

Department of Mathematics

University of Iowa

Iowa City, IA 52242

118 MLH

7:00 pm, May 13, 1999

**22M:072**

ELEMENTARY NUMERICAL ANALYSIS

FINAL EXAM

Name: \_\_\_\_\_

Score: \_\_\_\_\_

1. (5%) How to avoid the loss of significance error in computing  $f(x) = (e^{x^2} - 1)/(4x)$  for  $x$  close to 0?

2. (15%) What is the order of convergence of the iteration  $x_{n+1} = \frac{x_n(x_n^2 + 3a)}{3x_n^2 + a}$  as it converges to the fixed point  $\alpha = \sqrt{a}$ ?

3. (10%) Consider the function  $\sin x$  on  $[-\pi, \pi]$  and its approximation by an interpolating polynomial. For  $n \geq 1$ , let  $h = 2\pi/n$ ,  $x_j = -\pi + jh$ ,  $j = 0, 1, \dots, n$ , and let  $p_n(x)$  be the  $n$ th-degree polynomial interpolating  $\sin x$  at the nodes  $x_0, \dots, x_n$ . Prove that

$$\max_{-\pi \leq x \leq \pi} |\sin x - p_n(x)| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

4. (20%) Consider the linear system  $A\mathbf{x} = \mathbf{b}$  with  $A = \begin{pmatrix} 1 & 1.1 \\ 0.9 & 1 \end{pmatrix}$ . With  $\mathbf{b} = (1, 1)^T$ , the solution is  $\mathbf{x} = (-10, 10)^T$ . If we perturb  $\mathbf{b}$  to  $\hat{\mathbf{b}} = (1.1, 1)^T$ , then the solution becomes  $\hat{\mathbf{x}} = (0, 1)^T$ . Explain why a small change in  $\mathbf{b}$  causes a large change in  $\mathbf{x}$ .

5. (30%) Consider the system

$$\begin{pmatrix} 4 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

(1) Show that for any right side vector  $\mathbf{b} = (b_1, b_2, b_3)^T$ , the system has a unique solution  $\mathbf{x} = (x_1, x_2, x_3)^T$ .

(2) Apply the Jacobi and Gauss-Seidel methods to solve the system. Write down the iteration formulas.

(3) Do the Jacobi and Gauss-Seidel methods converge for the given system? Why?

6. (20%) Consider the initial value problem

$$Y'(x) = Y(x), \quad Y(0) = 1.$$

(1) What is the backward Euler's method for solving the problem?

(2) Denote the backward Euler solution at an arbitrary node  $x_n$  by  $y_h(x_n)$ , with  $h$  the step-size. Show that for fixed  $x = x_n$ , there holds the convergence  $\lim_{h \rightarrow 0^+} y_h(x) = Y(x)$ .