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THE STATIC BIFURCATION DIAGRAM FOR THE GRAY–SCOTT MODEL

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The singularity theorem is used to derive the deformations of the static bifurcation diagram of the Gray–Scott model.

1. Introduction

Gray, Scott and their collaborators proposed the model analyzed in our paper in the 80's. The model describes an autocatalytic mechanism in isothermal systems where the reaction rate is increased by strengthening the concentration of one of the products. The chemical reaction (Fig. 1) occurs in a continuously stirred tank reactor (CSTR) and it involves two reactants X and Y with concentrations x and y respectively. Y is produced both autocatalytically and by an uncatalyzed process and it degrades with first order kinetics. The corresponding mathematical model is the system of two first order ODEs

$$\begin{cases} \dfrac{dx}{dt} = k_0(x_0 - x) - k_1xy^2 - k_3x\,, \\ \dfrac{dy}{dt} = k_0(y_0 - y) + k_1xy^2 + k_3x - k_2y\,, \end{cases}$$

where x_0, y_0 are the total concentrations of the reactants X and Y and the k's are the positive rate constants. Using the nondimensionalization

$$t \mapsto tk_1x_0^2\,, \quad u = \frac{x}{x_0}\,, \quad w = 1 - u\,, \quad v = \frac{y}{x_0}\,,$$

$$a = \frac{k_0}{k_1x_0^2}\,, \quad b = \frac{k_3}{k_1x_0^2}\,, \quad c = \frac{y_0}{x_0}\,, \quad d = \frac{k_2}{k_1x_0^2}$$

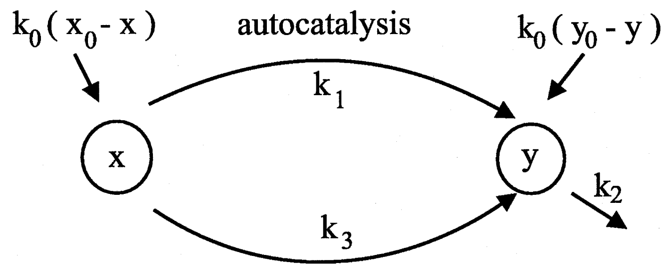

Fig. 1. The enzymatic reaction in the Gray–Scott model.

we obtain the system

$$\begin{cases} w' = -(a+b)w + v^2 - wv^2 + b\,, \\ v' = -bw - (a+d)v + v^2 - wv^2 + b + ac\,. \end{cases} \tag{1}$$

Here $'$ means the derivative with respect to the new t and all the parameters a, b, c and d are strictly positive.

Such a model has been proved to have applications in chemistry, biochemistry for enzyme reactions and also in mathematical biology for the spread of infections diseases or population growth [D'Anna *et al.*, 1986].

Gray, Scott and their collaborators [Gray, 1988; Gray & Scott, 1983, 1986] studied the system (1) numerically and proved that it exhibits multi-steady states with mushrooms and isolas [Murray, 1993]. They have also applied some singularity theory methods to prove analytically the existence of

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those mushrooms and isolas. Anyway, they have done the analysis only in the limit case $b = 0$.

In our paper we have constructed the parametric portrait of the Gray–Scott model for arbitrary values of the parameters. Namely, the transient set for the universal unfolding divides the parameter space into four disjoint regions in each of them the perturbations being persistent. Then the parametric portrait was supplemented by the corresponding static bifurcation diagrams.

The equilibrium points (w_e, v_e) of the system (1) are situated at the nullclines' intersection, therefore they satisfy the conditions $w' = v' = 0$, i.e.

$$\begin{cases} -(a+b)w + v^2 - wv^2 + b = 0\,, \\ -bw - (a+d)v + v^2 - wv^2 + b + ac = 0\,. \end{cases} \tag{2}$$

By subtracting $(2)_1$ from $(2)_2$ we get the relationship between w_e and v_e, namely $v_e = a(w_e + c)/(a + d)$. Then by using this result in $(2)_1$ we obtain a third order equation in the stationary solution w_e

$$w_e^3 + (2c-1)w_e^2 + \left(c^2 - 2c + \frac{(a+d)^2(a+b)}{a^2}\right) w_e - \left(c^2 + \frac{(a+d)^2 b}{a^2}\right) = 0\,. \tag{3}$$

In order to draw the static bifurcation diagram we must find the number of the real roots of (3) and the corresponding regions in the system parameter space.

2. Perturbed Static Bifurcation

Among several definitions we use the following

Definition 1. [Georgescu *et al.*, 1999]

(i) Consider the equation

$$F(\lambda, u) = 0\,, \quad F : \mathbf{B}_1 \times \mathbf{B}_2 \to \mathbf{B}_3\,, \tag{4}$$

where $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$ are Banach spaces, F is a nonlinear function, λ is a parameter and u is the unknown function.

We call the pair (λ, u) as a solution for Eq. (4) if it satisfies (4).

Let $S = \{(\lambda, u) \in \mathbf{B}_1 \times \mathbf{B}_2 : F(\lambda, u) = 0\}$ be the solution set.

(ii) The bifurcation point of (4) is a point $(\lambda_0, u_0) \in \overline{S}$ such that the topological or/and the smoothness properties of S are changed as (λ_0, u_0) is crossed. We say that λ is the *bifurcation parameter* and λ_0 is the *bifurcation value.*

We also call (λ_0, u_0) as a static bifurcation point since the solution (λ_0, u_0) is stationary (here (4) is not a time dependent equation; it is a "stationary" equation).

In the sequel we introduce the concept of a static bifurcation point for a dynamical system associated with the system of ODEs

$$x_i' = F_i(\lambda, x_1, x_2, \ldots, x_n, p)\,, \quad i = 1, \ldots, n\,, \tag{5}$$

where $\lambda \in \mathbf{R}$ is a scalar parameter and $p \in \mathbf{R}^s$ is the vector of all the other parameters in the system. The distinctly chosen parameter λ plays the role of the static bifurcation parameter. The other s parameters $p_1, \ldots, p_s$ are small and they produce perturbation.

The equilibrium points are the roots of the system of n algebraic equations, $x_i' = 0$, i.e. $F_i(\lambda, x_1, x_2, \ldots, x_n, p) = 0$, $i = 1, \ldots, n$. Usually this system can be reduced, by variables elimination (e.g. $x_2, \ldots, x_n$), to a single nonlinear equation $F(\lambda, x, p) = 0$ in the variable x_1, denoted here by x. For a constant p this equation is of exactly the type as Eq. (4). This means that the study of static bifurcation for the system (5) deals with certain topological properties of the solution $x = x(\lambda, p)$ of the equation $F(\lambda, x, p) = 0$. The graph of x as a function of λ for fixed p is called *the static bifurcation diagram.* In addition we are interested in the changes of the static bifurcation diagram as p is varied. This perturbed static bifurcation will be analyzed by using singularity theory.

The static bifurcation diagram changes its topological type in the neighborhood of the singular points, or shortly, singularities. A singularity of the function F is a point at which both F and one or more of its partial derivatives $\partial^k F/\partial x^k$ are equal to zero. The more such partial derivatives are zero, the higher is the singularity order.

Here we use the method of Golubitsky and Schaeffer [1985]. Its main idea is to determine the possible singularities which have the highest order. Thus in the parameter space we find certain points p_0 in the neighborhood of which the function F is topologically equivalent to a polynomial $G(u)$, called *the normal form of* F. A certain local behavior is specific to each possible normal form. Moreover, using the normal form we can construct the "universal unfolding" of F which characterizes the qualitative (topologically nonequivalent) changes in the static bifurcation diagram.

Definition 2. [Golubitsky & Schaeffer, 1985]

(i) Let us consider two static bifurcation problems

$$h(\lambda,\, x) = 0 \quad \text{and} \quad g(\lambda,\, x) = 0$$

about the point $(\lambda_0,\, x_0)$ and $(\Lambda_0,\, X_0)$ respectively. The two problems are called *equivalent* and we write $g \sim h$ if there exists a local diffeomorphism $(\lambda,\, x) \to (\Lambda(\lambda),\, X(\lambda,\, x))$ which takes $(\lambda_0,\, x_0)$ into $(\Lambda_0,\, X_0)$ and preserves the orientation on x and λ, and if there exists a strictly positive function $T(\lambda,\, x)$ such that $T(\lambda,\, x) \cdot g(\Lambda(\lambda),\, X(\lambda,\, x)) = h(\lambda,\, x)$.

(ii) Let $g(\lambda,\, x) = 0$ be a static bifurcation problem and let $G(\lambda,\, x,\, p) = G(\lambda,\, x,\, p_1, \ldots,\, p_s)$ be an s-parameter family of bifurcation problems. We say that G is a *perturbation* of g if $G(\lambda,\, x,\, 0, \ldots,\, 0) = g(\lambda,\, x)$ and, in the s-parameter function space, G belongs to a neighborhood of g corresponding to the origin of $\mathbf{R}^s$.

(iii) Such a function G is called the *universal unfolding* of g if for any perturbation $\varepsilon q(\lambda,\, x,\, \varepsilon)$ of g there exist some values for the parameters, $p_1(\varepsilon), \ldots,\, p_s(\varepsilon)$, such that $g + \varepsilon q \sim G(\cdot,\, p(\varepsilon))$ for sufficiently small ε.

(iv) The number of the parameters in the universal unfolding of g gives us the g-static bifurcation problem *codimension.*

In the Gray–Scott model the bifurcation study is made on Eq. (3) of the equilibrium point w_e. We choose a as the static bifurcation parameter since this is the control parameter of the original chemical system. The other (perturbation) parameters are b, c, d, hence p is three-dimensional. That is why we will be interested only in the normal forms corresponding to the codimension 3 or less singularities.

In their classification theorem, Golubitsky and Schaeffer [1985] have found what all the possible normal forms for codimension 3 or less singularities are and they have also described the conditions which characterize these normal forms. Thus the normal forms for the codimension 3 or less singularities are: the limit point (codim 0); the simple bifurcation, the isola center and the hysteresis (codim 1); the asymmetric cusp, the pitchfork, the quartic fold (codim 2); the winged cusp and other three types with no specific name (codim 3).

In the sequel we prove that the corresponding normal form for the Gray–Scott model is the winged cusp.

Let Σ be the *transient set* of the universal unfolding G. It consists of points of the $(p_1, \ldots,\, p_s)$ parameter space such that as they are crossed the topological type of the static bifurcation diagram is changed. In this space the set Σ delimits regions of the same topological type for these diagrams. By analogy with the dynamical bifurcation, the $(p_1, \ldots,\, p_s)$ space on which Σ is drawn is called the *parametric portrait.* The regions in it corresponding to topologically equivalent static bifurcation diagrams are called *strata.* The object of perturbed static bifurcation theory is to provide these strata and the corresponding static bifurcation diagrams.

Let $g(\lambda,\, x) = 0$, $g : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be a static bifurcation problem and let $G(\lambda,\, x,\, p)$, $p \in \mathbf{R}^s$ be the universal unfolding of g. In singularity theory the (static) bifurcation diagram of g is defined as the set

$$\{(\lambda,\, x) | G(\lambda,\, x,\, p) = 0\}\,, \quad \forall p \in \mathbf{R}^s \text{ fixed}\,.$$

Since even for smooth G the dependence on λ of the solution $x = x(\lambda)$ of the equation $g(\lambda,\, x) = 0$ can be nonsmooth, we are interested in the topologically nonequivalent changes of the static bifurcation diagram for g. In order to find them we first define in the p-parameter space four different sets: $\mathcal{D}$, *the double limit point set*; $\mathcal{H}$, *the hysteresis set*; $\mathcal{B}$, *the bifurcation set*; Σ, *the transient set*, as follows

$$\begin{aligned}
\mathcal{D} &= \{p \in \mathbf{R}^s | (\exists)(\lambda,\, x_1,\, x_2) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}, \\
&\qquad x_1 \neq x_2 : G = G_x = 0 \text{ at } (\lambda,\, x_i,\, p),\, i = 1,\, 2\}\,, \\
\mathcal{H} &= \{p \in \mathbf{R}^s | (\exists)(\lambda,\, x) \in \mathbf{R} \times \mathbf{R} : \\
&\qquad G = G_x = G_{xx} = 0 \text{ at } (\lambda,\, x,\, p)\}\,, \\
\mathcal{B} &= \{p \in \mathbf{R}^s | (\exists)(\lambda,\, x) \in \mathbf{R} \times \mathbf{R} : \\
&\qquad G = G_x = G_\lambda = 0 \text{ at } (\lambda,\, x,\, p)\}\,, \\
\Sigma &= \mathcal{D} \cup \mathcal{H} \cup \mathcal{B}\,.
\end{aligned}$$

In singularity theory it is proved that in each region defined by the transient set Σ, the perturbations are persistent. All the topologically nonequivalent changes take place only when crossing Σ from one region to another.

3. The Universal Unfolding in the Gray–Scott Model

We can rewrite Eq. (3) in the variable w_e as

$$(w_e + c)^2(w_e - 1) + \frac{(a + d)^2}{a^2}((a + b)w_e - b) = 0\,,$$

or, equivalently,

$$F \equiv (w + c)^2(w - 1) + \left(1 + \frac{d}{a}\right)^2 (a + b)w - \left(1 + \frac{d}{a}\right)^2 b = 0\,, \tag{6}$$

where, in order to simplify the notation, we dropped the index e. This means that Eq. (6) has the form $F(a;\, w;\, b,\, c,\, d) = 0$ where a plays the role of the bifurcation parameter λ and $(b,\, c,\, d)$ corresponds to the other parameters of vector p. Hence the static bifurcation codimension must be less or equal to 3. According to singularity theory, the highest possible order for the singularity should be five and so we could have maximum four zero partial derivatives, i.e. $F = F_w = F_{ww} = F_{www} = F_{wwww} = 0$. A short computation gives us

$$F_w = \left(1 + \frac{d}{a}\right)^2 (a + b) + (w + c)(3w + c - 2) = 0\,, \tag{7}$$

$$F_{ww} = 2(3w + 2c - 1) = 0\,, \tag{8}$$

$$F_{www} = 6 \neq 0\,, \quad \operatorname{sgn}(F_{www}) = 1\,.$$

In order to evaluate the singular point apart from the conditions (6)–(8) we need two other equations. Thus we consider in addition that $F_a = F_{wa} = 0$.

Let us see what we can get from Eqs. (6)–(8). First we find the expression of the singular point $w^\circ = (1 - 2c^\circ)/3$ and the relationship among all parameters b, c, d and a, i.e. $(c+1)^2 = 3(a+b)(1+d/a)^2$. Then (6) implies that $(1 - 8c)(c + 1)^2/27 = (1 + d/a)^2 b$, and combining the last two equations we find (as expected) the conditions characterizing the triple equilibrium point,

$$b^\circ = \frac{(1 - 8c^\circ)a^\circ}{8(c^\circ + 1)}\,, \qquad d^\circ = a^\circ \left(\sqrt{\frac{8(c^\circ + 1)^3}{27a^\circ}} - 1\right). \tag{9}$$

The equation $F_a = 0$ defined by the partial derivative F_a,

$$F_a = \left(1 + \frac{d}{a}\right)^2 w - \frac{2d}{a^2}\left(1 + \frac{d}{a}\right)(a + b)w + \frac{2d}{a^2}\left(1 + \frac{d}{a}\right) b = 0\,, \tag{10}$$

together with Eq. (9) leads to

$$a^\circ = \frac{8(c^\circ + 1)^3(4c^\circ + 1)^2}{3^5}\,, \quad b^\circ = \frac{(c^\circ + 1)^2(4c^\circ + 1)^2(1 - 8c^\circ)}{3^5}\,, \quad d^\circ = \frac{16(c^\circ + 1)^3(4c^\circ + 1)(1 - 2c^\circ)}{3^5}\,. \tag{11}$$

The last condition is $F_{wa} = 0$, i.e. $(1 + d/a)^2 = 2d(a+b)(1+d/a)/a^2$ and when we add it to those of (11) it follows that $c^\circ = 1/8$, so $w^\circ = 1/4$ and $a^\circ = 27/256$, $b^\circ = 0$, $d^\circ = 27/256$. Moreover, the partial derivative F_{aa} evaluated at $(a^\circ;\, w^\circ;\, b^\circ,\, c^\circ,\, d^\circ)$ is $64/27$, therefore $\operatorname{sgn}(F_{aa}) = 1$.

Thus, the corresponding normal form for the Gray–Scott model is $x^3 + \lambda^2$, i.e. the winged cusp. According to singularity theory [Golubitsky & Schaeffer, 1985], its associated universal unfolding reads

$$G(\lambda,\, x,\, p_1,\, p_2,\, p_3) = x^3 + \lambda^2 + p_1 + p_2 x + p_3 \lambda x\,,$$

and we know what are the possible "pictures" associated with this universal unfolding. Here by "pictures" we mean the graphs of the transient set and the graphs of the bifurcation diagrams persistent or not to the perturbations. All these are important when making numerical computations in order to construct the bifurcation diagrams.

3.1. *The transient set of the universal unfolding for the Gray–Scott model*

Below we construct the set Σ for $F(a;\, w;\, b,\, c,\, d) = 0$ defined by Eq. (6).

3.1.1. *The double limit point set $\mathcal{D}$*

Since F is a third order polynomial in w, the set

$$\mathcal{D} = \{(b,\, c,\, d) \in \mathbf{R}_+^{*3} | (\exists)(a,\, w_1,\, w_2) \in (0,\, \infty) \times (0,\, 1) \times (0,\, 1),\, w_1 \neq w_2 : F = F_w = 0 \text{ at } (a,\, w_i,\, b,\, c,\, d),\ i = 1,\, 2\}\,,$$

is empty, i.e. $\mathcal{D} = \emptyset$.

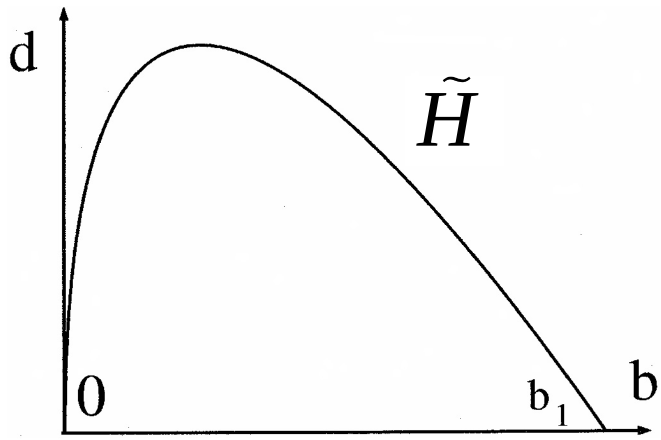

Fig. 2. The projection of the set $\mathcal{H}$ on the (b, d) plane for $c = 1/32 < 1/8$.

3.1.2. *The hysteresis set* $\mathcal{H}$

The set $\mathcal{H}$ is defined by

$$\mathcal{H} = \{(b, c, d) \in \mathbf{R}_+^{*3} | (\exists)(a, w) \in (0, \infty) \times (0, 1) : F = F_w = F_{ww} = 0\} .$$

Consequently, we have to solve the system with the three equations (6)–(8) which leads as we have already seen to the conditions (9).

Obviously $\mathcal{H} = \emptyset$ if $c \geq 1/8$. If $0 < c < 1/8$ by taking off a from (9) we find

$$\mathcal{H} = \left\{ (b, c, d) | d = \frac{8(c+1)}{1-8c} \left[\sqrt{\frac{b(1-8c)(c+1)^2}{27}} - b \right], \quad b, d > 0, c \in (0, 1/8) \right\} .$$

In Fig. 2 we have drawn the projection $\tilde{\mathcal{H}}$ of the set $\mathcal{H}$ on the (b, d) plane, for a fixed c between 0 and 1/8.

In the $c < 1/8$ case, at $b_1 = (1 - 8c)(c + 1)^2/27$ we get the intersection of $\tilde{\mathcal{H}}$ with the b-axis and the maximum is obtained at the point $((1 - 8c)(c + 1)^2/108, 2(c + 1)^3/27)$.

3.1.3. *The bifurcation set* $\mathcal{B}$

The set $\mathcal{B}$ is defined by

$$\mathcal{B} = \{(b, c, d) \in \mathbf{R}_+^{*3} | (\exists)(a, w) \in (0, \infty) \times (0, 1) : F = F_w = F_a = 0\} ,$$

i.e. we have to solve the system (6), (7) and (10). In order to simplify the computation we denote by D the ratio $(a + d)^2/a^2$. Thus Eq. (7) implies that

$$D(a + b) = (w + c)(2 - c - 3w) , \tag{12}$$

and this together with Eq. (6) implies that

$$Db = (w + c)(-2w^2 + w - c) . \tag{13}$$

Subtract (13) from (12) to get

$$Da = 2(w + c)(1 - w)^2 . \tag{14}$$

On the other hand Eq. (10) provides

$$d = \frac{a^2 w}{aw - 2b(1 - w)} \Leftrightarrow \frac{d}{a} = \frac{aw}{aw - 2b(1 - w)} . \tag{15}$$

From $Db/Da = b/a$ and Eqs. (13) and (14) it follows

$$b = \frac{(-2w^2 + w - c)a}{2(1 - w)^2} . \tag{16}$$

Then we introduce (16) in (15) and obtain the result

$$\begin{aligned} \frac{a + d}{a} &= 1 + \frac{d}{a} = 1 + \frac{aw}{aw - 2b(1 - w)} \\ &= 1 + \frac{1}{1 - \frac{2b(1 - w)}{aw}} \\ &= 1 + \frac{1}{1 - \frac{2(1 - w)}{w} \cdot \frac{(-2w^2 + w - c)}{2(1 - w)^2}} \\ &= \frac{w + c}{w^2 + c} , \end{aligned}$$

which, when introduced in (14), follows

$$a = \frac{2(w+c)(1-w)^2}{D} = \frac{2(w+c)(1-w)^2}{(w+c)^2}(w^2+c)^2 ,$$

i.e. $a = 2(1 - w)^2(w^2 + c)^2/(w + c)$.

Thus using (16) and (15) and the above expression for a, as a function only of w and c, we get

$$\begin{aligned} b &= \frac{(-2w^2 + w - c)(w^2 + c)^2}{w + c} , \\ d &= \frac{2w(1 - w)^3(w^2 + c)}{w + c} . \end{aligned} \tag{17}$$

Now for a fixed c, we can represent parametrically the graph of $\tilde{\mathcal{B}}$, projection of the set $\mathcal{B}$ on the

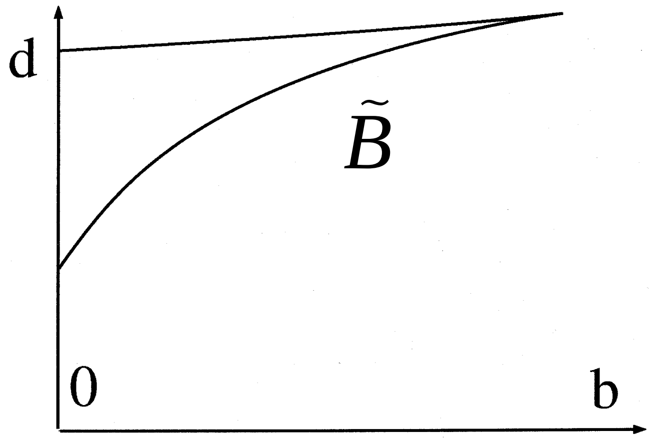

Fig. 3. The projection of the set $\mathcal{B}$ on the $(b,\ d)$ plane for $c = 1/32 < 1/8$.

$(b,\ d)$ plane. The parameter along the curve will be $w \in (0,\ 1)$.

The set $\mathcal{B}$ is empty if $c \geq 1/8$, while if $w \in (0,\ 1)$ we have $b < (c+1)^2(-2w^2+w-c)/c$ and the equation $-2w^2+w-c=0$ has the discriminant $\Delta_w = 1-8c$. This means that for $\Delta_w < 0$, i.e. $c > 1/8$, $(-2w^2+w-c)$ has a negative constant sign, therefore b should be negative which is false. If $c = 1/8$ we also have a contradiction, $b = -2(w-1/4)^2(w^2+1/8)/(w+1/8) \leq 0$. The only possible case is $0 < c < 1/8$, when $\Delta_w > 0$, therefore the sign of $(-2w^2+w-c)$ is positive for $w \in ((1-\sqrt{1-8c})/4,\ (1+\sqrt{1-8c})/4) \subset (0,\ 1)$. We can easily see that for $w \in (0,\ 1)$, d is always positive.

In other words, if $c \geq 1/8$, $\mathcal{B} = \emptyset$ and if $0 < c < 1/8$ we have

$$\mathcal{B} = \left\{ (b,\ c,\ d) | c \in (0,\ 1/8) \right.$$
$$\text{and } b,\ d \text{ given by Eqs. (17), with}$$
$$\left. w \in \left(\frac{1-\sqrt{1-8c}}{4},\ \frac{1+\sqrt{1-8c}}{4} \right) \right\}.$$

In Fig. 3 we have drawn the curve $b = b(w)$, $d = d(w)$, which describes the projection of the $\mathcal{B}$ set on the $(b,\ d)$ plane for $c = 1/32$.

3.1.4. *The transient set* Σ

Since the transient set Σ is the union $\Sigma = \mathcal{D} \cup \mathcal{H} \cup \mathcal{B} = \mathcal{H} \cup \mathcal{B}$, it will be the empty set for $c \geq 1/8$, $\Sigma = \emptyset$.

The projection Σ on the $(b,\ d)$ plane in the $c = 1/32 < 1/8$ case is shown in Fig. 4.

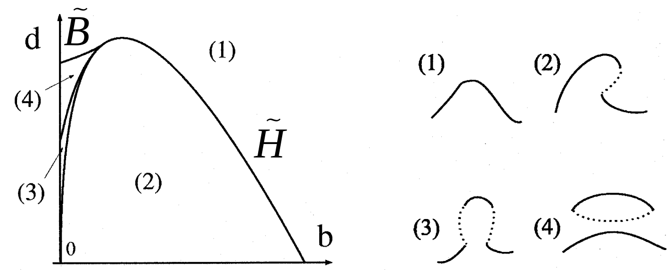

Fig. 4. The projection of the set Σ on the $(b,\ d)$ plane for $c = 1/32 < 1/8$.

The projection on the $(b,\ d)$ plane of the intersection $\mathcal{H} \cap \mathcal{B}$, for a fixed c, has only one element. It is given by the equations $F = F_w = F_{ww} = F_a = 0$, i.e. Eqs. (6)–(8) and (10) which, as we have already seen, implied (11). Consequently, for a fixed $c < 1/8$ the projection of the set $\mathcal{H} \cap \mathcal{B}$ on the $(b,\ d)$ plane is the set $\{(b^\circ,\ d^\circ)\}$ with b° and d° given by Eq. (11).

If $c < 1/8$, the graphs of $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{B}}$ divide the positive quadrant of the $(b,\ d)$ plane into four distinct regions (Fig. 4). Each region in the parameter plane corresponds to a different type of static bifurcation diagram persistent to perturbations. The region (4) is shrinking as c approaches $1/8$ because the lower branch of $\tilde{\mathcal{B}}$ is raised up. On the other hand, the region (3) is shrinking as c approaches 0 because the lower branch of $\tilde{\mathcal{B}}$ is folded down (Fig. 4).

4. The Static Bifurcation Diagrams in the Gray–Scott Model

In the sequel we have drawn a few static bifurcation diagrams. They were obtained numerically with Matlab by choosing some fixed values for the parameters c (we have taken $c = 1/32$) and b and taking different values for d. The static bifurcation parameter a has been chosen between 0.0001 and 0.33.

By singularity theory, in the case of the winged cusp singularity there exist seven topologically nonequivalent bifurcation diagrams corresponding to the seven possible regions in the parameter space. Nevertheless in the Gray–Scott model case, as we have already noticed (Fig. 4), in the parameter space there exist only four distinct regions therefore we can find only four distinct types of bifurcation diagrams. In the following we have considered certain points from each region and made the numerical computation.

The topologically nonequivalent static bifurcation diagrams $(a,\, w_e)$ for $c = 1/32$, $b = 0.001$.

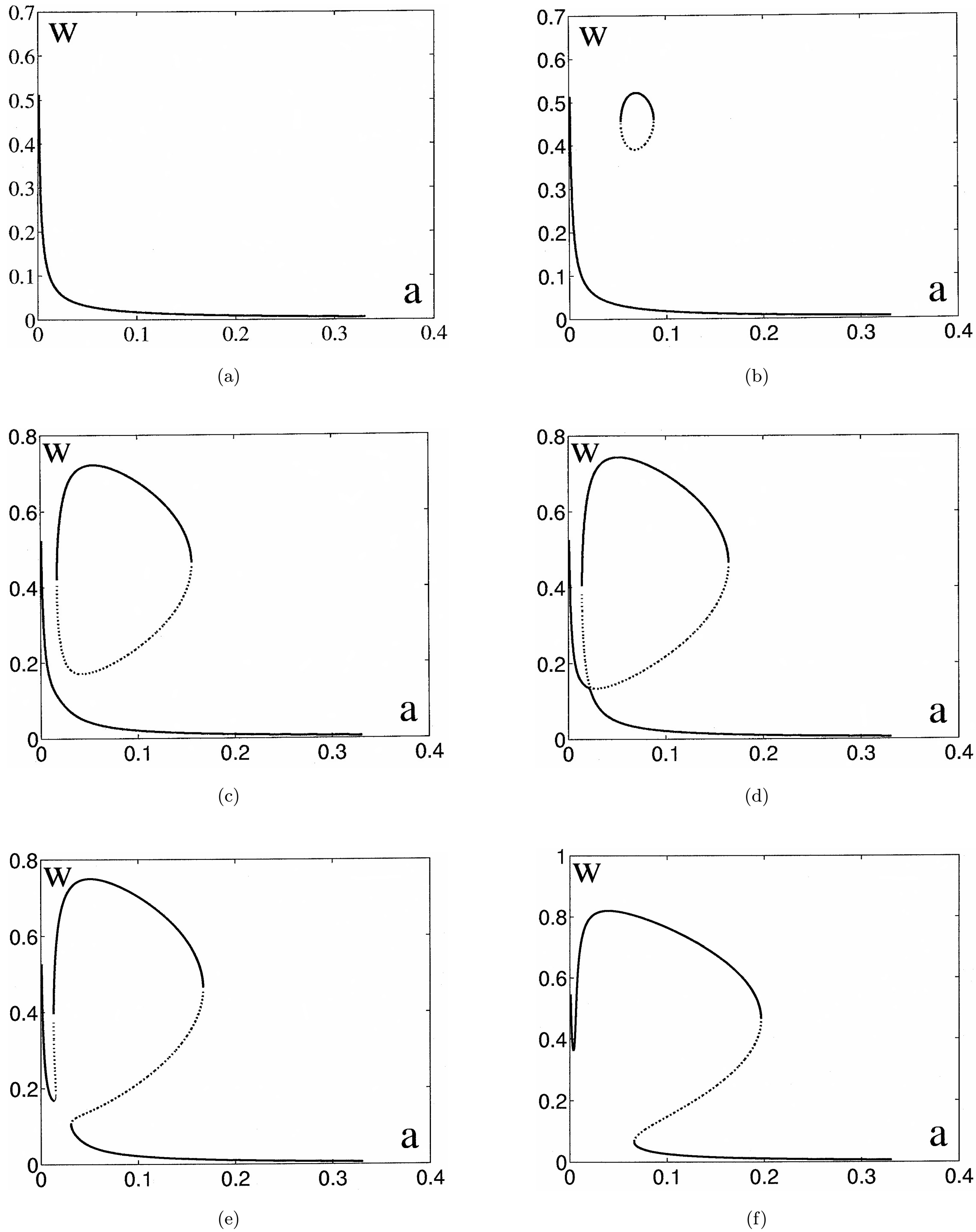

Fig. 5. The diagrams correspond to different regions as d varies at (a) $d = 0.0763$ — region (1); (b) $d = 0.071$ — region (4); (c) $d = 0.055$ — still region (4); (d) $d = 0.0521525$ — it corresponds to the lower branch of the $\mathcal{B}$ curve; (e) $d = 0.05125$ — region (3); (f) $d = 0.04$ — region (2).

The topologically nonequivalent static bifurcation diagrams $(a,\, w_e)$ for $c = 1/32$, $b = 0.01$.

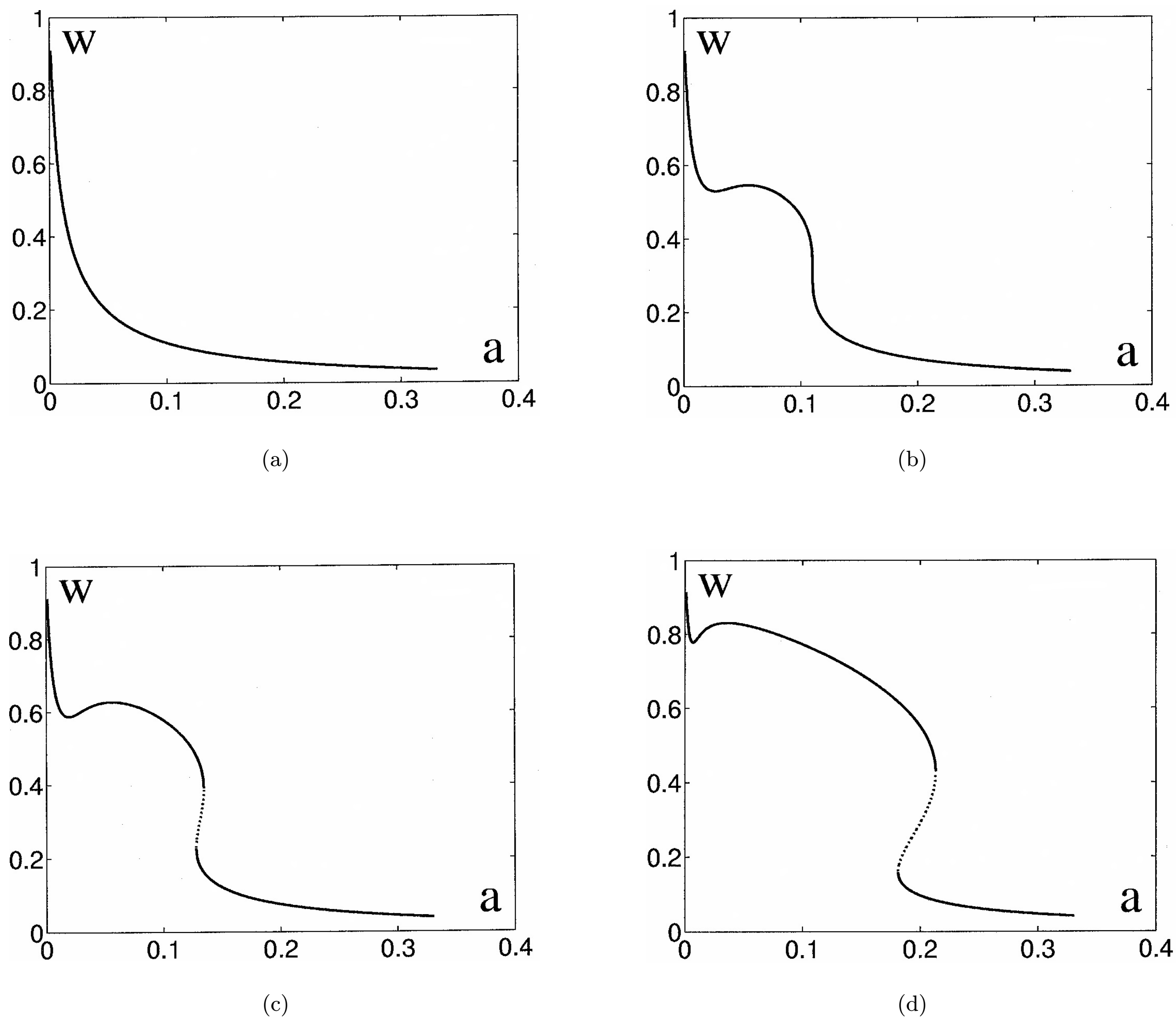

Fig. 6. The diagrams correspond to different regions as d varies at (a) $d = 0.2$ — region (1); (b) $d = 0.079$ — it corresponds to the $\mathcal{H}$ curve; (c) $d = 0.071$ — region (2); (d) $d = 0.04$ — still region (2).

In the limit case $b = 0$ we have found the same diagrams drawn numerically by Gray, Scott and their collaborators [D'Anna *et al.*, 1986; Gray, 1988; Gray & Scott, 1983, 1986]. In order to illustrate the above obtained results we have chosen two different positive values for b.

The vertical line of equation $b = 0.001$ intersects all four regions of the $(b,\, d)$ plane and the curves $\tilde{\mathcal{B}}$, $\tilde{\mathcal{H}}$ too. Thus, for $d = 0.0763$ we are in the region (1); $d = 0.072$ corresponds to the upper branch of $\tilde{\mathcal{B}}$; $d = 0.071$ and $d = 0.055$ to the region (4) where there is an isola. At $d = 0.0521525$ the lower branch of the curve $\tilde{\mathcal{B}}$ is reached and we pass into the region (3) (with a mushroom) for $d = 0.05125$. Around $d = 0.05$ the curve $\tilde{\mathcal{H}}$ is crossed up and we enter the region (2) where only one hysteresis (the right one) is preserved, e.g. $d = 0.04$ (Fig. 5).

The vertical line $b = 0.01$ intersects only two regions, the regions (1) and (2). Consequently, we can have only one equilibrium point in the region (1), e.g. for $d = 0.2$. Around $d = 0.079$ the curve $\tilde{\mathcal{H}}$ is crossed up and then for a smaller d, e.g. $d = 0.071$, $d = 0.04$, we will find in the region (2) one hysteresis, which means that here three different equilibrium points (Fig. 6) exist.

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