# 22m:033 Notes:
# 3.2 Properties of Determinants

Dennis Roseman
University of Iowa
Iowa City, IA

http://www.math.uiowa.edu/∼roseman

March 23, 2010

## 1 Determinants and row operations

**Proposition 1.1** *Let $A$ be a square matrix.*

1. *If we add a multiple of one row of $A$ to another and get matrix $B$, then* $\det B = \det A$
2. *If we interchange two rows of $A$ to another and get matrix $B$, then* $\det B = -\det A$
3. *If we multiply of one row of $A$ by $k$ to another and get matrix $B$, then* $\det B = k \det A$

**Remark 1.2** In the previous section we noted that it was easy to calculate the determinant of a triangular matrix. If we row reduce a square matrix to echelon form, we will have a triangular matrix. So we may use Proposition 1.1 to get an alternate method of calculating a determinant.

**Example 1.3** Here is one way to calculate

$$\begin{vmatrix} 1 & 2 & 11 & 13 \\ 1 & 7 & 9 & 11 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 14 \end{vmatrix}$$

Note that if we multiply row 3 by -1 of $\begin{pmatrix} 1 & 2 & 11 & 13 \\ 1 & 7 & 9 & 11 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 14 \end{pmatrix}$ and add to each of the other rows (a total of three row operations) we get:

$$\begin{pmatrix} 0 & 0 & 8 & 9 \\ 0 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 10 \end{pmatrix}.$$

If we next switch the the first row with the third row we get the triangular matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}.$$

We can easily calculate

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{vmatrix} = 1 \cdot 5 \cdot 8 \cdot 10 = 400$$

Reviewing the row operations we did, the first three did not change the determinant by 1 of Proposition 1.1, but the last one changed the sign as in part 2 of Proposition 1.1.

So we conclude that

$$\begin{vmatrix} 1 & 2 & 11 & 13 \\ 1 & 7 & 9 & 11 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 14 \end{vmatrix} = -400.$$

**Remark 1.4** We can always change a given square matrix into a triangular one using only the first two row operations so for this method of calculation, part 3 of Proposition 1.1 is never needed. But it is useful in some calculations as shown in the text and it is significant for other reasons we will soon see.

From the relationship of row reduction and inverses we can conclude:

**Proposition 1.5** *A square matrix $A$ is invertible if and only if* $\det A \neq 0$.

## 2 Column operations

Concerning the transpose of a matrix, we note:

**Proposition 2.1** *If $A$ is square matrix then* $\det A = \det A^T$.

**Remark 2.2** Our text points out that this means that we could also state a theorem similar to Proposition 1.1 which speaks of "column operations" rather than "row operations".

Although this is true I advise all but the best students to **ignore this observation completely**. It is not all that useful and tends to cause confusion since column operations are completely different from row operations.

## 3 The determinant of a product

**Proposition 3.1** *If $A$ and $B$ are two $n \times n$ matrices, then* $\det AB = \det A \det B$.