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On a family of discontinuous Galerkin fully-discrete schemes for the wave equation

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Abstract

In this paper, we study a family of discontinuous Galerkin (DG) fully discrete schemes for solving the second-order wave equation. The spatial variable discretization is based on an application of the DG method. The temporal variable discretization depends on a parameter $\theta \in [0, 1]$. Under suitable regularity hypotheses on the solution, optimal order error bounds are shown for the numerical schemes with $\theta \in [\frac{1}{2}, 1]$, unconditionally with respect to the spatial mesh-size and the time-step, and for the numerical schemes with $\theta \in [0, \frac{1}{2})$ where a Courant–Friedrichs–Lewy stability condition is satisfied relating the mesh-size and the time-step. The optimal order error estimates are derived for $H^1(\Omega)$ and $L^2(\Omega)$ norms. Simulation results are reported to provide numerical evidence of the optimal convergence orders predicted by the theory.

Keywords Wave equation · Discontinuous Galerkin methods · Fully discrete approximation · Optimal order error estimates

Mathematics Subject Classification 65N30 · 49J40

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1 Introduction

The wave equation plays an important role in a variety of applications, e.g., acoustic, elastic, electromagnetic, seismic waves, and so forth. As such, the numerical solution of the wave equation has attracted steady interest in the research community. For examples, one is referred to Britt et al. (2018), Gustafsson and Mossberg (2004), Kreiss et al. (2002) on finite difference methods for the wave equation; Baker (1976), French and Peterson (1996) on finite element methods; Cowsar et al. (1990), Bécache et al. (2000) on mixed finite element methods; Diaz and Grote (2009), Walkington (2014) on explicit and implicit time-stepping schemes; Adjerid and Temimi (2011) on a discontinuous Galerkin (DG) time-stepping scheme for the temporal discretization and classical finite elements for the spatial discretization.

In the past four decades, DG methods have been applied in solving a large number of problems (Arnold et al. 2002, Cockburn et al. 2000), for instance, convection–diffusion equations (Castillo et al. 2002), hyperbolic equations (Grote et al. 2006, Grote and Schötzau 2009), Navier–Stokes equations (Bassi and Rebay 1997, Cockburn et al. 2005), Hamilton–Jacobi equations (Hu and Shu 1999, Kornhuber et al. 2000), radiative transfer equation (Han et al. 2010, Gao and Zhao 2013), variational inequalities (Wang et al. 2010, 2011, 2014, 2019), and so on. Since approximation functions are less smooth or discontinuous across the element boundaries, DG methods can handle easily general meshes with hanging nodes and elements of different shapes, are capable of capturing various local features of the problems, are better suited for parallel implementation.

In Grote et al. (2006), a symmetric interior penalty DG method is applied to solve the wave equation, and optimal order error estimates are derived for the spatially semi-discrete scheme. In the sequel (Grote and Schötzau 2009), a fully discrete scheme for the wave equation is studied, and an optimal $L^2(\Omega)$ norm error estimate for the fully discrete solution is derived when a Courant–Friedrichs–Lewy (CFL) condition is satisfied. In Han et al. (2019), spatially semi-discrete schemes and fully discrete schemes are introduced and analyzed, and optimal order error estimates in the $H^1(\Omega)$ and the $L^2(\Omega)$ norms are derived without the need of a restrictive CFL constraint. In this paper, we extend the discussions of Han et al. (2019) by introducing and studying a family of DG fully discrete schemes for the scalar second-order wave equation; the schemes depend on a parameter $\theta \in [0, 1]$. The schemes studied in Grote et al. (2006), Grote and Schötzau (2009) correspond to the choice $\theta = 0$, whereas those studied in Han et al. (2019) correspond to $\theta = 1$. We show that under suitable regularity assumptions on the solution, the numerical schemes with $\theta \in [\frac{1}{2}, 1]$ have optimal convergence orders unconditionally with respect to the spatial mesh-size and the time-step, and the numerical schemes with $\theta \in [0, \frac{1}{2})$ converge with optimal orders if a CFL stability condition is satisfied on the mesh-size and the time-step. Optimal order error estimates are derived for the $H^1(\Omega)$-like norm and the $L^2(\Omega)$ norm.

An outline of this paper is as follows. In the next section, we present some preliminary materials on the continuous problem, notation, DG bilinear forms, and review error estimates for $L^2(\Omega)$-projection and Galerkin projection. In Sect. 3, we introduce a family of fully discrete schemes where the spatial discretization is carried out through one of four DG formulations, and the temporal discretization is done with a family of finite differences depending on a parameter $\theta \in [0, 1]$. With different θ values, we obtain the optimal order error bounds unconditionally or with a CFL stability condition in terms of the spatial mesh-size and the time-step. Our optimal order error estimates are derived for the fully discrete solutions in both $H^1(\Omega)$-like and $L^2(\Omega)$ norms. Finally in Sect. 4, we report simulation results for $\theta = 0.0, 0.25, 0.5, 0.75, 1.0$ on a model problem to show the numerical convergence orders; the numerical results match the theoretical error estimates.

2 Preliminaries

We start with a description of the initial-boundary value problem of the wave equation. To simplify the notation, our discussion is focused on the two-dimensional case. It is possible to extend all the discussion to the three-dimensional case. Let $\Omega \subset \mathbb{R}^2$ be an open bounded connected domain with a Lipschitz boundary Γ. For a non-negative integer m, we will use the notation $\|\cdot\|_m$ for the $H^m(\Omega)$ norm. When $m = 0$, $\|\cdot\|_0$ denotes the $L^2(\Omega)$ norm. For a given positive number T, $I = (0, T)$ will be the time interval of interest. Let there be given an external force density $f \in L^2(I; L^2(\Omega))$, an initial displacement $u_0 \in H_0^1(\Omega)$, and an initial velocity $v_0 \in L^2(\Omega)$. As in Grote et al. (2006), Grote and Schötzau (2009) and Han et al. (2019), we consider the following initial-boundary value problem of the scalar wave equation: Find $u(\boldsymbol{x}, t)$ such that

$$\partial_t^2 u - \nabla\cdot(b\,\nabla u) = f \quad \text{in } \Omega \times I, \tag{2.1}$$

$$u = 0 \quad \text{on } \Gamma \times I, \tag{2.2}$$

$$u|_{t=0} = u_0 \quad \text{in } \Omega, \tag{2.3}$$

$$\partial_t u|_{t=0} = v_0 \quad \text{in } \Omega. \tag{2.4}$$

Here, u represents the unknown displacement function, $\partial_t u$ and $\partial_t^2 u$ are its first and second order time derivative, respectively. Throughout the paper, we assume b is a given smooth function and for two positive constants $b_{\min}$ and $b_{\max}$,

$$b_{\min} \le b(\boldsymbol{x}) \le b_{\max}, \quad \boldsymbol{x} \in \overline{\Omega}. \tag{2.5}$$

The standard weak formulation of the problem (2.1)–(2.4) is as follows.

Problem 2.1 *Find $u \in L^2(I; H_0^1(\Omega))$ with $\partial_t u \in L^2(I; L^2(\Omega))$ and $\partial_t^2 u \in L^2(I; H^{-1}(\Omega))$ such that*

$$\langle \partial_t^2 u, v\rangle + a(u, v) = (f, v) \quad \forall\, v \in H_0^1(\Omega),\ \text{a.e. in } I, \tag{2.6}$$

and

$$u|_{t=0} = u_0,\ \partial_t u|_{t=0} = v_0 \quad \text{a.e. in } \Omega. \tag{2.7}$$

In Problem 2.1, the time derivatives are understood in the distributional sense, $\langle\cdot,\cdot\rangle$ is the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$, and

$$a(u, v) = \int_\Omega b\,\nabla u\cdot\nabla v\,\mathrm{d}\boldsymbol{x}, \quad u, v \in H_0^1(\Omega). \tag{2.8}$$

It is known, cf. e.g. [Lions and Magenes (1972), Chapter 3(8)], that Problem 2.1 has a unique solution and moreover, $u \in C(\overline{I}; H_0^1(\Omega))$ and $\partial_t u \in C(\overline{I}; L^2(\Omega))$.

To prepare for the presentation of the fully discrete schemes for solving Problem 2.1, we introduce some notation. Assume Ω is a convex polygon as in Grote and Schötzau (2009) and Han et al. (2019). Let $\{\mathcal{T}_h\}_h$ be a regular family of quasi-uniform finite element triangulations of $\overline{\Omega}$. Corresponding to a finite element mesh $\mathcal{T}_h$ in the family, denote by K a generic element, by $h_K = \mathrm{diam}(K)$ the diameter of K, and by $h = \max\{h_K : K \in \mathcal{T}_h\}$ the finite element mesh-size. Let $\mathcal{E}_h$ be the collection of all the edges of $\mathcal{T}_h$, $\mathcal{E}_h^i$ the set of all interior edges, and $\mathcal{E}_h^\partial = \mathcal{E}_h \backslash \mathcal{E}_h^i$ the set of all the edges on the boundary Γ. For a function v differentiable on each element of the mesh $\mathcal{T}_h$, we define the broken gradient operator ∇_h piecewise by the relation $\nabla_h v = \nabla v$ on any element $K \in \mathcal{T}_h$.

Let K_1 and K_2 be two neighboring elements sharing a common edge e. Denote by $\boldsymbol{n}_1 = \boldsymbol{n}|_{\partial K_1}$ and $\boldsymbol{n}_2 = \boldsymbol{n}|_{\partial K_2}$ the unit outward normal vectors on $\partial K_1 \cap e$ and $\partial K_2 \cap e$. For a piecewise smooth scalar-valued function v, let $v_1 = v|_{\partial K_1}$, $v_2 = v|_{\partial K_2}$ and define the average $\{v\}$ and the jump $[\![v]\!]$ on $\mathcal{E}_h^i$ as follows:

$$\{v\} = \frac{1}{2}(v_1 + v_2), \quad [\![v]\!] = v_1\boldsymbol{n}_1 + v_2\boldsymbol{n}_2 \quad \text{on } e \in \mathcal{E}_h^i.$$

For a piecewise smooth vector-valued function $\boldsymbol{w}$, denote $\boldsymbol{w}_1 = \boldsymbol{w}|_{\partial K_1}$, $\boldsymbol{w}_2 = \boldsymbol{w}|_{\partial K_2}$. Define the average $\{\boldsymbol{w}\}$ and the jump $[\boldsymbol{w}]$ of $\boldsymbol{w}$ on $\mathcal{E}_h^i$ as follows:

$$\{\boldsymbol{w}\} = \frac{1}{2}(\boldsymbol{w}_1 + \boldsymbol{w}_2), \quad [\boldsymbol{w}] = \boldsymbol{w}_1 \cdot \boldsymbol{n}_1 + \boldsymbol{w}_2 \cdot \boldsymbol{n}_2 \quad \text{on } e \in \mathcal{E}_h^i.$$

On a boundary edge, define

$$[\![v]\!] = v\boldsymbol{n}, \quad \{\boldsymbol{w}\} = \boldsymbol{w} \quad \text{on } e \in \mathcal{E}_h^\partial,$$

where $\boldsymbol{n}$ is the unit outward normal vector on Γ. For piecewise smooth functions v and $\boldsymbol{w}$, the following identity holds (Arnold et al. 2002):

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} v\boldsymbol{w} \cdot \boldsymbol{n}_K \, \mathrm{d}s = \int_{\mathcal{E}_h} [\![v]\!] \cdot \{\boldsymbol{w}\} \, \mathrm{d}s + \int_{\mathcal{E}_h^i} \{v\}[\boldsymbol{w}] \, \mathrm{d}s.$$

Let $p \geq 1$ be a positive integer, to be used as the local polynomial degree of the DG formulations. We introduce the following discontinuous finite element spaces:

$$\begin{aligned} V^h &= \{v^h \in L^2(\Omega) : \; v^h|_K \in P_p(K) \; \forall \; K \in \mathcal{T}_h\}, \\ \boldsymbol{W}^h &= \{\boldsymbol{w}^h \in [L^2(\Omega)]^2 : \; \boldsymbol{w}^h|_K \in [P_p(K)]^2 \; \forall \; K \in \mathcal{T}_h\}. \end{aligned}$$

For some of the DG formulations, we will need lifting operators $r : [L^2(\mathcal{E}_h)]^2 \to \boldsymbol{W}^h$ and $r_e : [L^2(e)]^2 \to \boldsymbol{W}^h$ defined by the relations (Arnold et al. 2002)

$$\begin{aligned} \int_\Omega r(\boldsymbol{q}) \cdot \boldsymbol{w}^h \mathrm{d}\boldsymbol{x} &= -\int_{\mathcal{E}_h} \boldsymbol{q} \cdot \{\boldsymbol{w}^h\} \, \mathrm{d}s, \\ \int_\Omega r_e(\boldsymbol{q}) \cdot \boldsymbol{w}^h \mathrm{d}\boldsymbol{x} &= -\int_e \boldsymbol{q} \cdot \{\boldsymbol{w}^h\} \, \mathrm{d}s \end{aligned}$$

for all $\boldsymbol{w}^h \in \boldsymbol{W}^h$.

We are now ready to introduce four choices of the DG bilinear form $a_h = a_h^{(j)}$, $1 \leq j \leq 4$, for the approximation of the bilinear form (2.8). The first DG bilinear form is related to the interior penalty (IP) method (Douglas and Dupont 1976, Arnold 1982), which is also the bilinear form of the DG method used in Grote and Schötzau (2009):

$$\begin{aligned} a_h^{(1)}(u, v) = & \int_\Omega b \, \nabla_h u \cdot \nabla_h v \, \mathrm{d}\boldsymbol{x} - \int_{\mathcal{E}_h} [\![u]\!] \cdot \{b \, \nabla_h v\} \, \mathrm{d}s \\ & - \int_{\mathcal{E}_h} \{b \, \nabla_h u\} \cdot [\![v]\!] \, ds + \int_{\mathcal{E}_h} b \, \eta [\![u]\!] \cdot [\![v]\!] \, ds, \end{aligned}$$

where the penalty weighting function $\eta : \mathcal{E}_h \to \mathbb{R}$ is given by $\eta_e h_e^{-1}$ on each edge $e \in \mathcal{E}_h$, η_e being a positive number.

The second DG bilinear form is (cf. Bassi et al. (1997))

$$a_h^{(2)}(u,v) = \int_\Omega b\,\nabla_h u \cdot \nabla_h v\,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{E}_h} [\![u]\!] \cdot \{b\,\nabla_h v\}\,\mathrm{d}s - \int_{\mathcal{E}_h} \{b\,\nabla_h u\} \cdot [\![v]\!]\,\mathrm{d}s + \sum_{e\in\mathcal{E}_h} \int_\Omega b\,\eta_e r_e([\![u]\!]) \cdot r_e([\![v]\!])\,\mathrm{d}\boldsymbol{x}.$$

The third DG bilinear form is (cf. Brezzi et al. (1997))

$$a_h^{(3)}(u,v) = \int_\Omega b\,\nabla_h u \cdot \nabla_h v\,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{E}_h} [\![u]\!] \cdot \{b\,\nabla_h v\}\,\mathrm{d}s - \int_{\mathcal{E}_h} \{b\,\nabla_h u\} \cdot [\![v]\!]\,\mathrm{d}s + \int_\Omega b\,r([\![u]\!]) \cdot r([\![v]\!])\,\mathrm{d}\boldsymbol{x} + \sum_{e\in\mathcal{E}_h} \int_\Omega b\,\eta_e r_e([\![u]\!]) \cdot r_e([\![v]\!])\,\mathrm{d}\boldsymbol{x}.$$

The fourth DG bilinear form is related to the simplified local DG (LDG) method in Cockburn and Shu (1998),

$$a_h^{(4)}(u,v) = \int_\Omega b\,\nabla_h u \cdot \nabla_h v\,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{E}_h} [\![u]\!] \cdot \{b\,\nabla_h v\}\,\mathrm{d}s - \int_{\mathcal{E}_h} \{b\,\nabla_h u\} \cdot [\![v]\!]\,\mathrm{d}s + \int_\Omega b\,r([\![u]\!]) \cdot r([\![v]\!])\mathrm{d}\boldsymbol{x} + \int_{\mathcal{E}_h} b\,\eta [\![u]\!] \cdot [\![v]\!]\,\mathrm{d}s.$$

For all the four DG bilinear forms, we have consistency, boundedness, and stability (Arnold et al. 2002, Wang et al. 2010, Han et al. 2019). To state these results, it is convenient to introduce the space $V(h) = V^h + H^2(\Omega) \cap H_0^1(\Omega)$ with a norm $\|\cdot\|_h$ defined by the relation

$$\|v\|_h^2 = \sum_{K\in\mathcal{T}_h} |v|_{1,K}^2 + \sum_{K\in\mathcal{T}_h} h_K^2 |v|_{2,K}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1} \|[\![v]\!]\|_{0,e}^2, \quad v \in V(h). \tag{2.9}$$

Lemma 2.2 (Consistency) *Assume the solution of Problem 2.1 has the regularity property $u \in L^2(0,T;H^2(\Omega))$. Then for $a_h = a_h^{(j)}$, $1 \le j \le 4$, we have for a.e. $t \in [0,T]$,*

$$(\partial_t^2 u, v^h) + a_h(u, v^h) = (f, v^h) \quad \forall\, v^h \in V^h. \tag{2.10}$$

Lemma 2.3 (Boundedness) *There exists a constant c such that for $a_h = a_h^{(j)}$, $1 \le j \le 4$,*

$$|a_h(u,v)| \le c\,\|u\|_h \|v\|_h \quad \forall\, u, v \in V(h).$$

Lemma 2.4 (Stability) *Let $\eta_0 = \inf_e \eta_e$ be sufficiently large for $j = 1, 2$, and $\eta_0 > 0$ for $j = 3, 4$. Then there exists a constant c such that for $a_h = a_h^{(j)}$, $1 \le j \le 4$,*

$$a_h(v,v) \ge c\,\|v\|_h^2 \quad \forall\, v \in V^h.$$

In the rest of the paper, we will assume the conditions stated in Lemma 2.4 are satisfied. Because of the boundedness and stability of the bilinear form $a_h(u,v)$, it makes sense to use the notation

$$\|v^h\|_{a_h}^2 = a_h(v^h, v^h), \quad v^h \in V^h;$$

this defines a norm that is equivalent to the norm $\|v^h\|_h$. In addition, we notice that

$$\|w\|_h \le c\,\|w\|_2 \quad \forall\, w \in H^2(\Omega). \tag{2.11}$$

We will need the L^2-projection and Galerkin projection in defining and/or analyzing the DG methods. The $L^2(\Omega)$-projection $P^h : L^2(\Omega) \to V^h$ is defined by

$$P^h w \in V^h, \quad (w - P^h w, v^h) = 0 \quad \forall\, v^h \in V^h$$

for $w \in L^2(\Omega)$. The following error bounds hold ([Grote et al. (2006), Lemmas 4.6 and 4.7]):

$$\|w - P^h w\|_0 \le c\, h^{\min\{p+1,m\}} \|w\|_m \quad \forall\, w \in H^m(\Omega)\ (m \ge 0), \tag{2.12}$$

$$\|w - P^h w\|_h \le c\, h^{\min\{p,m-1\}} \|w\|_m \quad \forall\, w \in H^m(\Omega)\ (m \ge 2). \tag{2.13}$$

Due to the equivalence of the two norms $\|\cdot\|_{a_h}$ and $\|\cdot\|_h$, we deduce from (2.13) that

$$\|w - P^h w\|_{a_h} \le c\, h^{\min\{p,m-1\}} \|w\|_m \quad \forall\, w \in H^m(\Omega). \tag{2.14}$$

The Galerkin projection $\Pi^h : V \to V^h$ is defined by

$$\Pi^h w \in V^h, \quad a_h(w - \Pi^h w, v^h) = 0 \quad \forall\, v^h \in V^h$$

for $w \in V$. The following error bounds hold ([Grote and Schötzau (2009), Lemma 4.1]):

$$\|w - \Pi^h w\|_0 \le c\, h^{\min\{p+1,m\}} \|w\|_m \quad \forall\, w \in H^m(\Omega)\ (m \ge 1), \tag{2.15}$$

$$\|w - \Pi^h w\|_h \le c\, h^{\min\{p,m-1\}} \|w\|_m \quad \forall\, w \in H^m(\Omega)\ (m \ge 2). \tag{2.16}$$

The convexity assumption of the domain Ω is used in proving (2.15). We deduce from (2.16) that

$$\|w - \Pi^h w\|_{a_h} \le c\, h^{\min\{p,m-1\}} \|w\|_m \quad \forall\, w \in H^m(\Omega). \tag{2.17}$$

If $u \in C^l(\overline{I}; H^m(\Omega))$, $l \ge 0$, then ([Grote and Schötzau (2009), Lemma 4.2])

$$\|\partial_t^l (u(t) - \Pi^h u(t))\|_0 \le c\, h^{\min\{p+1,m\}} \|\partial_t^l u(t)\|_m, \quad t \in \overline{I}. \tag{2.18}$$

Similarly,

$$\|\partial_t^l (u(t) - \Pi^h u(t))\|_{a_h} \le c\, h^{\min\{p,m-1\}} \|\partial_t^l u(t)\|_m, \quad t \in \overline{I}. \tag{2.19}$$

In deriving error bounds, we will make use of the following form of the discrete Gronwall inequality, which is a special case of [Han and Sofonea (2002) Lemma 7.26].

Lemma 2.5 *Let $T > 0$ be fixed. For any positive integer N, define $k = T/N$. Assume $\{g_n\}_{n=1}^N$ and $\{e_n\}_{n=1}^N$ are two sequences of non-negative numbers satisfying*

$$e_n \le c\, g_n + c\, k \sum_{j=1}^{n-1} e_j, \quad n = 1, 2, \ldots, N.$$

Then for some constant c,

$$\max_{1 \le n \le N} e_n \le c \max_{1 \le n \le N} g_n.$$

The following elementary relation will be used repeatedly (cf. e.g., [Han and Reddy (2013) Ch. 11]):

$$a,\ b,\ x \ge 0 \quad \text{and} \quad x^2 \le a\, x + b \Longrightarrow x^2 \le a^2 + 2\, b. \tag{2.20}$$

3 A family of fully discrete schemes

Spatially semi-discrete schemes for Problem 2.1, with DG formulations for the spatial discretization, are discussed in Han et al. (2019) where optimal order error estimates are derived. In this section, we introduce and study a family of fully discrete schemes. For simplicity in notation, we give the discussion only for the case of evenly spaced nodes so that the interval I is partitioned uniformly into subintervals of equal length. Thus, for a positive integer N, let $k = T/N$ be the step size, and denote $t_n = n\,k$ for $0 \le n \le N$, $I_n = (t_n, t_{n+1})$ for $n = 0, 1, \cdots, N-1$. For a function u continuous in t, we write $u_n = u(\cdot, t_n)$. To simplify the writing, define operators γ_k, δ_k and d_k as follows:

$$\gamma_k u_n = \frac{u_{n+1} + u_{n-1}}{2},$$
$$\delta_k u_n = \frac{u_{n+1} - u_{n-1}}{2k},$$
$$d_k u_n = \frac{u_{n+1} - 2u_n + u_{n-1}}{k^2}.$$

Let $a_h(\cdot,\cdot)$ be one of the bilinear forms $a_h^{(j)}(\cdot,\cdot)$ with $j = 1, \cdots, 4$, and let $\theta \in [0, 1]$ be a parameter. Let

$$f \in C(\overline{I}; L^2(\Omega));$$

this condition holds true under the solution regularity assumptions to be made in the statements of error bounds below. Then we consider the following fully discrete θ-scheme of Problem 2.1; recall that P^h is the $L^2(\Omega)$-projection onto the space V^h.

Problem 3.1 *Find* $\{u_n^{hk}\}_{n=0}^N \subset V^h$ *such that for* $1 \le n \le N-1$,

$$(d_k u_n^{hk}, v^h) + a_h(\theta\, \gamma_k u_n^{hk} + (1-\theta)\, u_n^{hk}, v^h) = (f_n, v^h) \quad \forall\, v^h \in V^h, \tag{3.1}$$

and

$$u_0^{hk} = P^h u_0, \tag{3.2}$$
$$u_1^{hk} = u_0^{hk} + k\, P^h v_0 + \frac{k^2}{2}\, \tilde{u}_0^h, \tag{3.3}$$

where

$$\tilde{u}_0^h \in V^h, \quad (\tilde{u}_0^h, v^h) = (f_0, v^h) - a_h(u_0, v^h) \quad \forall\, v^h \in V^h. \tag{3.4}$$

The goal of the rest of the section is to present optimal order error estimates for the numerical solution of the fully discrete θ-scheme. We start with error estimates in an $H^1(\Omega)$-like norm. Recall from [Grote and Schötzau (2009), Lemma 3.3] that there is a constant $c_b > 0$ such that

$$a_h(v, v) \le c_b\, b_{max}\, h^{-2} \|v\|_0^2 \quad \forall\, v \in V^h. \tag{3.5}$$

Theorem 3.2 *Let* u *and* u^{hk} *be the solutions of Problem 2.1 and Problem 3.1, respectively. Assume* $u \in C^2(\overline{I}; H^{p+1}(\Omega))$, $\partial_t^3 u \in C(\overline{I}; L^2(\Omega)) \cap L^2(I; H^2(\Omega))$, $\partial_t^4 u \in L^2(I; L^2(\Omega))$. *Then for* $\theta \in (1/2, 1]$, *we have the error bound*

$$\max_{0\le n\le N-1} k^{-1} \|(u_{n+1} - u_{n+1}^{hk}) - (u_n - u_n^{hk})\|_0 + \max_{0\le n\le N} \|u_n - u_n^{hk}\|_h \le c\left(h^p + k^2\right) \tag{3.6}$$

for some positive constant c depending on $\|u\|_{C^2(\bar{I};H^{p+1}(\Omega))}$, $\|\partial_t^3 u\|_{C(\bar{I};L^2(\Omega))}$, $\|\partial_t^3 u\|_{L^2(I;H^2(\Omega))}$, *and* $\|\partial_t^4 u\|_{L^2(I;L^2(\Omega))}$*. For* $\theta = 1/2$*, the error bound* (3.6) *is modified to*

$$\max_{0\le n\le N-1} k^{-1}\|(u_{n+1}-u_{n+1}^{hk})-(u_n-u_n^{hk})\|_0 + \max_{0\le n\le N-1}\|(u_{n+1}-u_{n+1}^{hk})+(u_n-u_n^{hk})\|_h \le c\left(h^p+k^2\right). \tag{3.7}$$

For $\theta \in [0, \frac{1}{2})$*, if the CFL stability condition*

$$k < \frac{\sqrt{2}\,h}{\sqrt{(1-\theta)\,c_b\,b_{max}}} \tag{3.8}$$

is satisfied, then the error estimate (3.6) *is valid.*

Proof As in Han et al. (2019), we write the error $e_n = u_n - u_n^{hk}$ at time t_n as

$$e_n = e_n^1 + e_n^2,$$

where

$$e_n^1 = u_n - \Pi^h u_n, \quad e_n^2 = \Pi^h u_n - u_n^{hk}.$$

For $0 \le n \le N-1$, we have

$$k^{-1}\|(u_{n+1}-u_{n+1}^{hk})-(u_n-u_n^{hk})\|_0 \le k^{-1}\|e_{n+1}^1-e_n^1\|_0 + k^{-1}\|e_{n+1}^2-e_n^2\|_0, \tag{3.9}$$

$$\|u_n-u_n^{hk}\|_h \le \|e_n^1\|_h + \|e_n^2\|_h, \tag{3.10}$$

where ([Han et al. (2019), (4.8), (4.9)])

$$k^{-1}\|e_{n+1}^1-e_n^1\|_0 \le c\,h^p\|\partial_t u\|_{C(\bar{I};H^p(\Omega))}, \tag{3.11}$$

$$\|e_n^1\|_h \le c\,h^p\|u\|_{C(\bar{I};H^{p+1}(\Omega))}. \tag{3.12}$$

To bound the second terms of the right-hand sides of (3.9) and (3.10), we start with a consideration of the expression

$$A_n = (d_k e_n^2, \delta_k e_n^2) + a_h(\theta\,\gamma_k e_n^2 + (1-\theta)\,e_n^2, \delta_k e_n^2).$$

By making use of the definitions of γ_k, δ_k, and d_k, we have

$$A_n = \frac{1}{2\,k^3}\left(\|e_{n+1}^2-e_n^2\|_0^2 - \|e_n^2-e_{n-1}^2\|_0^2\right) + \frac{\theta}{4\,k}\left(\|e_{n+1}^2\|_{a_h}^2 - \|e_{n-1}^2\|_{a_h}^2\right) + \frac{1-\theta}{2\,k}a_h(e_n^2, e_{n+1}^2-e_{n-1}^2). \tag{3.13}$$

Take $v^h = \delta_k e_n^2$ in (3.1) and (2.10), and subtract,

$$(\partial_t^2 u_n - d_k u_n^{hk}, \delta_k e_n^2) + a_h(u_n - \theta\,\gamma_k u_n^{hk} - (1-\theta)\,u_n^{hk}, \delta_k e_n^2) = 0.$$

Thus,

$$A_n = (d_k\Pi^h u_n - \partial_t^2 u_n, \delta_k e_n^2) + a_h(\theta\,\gamma_k\Pi^h u_n - u_n + (1-\theta)\,\Pi^h u_n, \delta_k e_n^2). \tag{3.14}$$

We combine (3.13) and (3.14), and multiply both sides by $2\,k$,

$$\frac{1}{k^2}\left(\|e_{n+1}^2-e_n^2\|_0^2 - \|e_n^2-e_{n-1}^2\|_0^2\right) + \frac{\theta}{2}\left(\|e_{n+1}^2\|_{a_h}^2 - \|e_{n-1}^2\|_{a_h}^2\right) + (1-\theta)\,a_h(e_n^2, e_{n+1}^2-e_{n-1}^2) = (\xi_n, e_{n+1}^2-e_{n-1}^2) + a_h(\eta_n, e_{n+1}^2-e_{n-1}^2), \tag{3.15}$$

where for $1 \le n \le N-1$,

$$\xi_n = d_k \Pi^h u_n - \partial_t^2 u_n, \tag{3.16}$$

$$\eta_n = \theta\, \gamma_k \Pi^h u_n - u_n + (1-\theta)\, \Pi^h u_n = \theta\, (\gamma_k \Pi^h u_n - u_n) + (1-\theta)\, (\Pi^h u_n - u_n). \tag{3.17}$$

Changing n to i in (3.15) and summing on the relation for $i = 1, \ldots, n-1$, we have

$$\frac{1}{k^2} \left(\|e_n^2 - e_{n-1}^2\|_0^2 - \|e_1^2 - e_0^2\|_0^2 \right) + \frac{\theta}{2} \left(\|e_n^2\|_{a_h}^2 + \|e_{n-1}^2\|_{a_h}^2 - \|e_1^2\|_{a_h}^2 - \|e_0^2\|_{a_h}^2 \right)$$
$$+ (1-\theta) \sum_{i=1}^{n-1} a_h(e_i^2, e_{i+1}^2 - e_{i-1}^2) = \sum_{i=1}^{n-1} (\xi_i, e_{i+1}^2 - e_{i-1}^2) + \sum_{i=1}^{n-1} a_h(\eta_i, e_{i+1}^2 - e_{i-1}^2). \tag{3.18}$$

Two of the sums in (3.18) can be handled as follows:

$$\sum_{i=1}^{n-1} a_h(e_i^2, e_{i+1}^2 - e_{i-1}^2) = a_h(e_{n-1}^2, e_n^2) - a_h(e_1^2, e_0^2), \tag{3.19}$$

$$\sum_{i=1}^{n-1} (\xi_i, e_{i+1}^2 - e_{i-1}^2) = (\xi_{n-1}, e_n^2 - e_{n-1}^2) + (\xi_1, e_1^2 - e_0^2) + \sum_{i=2}^{n-1} (\xi_{i-1} + \xi_i, e_i^2 - e_{i-1}^2), \tag{3.20}$$

while for the last sum,

$$\sum_{i=1}^{n-1} a_h(\eta_i, e_{i+1}^2 - e_{i-1}^2) = a_h(\eta_{n-1}, e_n^2) + a_h(\eta_{n-2}, e_{n-1}^2) - a_h(\eta_1, e_0^2)$$
$$- a_h(\eta_2, e_1^2) + \sum_{i=2}^{n-2} a_h(\eta_{i-1} - \eta_{i+1}, e_i^2) \tag{3.21}$$

or

$$\sum_{i=1}^{n-1} a_h(\eta_i, e_{i+1}^2 - e_{i-1}^2) = a_h(\eta_{n-1}, e_n^2 + e_{n-1}^2) - a_h(\eta_1, e_1^2 + e_0^2)$$
$$+ \sum_{i=2}^{n-1} a_h(\eta_{i-1} - \eta_i, e_i^2 + e_{i-1}^2). \tag{3.22}$$

We now distinguish three cases.

Case 1: $\frac{1}{2} < \theta \le 1$. By (3.18), (3.19)–(3.21),

$$\frac{1}{k^2} \left(\|e_n^2 - e_{n-1}^2\|_0^2 - \|e_1^2 - e_0^2\|_0^2 \right) + \frac{\theta}{2} \left(\|e_n^2\|_{a_h}^2 + \|e_{n-1}^2\|_{a_h}^2 - \|e_1^2\|_{a_h}^2 - \|e_0^2\|_{a_h}^2 \right)$$
$$= (1-\theta)\, a_h(e_1^2, e_0^2) - (1-\theta)\, a_h(e_{n-1}^2, e_n^2)$$
$$+ (\xi_{n-1}, e_n^2 - e_{n-1}^2) + (\xi_1, e_1^2 - e_0^2) + \sum_{i=2}^{n-1} (\xi_{i-1} + \xi_i, e_i^2 - e_{i-1}^2)$$
$$+ a_h(\eta_{n-1}, e_n^2) + a_h(\eta_{n-2}, e_{n-1}^2) - a_h(\eta_1, e_0^2) - a_h(\eta_2, e_1^2)$$
$$+ \sum_{i=2}^{n-2} a_h(\eta_{i-1} - \eta_{i+1}, e_i^2).$$

Denote

$$M_1 = \max_{0\le n\le N} \|e_n^2\|_{a_h}.$$

Then,

$$\begin{aligned}
&\frac{1}{k^2}\left(\|e_n^2 - e_{n-1}^2\|_0^2 - \|e_1^2 - e_0^2\|_0^2\right) + \frac{\theta}{2}\left(\|e_n^2\|_{a_h}^2 + \|e_{n-1}^2\|_{a_h}^2 - \|e_1^2\|_{a_h}^2 - \|e_0^2\|_{a_h}^2\right)\\
&\quad\le \frac{1-\theta}{2}\,(\|e_1^2\|_{a_h}^2 + \|e_0^2\|_{a_h}^2) + \frac{1-\theta}{2}\,(\|e_n^2\|_{a_h}^2 + \|e_{n-1}^2\|_{a_h}^2)\\
&\qquad + \frac{k^2}{2}\,\|\xi_{n-1}\|_0^2 + \frac{1}{2\,k^2}\|e_n^2 - e_{n-1}^2\|_0^2 + \frac{k^2}{2}\|\xi_1\|_0^2 + \frac{1}{2\,k^2}\|e_1^2 - e_0^2\|_0^2\\
&\qquad + \left(\sum_{i=2}^{n-1}\|\xi_{i-1} + \xi_i\|_0^2\right)^{1/2}\left(\sum_{i=2}^{n-1}\|e_i^2 - e_{i-1}^2\|_0^2\right)^{1/2}\\
&\qquad + \left(\|\eta_{n-1}\|_{a_h} + \|\eta_{n-2}\|_{a_h} + \|\eta_2\|_{a_h} + \|\eta_1\|_{a_h} + \sum_{i=2}^{n-2}\|\eta_{i-1} - \eta_{i+1}\|_{a_h}\right) M_1.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{2\,k^2}\,\|e_n^2 - e_{n-1}^2\|_0^2 + \frac{2\,\theta - 1}{2}\,(\|e_n^2\|_{a_h}^2 + \|e_{n-1}^2\|_{a_h}^2)\\
&\quad\le \frac{3}{2\,k^2}\,\|e_1^2 - e_0^2\|_0^2 + \frac{1}{2}\,(\|e_0^2\|_{a_h}^2 + \|e_1^2\|_{a_h}^2) + \frac{k^2}{2}\,(\|\xi_{n-1}\|_0^2 + \|\xi_1\|_0^2)\\
&\qquad + \frac{k}{2}\sum_{i=2}^{n-1}\|\xi_{i-1} + \xi_i\|_0^2 + k\sum_{i=2}^{n-1}\frac{1}{2\,k^2}\,\|e_i^2 - e_{i-1}^2\|_0^2\\
&\qquad + \left(\|\eta_{n-1}\|_{a_h} + \|\eta_{n-2}\|_{a_h} + \|\eta_2\|_{a_h} + \|\eta_1\|_{a_h} + \sum_{i=2}^{n-2}\|\eta_{i-1} - \eta_{i+1}\|_{a_h}\right) M_1.
\end{aligned} \tag{3.23}$$

Applying Lemma 2.5 to (3.23), we obtain

$$\begin{aligned}
&\frac{1}{k^2}\,\|e_n^2 - e_{n-1}^2\|_0^2 + (2\,\theta - 1)\,\|e_n^2\|_{a_h}^2\\
&\quad\le c\left(\|e_0^2\|_{a_h}^2 + \|e_1^2\|_{a_h}^2 + \frac{1}{k^2}\|e_1^2 - e_0^2\|_0^2 + k^2\max_n\|\xi_n\|_0^2 + k\sum_{i=2}^{N-1}\|\xi_{i-1} + \xi_i\|_0^2 + \varphi_\theta(\eta)\,M_1\right),
\end{aligned} \tag{3.24}$$

where

$$\varphi_\theta(\eta) = \|\eta_{N-1}\|_{a_h} + \|\eta_{N-2}\|_{a_h} + \|\eta_2\|_{a_h} + \|\eta_1\|_{a_h} + \sum_{i=2}^{N-2}\|\eta_{i-1} - \eta_{i+1}\|_{a_h}. \tag{3.25}$$

Then, we have

$$M_1^2 \le c\,g_1(h,k) \tag{3.26}$$

and

$$\frac{1}{k^2}\,\|e_n^2 - e_{n-1}^2\|_0^2 \le c\,g_1(h,k), \tag{3.27}$$

where

$$g_1(h,k) = \|e_0^2\|_{a_h}^2 + \|e_1^2\|_{a_h}^2 + \frac{1}{k^2}\|e_1^2 - e_0^2\|_0^2 + k^2 \max_n \|\xi_n\|_0^2$$
$$+k\sum_{i=2}^{N-1} \|\xi_{i-1}+\xi_i\|_0^2 + \varphi_\theta^2(\eta). \tag{3.28}$$

By (3.26) and (3.27), we obtain

$$\max_n \frac{1}{k^2}\ \|e_n^2 - e_{n-1}^2\|_0^2 + \max_n \|e_n^2\|_{a_h}^2 \le c\ g_1(h,k). \tag{3.29}$$

We proceed to bound $\varphi_\theta(\eta)$ defined in (3.25). First, by (3.17),

$$\eta_n = \theta\ \gamma_k(\Pi^h - I)u_n + \theta\ (\gamma_k u_n - u_n) + (1-\theta)\ (\Pi^h u_n - u_n).$$

By (2.17),

$$\|\gamma_k(\Pi^h - I)u_n\|_{a_h} \le \frac{1}{2}\left(\|(\Pi^h - I)u_{n+1}\|_{a_h} + \|(\Pi^h - I)u_{n-1}\|_{a_h}\right)$$
$$\le c\ h^p\left(\|u_{n+1}\|_{p+1} + \|u_{n-1}\|_{p+1}\right),$$

and by [Han et al. (2019), (4.37)],

$$\|\gamma_k u_n - u_n\|_{a_h} \le c\ k\int_{t_{n-1}}^{t_{n+1}} \|\partial_t^2 u(\cdot,s)\|_{a_h} \mathrm{d}s.$$

So

$$\|\eta_n\|_{a_h} \le \theta\ \|\gamma_k(\Pi^h - I)u_n\|_{a_h} + \theta\ \|\gamma_k u_n - u_n\|_{a_h} + (1-\theta)\ \|\Pi^h u_n - u_n\|_{a_h}$$
$$\le c\ h^p \|u\|_{C(\bar{I};H^{p+1}(\Omega))} + c\ k\int_{t_{n-1}}^{t_{n+1}} \|\partial_t^2 u(\cdot,s)\|_{a_h} \mathrm{d}s. \tag{3.30}$$

Note that

$$\eta_{i+1} - \eta_{i-1} = \theta\ \gamma_k(\Pi^h - I)(u_{i+1}-u_{i-1}) + \theta\ \left(\gamma_k(u_{i+1}-u_{i-1}) - (u_{i+1}-u_{i-1})\right)$$
$$+ (1-\theta)\ (\Pi^h - I)(u_{i+1}-u_{i-1}).$$

By [Han et al. (2019), (4.39)–(4.40)],

$$\|\gamma_k(\Pi^h - I)(u_{i+1}-u_{i-1})\|_{a_h} \le c\ h^p\int_{t_{i-2}}^{t_{i+2}} \|\partial_t u(\cdot,s)\|_{p+1} ds,$$

$$\|\gamma_k(u_{i+1}-u_{i-1}) - (u_{i+1}-u_{i-1})\|_{a_h} \le c\ k^2\int_{t_{i-2}}^{t_{i+2}} \|\partial_t^3 u(\cdot,\tau)\|_{a_h} d\tau,$$

and by (2.17),

$$\|(\Pi^h - I)(u_{i+1}-u_{i-1})\|_{a_h} \le c\ h^p\|u_{i+1}-u_{i-1}\|_{p+1} \le c\ h^p\int_{t_{i-1}}^{t_{i+1}} \|\partial_t u(\cdot,s)\|_{p+1} ds.$$

So

$$\|\eta_{i+1}-\eta_{i-1}\|_{a_h} \le \theta\ \|\gamma_k(\Pi^h - I)(u_{i+1}-u_{i-1})\|_{a_h} + \theta\ \|\gamma_k(u_{i+1}-u_{i-1}) - (u_{i+1}-u_{i-1})\|_{a_h}$$
$$+ (1-\theta)\|(\Pi^h - I)(u_{i+1}-u_{i-1})\|_{a_h}$$
$$\le c\ h^p\int_{t_{i-2}}^{t_{i+2}} \|\partial_t u(\cdot,s)\|_{p+1} ds + c\ k^2\int_{t_{i-2}}^{t_{i+2}} \|\partial_t^3 u(\cdot,\tau)\|_{a_h} d\tau. \tag{3.31}$$

Combining (3.30) and (3.31), we obtain

$$\varphi_\theta(\eta) \le c\, h^p \|u\|_{C(\overline{I};H^{p+1}(\Omega))} + c\, k^2 \|\partial_t^2 u\|_{C(\overline{I};H^{p+1}(\Omega))} + c\, h^p \int_0^T \|\partial_t u(\cdot,s)\|_{p+1} ds + c\, k^2 \int_0^T \|\partial_t^3 u(\cdot,\tau)\|_h d\tau. \tag{3.32}$$

Next, we bound the other terms of $g_1(h,k)$ in (3.28). By [Han et al. (2019), [(4.25), (4.27)–(4.29), (4.36)],

$$\|e_0^2\|_{a_h} \le \|\Pi^h u_0 - u_0\|_{a_h} + \|u_0 - P^h u_0\|_{a_h} \le c\, h^p \|u_0\|_{p+1}, \tag{3.33}$$

$$\|e_1^2\|_{a_h} \le c\, h^p \left(\|u_0\|_{p+1} + k\, \|v_0\|_{p+1} + k^2 \|\partial_t^2 u(0)\|_{p+1}\right) + c\, k^2 \int_0^T \|\partial_t^3 u(\cdot,s)\|_{a_h} \mathrm{d}s, \tag{3.34}$$

$$\frac{1}{k}\|e_1^2 - e_0^2\|_0 \le c\, h^{p+1} \left(\|v_0\|_{p+1} + k \|\partial_t^2 u(0)\|_{p+1}\right) + c\, k^2 \|\partial_t^3 u\|_{C(\overline{I};L^2(\Omega))}, \tag{3.35}$$

$$k\, \|\xi_n\|_0 \le c\, h^p \int_{t_{n-1}}^{t_{n+1}} \|\partial_t^2 u(\cdot,s)\|_p \mathrm{d}s + c\, k^2 \int_{t_{n-1}}^{t_{n+1}} \|\partial_t^4 u(\cdot,s)\|_0 ds, \tag{3.36}$$

$$\left(k \sum_{i=2}^{N-1} \|\xi_{i-1} + \xi_i\|_0^2\right)^{1/2} \le c\, h^p \|\partial_t^2 u\|_{L^2(I;H^p(\Omega))} + c\, k^2 \|\partial_t^4 u\|_{L^2(I;L^2(\Omega))}. \tag{3.37}$$

Using the bounds (3.32)–(3.37) in (3.28), we have

$$g_1^{1/2}(h,k) \le c\, h^p \|u\|_{C^2(\overline{I};H^{p+1}(\Omega))} + c\, k^2 \left(\|\partial_t^3 u\|_{L^2(I;H^2(\Omega))} + \|\partial_t^3 u\|_{C(\overline{I};L^2(\Omega))} + \|\partial_t^4 u\|_{L^2(I;L^2(\Omega))}\right) \tag{3.38}$$

Then from (3.29),

$$\max_n \frac{1}{k}\, \|e_n^2 - e_{n-1}^2\|_0 + \max_n \|e_n^2\|_{a_h} \le c\, h^p \|u\|_{C^2(\overline{I};H^{p+1}(\Omega))} + c\, k^2 \left(\|\partial_t^3 u\|_{L^2(I;H^2(\Omega))} + \|\partial_t^3 u\|_{C(\overline{I};L^2(\Omega))} + \|\partial_t^4 u\|_{L^2(I;L^2(\Omega))}\right). \tag{3.39}$$

Combining (3.9)–(3.12) and (3.39), we arrive at the error bound (3.6) for $\theta \in (1/2, 1]$.

Case 2: $\theta = \frac{1}{2}$. For $0 \le n \le N-1$, we have

$$\|(u_{n+1} - u_{n+1}^{hk}) + (u_n - u_n^{hk})\|_h \le \|e_{n+1}^1 + e_n^1\|_h + \|e_{n+1}^2 + e_n^2\|_h. \tag{3.40}$$

For the first terms of the right-hand sides of (3.9) and (3.40), we have (3.11), and by (3.12),

$$\|e_{n+1}^1 + e_n^1\|_h \le c\, h^p \|u\|_{C(\overline{I};H^{p+1}(\Omega))}. \tag{3.41}$$

Consider the second terms of the right-hand sides of (3.9) and (3.40). By (3.18) for $\theta = \frac{1}{2}$, we have

$$\frac{1}{k^2}\left(\|e_n^2 - e_{n-1}^2\|_0^2 - \|e_1^2 - e_0^2\|_0^2\right) + \frac{1}{4}\left(\|e_n^2\|_{a_h}^2 + \|e_{n-1}^2\|_{a_h}^2 - \|e_1^2\|_{a_h}^2 - \|e_0^2\|_{a_h}^2\right) + \frac{1}{2}\left(a_h(e_{n-1}^2, e_n^2) - a_h(e_1^2, e_0^2)\right) = \sum_{i=1}^{n-1}(\xi_i, e_{i+1}^2 - e_{i-1}^2) + \sum_{i=1}^{n-1} a_h(\eta_i, e_{i+1}^2 - e_{i-1}^2). \tag{3.42}$$

By (3.20) and (3.22) in (3.42),

$$
\begin{aligned}
&\frac{1}{k^2}(\|e_n^2 - e_{n-1}^2\|_0^2 - \|e_1^2 - e_0^2\|_0^2) + \frac{1}{4}\left(\|e_n^2\|_{a_h}^2 + \|e_{n-1}^2\|_{a_h}^2 + 2\,a_h(e_{n-1}^2, e_n^2)\right)\\
&\quad - \frac{1}{4}\left(\|e_1^2\|_{a_h}^2 + \|e_0^2\|_{a_h}^2\right)\\
&\quad = \frac{1}{2}\,a_h(e_1^2, e_0^2) + (\xi_{n-1}, e_n^2 - e_{n-1}^2) + (\xi_1, e_1^2 - e_0^2) + \sum_{i=2}^{n-1}(\xi_{i-1} + \xi_i, e_i^2 - e_{i-1}^2)\\
&\qquad + a_h(\eta_{n-1}, e_n^2 + e_{n-1}^2) - a_h(\eta_1, e_1^2 + e_0^2) + \sum_{i=2}^{n-1} a_h(\eta_{i-1} - \eta_i, e_i^2 + e_{i-1}^2).
\end{aligned}
$$

Denote

$$
M_2 = \max_{0\le n\le N} \|e_n^2 + e_{n-1}^2\|_{a_h}.
$$

Then,

$$
\begin{aligned}
&\frac{1}{k^2}(\|e_n^2 - e_{n-1}^2\|_0^2 - \|e_1^2 - e_0^2\|_0^2) + \frac{1}{4}\left(\|e_n^2\|_{a_h}^2 + \|e_{n-1}^2\|_{a_h}^2 + 2\,a_h(e_{n-1}^2, e_n^2)\right)\\
&\quad - \frac{1}{4}\left(\|e_1^2\|_{a_h}^2 + \|e_0^2\|_{a_h}^2\right)\\
&\quad \le \frac{1}{4}\,(\|e_1^2\|_{a_h}^2 + \|e_0^2\|_{a_h}^2) + \frac{k^2}{2}\,\|\xi_{n-1}\|_0^2 + \frac{1}{2\,k^2}\|e_n^2 - e_{n-1}^2\|_0^2 + \frac{k^2}{2}\|\xi_1\|_0^2 + \frac{1}{2\,k^2}\|e_1^2 - e_0^2\|_0^2\\
&\qquad + \frac{k}{2}\sum_{i=2}^{n-1}\|\xi_{i-1} + \xi_i\|_0^2 + k\sum_{i=2}^{n-1}\frac{1}{2k^2}\|e_i^2 - e_{i-1}^2\|_0^2\\
&\qquad + \left(\|\eta_{n-1}\|_{a_h} + \|\eta_1\|_{a_h} + \sum_{i=2}^{n-1}\|\eta_{i-1} - \eta_i\|_{a_h}\right) M_2.
\end{aligned}
$$

So

$$
\begin{aligned}
&\frac{1}{2\,k^2}\|e_n^2 - e_{n-1}^2\|_0^2 + \frac{1}{4}\|e_n^2 + e_{n-1}^2\|_{a_h}^2\\
&\quad \le \frac{3}{2\,k^2}\|e_1^2 - e_0^2\|_0^2 + \frac{1}{2}\,(\|e_1^2\|_{a_h}^2 + \|e_0^2\|_{a_h}^2) + \frac{k^2}{2}\,(\|\xi_{n-1}\|_0^2 + \|\xi_1\|_0^2)\\
&\qquad + \frac{k}{2}\sum_{i=2}^{n-1}\|\xi_{i-1} + \xi_i\|_0^2 + k\sum_{i=2}^{n-1}\frac{1}{2k^2}\|e_i^2 - e_{i-1}^2\|_0^2\\
&\qquad + \left(\|\eta_{n-1}\|_{a_h} + \|\eta_1\|_{a_h} + \sum_{i=2}^{n-1}\|\eta_{i-1} - \eta_i\|_{a_h}\right) M_2.
\end{aligned}
\tag{3.43}
$$

Applying Lemma 2.5 to (3.43), and using the relation (2.20), we get

$$
\max_n \frac{1}{k^2}\,\|e_n^2 - e_{n-1}^2\|_0^2 + \max_n \|e_n^2 + e_{n-1}^2\|_{a_h}^2 \le c\,g_2(h,k), \tag{3.44}
$$

where

$$g_2(h,k) = \|e_0^2\|_{a_h}^2 + \|e_1^2\|_{a_h}^2 + \frac{1}{k^2}\|e_1^2 - e_0^2\|_0^2 + k^2 \max_n \|\xi_n\|_0^2 + k\sum_{i=2}^{N-1} \|\xi_{i-1} + \xi_i\|_0^2 + \phi_\theta^2(\eta), \tag{3.45}$$

$$\phi_\theta(\eta) = \|\eta_{N-1}\|_{a_h} + \|\eta_1\|_{a_h} + \sum_{i=2}^{N-1} \|\eta_{i-1} - \eta_i\|_{a_h}. \tag{3.46}$$

Combining (3.30) and (3.31), we have

$$\phi_\theta(\eta) \le c\, h^p \|u\|_{C(\bar{I};H^{p+1}(\Omega))} + c\, k^2 \|\partial_t^2 u\|_{C(\bar{I};H^{p+1}(\Omega))} + c\, h^p \int_0^T \|\partial_t u(\cdot,s)\|_{p+1} ds + c\, k^2 \int_0^T \|\partial_t^3 u(\cdot,\tau)\|_h d\tau. \tag{3.47}$$

Collecting (3.33)–(3.37), (3.47) in (3.46) and noting (2.11), we obtain

$$g_2^{1/2}(h,k) \le c\, h^p \|u\|_{C^2(\bar{I};H^{p+1}(\Omega))} + c\, k^2 \left(\|\partial_t^3 u\|_{L^2(I;H^2(\Omega))} + \|\partial_t^3 u\|_{C(\bar{I};L^2(\Omega))} + \|\partial_t^4 u\|_{L^2(I;L^2(\Omega))} \right). \tag{3.48}$$

Then, by (3.44),

$$\max_n \frac{1}{k}\, \|e_n^2 - e_{n-1}^2\|_0 + \max_n \|e_n^2 + e_{n-1}^2\|_{a_h} \le c\, h^p \|u\|_{C^2(\bar{I};H^{p+1}(\Omega))} + c\, k^2 \left(\|\partial_t^3 u\|_{L^2(I;H^2(\Omega))} + \|\partial_t^3 u\|_{C(\bar{I};L^2(\Omega))} + \|\partial_t^4 u\|_{L^2(I;L^2(\Omega))} \right). \tag{3.49}$$

From (3.9), (3.11), (3.40)–(3.41) and (3.49), we arrive at the error bound (3.7) for $\theta = \frac{1}{2}$.

Case 3: $0 \le \theta < \frac{1}{2}$. From (3.18), (3.19)–(3.21),

$$\frac{1}{k^2}\left(\|e_n^2 - e_{n-1}^2\|_0^2 - \|e_1^2 - e_0^2\|_0^2\right) + \frac{\theta}{2}\left(\|e_n^2\|_{a_h}^2 + \|e_{n-1}^2\|_{a_h}^2 - \|e_1^2\|_{a_h}^2 - \|e_0^2\|_{a_h}^2\right) + (1-\theta)\, a_h(e_{n-1}^2, e_n^2)$$
$$= (1-\theta)\, a_h(e_1^2, e_0^2) + (\xi_{n-1}, e_n^2 - e_{n-1}^2) + (\xi_1, e_1^2 - e_0^2) + \sum_{j=2}^{n-1} (\xi_{j-1} + \xi_j, e_j^2 - e_{j-1}^2)$$
$$+ a_h(\eta_{n-1}, e_n^2) + a_h(\eta_{n-2}, e_{n-1}^2) - a_h(\eta_1, e_0^2) - a_h(\eta_2, e_1^2) + \sum_{j=2}^{n-2} a_h(\eta_{j-1} - \eta_{j+1}, e_j^2).$$

Then,

$$\frac{1}{k^2}\left(\|e_n^2 - e_{n-1}^2\|_0^2 - \|e_1^2 - e_0^2\|_0^2\right) + \frac{\theta}{2}\left(\|e_n^2\|_{a_h}^2 + \|e_{n-1}^2\|_{a_h}^2 - \|e_1^2\|_{a_h}^2 - \|e_0^2\|_{a_h}^2\right) + (1-\theta)\, a_h(e_{n-1}^2, e_n^2)$$
$$\le \frac{1-\theta}{2}\left(\|e_1^2\|_{a_h}^2 + \|e_0^2\|_{a_h}^2\right) + \frac{k^2}{2}\, \|\xi_{n-1}\|_0^2 + \frac{1}{2\,k^2}\|e_n^2 - e_{n-1}^2\|_0^2$$
$$+ \frac{k^2}{2}\, \|\xi_1\|_0^2 + \frac{1}{2\,k^2}\, \|e_1^2 - e_0^2\|_0 + \left(\sum_{j=2}^{n-1} \|\xi_{j-1} + \xi_j\|_0^2\right)^{1/2} \left(\sum_{j=2}^{n-1} \|e_j^2 - e_{j-1}^2\|_0^2\right)^{1/2}$$

$$+\left(\|\eta_{n-1}\|_{a_h}+\|\eta_{n-2}\|_{a_h}+\|\eta_2\|_{a_h}+\|\eta_1\|_{a_h}+\sum_{j=2}^{n-2}\|\eta_{j-1}-\eta_{j+1}\|_{a_h}\right)M_1$$

and

$$\begin{aligned}
&\frac{1}{2\,k^2}\,\|e_n^2-e_{n-1}^2\|_0^2+\frac{\theta}{2}\,(\|e_n^2\|_{a_h}^2+\|e_{n-1}^2\|_{a_h}^2)+(1-\theta)\,a_h(e_{n-1}^2,e_n^2)\\
&\quad\le\frac{3}{2\,k^2}\,\|e_1^2-e_0^2\|_0^2+\frac{1}{2}\,(\|e_0^2\|_{a_h}^2+\|e_1^2\|_{a_h}^2)+\frac{k^2}{2}\,(\|\xi_{n-1}\|_0^2+\|\xi_1\|_0^2)\\
&\qquad+\frac{k}{2}\sum_{j=2}^{n-1}\|\xi_{j-1}+\xi_j\|_0^2+k\sum_{j=2}^{n-1}\frac{1}{2\,k^2}\,\|e_j^2-e_{j-1}^2\|_0^2\\
&\qquad+\left(\|\eta_{n-1}\|_{a_h}+\|\eta_{n-2}\|_{a_h}+\|\eta_2\|_{a_h}+\|\eta_1\|_{a_h}+\sum_{j=2}^{n-2}\|\eta_{j-1}-\eta_{j+1}\|_{a_h}\right)M_1.
\end{aligned}\tag{3.50}$$

By the symmetry and the stability properties of a_h,

$$\begin{aligned}
a_h(e_{n-1}^2,e_n^2)&=a_h\Big(\frac{e_n^2+e_{n-1}^2}{2},\frac{e_n^2+e_{n-1}^2}{2}\Big)-a_h\Big(\frac{e_n^2-e_{n-1}^2}{2},\frac{e_n^2-e_{n-1}^2}{2}\Big)\\
&\ge-\frac{1}{4}a_h(e_n^2-e_{n-1}^2,e_n^2-e_{n-1}^2).
\end{aligned}$$

By (3.5),

$$a_h(e_n^2-e_{n-1}^2,e_n^2-e_{n-2}^2)\le b_{max}\,c_b\,h^{-2}\,\|e_n^2-e_{n-1}^2\|_0^2.$$

So

$$\begin{aligned}
&\frac{C^\star}{k^2}\,\|e_n^2-e_{n-1}^2\|_0^2+\frac{\theta}{2}(\|e_n^2\|_{a_h}^2+\|e_{n-1}^2\|_{a_h}^2)\\
&\quad\le\frac{3}{2\,k^2}\|e_1^2-e_0^2\|_0^2+\frac{1}{2}(\|e_1^2\|_{a_h}^2+\|e_0^2\|_{a_h}^2)+\frac{k^2}{2}\,(\|\xi_{n-1}\|_0^2+\|\xi_1\|_0^2)\\
&\qquad+\frac{k}{2}\sum_{j=2}^{n-1}\|\xi_{j-1}+\xi_j\|_0^2+k\sum_{j=2}^{n-1}\frac{1}{2\,C^\star}\,\frac{C^\star}{k^2}\|e_j^2-e_{j-1}^2\|_0^2\\
&\qquad+\left(\|\eta_{n-1}\|_{a_h}+\|\eta_{n-2}\|_{a_h}+\|\eta_2\|_{a_h}+\|\eta_1\|_{a_h}+\sum_{j=2}^{n-2}\|\eta_{j-1}-\eta_{j+1}\|_{a_h}\right)M_1,
\end{aligned}$$

where

$$C^\star=\frac{1}{2}-\frac{1-\theta}{4}\,k^2\,b_{max}\,c_b\,h^{-2}>0$$

by the CFL condition (3.8). Applying Lemma 2.5,

$$\begin{aligned}
&\frac{C^\star}{k^2}\,\|e_n^2-e_{n-1}^2\|_0^2+\frac{\theta}{2}(\|e_n^2\|_{a_h}^2+\|e_{n-1}^2\|_{a_h}^2)\\
&\quad\le c\left(\|e_0^2\|_{a_h}^2+\|e_1^2\|_{a_h}^2+\frac{1}{k^2}\|e_1^2-e_0^2\|_0^2+k^2\ \max_n\|\xi_n\|_0^2\right.
\end{aligned}$$

$$+k \sum_{j=2}^{N-1} \|\xi_{j-1} + \xi_j\|_0^2 + \varphi_\theta(\eta) M_1 \Bigg) . \tag{3.51}$$

Starting with (3.51), similar to the derivation based on (3.24), we have

$$\max_n \frac{1}{k^2} \|e_n^2 - e_{n-1}^2\|_0^2 + \max_n \|e_n^2\|_{a_h}^2 \le c\, g_1(h,k). \tag{3.52}$$

We then apply (3.9)–(3.12), (3.38) to obtain the error bound (3.6) for the case $0 \le \theta < \frac{1}{2}$ under the CFL stability condition (3.8). □

We now proceed to derive an optimal L^2 error estimate for the fully discrete θ-schemes.

Theorem 3.3 *Let u and u^{hk} be the solutions of Problem 2.1 and Problem 3.1, respectively. Assume $u \in C^2(\bar{I}; H^{p+1}(\Omega))$, $\partial_t^3 u \in C(\bar{I}; L^2(\Omega)) \cap L^2(I; H^2(\Omega))$, $\partial_t^4 u \in C(\bar{I}; L^2(\Omega))$. Then, for $\theta \in [\frac{1}{2}, 1]$, or for $\theta \in [0, \frac{1}{2})$ and if the CFL stability (3.8) condition holds, we have the following error bound holds*

$$\max_{0\le n\le N-1} \|u_n - u_n^{hk}\|_0 \le c\left(h^{p+1} + k^2\right) \tag{3.53}$$

for some constant c depending on $\|u\|_{C^2(\bar{I};H^{p+1}(\Omega))}$, $\|\partial_t^3 u\|_{C(\bar{I};L^2(\Omega))}$, and $\|\partial_t^4 u\|_{C(\bar{I};L^2(\Omega))}$.

Proof For $0 \le n \le N-1$, similar to the error decomposition in the proof of Theorem 3.2, we have

$$\|u_n - u_n^{hk}\|_0 = \|u_n - \Pi^h u_n + \Pi^h u_n - u_n^{hk}\|_0 \le \|e_n^1\|_0 + \|e_n^2\|_0. \tag{3.54}$$

By (2.15), as in [Han et al. (2019), (4.45)],

$$\|e_n^1\|_0 \le c\, h^{p+1} \|u\|_{C(\bar{I};H^{p+1}(\Omega))}, \tag{3.55}$$

and considering the truncation errors due to the time discretization for $n = 1, 2, \ldots, N-1$,

$$r_n = d_k u_n - \nabla\cdot(b\,\nabla(\theta\,\gamma_k u_n + (1-\theta)\,u_n)) - f_n = d_k u_n - \partial_t^2 u_n - \frac{\theta}{2}\,k^2\,\nabla\cdot(b\,\nabla d_k u_n)\,.$$

Under the regularity assumptions in the theorem, we obtain

$$\|r_n\|_0 \le c\,k^2 \left(\|\partial_t^2 u\|_{C(\bar{I};H^2(\Omega))} + \|\partial_t^4 u\|_{C(\bar{I};L^2(\Omega))}\right). \tag{3.56}$$

By the definition of r_n, we have, for $n = 1, 2, \ldots, N-1$,

$$(d_k u_n, v^h) + a_h(\theta\,\gamma_k u_n + (1-\theta)\,u_n, v^h) = (f_n, v^h) + (r_n, v^h) \quad \forall\, v^h \in V^h. \tag{3.57}$$

Next, we subtract the equation (3.1) from the equation (3.57),

$$(d_k u_n - d_k u_n^{hk}, v^h) + a_h(\theta\,\gamma_k u_n - \theta\,\gamma_k u_n^{hk} + (1-\theta)\,(u_n - u_n^{hk}), v^h) = (r_n, v^h) \quad \forall\, v^h \in V^h.$$

By the definition of the Galerkin projection, $a_h(u_n - \Pi^h u_n, v^h) = 0$,

$$(d_k e_n^2, v^h) + a_h(\theta\,\gamma_k e_n^2 + (1-\theta)\,e_n^2, v^h) = (r_n, v^h) - (d_k e_n^1, v^h). \tag{3.58}$$

We now change n to i in (3.58) and sum for $i = 1, 2, \ldots, m$ and multiply both sides by k, to obtain

$$\left(\frac{e_{m+1}^2 - e_m^2}{k}, v^h\right) - \left(\frac{e_1^2 - e_0^2}{k}, v^h\right) + k\sum_{i=1}^m a_h(\theta\,\gamma_k e_i^2 + (1-\theta)\,e_i^2, v^h)$$

$$= k\sum_{i=1}^{m}(r_i, v^h) - k\sum_{i=1}^{m}(d_k e_i^1, v^h). \tag{3.59}$$

Define

$$\Phi^m = k\sum_{i=1}^{m} e_i^2, \quad \Phi^0 = 0,$$
$$R^m = k\sum_{i=0}^{m} r_i, \quad r_0 = \frac{e_1^2 - e_0^2}{k^2},$$
$$A^m = k\sum_{i=1}^{m} d_k e_i^1, \quad A^0 = 0.$$

So (3.59) is rewritten as

$$\left(\frac{e_{m+1}^2 - e_m^2}{k}, v^h\right) + a_h(\theta\, \gamma_k \Phi^m + (1-\theta)\, \Phi^m, v^h) = (R^m, v^h) - (A^m, v^h) \quad \forall v^h \in V^h.$$

Take $v^h = e_{m+1}^2 + e_m^2 \in V^h$ and multiply the resulting expression by k, for $0 \le m \le N-1$,

$$\|e_{m+1}^2\|_0^2 - \|e_m^2\|_0^2 + k a_h(\theta\, \gamma_k \Phi^m + (1-\theta)\, \Phi^m, e_{m+1}^2 + e_m^2) = k(R^m - A^m, e_{m+1}^2 + e_m^2).$$

Making a summation from $m = 0$ to $m = n-1$, for $1 \le n \le N$, we obtain

$$\|e_n^2\|_0^2 - \|e_0^2\|_0^2 + k\sum_{m=0}^{n-1} a_h(\theta\, \gamma_k \Phi^m + (1-\theta)\, \Phi^m, e_{m+1}^2 + e_m^2) = k\sum_{m=0}^{n-1}(R^m - A^m, e_{m+1}^2 + e_m^2).$$

Note that

$$\Phi^0 = 0,$$
$$\Phi^{m+1} - \Phi^{m-1} = k(e_{m+1}^2 + e_m^2), \quad m = 1, 2, \ldots, N-1.$$

By the symmetry of a_h,

$$\begin{aligned} & k\sum_{m=0}^{n-1} a_h(\theta\, \gamma_k \Phi^m + (1-\theta)\, \Phi^m, e_{m+1}^2 + e_m^2) \\ &= \sum_{m=1}^{n-1} a_h(\theta\, \gamma_k \Phi^m + (1-\theta)\, \Phi^m, \Phi^{m+1} - \Phi^{m-1}) \\ &= \frac{\theta}{2}(\|\Phi^n\|_{a_h}^2 + \|\Phi^{n-1}\|_{a_h}^2 - k^2\, \|e_1^2\|_{a_h}^2) + (1-\theta)\, a_h(\Phi^{n-1}, \Phi^n), \end{aligned}$$

where

$$\sum_{m=1}^{n-1} a_h(\Phi^m, \Phi^{m+1} - \Phi^{m-1}) = a_h(\Phi^{n-1}, \Phi^n).$$

Therefore, for $1 \le n \le N$,

$$\|e_n^2\|_0^2 + \frac{\theta}{2}(\|\Phi^n\|_{a_h}^2 + \|\Phi^{n-1}\|_{a_h}^2) + (1-\theta)\, a_h(\Phi^{n-1}, \Phi^n)$$

$$= \|e_0^2\|_0^2 + \frac{\theta}{2}\, k^2\, \|e_1^2\|_{a_h}^2 + k \sum_{m=0}^{n-1} (R^m - A^m, e_{m+1}^2 + e_m^2), \quad 1 \le n \le N. \tag{3.60}$$

Now we distinguish two cases.
Case 1: $\frac{1}{2} \le \theta \le 1$. By Lemma 2.3 in (3.60), we have

$$(1-\theta)\, a_h(\Phi^{n-1}, \Phi^n) \ge -\frac{(1-\theta)}{2} (\|\Phi^n\|_{a_h}^2 + \|\Phi^{n-1}\|_{a_h}^2).$$

Then,

$$\begin{aligned} &\|e_n^2\|_0^2 + \frac{2\,\theta - 1}{2} (\|\Phi^n\|_{a_h}^2 + \|\Phi^{n-1}\|_{a_h}^2) \\ &\le \|e_0^2\|_0^2 + \frac{\theta}{2}\, k^2\, \|e_1^2\|_{a_h}^2 + k \sum_{m=0}^{n-1} (R^m - A^m, e_{m+1}^2 + e_m^2). \end{aligned} \tag{3.61}$$

By applying the Cauchy–Schwarz inequality and Young inequality $ab \le \epsilon a^2 + \frac{1}{4\epsilon} b^2$ $(\epsilon = 1)$ to the third term of the right-hand side in (3.61), we obtain

$$\|e_n^2\|_0^2 \le \|e_0^2\|_0^2 + \frac{\theta}{2}\, k^2\, \|e_1^2\|_{a_h}^2 + 2 \left[\frac{1}{4} \max_n \|e_n^2\|_0^2 + \left(k \sum_{m=0}^{n-1} \|R^m\|_0 + k \sum_{m=0}^{n-1} \|A^m\|_0 \right)^2 \right].$$

So

$$\max_n \|e_n^2\|_0 \le \sqrt{2} \|e_0^2\|_0 + \frac{\theta}{2}\, k^2 + \frac{1}{2} \|e_1^2\|_{a_h}^2 + 2 \left(k \sum_{m=0}^{n-1} \|R^m\|_0 + k \sum_{m=0}^{n-1} \|A^m\|_0 \right). \tag{3.62}$$

By (2.12), (2.15),

$$\|e_0^2\|_0 \le \|\Pi^h u_0 - u_0\|_0 + \|u_0 - P^h u_0\|_0 \le c\, h^{p+1} \|u_0\|_{p+1}. \tag{3.63}$$

By (3.35), (3.56),

$$\begin{aligned} k \sum_{m=0}^{n-1} \|R^m\|_0 &= k \sum_{m=0}^{n-1} \|k \sum_{i=1}^{m} r_i + k r_0\|_0 \le k \sum_{m=0}^{n-1} \sum_{i=1}^{m} k\, \|r_i\|_0 + \sum_{m=0}^{n-1} \|e_1^2 - e_0^2\|_0 \\ &\le c\, k^2 \left(\|\partial_t^2 u\|_{C(\bar{I}; H^2(\Omega))} + \|\partial_t^3 u\|_{C(\bar{I}; L^2(\Omega))} + \|\partial_t^4 u\|_{C(\bar{I}; L^2(\Omega))} \right) \\ &\quad + c\, h^{p+1} \left(\|v_0\|_{p+1} + k \|\partial_t^2 u(0)\|_{p+1} \right), \end{aligned} \tag{3.64}$$

and by [Han et al. (2019), (4.54)],

$$\begin{aligned} k \sum_{m=0}^{n-1} \|A^m\|_0 &\le k^2 \sum_{m=0}^{n-1} \sum_{i=1}^{m} \|d_k e_i^1\|_0 = k^2 \sum_{m=0}^{n-1} \sum_{i=1}^{m} \|d_k (I - \Pi^h) u_i\|_0 \\ &\le c\, h^{p+1} \int_0^T \|\partial_t^2 u(\cdot, s)\|_{p+1} \mathrm{d}s. \end{aligned} \tag{3.65}$$

Applying (3.34) and (3.63)–(3.65) to (3.62), we have

$$\begin{aligned} \max_n \|e_n^2\|_0 &\le c\, k^2 \left(\|\partial_t^2 u\|_{C(\bar{I}; H^2(\Omega))} + \|\partial_t^3 u\|_{L^2(I; H^2(\Omega))} + \|\partial_t^3 u\|_{C(\bar{I}; L^2(\Omega))} \right. \\ &\quad \left. + \|\partial_t^4 u\|_{C(\bar{I}; L^2(\Omega))} \right) + c\, h^{p+1} \|u\|_{C^2(\bar{I}; H^{p+1}(\Omega))}. \end{aligned} \tag{3.66}$$

The error bound (3.53) now follows from (3.54), (3.55) and (3.66) for $\frac{1}{2} \le \theta \le 1$.

Case 2: $0 \le \theta < \frac{1}{2}$. From (3.60), we see that

$$\|e_n^2\|_0^2 + \frac{\theta}{2}\|\Phi^n + \Phi^{n-1}\|_{a_h}^2 + (1-2\theta)\, a_h(\Phi^{n-1}, \Phi^n)$$
$$= \|e_0^2\|_0^2 + \frac{\theta}{2}\, k^2\, \|e_1^2\|_{a_h}^2 + k\sum_{m=0}^{n-1}(R^m - A^m, e_{m+1}^2 + e_m^2), \quad 1 \le n \le N. \tag{3.67}$$

By the symmetry and the stability properties of a_h, and since $\Phi^n - \Phi^{n-1} = k\, e_n^2$, we find that

$$a_h(\Phi^{n-1}, \Phi^n) = a_h\left(\frac{\Phi^{n-1}+\Phi^n}{2}, \frac{\Phi^{n-1}+\Phi^n}{2}\right) - a_h\left(\frac{\Phi^n-\Phi^{n-1}}{2}, \frac{\Phi^n-\Phi^{n-1}}{2}\right)$$
$$\ge -\frac{k^2}{4} a_h(e_n^2, e_n^2).$$

By (3.5), we get

$$a_h(e_n^2, e_n^2) \le b_{max}\, c_b\, h^{-2}\, \|e_n^2\|_0^2.$$

So

$$C^\star\|e_n^2\|_0^2 + \frac{\theta}{2}\|\Phi^n + \Phi^{n-1}\|_{a_h}^2 \le \|e_0^2\|_0^2 + \frac{\theta}{2}\, k^2\, \|e_1^2\|_{a_h}^2 + k\sum_{m=0}^{n-1}(R^m - A^m, e_{m+1}^2 + e_m^2). \tag{3.68}$$

where

$$C^\star = 1 - \frac{1-2\theta}{4}\, k^2\, b_{max}\, c_b\, h^{-2} > 0$$

by the CFL condition (3.8). By applying the Cauchy–Schwarz inequality and the modified Young inequality $ab \le \varepsilon a^2 + \frac{1}{4\,\varepsilon} b^2$ with $\varepsilon = \frac{C^\star}{4}$ to the third term of the right-hand side in (3.68), we obtain

$$C^\star\, \|e_n^2\|_0^2 \le \|e_0^2\|_0^2 + \frac{\theta}{2}\, k^2\, \|e_1^2\|_{a_h}^2 + k\sum_{m=0}^{n-1}\left(\|R^m\|_0 + \|A^m\|_0\right)\left(\|e_{m+1}^2\|_0 + \|e_m^2\|_0\right)$$
$$\le \|e_0^2\|_0^2 + \frac{\theta}{2}\, k^2\, \|e_1^2\|_{a_h}^2 + 2\left(\max_n \|e_n^2\|_0\right)\left(k\sum_{m=0}^{n-1}\|R^m\|_0 + k\sum_{m=0}^{n-1}\|A^m\|_0\right)$$
$$\le \|e_0^2\|_0^2 + \frac{\theta}{2}\, k^2\, \|e_1^2\|_{a_h}^2 + 2\left[\frac{C^\star}{4}\max_n \|e_n^2\|_0^2 + \frac{1}{C^\star}\left(k\sum_{m=0}^{n-1}\|R^m\|_0 + k\sum_{m=0}^{n-1}\|A^m\|_0\right)^2\right]$$
$$= \|e_0^2\|_0^2 + \frac{\theta}{2}\, k^2\, \|e_1^2\|_{a_h}^2 + \frac{C^\star}{2}\max_n \|e_n^2\|_0^2 + \frac{2}{C^\star}\left(k\sum_{m=0}^{n-1}\|R^m\|_0 + k\sum_{m=0}^{n-1}\|A^m\|_0\right)^2.$$

So

$$\max_n \|e_n^2\|_0 \le \sqrt{\frac{2}{C^\star}}\|e_0^2\|_0 + \frac{\theta}{2\,C^\star}\, k^2 + \frac{1}{2}\|e_1^2\|_{a_h}^2 + 2\left(k\sum_{m=0}^{n-1}\|R^m\|_0 + k\sum_{m=0}^{n-1}\|A^m\|_0\right). \tag{3.69}$$

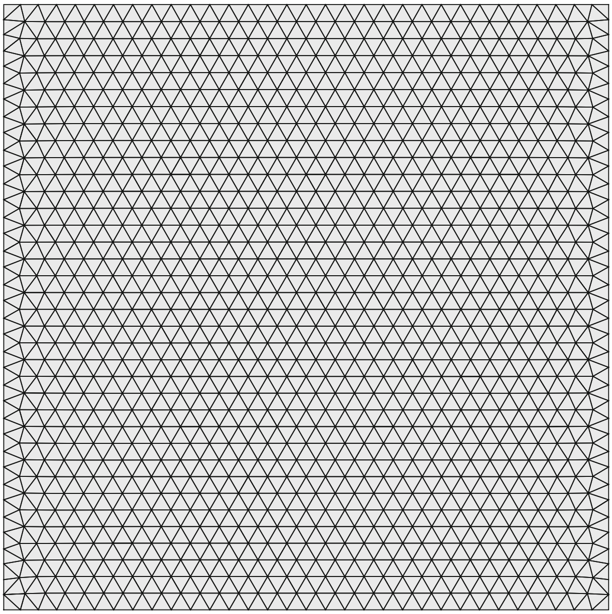

Fig. 1 Quasi-uniform triangulation with $h = 1/32$

Combining (3.54), (3.55), (3.66) and (3.69), we obtain the error bound (3.53) for $0 \le \theta < \frac{1}{2}$ under the CFL condition (3.8). □

4 Numerical results

In this section, we present numerical results to illustrate numerical convergence orders of the proposed numerical schemes, paying particular attention to their dependence on the parameter θ. Let $\Omega := (0, 1)^2$, $b = 1$, and choose f such that the exact solution is

$$u(x, y, t) = (e^{x^2-x} - 1)(e^{y^2-y} - 1)(e^{t^2} - 1).$$

Correspondingly, the initial values are $u_0 = 0$, $v_0 = 0$. We use a sequence of quasi-uniform triangulations $\{\mathcal{T}_h\}$ of the type shown in Fig. 1 to partition $\overline{\Omega}$. The polynomial degrees for the numerical schemes will be $p = 1, 2, 3$. The fully discrete scheme (3.1)–(3.4) with the IPDG method will be used, and the penalty parameter $\eta_e = 200\ (p + 1)^2$.

First we explore the dependence of numerical solution errors on the mesh size h. The schemes are unconditionally convergent with $\theta = 0.5,\ 0.75,\ 1.0$. We use a sufficiently small fixed time step for these values of the parameter $\theta : k = 0.01$ for $p = 1$, $k = 0.001$ for $p = 2$ and $k = 0.0001$ for $p = 3$. We take $h = 2^{-2}, \ldots, 2^{-5}$ and the numerical errors and convergence orders in the $H^1(\Omega)$ norm and $L^2(\Omega)$ norm at $t = 1$ are summarized in Tables 1 and 2. For $\theta \in [0, 0.5)$, a CFL stability condition of the form $k \le c_0\ h$ is needed to ensure convergence of the numerical solutions. To illustrate this point, we consider two values of $\theta :$ 0.25 and 0.0. Numerical experiments suggest that for $\theta = 0.25$, the scheme performs well if we use the time step $k \le \frac{1}{68}h$ for $p = 1$, $k \le \frac{1}{137}h$ for $p = 2$ and $k \le \frac{1}{230}h$ for $p = 3$; whereas for $\theta = 0.0$, we have convergent numerical results if $k \le \frac{1}{96}h$ for $p = 1$,

Table 1 H^1 norm errors at $t = 1.0$ for varying h and a suitably chosen k

θ	h	Errors for $p = 1$	Order	Errors for $p = 2$	Order	Errors for $p = 3$	Order
0.5	2^{-2}	5.2703e−02	–	1.1401e−02	–	7.6622e−04	–
	2^{-3}	2.4827e−02	1.0860	2.3818e−03	2.2590	7.4390e−05	3.3646
	2^{-4}	1.0974e−02	1.1778	5.9296e−04	2.0060	9.1617e−06	3.0214
	2^{-5}	5.1302e−03	1.0970	1.4188e−04	2.0633	1.0891e−06	3.0725
0.75	2^{-2}	5.2705e−02	–	1.1401e−02	–	7.6622e−04	–
	2^{-3}	2.4827e−02	1.0860	2.3818e−03	2.2590	7.4390e−05	3.3646
	2^{-4}	1.0974e−02	1.1778	5.9296e−04	2.0060	9.1617e−06	3.0214
	2^{-5}	5.1306e−03	1.0969	1.4189e−04	2.0632	1.0891e−06	3.0725
1.0	2^{-2}	5.2706e−02	–	1.1401e−02	–	7.6622e−04	–
	2^{-3}	2.4828e−02	1.0860	2.3818e−03	2.2590	7.4390e−05	3.3646
	2^{-4}	1.0975e−02	1.1777	5.9296e−04	2.0060	9.1617e−06	3.0214
	2^{-5}	5.1311e−03	1.0969	1.4189e−04	2.0632	1.0891e−06	3.0725

Table 2 L^2 norm errors at $t = 1.0$ for varying h and a suitably chosen k

θ	h	Errors for $p = 1$	Order	Errors for $p = 2$	Order	Errors for $p = 3$	Order
0.5	2^{-2}	8.2617e−03	–	3.4044e−04	–	2.4000e−05	–
	2^{-3}	1.9166e−03	2.1079	3.5495e−05	3.2617	9.3505e−07	4.6818
	2^{-4}	4.3166e−04	2.1506	4.1020e−06	3.1132	6.0865e−08	3.9414
	2^{-5}	1.0824e−04	1.9957	5.0441e−07	3.0237	3.3401e−09	4.1876
0.75	2^{-2}	8.2644e−03	–	3.4045e−04	–	2.4000e−05	–
	2^{-3}	1.9197e−03	2.1060	3.5496e−05	3.2617	9.3507e−07	4.6818
	2^{-4}	4.3497e−04	2.1479	4.1038e−06	3.1126	6.0888e−08	3.9408
	2^{-5}	1.1165e−04	1.9619	5.1252e−07	3.0013	3.4734e−09	4.1317
1.0	2^{-2}	8.2670e−03	–	3.4045e−04	–	2.4000e−05	–
	2^{-3}	1.9228e−03	2.1042	3.5498e−05	3.2616	9.3509e−07	4.6818
	2^{-4}	4.3828e−04	2.1333	4.1059e−06	3.1120	6.0913e−08	3.9403
	2^{-5}	1.1509e−04	1.9291	5.2316e−07	2.9724	3.6397e−09	4.0649

$k \le \frac{1}{194}h$ for $p = 2$ and $k \le \frac{1}{325}h$ for $p = 3$. We take $h = 2^{-2}, \cdots, 2^{-5}$ and compute numerical solutions. The numerical errors and convergence orders in the $H^1(\Omega)$ norm and $L^2(\Omega)$ norm at $t = 1$ are listed in Tables 3 and 4, where for $\theta = 0.25$, we choose $k = \frac{1}{69}h$ for $p = 1$, $k = \frac{1}{139}h$ for $p = 2$ and $k = \frac{1}{232}h$ for $p = 3$, and for $\theta = 0.0$, we choose $k = \frac{1}{97}h$ for $p = 1$, $k = \frac{1}{196}h$ for $p = 2$ and $k = \frac{1}{328}h$ for $p = 3$. We observe that the numerical convergence orders for H^1 norm and L^2 norm are around p and $p + 1$, respectively, which matches well with the theoretical prediction.

However, if the CFL condition is slightly violated, then the numerical results are not good. For example, for parameter $\theta = 0.25$, we take the time step $k = \frac{1}{60}h$ for $p = 1$, $k = \frac{1}{120}h$ for $p = 2$ and $k = \frac{1}{200}h$ for $p = 3$; for parameter $\theta = 0.0$, we take the time step $k = \frac{1}{90}h$

Table 3 H^1 norm errors at $t = 1.0$ for varying h and k with stability

θ	h	Errors for $p = 1$	Order	Errors for $p = 2$	Order	Errors for $p = 3$	Order
0.0	2^{-2}	5.2700e−02	–	1.1401e−02	–	7.6622e−04	–
	2^{-3}	2.4825e−02	1.0860	2.3818e−03	2.2590	7.4390e−05	3.3646
	2^{-4}	1.0973e−02	1.1778	5.9296e−04	2.0060	9.1617e−06	3.0214
	2^{-5}	5.1295e−03	1.0971	1.4188e−04	2.0633	1.0891e−06	3.0725
0.25	2^{-2}	5.2701e−02	–	1.1401e−02	–	7.6623e−04	–
	2^{-3}	2.4825e−02	1.0860	2.3818e−03	2.2590	7.4391e−05	3.3646
	2^{-4}	1.0973e−02	1.1778	5.9296e−04	2.0060	9.1618e−06	3.0214
	2^{-5}	5.1295e−03	1.0971	1.4188e−04	2.0633	1.0891e−06	3.0725

Table 4 L^2 norm errors at $t = 1.0$ for varying h and k with stability

θ	h	Errors for $p = 1$	Order	Errors for $p = 2$	Order	Errors for $p = 3$	Order
0.0	2^{-2}	8.2555e−03	–	3.4043e−04	–	2.4000e−05	–
	2^{-3}	1.9094e−03	2.1122	3.5491e−05	3.2618	9.3510e−07	4.6818
	2^{-4}	4.2394e−04	2.1712	4.0992e−06	3.1140	6.0843e−08	3.9420
	2^{-5}	1.0034e−04	2.0790	4.9591e−07	3.0472	3.1880e−09	4.2544
0.25	2^{-2}	8.2559e−03	–	3.4045e−04	–	2.4002e−05	–
	2^{-3}	1.9095e−03	2.1122	3.5493e−05	3.2618	9.3587e−07	4.6807
	2^{-4}	4.2397e−04	2.1712	4.0995e−06	3.1140	6.1080e−08	3.9375
	2^{-5}	1.0035e−04	2.0789	4.9594e−07	3.0472	3.3522e−09	4.1875

Table 5 H^1 norm errors at $t = 1.0$ for varying h and k without stability

θ	h	Errors for $p = 1$	Order	Errors for $p = 2$	Order	Errors for $p = 3$	Order
0.0	2^{-2}	3.1797e+29	–	3.5299e+124	–	Inf	–
	2^{-3}	8.0694e+43	− 47.851	Inf	− Inf	NaN	NaN
	2^{-4}	Inf	− Inf	NaN	NaN	NaN	NaN
	2^{-5}	NaN	NaN	NaN	NaN	NaN	NaN
0.25	2^{-2}	1.3762e+43	–	2.8038e+106	–	Inf	–
	2^{-3}	5.1719e+91	− 161.36	Inf	− Inf	NaN	NaN
	2^{-4}	Inf	− Inf	NaN	NaN	NaN	NaN
	2^{-5}	NaN	NaN	NaN	NaN	NaN	NaN

for $p = 1$, $k = \frac{1}{180}h$ for $p = 2$ and $k = \frac{1}{300}h$ for $p = 3$. Then we take $h = 2^{-2}, \cdots, 2^{-5}$ and compute numerical results are demonstrated in Tables 5 and 6, respectively.

Next, to examine the orders of convergence concerning the time step k, and to keep the test simple, we consider cubic element ($p = 3$) and fix a mesh size $h = 1/128$ for $\theta = 0.5, 0.75, 1.0$, respectively. Then the L^2, H^1 error accuracy of time are presented, cf. the results in Table 7. We observe that the numerical convergence orders in time $T = 1$ are

Table 6 L^2 norm errors at $t = 1.0$ for varying h and k without stability

θ	h	Errors for $p = 1$	Order	Errors for $p = 2$	Order	Errors for $p = 3$	Order
0.0	2^{-2}	4.8416e+27	–	9.5969e+122	–	Inf	–
	2^{-3}	5.0302e+41	– 46.562	Inf	– Inf	NaN	NaN
	2^{-4}	Inf	– Inf	NaN	NaN	NaN	NaN
	2^{-5}	NaN	NaN	NaN	NaN	NaN	NaN
0.25	2^{-2}	2.0955e+41	–	7.6229e+104	–	Inf	–
	2^{-3}	3.2240e+89	– 160.07	Inf	– Inf	NaN	NaN
	2^{-4}	Inf	– Inf	NaN	NaN	NaN	NaN
	2^{-5}	NaN	NaN	NaN	NaN	NaN	NaN

Table 7 Numerical convergence orders in L^2 and H^1 norms at $t = 1.0$ for $h = 1/128$, $p = 3$ with varying k

θ	k	L^2 errors	Order	H^1 errors	Order
0.5	2^{-2}	5.6746e−03	–	2.6140e−02	–
	2^{-3}	1.4133e−03	2.0055	6.5643e−03	1.9935
	2^{-4}	3.4834e−04	2.0205	1.6246e−03	2.0146
	2^{-5}	8.6296e−05	2.0131	4.0289e−04	2.0116
0.75	2^{-2}	7.8414e−03	–	3.6148e−02	–
	2^{-3}	2.0088e−03	1.9648	9.3211e−03	1.9553
	2^{-4}	4.9654e−04	2.0164	2.3182e−03	2.0075
	2^{-5}	1.2300e−04	2.0133	5.7531e−04	2.0106
1.0	2^{-2}	9.8085e−03	–	4.5382e−02	–
	2^{-3}	2.5974e−03	1.9170	1.2041e−02	1.9142
	2^{-4}	6.4451e−04	2.0108	3.0099e−03	2.0002
	2^{-5}	1.5970e−04	2.0128	7.4767e−04	2.0092

nearly all of $O(k^2)$, which is in agreement with the theoretical results in Theorem 3.2 and Theorem 3.3.

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