation***) If $a_F \neq 0$ and $a_{10}^2 + a_{01}^2 \neq 0$ where $a_F$, $a_{10}, a_{01}$ are defined by:

$$a_F = p \cdot B(q,q), \quad a_{10} = p \cdot V_1, \quad a_{01} = p \cdot V_2, \tag{8}$$

then the dynamical system associated with $\alpha_1 = \alpha_2 = 0$ is a saddle-node bifurcation in the family of dynamical systems generated by (7). The miniversal unfolding of the dynamical system corresponding to $(\alpha_1, \alpha_2) = (0,0)$ is $\dot\eta = \beta + \eta^2$ if $a_F > 0$ and $\dot\eta = \beta - \eta^2$ if $a_F < 0$.

*ii*) (***Cusp bifurcation***) If $a_F = 0$, $b_F \neq 0$ and $a_{10}b_{01} - a_{01}b_{10} \neq 0$ where $a_F$, $a_{10}$ and $a_{01}$ are defined by Eq. (8), and $b_F$, $b_{01}$, $b_{10}$ by:

$$b_F = p \cdot [2B(q,h_2) + C(q,q,q)], \; b_{10} = p \cdot [A_1q + 2B(q,k_{100})], \; b_{01} = p \cdot [A_2q + 2B(q,k_{010})], \tag{9}$$

where $h_2, k_{100}, k_{010} \in \mathbb{R}^n$ are the unique solutions of equations $A_0h_2 = a_Fq - B(q,q)$, $h_2 \cdot p = 0$; $A_0k_{100} = a_{10}q - V_1$, $k_{100} \cdot p = 0$ and $A_0k_{010} = a_{01}q - V_2$, $k_{010} \cdot p = 0$, then the dynamical system corresponding to $(\alpha_1, \alpha_2) = (0,0)$ is a cusp bifurcation in the family of dynamical systems generated by (7). The miniversal unfolding of the dynamical system corresponding to $(\alpha_1, \alpha_2) = (0,0)$ is $\dot\eta = \beta_1 + \beta_2\eta + \eta^3$ if $b_F > 0$ and $\dot\eta = \beta_1 + \beta_2\eta - \eta^3$ if $b_F < 0$.

In the following we prove that the points of the manifold $S \setminus \mathcal{DZ} = [S_1 \cup S_2] \setminus [\mathcal{T} \cup \mathcal{DZ}]$ correspond to a saddle-node bifurcation for the dynamical scheme associated with (1). For some $\mu_S = (a_S, b_S, c_S, d_S) \in [S_1 \cup S_2] \setminus [\mathcal{T} \cup \mathcal{DZ}]$ the system (1) has a nonhyperbolic equilibrium $(w_{SN}, v_{SN})$ with a zero eigenvalue. Let us now consider a system topologically equivalent with (1) and obtained by the translation $w = W + w_{SN}$ and $v = V + v_{SN}$, that is:

$$\begin{aligned}
\dot W &= -(a + b + v_{SN}^2)W + 2v_{SN}(1 - w_{SN})V + (1 - w_{SN})V^2 - 2v_{SN}WV - WV^2 \\
&\quad + b - (a + b)w_{SN} + v_{SN}^2(1 - w_{SN}), \\
\dot V &= -(b + v_{SN}^2)W + [-(a + d) + 2v_{SN}(1 - w_{SN})]V + (1 - w_{SN})V^2 - 2v_{SN}WV \\
&\quad - WV^2 + b + ac + v_{SN}^2(1 - w_{SN}) - bw_{SN} - (a + d)v_{SN}.
\end{aligned} \tag{10}$$

For the bifurcation study we fix $b = b_S$ and $c = c_S$ and take $\alpha_1 = a - a_S$ and $\alpha_2 = d - d_S$ as the bifurcation parameters. Therefore the system (10) takes the matrix form (7) with:

$$X = (W, V)^T, \; V_1 = (-w_{SN}, c - v_{SN})^T, \; V_2 = (0, -v_{SN})^T,$$
$$B(X_1, X_2) = [(1 - w_{SN})V_1V_2 - v_{SN}(W_1V_2 + W_2V_1)]\,(1,1)^T,$$
$$C(X_1, X_2, X_3) = -\frac{1}{3}[W_1V_2V_3 + W_2V_1V_3 + W_3V_1V_2]\,(1,1)^T,$$

and

$$A_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},\ A_0 = \begin{pmatrix} -(a_S + b + v_{SN}^2) & 2v_{SN}(1-w_{SN}) \\ -(b+v_{SN}^2) & -(a_S+d_S)+2v_{SN}(1-w_{SN}) \end{pmatrix}. \quad (11)$$

**Theorem 1.** (***Saddle-node bifurcation points***) Assume that $c \in \left(0, \frac{1}{8}\right)$ and $b \in (0, b_f)$ with $b_f$ defined by (3), and $\mu_S = (a_S, b, c, d_S)$ cu $(a_S, d_S) \in [S_1 \cup S_2] \setminus [\mathcal{T} \cup \mathcal{DZ}]$. At $\mu = \mu_S$ the dynamical system associated with (1) has a nonhyperbolic equilibrium point $(w_{SN}, v_{SN})$ with a zero eigenvalue. The point $(w_{SN}, v_{SN}; \mu_S)$ is a saddle-node bifurcation in the family of Gray-Scott dynamical scheme (1).

***Proof.*** On the basis of Proposition 2, there exist unique parameter values $c \in \left(0, \frac{1}{8}\right)$, $w \in \left(\frac{1-\sqrt{1-8c}}{4}, \frac{1+\sqrt{1-8c}}{4}\right)$, $w \neq \frac{1-2c}{3}$, and $p \in (0,1)$, that characterize $\mu_S \in [S_1 \cup S_2] \setminus \mathcal{T}$. Moreover, since $\mu_S$ does not belong to $\mathcal{DZ}$, we also have $p \neq w$.

The nonhyperbolic equilibrium has the coordinates $w_{SN} = w$ and $v_{SN} = p(w + c)$ and the corresponding eigenvalues $\lambda_1 = 0$, $\lambda_2 = t_{SN} = 2p(1-w)(w+c)(w-p) \neq 0$. Thus, these are also the eigenvalues of the matrix $A_0$ for which we choose $q = (2\, v_{SN}(1-w_{SN}),\ a_S + b + v_{SN}^2)^T$ and $p = (w_{SN} / t_{SN}, -1 / t_{SN})^T$ as eigenvectors that satisfy $A_0\, q = 0$, $A_0^T p = 0$, $p \cdot q = 1$.

Then we apply Proposition 4 ($i$) with $a_{01} = p \cdot V_2 = v_{SN} / t_{SN} = \dfrac{1}{2(1-w)(w-p)}$ and

$$a_F = p \cdot B(q, q) = -\frac{1}{t_{SN}}(1-w_{SN})^2(a_S+b+v_{SN}^2)(a_S+b-3v_{SN}^2)$$ that is by Eqs. (5),

$$a_F = \frac{6p^3(w+c)(1-w)^2}{w-p}\left(\frac{1-2c}{3} - w\right) \neq 0.$$ ■

**Corollary 1.** In the hypotheses of Theorem 1, about the nonhyperbolic equilibrium $(w_{SN}, v_{SN})$ the system (1) is topologically equivalent to $\dot{\eta} = a_F \eta^2$, $\dot{\xi} = t_{SN}\xi$.

***Proof.*** The conclusion follows immediately from the reduction principle with $t_{SN}$ as the nonzero eigenvalue and $\dot{\eta} = a_F\eta^2 + O(\eta^3), \eta \to 0$, as bifurcation equation on the central manifold for $\alpha_1 = \alpha_2 = 0$ that occurs at $\mu_S \in [S_1 \cup S_2] \setminus [\mathcal{T} \cup \mathcal{DZ}]$. ■

The points of the manifold $\mathcal{T} \setminus \mathcal{T}^*$ correspond to a cusp bifurcation of the dynamical scheme (1). For some $\mu_{\mathcal{T}} = (a_{\mathcal{T}}, b_{\mathcal{T}}, c_{\mathcal{T}}, d_{\mathcal{T}}) \in \mathcal{T} \setminus \mathcal{T}^*$ the system (1) has a nonhyperbolic equilibrium $(w_{\mathcal{T}}, v_{\mathcal{T}})$ with a zero eigenvalue. We will consider a system topologically equivalent with (1) and obtained by the translation $w = W + w_{\mathcal{T}}$ and $v = V + v_{\mathcal{T}}$, that is (10) with the index changed from $SN$ to $\mathcal{T}$.

Similar to the above theorem, we fix now $b = b_{\mathcal{T}}$ and $c = c_{\mathcal{T}}$, and take $\alpha_1 = a - a_{\mathcal{T}}$ and $\alpha_2 = d - d_{\mathcal{T}}$ as bifurcation parameters. The matrices $A_0$, $A_1$, $A_2$, $V_1$, $V_2$, $B$ and $C$ in (7) are re-computed accordingly.

**Theorem 2.** (***Cusp bifurcation points***) Assume that $c \in \left(0, \frac{1}{8}\right)$ and $b \in (0, b_f)$, $b \neq b^*$, with $b_f$ and $b^*$ defined by Eq. (3), and consider $\mu_{\mathcal{T}} = (a_{\mathcal{T}}, b, c, d_{\mathcal{T}})$ with $(a_{\mathcal{T}}, d_{\mathcal{T}}) = \mathcal{T}$. At $\mu = \mu_{\mathcal{T}}$ the dynamical system associated with (1) has a nonhyperbolic equilibrium point $(w_{\mathcal{T}}, v_{\mathcal{T}})$ with a zero eigenvalue. The point $(w_{\mathcal{T}}, v_{\mathcal{T}}; \mu_{\mathcal{T}})$ is a cusp bifurcation in the family of Gray-Scott dynamical systems (1).

***Proof.*** On the basis of Proposition 1, there exist unique parameter values $c \in \left(0, \frac{1}{8}\right)$ and $p \in (0, 1)$ that characterize $\mu_{\mathcal{T}} \in \mathcal{T}$. Moreover, since $\mu_{\mathcal{T}}$ does not belong to $\mathcal{T}^*$, we have $p \neq \frac{1-2c}{3}$.

The nonhyperbolic equilibrium has the coordinates $w_{\mathcal{T}} = \frac{1-2c}{3}$ and $v_{\mathcal{T}} = \frac{(c+1)p}{3}$ and the corresponding eingenvalues $\lambda_1 = 0$, $\lambda_2 = t_{\mathcal{T}} = \frac{4p(c+1)^2}{9}\left(\frac{1-2c}{3} - p\right) \neq 0$ as in Proposition 1. Thus, these are also the eingenvalues of the matrix $A_0$, for which we choose $q = (2\, v_{\mathcal{T}}(1 - w_{\mathcal{T}}),\ a_{\mathcal{T}} + b + v_{\mathcal{T}}^2)^T$ and $p = (w_{\mathcal{T}}/t_{\mathcal{T}}, -1/t_{\mathcal{T}})^T$ as eingenvalues that satisfy $A_0\, q = 0$ and $A_0^T p = 0$, $p \cdot q = 1$. By Eq. (2) they are $q = \frac{4p(c+1)^2}{9}(1, p)^T$ and $p = \frac{1}{t_{\mathcal{T}}}\left(\frac{1-2c}{3}, -1\right)^T$.

We apply Proposition 4 (*ii*) with $B(q, q) = 0$ that implies $a_F = p \cdot B(q, q) = 0$. Since in this case the vector $h_2 \in \mathbb{R}^2$ is a solution of $A_0 h_2 = 0$ with $h_2 \cdot p = 0$, we have $h_2 = 0$, and therefore $b_F = p \cdot C(q, q, q)$, *i.e.* $b_F = \frac{32p^4(c+1)^5}{81(1-2c-3p)} \neq 0$.

Similarly, by direct computation we obtain $a_{10} = \frac{(c+1)[3p-(4c+1)]}{9t_{\mathcal{T}}}$ and $a_{01} = \frac{p(c+1)}{3t_{\mathcal{T}}}$.

Then $a_{10}k_{010} - a_{01}k_{100} = -\frac{p(c+1)(1-2c)}{9t_{\mathcal{T}}^2}\left(1, \frac{1-2c}{3}\right)^T$ and $a_{10}b_{01} - a_{01}b_{10} = -\frac{8c(c+1)^3p^2}{27\,t_{\mathcal{T}}^2} + 2p \cdot B(q, a_{10}k_{010} - a_{01}k_{100})$.

Therefore $a_{10}b_{01} - a_{01}b_{10} = \frac{9(1-8c)}{4(c+1)(1-2c-3p)^2} \neq 0$.

**Corollary 2.**

i) If $c \in \left(0, \frac{1}{8}\right)$ and $0 < b < b^*$ then about the cusp bifurcation point $(w_{\mathcal{T}}, v_{\mathcal{T}}; \mu_{\mathcal{T}})$, the Gray-Scott system (1) has the miniversal unfolding $\dot{\eta} = \beta_1 + \beta_2\eta + \eta^3$, $\dot{\xi} = \xi$.

ii) If $c \in \left(0, \frac{1}{8}\right)$ and $b^* < b < b_f$ then about the cusp bifurcation point $(w_{\mathcal{T}}, v_{\mathcal{T}}; \mu_{\mathcal{T}})$, the Gray-Scott system (1) has the miniversal unfolding $\dot{\eta} = \beta_1 + \beta_2\eta - \eta^3$, $\dot{\xi} = -\xi$.

***Proof.*** The result follows from $b < b^*$ iff $p < \frac{1-2c}{3}$, Proposition 1 and Proposition 4 (*ii*). ■

In conclusion, in this paper we analyzed the Gray-Scott model in the particular case of homogenous compositions of reactants and in the presence of uncatalyzed conversion. The resulting system is two-dimensional and possesses four positive parameters. We used the parametric representation of the manifolds $\mathcal{T}$, $S$ and $\mathcal{DZ}$ to study the system from dynamical bifurcation point of view. Those sets correspond respectively to a triple, a double equilibrium, and a double-zero eigenvalue equilibrium. A criterion that connects the coefficients of the system (1) to the coefficients of the bifurcation equation on the associated center manifold proved to be very useful to our investigation. Thus we were able to prove the saddle-node and cusp bifurcations in the system (1), the bifurcation points belonging to $S \setminus \mathcal{DZ}$ (for the saddle-node) and $\mathcal{T} \setminus \mathcal{T}^*$ (for the cusp).

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