Department of Mathematics University of Iowa Iowa City, IA 52242

110 MLH 12:00 pm--2:00 pm, May 13, 2009

22M:171, Final Exam

Name:________________ Score: ________

1. (5%) Let A be a square matrix of order n, and let U and V be unitary of the same order. Use the definition of the matrix $\|\cdot\|_2$-norm to show that $\|UAV\|_2 = \|A\|_2$.

$$\|UAV\|_2^2 = \sup_{x\neq 0} \frac{\|UAVx\|_2^2}{\|x\|_2^2} = \sup_{x\neq 0} \frac{x^H V^H A^H A V x}{\|x\|_2^2} = \sup_{y\neq 0} \frac{\|Ay\|_2^2}{\|y\|_2^2} = \|A\|_2^2$$

2. (15%) Consider two real systems

$$Q\boldsymbol{x} = \boldsymbol{b}, \quad (Q + E)\,\hat{\boldsymbol{x}} = \boldsymbol{b},$$

where Q is an orthogonal matrix. Assume $\|E\|_2 < 1$. Prove the error bound

$$\frac{\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2} \leq \frac{\|E\|_2}{1 - \|E\|_2}.$$

$$x - \hat{x} = -E\hat{x}$$

$$\hat{x} = (I + Q^{-1}E)^{-1} Q^{-1} b$$

$$\|x\|_2 = \|b\|_2$$

$$\therefore \frac{\|\hat{x} - x\|_2}{\|x\|_2} = \frac{\|E\hat{x}\|_2}{\|b\|_2} \leq \|E\|_2 \cdot \|(I + Q^{-1}E)^{-1}\|_2 \leq \frac{\|E\|_2}{1 - \|E\|_2}.$$

3. (10%) Compute the LU factorization of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 4 & 2 \\ 0 & 6 & 9 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & & \\ 2 & 1 & \\ 0 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ & 2 & 2 \\ & & 3 \end{pmatrix}$$

4. (15%) Let A be a symmetric, positive definite matrix. Recall that the CG method for solving $A\boldsymbol{x} = \boldsymbol{b}$ is the following:

Initialization. Given $\boldsymbol{x}_0$, compute $\boldsymbol{r}_0 = A\boldsymbol{x}_0 - \boldsymbol{b}$ and set $\boldsymbol{p}_0 = -\boldsymbol{r}_0$ and $k = 0$.

Iteration. For $k \geq 0$, while $\boldsymbol{r}_k \neq \boldsymbol{0}$, compute $\alpha_k = \boldsymbol{r}_k^T\boldsymbol{r}_k/\boldsymbol{p}_k^T A\boldsymbol{p}_k$, $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \alpha_k\boldsymbol{p}_k$,

$$\boldsymbol{r}_{k+1} = \boldsymbol{r}_k + \alpha_k A\boldsymbol{p}_k,\ \beta_{k+1} = \boldsymbol{r}_{k+1}^T\boldsymbol{r}_{k+1}/\boldsymbol{r}_k^T\boldsymbol{r}_k,\ \boldsymbol{p}_{k+1} = -\boldsymbol{r}_{k+1} + \beta_{k+1}\boldsymbol{p}_k.$$

Suppose for some nonsingular matrix C, the condition number of $C^{-T}AC^{-1}$ is much smaller than that of A. It then makes sense to apply the CG method to the system $(C^{-T}AC^{-1})(C\boldsymbol{x}) = C^{-T}\boldsymbol{b}$. Derive formulas for such a preconditioned CG method, with the requirement that the only additional operations are solving systems of the form $M\boldsymbol{y} = \boldsymbol{g}$ where $M = C^TC$.

Initialization. Given x_0, Compute $r_0 = Ax_0 - b$, solve $My_0 = r_0$, $p_0 = -y_0$, $k=0$.

Iteration. For $k \geq 0$, if $r_k \neq 0$, then

$$\alpha_k = \frac{r_k^T y_k}{p_k^T A p_k}, \quad x_{k+1} = x_k + \alpha_k p_k,$$

$$r_{k+1} = r_k + \alpha_k A p_k, \quad \text{solve } My_{k+1} = r_{k+1},$$

$$\beta_{k+1} = \frac{r_{k+1}^T y_{k+1}}{r_k^T y_k}, \quad p_{k+1} = -y_{k+1} + \beta_{k+1} p_k.$$

5. (20%) Consider the system

$$\begin{pmatrix} 4 & -2 & 0 \\ -1 & 4 & -1 \\ 0 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

(1) Show that for any right side, the system has a unique solution.
(2) Apply the Jacobi and GS methods to solve the system. Write down the iteration formulas.
(3) Can you determine if the Jacobi and GS methods converge for the given system? Explain.

(1) A is sym. pos. def.

(2) Initial guess, then iteration formulas.

(3) Converge, since A is diagonally dominant.

6. (15%) Apply the Sturm theory (Theorem 9.5 of the textbook on eigenvalues of a real symmetric tri-diagonal matrix) to separate the eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

$f_0(\lambda) = 1, \quad f_1(\lambda) = 1-\lambda, \quad f_2(\lambda) = \lambda^2 - 2\lambda, \quad f_3(\lambda) = (2-\lambda) f_2(\lambda) - f_1(\lambda)$

By Gerschgorin, $|\lambda - 1| \le 1, \ |\lambda - 1| \le 2, \ |\lambda - 2| \le 1. \Rightarrow -1 \le \lambda \le 3$

λ	f_0	f_1	f_2	f_3	s	Comment
-1	+	+	+	+	3	$-1 < \lambda_3$
3	+	−	+	−	0	$\lambda_1 < 3$
0	+	+	0(−)	−	2	$-1 < \lambda_3 < 0$
1	+	0(−)	−	−	2	no eigenvalue in [0,1]
2	+	−	0(+)	+	1	$1 < \lambda_2 < 2, \ 2 < \lambda_1 < 3$

$s(\lambda)$	3	2	2	1	0
λ	-1	0	1	2	3

7. (20%) Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

(a) Find the eigenvalues.

(b) Let us choose $\boldsymbol{z}^{(0)} = (1, -1)^T$ as an initial guess and apply the power method. Find the sequences $\boldsymbol{w}^{(m)}$, $\boldsymbol{z}^{(m)}$ and $\lambda^{(m)} = w_1^{(m)}/z_1^{(m-1)}$. Does $\{\lambda^{(m)}\}$ converge to the dominant eigenvalue? Explain if this example contradicts the convergence theory of the power method presented in Section 9.2.

(a) $\begin{vmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{vmatrix} = (\lambda-2)^2 - 1 = 0, \quad \lambda - 2 = \pm 1, \quad \lambda_1 = 3, \ \lambda_2 = 1.$

(b) $A z^{(0)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. So $A^m z^{(0)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$,

$w^{(m)} = z^{(m)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\lambda^{(m)} = 1$ not the dominant eigenvalue.

$z^{(0)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector for $\lambda = 1$.