22m:033 Notes: 1.4 The Matrix Equation $A\overrightarrow{x} = \overrightarrow{b}$

Dennis Roseman
University of Iowa
Iowa City, IA

http://www.math.uiowa.edu/∼roseman

February 2, 2010

1 How to multiply a matrix and a vector

Definition 1.1 *Let* A *be an* $m \times n$ *matrix written* (a_{ij}) *where* $1 \leq i \leq m$ *and* $1 \leq j \leq n$ *and* $\overrightarrow{x}$ *an* n*-dimensional vector written* $\overrightarrow{x} = (x_i)$ *where* $1 \leq i \leq m$*. Then we define*

$$A\overrightarrow{x} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \ldots + a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 \ldots + a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 \ldots + a_{nm}x_m \end{pmatrix}$$

Example 1.2

Remark 1.3 If we view the i-th column of the matrix A as a vector $\overrightarrow{a_i}$ we can express the above multiplication as a linear combination of these vectors with weights (x_i) as follows:

$$A\overrightarrow{x} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} + \cdots + x_m \begin{pmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{pmatrix}$$

Definition 1.4 *An equation that involves vectors is called a* ***vector equation****. An equation that involves matrices is called a* ***matrix equation****.*

Example 1.5 So if A is a matrix, $A\overrightarrow{x} = \overrightarrow{b}$ is a matrix equation and $\overrightarrow{a} + \overrightarrow{x} = \overrightarrow{b}$ is a vector equation.

Proposition 1.6 *If A is an $m \times n$ matrix with columns $\overrightarrow{a_1}, \dots \overrightarrow{a_n}$ and if $\overrightarrow{b} \in R^n$ then the matrix equation $A\overrightarrow{x} = \overrightarrow{b}$ has the same solution set as the vector equation*

$$x_1\overrightarrow{a_1} + \cdots + x_n\overrightarrow{a_n} = \overrightarrow{b}$$

which is also the same as the solution of the linear system whose augmented matrix is

$$\left(\overrightarrow{a_1} \cdots \overrightarrow{a_n}\,\overrightarrow{b}\right)$$

Remark 1.7 $A\overrightarrow{x} = \overrightarrow{b}$ has a solution if and only if $\overrightarrow{b}$ is a linear combination of the vectors $\overrightarrow{a_1}, \dots \overrightarrow{a_n}$.

Proposition 1.8 *If A is an $m \times n$ matrix with columns $\overrightarrow{a_1}, \dots, \overrightarrow{a_n}$ then the following are equivalent:*

1. *For each $\overrightarrow{b} \in R^n$, the equation $A\overrightarrow{x} = \overrightarrow{b}$ has a solution*

2. *Each* $\overrightarrow{b} \in R^n$ *is a linear combination of* $\overrightarrow{a_1}, \ldots \overrightarrow{a_n}$

3. *The span of* $\overrightarrow{a_1}, \ldots, \overrightarrow{a_n}$ *is all of* R^n

4. *A has a pivot position in every row*

Remark 1.9 The last item on the above list might be better put—"When we put A in row reduced echelon form, each row has a leading 1".

Remark 1.10 IMPORTANT. Proposition 1.8 refers to the matrix A and *not* an augmentation of A.

Example 1.11 If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. Does $A\overrightarrow{x} = \overrightarrow{b}$ have a solution for any $\overrightarrow{b}$?

Let $\overrightarrow{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. We will have a solution if the corresponding system of linear equations is not inconsistent. So the question is—if we take the augmented matrix when (if ever) will we get a row of all zeros followed by a non-zero?

So we write the augmented matrix:

$$\begin{pmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{pmatrix}$$

and row reduce:

$$\begin{pmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 7 & 8 & 9 & b_3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & -6 & -12 & b_3 - 7b_1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & b_1 \\ 0 & 1 & 2 & -\frac{1}{3}(b_2 - 4b_1) \\ 0 & 1 & 2 & -\frac{1}{6}(b_3 - 7b_1) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & b_1 \\ 0 & 1 & 2 & -\frac{1}{3}(b_2 - 4b_1) \\ 0 & 0 & 0 & \frac{1}{3}(b_2 - 4b_1) - \frac{1}{6}(b_3 - 7b_1) \end{pmatrix}$$

So we will get consistent equations if $\frac{1}{3}(b_2 - 4b_1) - \frac{1}{6}(b_3 - 7b_1) = 0$ and cleaning this up we see that this is $b_1 + b_3 = 2b_2$. So there are infinitely many values of $\overrightarrow{b}$ that will work such as $b_1 = 1, b_2 = 1, b_3 = 1$ or $b_1 = 2, b_2 = 4, b_3 = 3$ and also infinitely many that will not such as $b_1 = 1, b_2 = 1, b_3 = 0$.

2 Properties of the Matrix-vector Multiplication

Definition 2.1 *For any n, the* ***identity matrix of dimension*** *n is the $n \times n$ matrix $I_n = (a_{ij})$ where $a_{ii} = 1$ for all $1 \leq i \leq n$ and $a_{ij} = 0$ if $i \neq j$*

Definition 2.2 *If $A = (a_{ij})$ is $n \times n$ matrix the entries $a_{ii} = 1$ for all $1 \leq i \leq n$ is called the* ***diagonal of*** *A.*

Example 2.3 So an identity matrix " has ones along the diagonal and zeros elsewhere":

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \cdots$$

Proposition 2.4 *If I is an $n \times n$ identity matrix and $\overrightarrow{x}$ is an n-dimensional vector then $I\overrightarrow{x} = \overrightarrow{x}$.*

Proposition 2.5 *If A is an $m \times n$ matrix, $\overrightarrow{x}$ and $\overrightarrow{y}$ are an n-dimensional vectors, c a number then*

1. $A(\overrightarrow{x} + \overrightarrow{y}) = A\overrightarrow{x} + A\overrightarrow{y}$
2. $A(c\overrightarrow{x}) = c(A\overrightarrow{x})$