Chapter 7
Related Rates and Implicit Derivatives

This chapter gives some basic applications of the Chain Rule but also shows why it is important to learn to work with parameters and variables other than x and y. Most of this chapter is independent of the next few, so it could be skipped now in favor of other topics. If you skip this chapter now, return to implicit differentiation later when it arises in another application.

The main new topic in this chapter is an application of the Chain Rule called "related rates problems" given in the Section 7.4. When functions are chained or composed, the rate of change of the first output variable changes the second output variable: Their rates are related. This idea generalizes to implicitly linked variables.

Implicitly linked variables change with one another, but there are no explicit functions connecting them, only a formula involving both variables. Implicit differentiation is often an easier way to solve related rate, max - min, or other problems later in the course. Essentially, this method is easier because implicit differentiation "treats all variables equally."

7.1 Differentiation with Parameters

You just learned the $\frac{dy}{dx}$ versions of the rules for differentiation. However, it is important to work with parameters (letters you treat as constants) and other variable names. This section has examples to show you why.

In the Chain Rule, we asked you to use a different variable name u and find $\frac{dy}{du}$ with formulas you just learned for $\frac{dy}{dx}$. A few exercises in Chapter 6 were also written in terms of other variables. Usually students do not like this at first which is an understandable reaction. However, there are times mathematically when you have already used the variable x for something but need to vary something else. Here is an oversimplified example to illustrate the point:

Example 7.1 *Approximating a Root x as b Varies*

Suppose we are finding a root of the quadratic equation

$$a x^2 + b x + c = 0$$

where the coefficient b is a measured quantity and not known with perfect accuracy. We want to know how sensitive the largest root of the equation is to errors in measuring b. The largest root of the quadratic equation above can be written as a function of b, including the parameters a and c:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

The derivative $\frac{dx}{db}$ measures the rate of change of x with respect to b. A small change in b denoted db produces the approximate change in x of

$$dx = x'[b]\ db$$

by the microscope approximation (or meaning of derivative). An exercise below has you explore this approximation. We return to this in Example 7.12.

Example 7.2 *The Role of Parameters*

Much later in the course (in the project on resonance) we will see that the amplitude, A, of a certain kind of oscillation is given by the formula

$$A[\omega] = \sqrt{\frac{1}{(s - m\,\omega^2)^2 + (c\,\omega)^2}}$$

where s, m and c are measured quantities of a particular system. If we think of the oscillator as the front suspension of an old car with weak shocks, m is the mass, s is the strength of the spring, and c measures the strength left in the shocks. These are fixed for any particular car and we want to see how the peak of A depends on these parameters. This will tell us the frequency of the most violent shaking of a car in terms of m, s, and c. The graph typically looks like Figure 7.1:1

The peak response (called the “resonant frequency”) is at the frequency ω_r where $A'[\omega_r] = 0$.

We compute the derivative of A with respect to ω treating the other letters as constants:

$$\begin{aligned} A'[\omega] &= \frac{d(((s - m\,\omega^2)^2 + (c\,\omega)^2)^{(-1/2)})}{d\omega} \\ &= -\frac{1}{2}((s - m\,\omega^2)^2 + (c\,\omega)^2)^{(-3/2)} \times (2(s - m\,\omega^2) \times 2m\,\omega + 2c^2\,\omega) \\ &= -\omega \frac{2m^2\,\omega^2 + (c^2 - 2m\,s)}{((s - m\,\omega^2)^2 + (c\,\omega)^2)^{(3/2)}} \end{aligned}$$

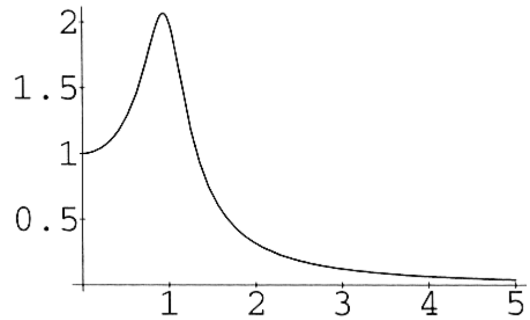

Figure 7.1:1: Amplitude A as a function of frequency ω

This is very messy, but we can check our work with the computer.

Notice that this derivative is zero, $A'[\omega] = 0$, when $\omega = 0$ (look at the graph) or ω is a positive root (the peak) of the numerator:

$$
\begin{aligned}
2m^2\,\omega^2 + (c^2 - 2m\,s) &= 0 \\
2m^2\,\omega^2 &= (2m\,s - c^2) \\
\sqrt{2}m\,\omega &= \sqrt{2m\,s - c^2} \\
\omega &= \frac{\sqrt{2m\,s - c^2}}{\sqrt{2}m} = \frac{\sqrt{4m\,s - 2c^2}}{2m}
\end{aligned}
$$

so the resonant frequency is

$$\omega_r = \frac{\sqrt{4m\,s - 2c^2}}{2m}$$

Given m, s, and c, we just "plug in" and find the frequency. The important point is that we find the max BEFORE we know the actual values of these constants.

Exercise Set 7.1

The next exercise is solved with implicit differentiation in Example 7.12. Solve it explicitly now so you can compare the explicit and implicit methods.

1. *Approximate Roots*

   (a) *Compute the derivative* $\frac{dx}{db}$ *(considering* a *and* c *as parameters). When is this defined? In the cases when it is not defined, what is going on in the original root-finding problem? Consider some special cases to help such as* $(a, b, c) = (1, -3, 2)$, $(a, b, c) = (1, -2, 1)$, $(a, b, c) = (0, -2, 1)$.

(b) *Consider the case $(a, b, c) = (1, -3, 2)$ and denote the error in measuring b by db. Suppose the magnitude of db could be as large as 0.01. Use the differential approximation to estimate the resulting error in the root.*

The following are for practice differentiating with respect to different letters:

(c) *Compute the derivative $\frac{dx}{da}$ (considering b and c as parameters). When is this defined?*

(d) *Compute the derivative $\frac{dx}{dc}$ (considering a and b as parameters). When is this defined?*

(e) *Check your differentiation with the computer.*

It is customary in physics to let the Greek letter omega, ω, denote frequency and T denote absolute temperature. Certainly, the capital T suggests the word that the variable measures. Whether you like ω or not, it is almost impossible to read the physics literature without working with it. Here is an example to test your skills.

2. *Planck's radiation law can be written*

$$I = \frac{a\omega^3}{e^{b\omega/T} - 1}$$

for constants a and b. This expresses the intensity I of radiation at frequency ω for a body at absolute temperature T. Suppose T is also fixed. Express I as a chain or composition of functions (of the variable ω with parameters) and products of functions to which the rules of this chapter apply. For example, you can use the Exponential Rule, $\frac{d(e^u)}{du} = e^u$. What is the formula for $\frac{dI}{d\omega}$? Check your work with the DfDx program.

There is a project on Planck's law studying the interaction between calculus and graphs and between calculus and maximization. Planck's Law was one of the first big achievements of quantum mechanics.

We postpone the exercises on related rates until we have shown you implicitly linked variables and the method of implicit differentiation. In those problems, you have your choice of solving for explicit nonlinear equations or using implicit differentiation.

7.2 Implicit Differentiation

Implicit equations have many powerful uses. We can differentiate them directly simply by treating all variables equally.

A unit circle in the plane is given by the set of (x, y)-points satisfying the implicit equation

$$x^2 + y^2 = 1$$

This equation is called "implicit" because, if we treat x as given, then y is only implicitly given by the equation. Two values of y satisfy the equation, but the implicit equation does not give a direct "explicit" way to compute them. The explicit equation

$$y = -\sqrt{1 - x^2}$$

gives a direct way to compute y on the lower half-circle, whereas the implicit equation does not but does give the whole circle.

Example 7.3 *Implicit Tangent to the Circle*

The differential of $u^2 + b$, when u is the variable and b is a parameter, is $2\,u\;du$. Treating x as the variable and y^2 as a parameter in $x^2 + y^2$, we have the differential $2\,x\;dx$. Treating y as the variable and x^2 as a parameter in $x^2 + y^2$, we have the differential $2\,y\;dy$. Adding the two, we obtain $2\,x\;dx + 2\,y\;dy$. Since the differential of the constant 1 is zero, the total differential of the implicit equation becomes

$$x^2 + y^2 = 1 \qquad \Rightarrow \qquad 2\,x\;dx + 2\,y\;dy = 0$$

We may view the total differential as an implicit equation for the tangent line to the circle in the local variables (dx, dy) when (x, y) is a fixed point on the circle.

This may be solved for $\frac{dy}{dx}$ as follows:

$$\begin{aligned} 2x\;dx + 2y\;dy &= 0 \\ x\;dx + y\;dy &= 0 \\ y\;dy &= -x\;dx \\ dy &= -\frac{x}{y}dx \\ \frac{dy}{dx} &= -\frac{x}{y} \end{aligned}$$

This is a valid formula for the slope of the tangent to a circle. However, this expression uses both variables so that to use it, we need to know both x and y. For example, the point $(1/2, -\sqrt{3}/2)$ lies on the lower half of the circle as shown in Figure 7.2:2. At this point the slope is

$$\frac{dy}{dx} = -\frac{x}{y} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$$

Let us compare this computation of the slope of the circle at $(1/2, -\sqrt{3}/2)$ to the computation with the explicit equation.

$$y = -\sqrt{1 - x^2}$$

$$y = -\sqrt{u} = -u^{1/2} \qquad u = 1 - x^2$$

$$\tfrac{dy}{du} = -u^{-1/2} = -\tfrac{1}{2\sqrt{u}} \qquad \tfrac{du}{dx} = -2x$$

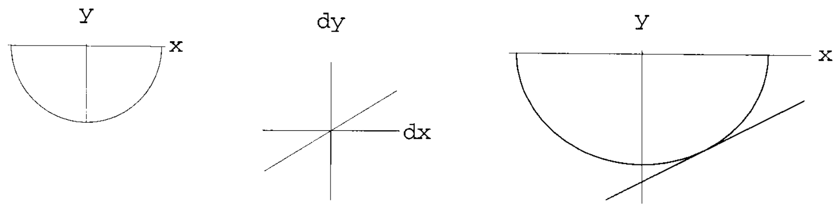

Figure 7.2:2: $y = -\sqrt{1-x^2}$ and $dy = dx/\sqrt{3} \Leftrightarrow y + \frac{\sqrt{3}}{2} = \left(x - \frac{1}{2}\right)/\sqrt{3}$

so the Chain Rule gives

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{x}{2\sqrt{u}} = \frac{x}{2\sqrt{1-x^2}}$$

and when $x = 1/2$

$$\frac{dy}{dx} = \frac{x}{2\sqrt{1-x^2}} = \frac{1/2}{2\sqrt{1-(1/2)^2}} = \frac{1}{\sqrt{3}}$$

The implicit computation

$$x^2 + y^2 = 1 \qquad \Rightarrow \qquad 2\,x\;dx + 2\,y\;dy = 0$$

is certainly simpler than solving and performing the four lines of computation above.

The idea of implicit differentiation is to differentiate everything with respect to x and multiply by dx, then to differentiate everything with respect to y and multiply by dy, and finally to add all the differentials together. This description is a little vague, but the drill exercise below should be enough to give you the idea. This works with any variables.

Example 7.4 *Different Letters*

Find the implicit equation of the tangent to the circle $w^2 + h^2 = 25^2$ at $(20, 15)$.

$$w^2 + h^2 = 25^2$$
$$2\,w\;dw + 2\,h\;dh = 0$$

The equation $w^2 + h^2 = 25^2$ is an implicit equation in w and h. When we consider w and h fixed and located somewhere on the circle, the equation in dw and dh, $w\,dw + h\,dh = 0$, is an implicit

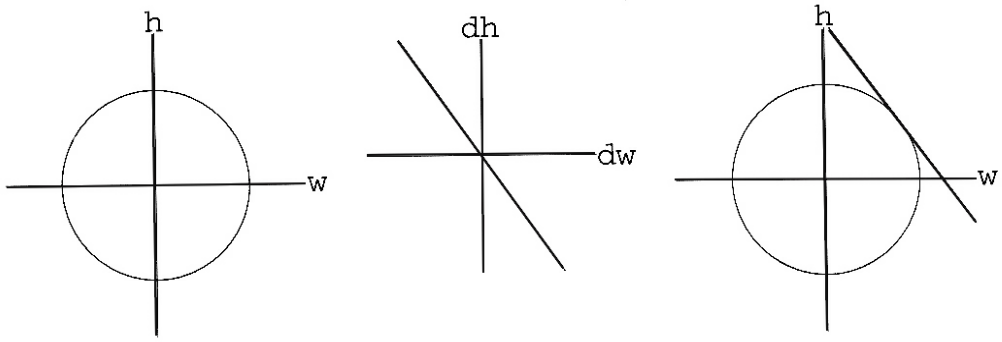

Figure 7.2:3: $20\ dw + 15\ dh = 0$

equation for a line. The point $(20, 15)$ lies on the circle. The implicit equation of its tangent at this point is shown in Figure 7.2:3.

Implicit curve	Implicit tangent line
$w^2 + h^2 = 25^2$	$w\ dw + h\ dh = 0$

The formula $w\,dw + h\,dh = 0$ has the advantage that we may think of either variable as the independent variable. When $w = 25$, we have $h = 0$. The circle is smooth, but the tangent is vertical, $25\,dw + 0\,dh = 0$ or $dw = 0$. (This is simply the local variable equation for the dh-axis.) The explicit formula $h = \sqrt{25^2 - w^2}$, with derivative $\frac{dh}{dw} = -w/\sqrt{25^2 - w^2}$, is undefined at $w = 25$.

Exercise Set 7.2 *Implicit Drill*

1. *Verify the following implicit total differential calculations:*

$$3x^2 + y^2/5 = 1 \quad \Rightarrow \quad 6x\,dx + \tfrac{2}{5}y\,dy = 0$$

$$xy = 1 \quad \Rightarrow \quad y\,dx + x\,dy = 0$$

$$x + xy = 2y \quad \Rightarrow \quad dx + x\,dy + y\,dx = 2\,dy$$

$$y + \sqrt{y} = x + x^2 \quad \Rightarrow \quad dy + \tfrac{1}{2\sqrt{y}}dy = dx + 2x\,dx$$

$$e^x = \text{Log}\,[y] \quad \Rightarrow \quad e^x\,dx = \tfrac{1}{y}dy$$

$$x = \text{Sin}\,[xy] \quad \Rightarrow \quad dx = y\,\text{Cos}\,[xy]\ dx + x\,\text{Cos}\,[xy]\ dy$$

$$x = \text{Log}\,[\text{Cos}\,[3x + 5y]] \quad \Rightarrow \quad dx = -3\,\text{Tan}\,[3x + 5y]\ dx - 5\,\text{Tan}\,[3x + 5y]\ dy$$

2. More Implicit Differentiation Drill

Find the total differential and solve for $\frac{dy}{dx}$

a) $x^2 - y^2 = 3$	*b)* $y + \sqrt{y} = \frac{1}{x}$	*c)* $xy = 4$
d) $y = \text{Cos}[xy]$	*e)* $\text{Sin}[x]\,\text{Cos}[y] = \frac{1}{2}$	*f)* $y = \text{Sin}[x + y]$
g) $y = e^{xy}$	*h)* $e^x e^y = 1$	*i)* $e^x = x + y^2$
j) $x = \text{Log}[xy]$	*k)* $y = \text{Log}\left[x^2 y\right]$	*l)* $x = \text{Log}[x + y]$

7.3 Implicit Tangents and Derivatives

Implicit differentiation can be used directly to find tangents and derivatives.

Example 7.5 *The General Implicit Slope of a Circle*

The circle of radius r (centered at the origin) is the set of (x, y) points satisfying

$$x^2 + y^2 = r^2$$

Because we are thinking of r as a constant, its differential is zero and

$$2x\ dx + 2y\ dy = 0$$

If we solve the equation $2\,x\ dx + 2\,y\ dy = 0$ for the slope of the tangent line,

$$\frac{dy}{dx} = -\frac{x}{y}$$

The slope of a radial line from the center to any point (x, y) on the circle is y/x. The negative reciprocal $-x/y$ is the slope of a perpendicular line. We have just shown that it is the slope of the tangent, so this shows that the tangent to a circle is perpendicular to a radius at a general point as shown on Figure 7.3:4

Example 7.6 *Implicit Tangent to an Ellipse*

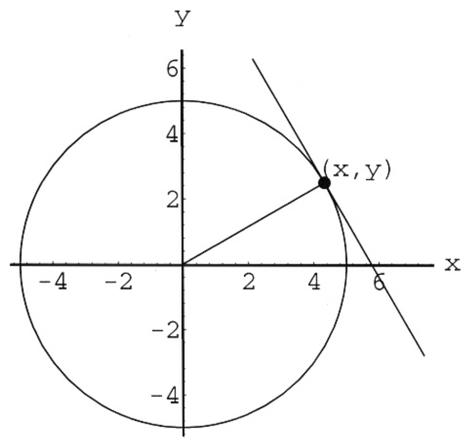

Figure 7.3:4: Circle and Tangent

The differential of the equation of an ellipse $(\frac{x}{2})^2 + (\frac{y}{3})^2 = 1$ is computed as follows:

$$
\begin{aligned}
\frac{x^2}{4} \ + \ \frac{y^2}{9} &= 1 \\
\frac{2x\ dx}{4} + \frac{2y\ dy}{9} &= 0 \\
\frac{1}{2}x\ dx + \frac{2}{9}y\ dy &= 0 \\
x\ dx + \frac{4}{9}y\ dy &= 0
\end{aligned}
$$

The result of this computation, shown in Figure 7.3:5, is not geometrically as obvious as the tangent to a circle because a line from the center is no longer perpendicular to the tangent. However, if we want to sketch the tangent at a point, for example where $x = 8/5 \approx 1.6$ and $y = 9/5 \approx 1.8$, so $(\frac{x}{2})^2 + (\frac{y}{3})^2 = \frac{4}{5}^2 + \frac{3}{5}^2 = 1$ is on the ellipse, we can use the local (dx, dy) coordinates (as in Chapter 1) in the implicit form. The specific tangent has equation $\frac{8}{5}\ dx + \frac{4}{9}\frac{9}{5}\ dy = 0$ or $2\ dx + dy = 0$.

7.3.1 The Chain Rule as Substitution in Differentials

Suppose we have a chain

$$y = f[u] \qquad \& \qquad u = g[x]$$

If we calculate the differentials

$$dy = f'[u]\ du \qquad \& \qquad du = g'[x]\ dx$$

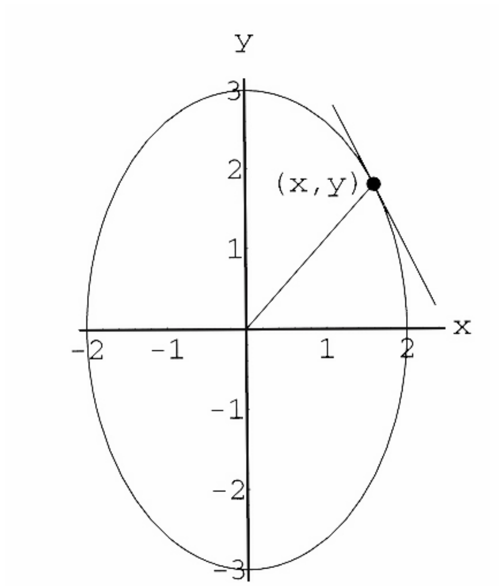

Figure 7.3:5: $\left(\frac{x}{2}\right)^2 - \left(\frac{y}{3}\right)^2 = 1$ and $2\,dx + dy = 0$ at $(x, y) = \left(\frac{8}{5}, \frac{9}{5}\right)$

and make the substitution for du,

$$dy = f'[u]\ du = f'[u](g'[x]\ dx) = f'[g[x]]g'[x]\ dx$$

we have in effect done a Chain Rule computation. In specific contexts, it looks simpler.

Example 7.7 *The Differential of* $y = e^{x^2}$

We compute the differential of $y = e^{x^2}$ using the decomposition

$$y = e^u \qquad \& \qquad u = x^2$$

The differentials are

$$dy = e^u\ du \qquad \& \qquad du = 2x\ dx$$

Substitution of du gives

$$dy = e^u\ du = e^u\ (2x\ dx) = 2xe^{x^2}\ dx$$

Example 7.8 *The S-I-R Invariant and Differentials*

In Chapter 2, you may recall that we claimed that the quantity

$$s + i - \frac{b}{a}\,\mathrm{Log}[s] = k$$

was constant, where s and i are the susceptible and infectious fraction of a population, wheras a and b are constants. The differential of this equation is

$$\begin{aligned} ds + di - \frac{b}{a}\frac{1}{s}\; ds &= dk \\ \left(1 - \frac{b}{a}\frac{1}{s}\right) ds + di &= dk \end{aligned}$$

In the S-I-R model of an epidemic, the variables s and i are both functions of time,

$$\begin{array}{ccc} s = s\,[t] & & i = i\,[t] \\ & \& & \\ ds = s'\,[t]\;dt & & di = i'\,[t]\;dt \end{array}$$

Substituting these in the first differential gives

$$(\left(1 - \frac{b}{a}\frac{1}{s}\right) s'[t] + i'[t])\; dt = dk$$

Carrying this one step further, recall that $s'[t] = -a\; s[t]\; i[t]$ and $i'[t] = as[t]i[t] - bi[t]$. Substituting these into the expression gives

$$\begin{aligned} dk &= (\left(1 - \frac{b}{a}\frac{1}{s}\right)(-asi) + (asi - bi))\; dt \\ dk &= ((-asi + bi) + (asi - bi))\;\; dt \\ \frac{dk}{dt} &= 0 \end{aligned}$$

Since the derivative of k with respect to t is zero, k is constant.

7.3.2 Derivatives of Inverse Functions

The method of implicit differentiation applies to inverse functions. This case is treated in detail in the Project on Inverse Functions.

Example 7.9 *Derivative of* Log[y]

Consider an example for the inverse pair of equations

$$y = e^x \qquad \Leftrightarrow \qquad x = \mathrm{Log}[y]$$

The differential of the exponential equation is $dy = e^x\ dx$, but we may use the fact that $e^x = y$ to write

$$dy = y\ dx$$

Solving for the derivative of x with respect to y gives us the derivative of the logarithm,

$$\frac{d(\mathrm{Log}[y])}{dy} = \frac{dx}{dy} = \frac{1}{y}$$

Example 7.10 *Derivative of* ArcTan$[y]$

Computation of inverse function derivatives this way can present computational difficulties, such as the following:

$$y = \mathrm{Tan}[x] \qquad \Leftrightarrow \qquad x = \mathrm{ArcTan}[y]$$

You computed the derivative of tangent directly in Chapter 5 and using rules in Chapter 6, obtaining the following answer both ways:

$$dy = \frac{1}{(\mathrm{Cos}[x])^2}\, dx$$

Consequently, we have a formula for the derivative of the arctangent

$$\frac{d(\mathrm{ArcTan}[y])}{dy} = \frac{dx}{dy} = (\mathrm{Cos}[x])^2$$

Unfortunately, this form of the equation is in terms of the dependent variable for arctangent, so some trig tricks are needed to put it in the form

$$\frac{d(\mathrm{ArcTan}[y])}{dy} = \frac{1}{1+y^2}$$

See the project on Inverse Functions. Additional examples are included in that project along with complete justification of this method of computing derivatives of inverse functions. The justification is in the form of a procedure you can use to compute the actual nonlinear inverse function. In other words, it is a “practical” proof.

Exercise Set 7.3

1. *Use implicit differentiation to find the equation of the tangent line to*

$$(5x)^2 - (4y)^2 = 3^2$$

at the point $(x, y) = (1, 1)$*. Use the local coordinates* (dx, dy) *centered at* $(x, y) = (1, 1)$*.*

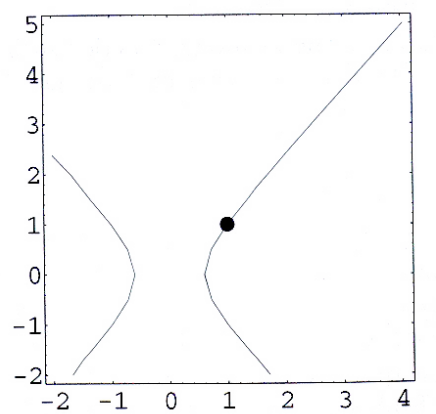

Figure 7.3:6: $\left(\frac{5}{3}x\right)^2 - \left(\frac{4}{3}x\right)^2 = 1$

7.4 Related Rates

Many applications of calculus involve different quantities that vary with time but are "linked" with one another; the time rate of change of one variable determines the time rate of change of the other. This section illustrates the idea with examples including the falling ladder.

Example 7.11 *The Fast Lighthouse*

A lighthouse is 1 mile off a straight shore with its beacon revolving 3 times per minute. How fast does the beam of light sweep down the beach at the points that are 2 miles from the point closest to the lighthouse?

The "chain" in this exercise is that the angle of the beacon depends on time and the distance to the point of contact down the beach depends on the angle. We will assume that the light revolves clockwise in the diagram Figure 7.4:7. We want to know the rate of change of distance with respect to time.

We introduce the variables shown in the diagram Figure 7.4:7:

$$\begin{aligned} t &= \text{the time from } t = 0 \text{ when the beam is perpendicular to the shore } (\textit{minutes}) \\ \theta &= \text{the angle the beam makes from the perpendicular line } (\textit{radians}) \\ D &= \text{the distance along the shore from the perpendicular point } (\textit{miles}) \end{aligned}$$

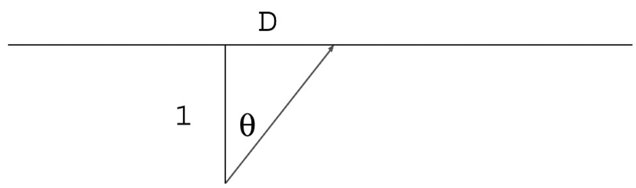

Figure 7.4:7: A lighthouse beam sweeps the shore

The link between the angle and the distance is

$$\mathrm{Tan}[\theta] = D$$

because $D/1$ is the opposite over the adjacent sides. (SOH-CAH-TOA).

The link between the time and the angle can be expressed as an explicit function, but what is important is to know the rate of change,

$$\frac{d\theta}{dt} = 6\,\pi = \frac{3 \times 2\,\pi}{1} = \frac{3 \text{ revolutions in radians}}{1 \text{ minute}}$$

Because the derivative is constant, this is equivalent to $\theta = 6\,\pi\; t$, because $t = 0$ when $\theta = 0$.

The question in this problem is

FIND:

$$\frac{dD}{dt} \quad \text{when } D = 2$$

SOLUTION: Distance as an explicit function of time is

$$D = \mathrm{Tan}[\theta[t]] \qquad \text{where} \qquad \theta = \theta[t] = 6\,\pi\,t$$

The Chain Rule gives us

$$\begin{aligned}\frac{dD}{dt} &= \frac{dD}{d\theta}\cdot\frac{d\theta}{dt}\\ &= \frac{d(\mathrm{Tan}[\theta])}{d\theta}\cdot\frac{d\theta}{dt}\\ &= \frac{1}{(\mathrm{Cos}[\theta])^2}\cdot\frac{d\theta}{dt} = \frac{6\,\pi}{(\mathrm{Cos}[\theta])^2}\end{aligned}$$

This formula tells us the speed of the light's motion in terms of θ. Notice that θ is not the independent variable t, but instead is the link variable or output from the first function in the chain. We could use the first function $\theta = 6\pi\,t$ to express the speed in terms of t, but this is neither necessary nor desirable (unless we want to know when the light gets to the point 2 miles down the

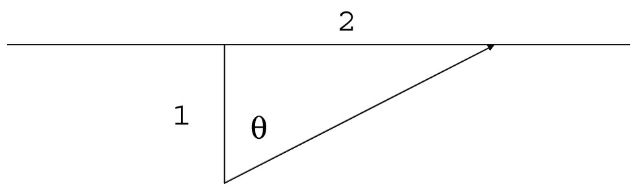

Figure 7.4:8: The angle when $D = 2$

beach.) In fact, it is not even necessary to know the derivative $\frac{d\theta}{dt}$ at any other time; it need not be constant as long as we know that $\frac{dD}{dt} = \frac{1}{(\mathrm{Cos}[\theta])^2} \cdot \frac{d\theta}{dt}$.

The specific position when $D = 2$ is shown in Figure 7.4:8.

The Pythagorean Theorem applied to Figure 7.4:8 tells us that the hypotenuse of the right triangle shown is $\sqrt{2^2 + 1^2} = \sqrt{5}$. SOH-CAH-TOA tells us that

$$\mathrm{Cos}[\theta] = \frac{1}{\sqrt{5}}$$

Finally, when $D = 2$

$$\frac{dD}{dt} = \frac{6\,\pi}{(\mathrm{Cos}[\theta])^2} = 6\,\pi \cdot (\sqrt{5})^2 = 30\pi \approx 94.25 \text{ (mi/min)} \approx 5656 \text{ mph}$$

Common error: A common error in related rate problems is to fix a quantity too soon. In the example above, we want the speed $\frac{dD}{dt}$ when $D = 2$; but, if we fix this position at $D = 2$ miles down the shore before we differentiate, we get $\mathrm{Tan}[\theta] = 2$ so θ is constant and there is nothing to differentiate. You must think about the drill problems in their general variable cases, differentiate, and then fix the quantities at the specific values.

The Lightspeed Lighthouse

The beam of light in the previous example moves down the beach faster and faster, but real light can only move at the speed of light. The Scientific Project on the Lightspeed Lighthouse examines the example taking into account that the light must travel from the lighthouse to the shore at the speed of light.

7.5 Implicitly Linked Variables

Implicit solutions of related rates problems often are simpler and more revealing than first solving for a quantity explicitly.

Example 7.12 *Implicit Solution of Exercise 7.11*

After you solve Exercise 7.11, you should compare your solution to the following implicit method. We begin with the equation

$$a\,x^2 + b\,x + c = 0$$

We are told that a and c are known exactly but that b is measured and may have some error when we determine the largest root x satisfying the equation above. In other words, the independent variable b implicitly determines the variable x. The variables in this problem are b and x, and x is the dependent variable.

The total differential of the equation is

$$(2\,a\,x + b)\ dx + x\ db = 0$$

The first term is the familiar derivative with respect to x and the second is the derivative with respect to b, thereby treating all other letters, including x, as parameters. The rate of change of x with respect to b is obtained by solving

$$\begin{aligned}(2\,a\,x + b)\ dx + x\ db &= 0\\ (2\,a\,x + b)\ dx &= -x\ db\\ dx &= -\frac{x}{2\,a\,x + b}\ db\\ \frac{dx}{db} &= -\frac{x}{2\,a\,x + b}\end{aligned}$$

When $(a, b, c) = (1, -3, 2)$, the largest root of the original equation is $x = 2$, so as b varies from -3,

$$\frac{dx}{db} = -\frac{x}{2\,a\,x + b} = -\frac{2}{4 - 3} = -2$$

For example, if $b{+}db = -3{+}.001 = -2.99$, then $dx \approx -2\ db = -2{\times}.001 = -.002$ and $x{+}dx = 1.998$. (The exact solution is $(2999 + \sqrt{994001})/2000 \approx 1.99799799397791$.)

When $(a, b, c) = (1, -2, 1)$, the only root of the original equation is $x = 1$. In this case, the implicit differential breaks down

$$\begin{aligned}(2\,a\,x + b)\ dx + x\ db &= 0\\ (2 - 2)\ dx + db &= 0\\ 0\ dx + db &= 0\end{aligned}$$

This equation cannot be solved for dx. Implicit differentiation of $\frac{dx}{db}$ fails in this case. We still can graph the line $0\ dx + db = 0$ in the (db, dx)-plane. The equation $db = 0$ is the vertical dx-axis, and vertical lines do not have slopes.

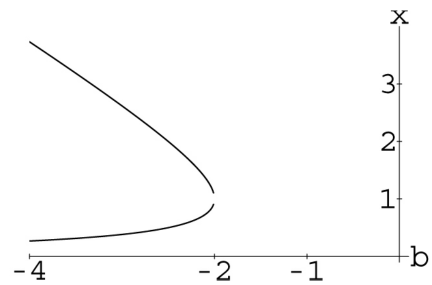

Figure 7.5:9: $x^2 + bx + 1 = 0$

Of course, this degeneracy is only a hint of the trouble in the original implicit equation. If we change b to -1.99, there are no real solutions to the original problem. The equation "branches" here between the positive discriminant and negative one, $x = -b/2 + \sqrt{b^2 - 4}/2$ and $x = -b/2 - \sqrt{b^2 - 4}/2$. The vertical tangent we just computed is the tangent touching these two branches shown in Figure 7.5:9.

Differentiating the explicit function $x[b]$ is more complicated, as you saw in Exercise 7.11:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ \frac{dx}{db} &= \frac{1}{2a}[-1 \pm \frac{b}{\sqrt{b^2 - 4ac}}] \\ dx &= \frac{1}{2a}[-1 \pm \frac{b}{\sqrt{b^2 - 4ac}}]\, db \end{aligned}$$

The implicit and explicit formulas for dx are the same when $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

Example 7.13 *Implicit Solution of the Balloon Exercise 7.4*

In Exercise 7.4.1, you calculate the rate of change of surface area of a balloon that is being blown up so that its volume is increasing at the rate of 2 cubic inches per second. We want to know how the rate of change of volume effects the rate of change of surface area. In the exercise, we ask you to solve explicitly for surface area as a function of volume. The implicit solution of that problem does not require that we find the explicit function.

Here is the implicit solution of the problem:

$$V = \tfrac{4}{3}\pi\, r^3 \qquad S = 4\pi\, r^2$$

$$dV = 4\pi\, r^2\, dr \qquad dS = 8\pi\, r\, dr$$

We solve the volume differential for dr, $dr = dV/(4\pi r^2)$ and substitute into the surface differential, $dS = 8\pi\, r\; dV/(4\pi\, r^2)$, obtaining

$$\begin{aligned} dS &= \frac{2}{r}\; dV \\ \frac{dS}{dt} &= \frac{2}{r}\;\frac{dV}{dt} \\ \frac{dS}{dt} &= \frac{2}{6}\cdot 2 = \frac{2}{3} \qquad (m^2/sec) \end{aligned}$$

Compare this solution with your explicit solution from Exercise 7.4 that first solves

$$S[V] = (??\, V^2)^{1/3}$$

and uses the Chain Rule to compute

$$\frac{dS}{dt} = \frac{dS}{dV}\cdot\frac{dV}{dt} = 2\,(???)^{1/3}\;\frac{dV}{dt}$$

Example 7.14 *The Fast Ladder*

A ladder of length L rests against a vertical wall. The bottom of the ladder is pulled out horizontally at a constant rate r (m/sec). What is the vertical speed of the tip that rests against the wall?

We introduce the variables shown in Figure 7.5:10:

$$\begin{aligned} x &= \text{the horizontal distance from the corner to the bottom of the ladder (meters)} \\ y &= \text{the vertical distance from the corner to the top of the ladder (meters)} \\ t &= \text{the time from some starting time } t = 0 \quad \text{(seconds)} \end{aligned}$$

The length L is a parameter, fixed, but not known to us. We wish to

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FIND:

$$y'[t] = \frac{dy}{dt}$$

in terms of $x'[t] = \frac{dx}{dt}$ and the other variables.

The fact that the length of the ladder is fixed together with the Pythagorean Theorem gives us the implicit linkage between x and y:

$$x^2 + y^2 = L^2 \qquad \text{a constant}$$

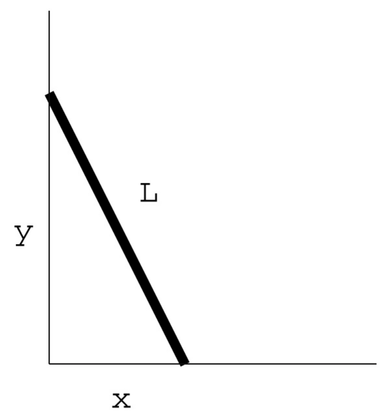

Figure 7.5:10: Ladder and wall

The fact that there is some implicit relationship is clear physically. If we place the base of the ladder at a given x, we know that there is a unique place, y, where it will rest against the wall. We could solve the equation above for y, since only the positive solution makes physical sense, but we will leave that approach to the problem to an exercise below.

The total differential of the implicit equation is

$$2\,x\;dx + 2\,y\;dy = 0$$

which we can solve for $dy = -\frac{x}{y}\;dx$.

In this problem, we also know that x and y are functions of time. This means that the differentials can be expanded

$$dx = x'[t]\;dt \qquad \& \qquad dy = y'[t]\;dt$$

Substitution of the time expressions of the differentials into the equation above yields the solution to our question:

$$\begin{aligned} dy &= -\frac{x}{y}\;dx \\ y'[t]\;dt &= -\frac{x[t]}{y[t]}\;x'[t]\;dt \\ y'[t] &= -\frac{x[t]}{y[t]}\;x'[t] \end{aligned}$$

The time rate of change of y equals the time rate of change of x times x over y and the motion is down, or $y'[t]$ is negative.

Suppose we have an enormous ladder of length 5 m. If the base rests 3 meters from the corner, then $y = 4$. If we are pulling the base out from the wall at the rate of 1/7 m/sec, then the top is moving down the wall at the rate

$$\begin{aligned} y'[t] &= -\frac{x[t]}{y[t]}\, x'[t] \\ &= -\frac{3}{4} \cdot \frac{1}{7} \quad (m/sec) \end{aligned}$$

If the base of this same ladder is moved to 4 meters from the corner so that $y = 3$ and if the base is still pulled out at the speed $x'[t] = 1/7$, then the speed down the wall is

$$\begin{aligned} y'[t] &= -\frac{x[t]}{y[t]}\, x'[t] \\ &= -\frac{4}{3} \cdot \frac{1}{7} \quad (m/sec) \end{aligned}$$

Example 7.15 *A Limiting Case*

As we pull the base of the ladder toward the point $x = L$ away from the wall, what happens to the speed with which the other tip moves down the wall? We can see from the diagram, Figure 7.5:10, that when $x \to L$, $y \to 0$. The formula for the vertical speed tends to minus infinity.

$$y'[t] = -\frac{x[t] \to L}{y[t] \to 0}\, r \to -\infty$$

Humm, this calculation is a little mysterious. Here is another way to look at it: Solve the equation $x^2 + y^2 = L^2$ for $y = \sqrt{L^2 - x^2}$ and substitute into the expression for vertical speed,

$$\begin{aligned} y'[t] &= -\frac{x[t]}{y[t]}\, x'[t] \\ &= -\frac{x[t]}{\sqrt{L^2 - (x[t])^2}}\, x'[t] \end{aligned}$$

Clearly, as x tends to L, $\frac{x}{\sqrt{L^2-x^2}}$ tends to infinity. This is correct mathematically, and we want you to verify it in Exercise 7.4.2.

Exercise Set 7.4

1. *A child is blowing up a balloon by adding air at the rate of 2 cubic inches per second. Well before it bursts, its radius is 6 inches. Assume that the balloon is a perfect sphere and find the surface area S as an explicit function of volume V. Use your explicit function to say how fast the surface is stretching at this point.*

HINTS: *Give variables for surface area, volume, time, and whatever else you need. Solve for S in terms of V. The "chain" in this exercise is thta volume depends on time (through the child's efforts), $V = V[t]$ and surface area depends on volume, $S = S[V]$. We want to know how area changes with time, $\frac{dS}{dt}$, where $S = S[V[t]]$. (See review Exercise in Chapter 28 to express surface area as an explicit function of volume.)*

2. The Explicit Fast Ladder

Solve the equation $x^2 + y^2 = L^2$ for $y = \sqrt{L^2 - x^2}$. Use the fact that $x = x[t]$ is a function of time together with the Chain Rule to show that

$$y'[t] = \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{-x[t]}{\sqrt{L^2 - (x[t])^2}} \cdot x'[t]$$

Verify that this model predicts that as x approaches L, the speed of the tip resting on the wall tends to infinity.

The result of the previous exercise is wrong for a real ladder. The tip of such a ladder cannot go faster than the speed of light. There is a physical condition that this simple mathematical model does not take into account. The real falling ladder is explored in the Scientific Projects Chapter on Mechanics. It uses Galileo's Law of Gravity from Chapter 9.

3. Related Rates Drill

(a) Each edge of a cube is expanding at the rate of 3 inches per second. How fast is the volume expanding at the point where the volume equals 64 cubic inches?

(b) Each edge of a cube is expanding at the rate of 3 inches per second. How fast is the surface area expanding at the point where the volume equals 64 cubic inches?

(c) A 6-foot man walks away from an 8-foot lamp at the rate of 5 feet per second. How fast is his shadow growing at the point where he is 7 feet from the lamp?

(d) A snowball melts at a rate proportional to its surface area, that is, it loses volume per unit time in this proportion. Say the constant of proportionality is k. What is $\frac{dr}{dt}$, the rate of change of radius with respect to time?

(e) Your snow-cone has melted filling the conical holder with sticky liquid. The liquid is dripping from the point at the bottom at the rate of 1 liter per hour. If the cone is 7 centimeters high and 5 centimers in diameter at the top, what is the rate of change of the height of the liquid when half of it has leaked out? (Note: The volume of a cone of height h and base radius r is $V = \frac{\pi}{3} r^2 h$.)

7.6 Projects

7.6.1 Dad's Disaster

Dad is painting the garage when Pooch gets her leash tangled around the bottom of the ladder and starts pulling it away from the wall. The Fast Ladder example suggests that Dad will break the sound barrier before he crashes to the sidewalk. Is this so? The Falling Ladder Project helps you find out.

7.6.2 The Inverse Function Rule

The project on finding the derivative an inverse function such as ArcTan[y] and computing values of the inverse itself. This project relies on basic graphical understanding and the microscope idea.