22m:033 Notes:
2.8 Subspaces of R^n

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1 Subspaces

Up to this point we have been speaking of solutions *sets* of equation sets. The first observation is that these sets have additional algebraic properties. Sets with these properties will be called *subspaces*.

Definition 1.1 *A subset $H \subseteq R^n$ **is a subspace of** R^n if*

1. $\overrightarrow{0} \in H$
2. *If* $\overrightarrow{u} \in H$ *and* $\overrightarrow{v} \in H$ *then* $\overrightarrow{u} + \overrightarrow{v} \in H$
3. *If If* $\overrightarrow{u} \in H$ *and* $c \in R$ *then* $c\overrightarrow{u} \in H$

Remark 1.2 The set consisting of the single zero vector of R^n a subspace of R^n. This is called the **zero subspace**.

Remark 1.3 On the other extreme, the set consisting of *all vectors* of R^n a subspace of R^n. In other words "R^n is a subspace of itself".

Remark 1.4 If a subspace is not the zero subspace, it contains at least one non-zero vector $\overrightarrow{v}$ and also all the vectors $c\overrightarrow{v}$ where $c \in R$.

Thus any non-zero subspace is an infinite subset of R^n.

So any non empty *finite* set of non-zero vectors is *not* a subspace.

Remark 1.5 There are many infinite sets of vectors that are not subspaces. For example all vectors in the plane $\begin{pmatrix} x \\ y \end{pmatrix}$ with $x^2 + y^2 = 1$ (the unit circle) satisfies none of the conditions of definition of subspace (Definition 1.1).

Remark 1.6 Suppose S is all vectors in the plane $\begin{pmatrix} x \\ y \end{pmatrix}$ with $0 \leq xy$. In other words, x and y have the same sign—so these are the vectors in the first and third quadrants. This set satisfies the first and the third condition of Definition 1.1, but not the second.

Here is an example of two vectors in S whose sum is

not in S:
$$\begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$
in R^2.

Remark 1.7 If S is any set of vectors in R^n, the set H of *all* linear combinations of vectors in S is a subspace of R^n.

For example consider the plane H containing the vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$. H is a subspace since we can describe it as "all linear combinations of the two given vectors.

In fact all subspaces H can be described as "all linear combinations of *some* set of vectors". ∎

Definition 1.8 *The* ***column space of a matrix*** *A is the set of all linear combinations of the column vectors of A.*

Definition 1.9 *The* ***null space of a matrix*** *A is the set of all solutions of the homogeneous equation $A\overrightarrow{x} = \overrightarrow{0}$ and is denoted by $Nul(A)$.*

Proposition 1.10 *The null space of an $m \times n$ matrix A is a subspaces of R^n.*

Remark 1.11 The column space and null space of a matrix are very different things. For example if

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

the null space of A is a set of *three-dimensional* vectors and the column space is a set of *two-dimensional* vectors.

Having said that we will soon see there is a relationship of some sort involving these two ideas,

2 Basis

In Remark 1.7 we noted that any subspace is all linear combinations of some set S of vectors. However this set might be large. The concept of basis captures the idea of "the smallest set of vectors we need to do this".

Definition 2.1 *A* ***basis for a subspace*** *$H \subseteq R^n$ is a linearly independent set in H that spans H*

Example 2.2 Let B consist of the two vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This is a basis for R^2 since we can write:

$$\begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This is called **the standard basis for** R^2.

Example 2.3 Let B consist of the two vectors

$$b_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } b_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This is a basis for R^2 they are clearly linearly independent and since we can write:

$$\begin{pmatrix} x \\ y \end{pmatrix} = (x - y) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This is a basis for R^2.

So how did we discover the above formula? We were looking for numbers, say a and b so that we can write:

$$\begin{pmatrix} x \\ y \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This gives rise to a set of non-homogeneous equations with augmented matrix:

$$\begin{pmatrix} 1 & 1 & x \\ 0 & 1 & y \end{pmatrix}.$$

This row reduces quickly to:

$$\begin{pmatrix} 1 & 0 & x-y \\ 0 & 1 & y \end{pmatrix}.$$

Thus a solution is $a = x - y$ and $b = y$

Definition 2.4 *Let $\overrightarrow{e_i}$ denote the i-the column vector of the identity matrix I_n. The collection of vectors $\{\overrightarrow{e_1}, \overrightarrow{e_2}, \ldots, \overrightarrow{e_n}\}$ is called the* ***standard basis for*** *R^n.*

3 How to find a basis for the null space of A

The methods we have used for expressing solutions in parametric vector form of the homogeneous equations $A\overrightarrow{x} = \overrightarrow{0}$ gives a basis.

Example 3.1 We will find a basis for the null space of

$$A = \begin{pmatrix} 1 & -2 & 3 & -4 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

This is actually the coefficient matrix corresponding the the equations

$$\begin{aligned} x - 2y + 3z - 4w &= 0 \\ z + w &= 0 \end{aligned}$$

which were considered as an example in the class notes for Chapter 1 Section 5.

When we solved this system, we expressed the solution set in vector parametric form as

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = t \begin{pmatrix} 7 \\ 0 \\ -1 \\ 1 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

.

Let B consist of the pair of vectors $\begin{pmatrix} 7 \\ 0 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

In terms we have learned since that section, we showed that the vectors in B span the solution space of $A\overrightarrow{x} = \overrightarrow{0}$. Observe that these two vectors are linearly independent. Therefore B is a basis for the null space of A. ∎

In general we can show that this process we have been using, based on row reduction of matrices will always result in a set of linearly independent vectors and thus give a basis for the null space.

4 How to find a basis for the column space of A

Warning: The method for doing this is easy but subtle so you must pay close attention to the details. This is one of the few times we make use of the echelon form of A rather than the *reduced* echelon form—that is why the book considers this "easy".

Here is the procedure for finding a basis for the column space of A:

1. Find an echelon form E of A.
2. For each row of E unless it is an all zero row will contain a first non-zero entry. Just pay attention to

which column this entry is in. The corresponding column of A will be one of our basis vectors.

3. Do this procedure for each row and we will end up with a basis of the column space.

Example 4.1 Using the same A from the previous problem we note that it is already in reduced form. In the first row the leading non-zero entry is in the first column. In the second row, the leading non-zero entry is in the third column.

So a basis for the column space will consist of the two vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Example 4.2 We find a basis for the column space of

$$A = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 7 & 8 & 9 \\ 2 & 3 & 4 & 3 & 4 \end{pmatrix}.$$

Subtracting the first row from the second and then the third gives us a echelon form:

$$E = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & -2 & -2 \end{pmatrix}.$$

. The leading non-zero entries of E are found in the first, third and fourth columns. So we let our basis B be the first, third and fourth columns of A. This will give us a basis for the column space of A.

In other words this basis

$$B = \{ \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 8 \\ 3 \end{pmatrix} \}$$

Warning: it is important that we use this to chose columns from A and not from E.