22m:033 Notes:
1.5 Solutions Sets of Linear Systems

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1 Planes and lines as sets of vectors

We will see that solution sets of sets of linear equations are simple subsets such and lines, planes (and in higher dimensions things that are similar sometimes called hyperplanes). The "free variables" used in solution sets are called "parameters" when describing the geometric sets.

1.1 Lines

In precalculus, a linear equation is something like $y = mx + b$ where m and b are fixed numbers. Here b gives you a point on the line and m gives the direction. Also x is a (free) variable— it could be any number.

Actually we use the letter t in place of x and write this $y = mt + b$ or even $y = tm + b$

Now in R^2 given two fixed vectors $\overrightarrow{m}$ and $\overrightarrow{b}$ consider

$$\overrightarrow{y} = t\overrightarrow{m} + \overrightarrow{b}.$$

This is a vector equation of a line in the plane that goes through the point $\overrightarrow{b}$ (when $t = 0$) and has direction determined by $\overrightarrow{m}$.

In R^3 given two fixed vectors $\overrightarrow{m}$ and $\overrightarrow{b}$ consider

$$\overrightarrow{y} = t\overrightarrow{m} + \overrightarrow{b}.$$

This is a vector equation of a line in space that goes through the point $\overrightarrow{b}$ (when $t = 0$) and has direction determined by $\overrightarrow{m}$.

If $\overrightarrow{b} = \overrightarrow{0}$, the line goes through the origin.

More generally in R^n:

Definition 1.1 *In R^n given two fixed vectors $\overrightarrow{m}$ and $\overrightarrow{b}$ consider*

$$\overrightarrow{y} = t\overrightarrow{m} + \overrightarrow{b}.$$

*The variable t is called the **parameter** and this equation is called the **parametric equation of a line**.*

1.2 Planes

In R^3 suppose $\overrightarrow{u}$ and $\overrightarrow{v}$ are fixed vectors (non zero and neither a multiple of the other) and s and t are independently free variables. Consider all linear combinations

$$\overrightarrow{x} = s\overrightarrow{u} + t\overrightarrow{v}.$$

As a set this corresponds to a plane. If s and t are both 0, we see that the plane the contains $\overrightarrow{0}$.

If $\overrightarrow{p}$ is another fixed vector, then the set of vectors:

$$\overrightarrow{x} = s\overrightarrow{u} + t\overrightarrow{v} + \overrightarrow{p}$$

will correspond to a plane that contains the point $\overrightarrow{p}$.

Definition 1.2 *In R^n given three fixed vectors $\overrightarrow{u}$, $\overrightarrow{v}$ and $\overrightarrow{p}$ consider*

$$\overrightarrow{x} = s\overrightarrow{u} + t\overrightarrow{v} + \overrightarrow{p}.$$

The variables t and s are called the ***parameters*** *and this equation is called the* ***parametric equation of a plane****.*

2 Homogeneous System

Definition 2.1 *Let A be an $m \times n$ matrix and $\overrightarrow{x}$ a column matrix of length m. The set of equations corresponding to $A\overrightarrow{x} = \overrightarrow{0}$ is called a* ***homogeneous set of linear equations****.*

Example 2.2 If $A = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}$

and $\overrightarrow{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, then the equation $A\overrightarrow{x} = \overrightarrow{0}$ is

$$\begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The corresponding set of equations is:

$$\begin{aligned} x + 4y &= 0 \\ 2x + 8y &= 0 \end{aligned}$$

So if we use a "free variable" t the solutions are of the form $y = t$ and $x = -4t$. In terms of vectors, the solution set is a line through the origin consisting of all vectors of the form $t \begin{pmatrix} -4 \\ 1 \end{pmatrix}$.

Remark 2.3 Any set of homogeneous equations always has a solution: namely $\overrightarrow{x} = \overrightarrow{0}$.

Geometrically an equation ax+by=0 is a line through (0,0), so the intersection of any two lines must contain (0,0).

Definition 2.4 *The solution $\overrightarrow{x} = \overrightarrow{0}$ to $A\overrightarrow{x} = \overrightarrow{0}$ is called the **trivial solution**. Any other solution is called a **non-trivial solution**.*

So when solving homogeneous equations, the main question is "Are there any non-trivial solutions?".

In Example 2.2, any vector of the form $t\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ will be a non-trivial solution.

Remark 2.5 If a system of homogeneous equations has one non-trivial solution it must have infinitely many. In fact if $\overrightarrow{x_0}$ is a solution, then so is $c\overrightarrow{x_0}$ for any number c.

3 Non-homogeneous System

A non-homogeneous system is one that corresponds to $A\overrightarrow{x} = \overrightarrow{b}$ where $\overrightarrow{b} \neq \overrightarrow{0}$. There might not be solutions to this set of equations.

But IF there are solutions (NOTE THE "IF") there is a nice relation between the solution set of $A\overrightarrow{x} = \overrightarrow{b}$ and the solution set of $A\overrightarrow{x} = \overrightarrow{0}$.

Proposition 3.1 *(THEOREM 6 in text) IF the equation* $A\overrightarrow{x} = \overrightarrow{b}$ *is consistent for some given* $\overrightarrow{b}$*, let* $\overrightarrow{p}$ *be a solution. Then the solution set of* $A\overrightarrow{x} = \overrightarrow{b}$ *is the set of all vectors of the form* $\overrightarrow{p} + \overrightarrow{v}_h$*, where* $\overrightarrow{v}_h$ *is any solution of the homogeneous equation* $A\overrightarrow{x} = \overrightarrow{0}$*.*

Remark 3.2 Take a line L through the origin in any R^n and take any vector $\overrightarrow{p}$. The set of points $\overrightarrow{l} + \overrightarrow{p}$ where $l \in L$ is a line parallel to L.

Geometrically, this means that if a non-homogeneous solution set is infinite, that the solution set is parallel to the solution set of the corresponding homogenous equations.

Example 3.3 If $A = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}$

and $\overrightarrow{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, then the equation $A\overrightarrow{x} = \overrightarrow{0}$ is

$$\begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The corresponding set of equations is:

$$\begin{aligned} x + 4y &= 1 \\ 2x + 8y &= 2 \end{aligned}$$

So if we use a "free variable" t the solutions are of the form $y = t$ and $x = 1 - 4t$. In terms of vectors, the solution set is a line through the origin consisting of all vectors of the form $t\begin{pmatrix} -4 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Example 3.4 Consider the following set of two equations with four variables:

$$\begin{aligned} x - 2y + 3z - 4w &= 0 \\ z + w &= 0 \end{aligned}$$

If we let $w = t$ then $z = -t$, Next let $y = s$ and we see that

$$x - 2s + 3(-t) - 4t = 0$$

so $x = 7t + 2s$.

We can write this solution set as all vectors of the form

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = t \begin{pmatrix} 7 \\ 0 \\ -1 \\ 1 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Remark 3.5 Note that this means that Span of the vectors

$$\begin{pmatrix} 7 \\ 0 \\ -1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

is the solution set of these homogeneous equations.

Next a set of non homogeneous equations with the same coefficient matrix

Example 3.6 Consider the following set of two equations with four variables:

$$\begin{aligned} x - 2y + 3z - 4w &= 4 \\ z + w &= 1 \end{aligned}$$

If we let $w = t$ then $z = 1 - t$, Next let $y = s$ and we see that

$$x - 2s + 3(1 - t) - 4t = 4$$

so $x = 7t + 2s + 1$.

We can write this solution set as all vectors of the form

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = t \begin{pmatrix} 7 \\ 0 \\ -1 \\ 1 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Note how this illustrates Proposition 3.1.