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The Nonconforming Virtual Element Method for a Stationary Stokes Hemivariational Inequality with Slip Boundary Condition

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Abstract
In this paper, the nonconforming virtual element method is studied to solve a hemivariational inequality problem for the stationary Stokes equations with a nonlinear slip boundary condition. The nonconforming virtual elements enriched with polynomials on slip boundary are used to discretize the velocity, and discontinuous piecewise polynomials are used to approximate the pressure. The inf-sup condition is shown for the nonconforming virtual element method. An error estimate is derived under appropriate solution regularity assumptions, and the error bound is of optimal order when lowest-order virtual elements for the velocity and piecewise constants for the pressure are used. A numerical example is presented to illustrate the theoretically predicted convergence order.

Keywords Virtual element method · Hemivariational inequality · Stokes problem · Slip boundary condition

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1 Introduction

Hemivariational inequality (HVI), concerning nonsmooth and nonconvex functionals, represents a powerful tool in the study of a large number of nonlinear boundary value problems. Mathematical theory, numerical approximations and applications of hemivariational inequalities can be found in several comprehensive References [11,20,24,26,27]. Recently, optimal order error estimates are derived for the linear finite element solutions of hemivariational inequalities, see [18,19] for a summary account. In [30], the interior penalty discontinuous Galerkin method is studied for solving an elliptic hemivariational inequality for semipermeable media, and optimal convergence order is proved for the linear element.

The conforming virtual element method (VEM) for a second-order elliptic problem was initially introduced in [3] as a generalization of the classical finite element method to accommodate arbitrary element-geometry. The nonconforming VEM for the same problem was constructed later in [2], where the corresponding virtual element can be viewed as an extension of the Crouzeix–Raviart element to general polygonal meshes. Because of its flexibility in mesh handling and properties of avoiding an explicit construction of shape functions, the virtual element method has been applied to solve a variety of partial differential equations, e.g., [4,17] for linear elasticity systems, [1] for the Cahn–Hilliard equation, [5,10,23] for the Stokes equations, [6,22] for the Navier–Stokes equations, [29] for the Darcy and Brinkman equations, [28] for the Helmholtz equation, [34] for the plate bending problem, [12] for elliptic interface problems, [15,31–33] for elliptic variational inequalities, and [16] for elliptic hemivariational inequalities.

This paper is devoted to the nonconforming virtual element method to solve a hemivariational inequality problem for the stationary Stokes equations with a nonlinear slip boundary condition. Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a simply connected polygonal/polyhedral domain with a Lipschitz boundary Γ that is split into two non-trivial parts Γ_D and Γ_S: $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_S$, $\Gamma_D \cap \Gamma_S = \emptyset$. Throughout this paper, we use boldface symbols for vector-valued variables and their spaces. Denote by $\boldsymbol{n}$ the unit outward normal to Γ. For a vector $\boldsymbol{u}$, denote its normal component and tangential component by $u_n = \boldsymbol{u} \cdot \boldsymbol{n}$ and $\boldsymbol{u}_\tau = \boldsymbol{u} - u_n \boldsymbol{n}$ on the boundary. We consider the Stokes equations

$$\begin{cases} -\nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

with the following boundary conditions

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_D, \tag{1.2}$$

$$u_n = 0, \quad -\boldsymbol{\sigma}_\tau \in \partial j(\boldsymbol{u}_\tau) \quad \text{on } \Gamma_S. \tag{1.3}$$

Here, the unknowns are the fluid velocity $\boldsymbol{u}$ and the pressure p, ν is the viscosity coefficient, $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ is a given force density function, and $\boldsymbol{\sigma}_\tau = \nu\, \partial \boldsymbol{u}_\tau / \partial \boldsymbol{n}$ is the tangential component of stress tensor defined on Γ_S. We use ":" for the canonical inner product on the space of second order tensors on $\mathbb{R}^d$. $j : \Gamma_S \times \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz continuous with respect to its second argument. To simplify the notation, we write $j(\boldsymbol{u}_\tau)$ for $j(\boldsymbol{x}, \boldsymbol{u}_\tau)$, and denote by ∂j for Clarke's generalized subdifferential of j with respect to its second argument. The condition (1.3) is known as a slip boundary condition. The first part $u_n = 0$ means that the fluid can not pass through Γ_S outside the domain. The second part represents a friction condition, relating the frictional force $\boldsymbol{\sigma}_\tau$ with the tangential velocity $\boldsymbol{u}_\tau$. This relation is of nonmonotone type when the potential j is not a convex function. The problem of the Stokes equations with a similar nonlinear slip boundary condition has been studied in [14], where the well-posedness of the problem is shown and an optimal order error estimate is derived for the mini finite element solution under suitable regularity assumptions.

In this paper, we modify the nonconforming virtual element method developed in [2] to solve a hemivariational inequality problem for the stationary Stokes equations with a nonlinear slip boundary condition by introducing a slightly different virtual element space on the slip boundary for the velocity, while the pressure is approximated by discontinuous piecewise polynomials. Moreover, we prove the discrete inf-sup condition for this method to obtain its solvability and stability. In addition, we present an error estimate for the velocity and pressure, achieving the optimal order for the lowest-order elements.

The paper is organized as follows. In Sect. 2, we introduce a hemivariational inequality formulation for the problem (1.1)–(1.3). In Sect. 3, we present the VEM discrete scheme. In Sect. 4, we derive a priori error estimate, which is optimal with lowest-order nonconforming virtual elements for the velocity and piecewise constants for the pressure. In Sect. 5, we show a numerical example and provide numerical evidence of the theoretically predicted convergence order.

2 The Hemivariational Inequality Formulation

In this section, we present the hemivariational inequality formulation for the problem (1.1)–(1.3). First, we recall the following definition.

Definition 2.1 Let V be a normed space and V^* be its dual. Let $\psi : V \to \mathbb{R}$ be a locally Lipschitz functional. The generalized (Clarke) directional derivative of ψ at $u \in V$ in the direction $v \in V$ is defined by

$$\psi^0(u; v) = \limsup_{w \to u,\, \lambda \downarrow 0} \frac{\psi(w + \lambda v) - \psi(w)}{\lambda}.$$

The generalized gradient (subdifferential) of ψ at u is defined by

$$\partial\psi(u) = \{\zeta \in V^* : \psi^0(u; v) \geq \langle \zeta, v\rangle \ \forall\, v \in V\}.$$

Knowing the generalized subdifferential, we can compute the generalized directional derivative through the formula [13]

$$\psi^0(u; v) = \max\{\langle \zeta, v\rangle : \zeta \in \partial\psi(u)\}.$$

Introduce function spaces

$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_D,\ v_n = 0 \text{ a.e. on } \Gamma_S\},$$
$$Q = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q(\boldsymbol{x})\, dx = 0 \right\}$$

for the velocity and pressure variables. We assume meas $(\Gamma_D) > 0$. Then the seminorm $|\cdot|_{1,\Omega}$ is a norm over $\boldsymbol{V}$, which is equivalent to the standard $\boldsymbol{H}^1(\Omega)$ norm. In this paper, we will use $|\cdot|_{1,\Omega}$ for the norm on $\boldsymbol{V}$ and use $\|\cdot\|_{0,\Omega}$ for the norm on Q.

The hemivariational inequality formulation of the problem (1.1)–(1.3) is to find $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ such that

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + \int_{\Gamma_S} j^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\, ds \geq (\boldsymbol{f}, \boldsymbol{v}) & \forall\, \boldsymbol{v} \in \boldsymbol{V}, \\ b(\boldsymbol{u}, q) = 0 & \forall\, q \in Q, \end{cases} \tag{2.1}$$

where

$$a(\boldsymbol{u},\boldsymbol{v}) = \nu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v}\, dx,$$
$$b(\boldsymbol{v},q) = -\int_\Omega q\, \mathrm{div}\boldsymbol{v}\, dx,$$
$$(\boldsymbol{f},\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\, dx.$$

Obviously, the bilinear form $a(\cdot,\cdot)$ is continuous and coercive:

$$|a(\boldsymbol{u},\boldsymbol{v})| \le \nu|\boldsymbol{u}|_{1,\Omega}|\boldsymbol{v}|_{1,\Omega} \quad \text{and} \quad a(\boldsymbol{v},\boldsymbol{v}) = \nu|\boldsymbol{v}|^2_{1,\Omega} \quad \forall\, \boldsymbol{u},\boldsymbol{v}\in \boldsymbol{V}.$$

It is well known that the bilinear form $b(\cdot,\cdot)$ is continuous and satisfies the inf-sup condition:

$$|b(\boldsymbol{v},q)| \lesssim |\boldsymbol{v}|_{1,\Omega}\|q\|_{0,\Omega} \quad \text{and} \quad \sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{b(\boldsymbol{v},q)}{|\boldsymbol{v}|_{1,\Omega}} \ge \beta_0\|q\|_{0,\Omega} \quad \forall\,(\boldsymbol{v},q)\in \boldsymbol{V}\times Q.$$

Here and below, the abbreviation $a \lesssim b$ stands for the inequality $a \le Cb$, where $C > 0$ denotes a generic constant, which may take different values at different occurrences. For the function $j : \Gamma_S \times \mathbb{R}^d \to \mathbb{R}$, we assume

$$\begin{cases} \text{(a)}\, j(\cdot,\boldsymbol{\xi}) \text{ is measurable on } \Gamma_S \text{ for all } \boldsymbol{\xi}\in\mathbb{R}^d \text{ and } j(\cdot,\boldsymbol{e}(\cdot))\in L^1(\Gamma_S) \text{ for some } \boldsymbol{e}\in \boldsymbol{L}^2(\Gamma_S);\\ \text{(b)}\, j(\boldsymbol{x},\cdot) \text{ is locally Lipschitz on } \mathbb{R}^d \text{ for a.e. } \boldsymbol{x}\in\Gamma_S;\\ \text{(c) there exist constants } c_0, c_1 > 0 \text{ such that}\\ \quad |\partial j(\boldsymbol{x},\boldsymbol{\xi})| \le c_0 + c_1|\boldsymbol{\xi}| \quad \text{a.e. } \boldsymbol{x}\in\Gamma_S,\ \forall\,\boldsymbol{\xi}\in\mathbb{R}^d;\\ \text{(d) there exists a constant } m_\tau \text{ such that}\\ \quad j^0(\boldsymbol{x},\boldsymbol{\xi}_1;\boldsymbol{\xi}_2-\boldsymbol{\xi}_1) + j^0(\boldsymbol{x},\boldsymbol{\xi}_2;\boldsymbol{\xi}_1-\boldsymbol{\xi}_2) \le m_\tau|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|^2 \quad \text{a.e. } \boldsymbol{x}\in\Gamma_S,\ \forall\,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2\in\mathbb{R}^d. \end{cases} \tag{2.2}$$

The Sobolev trace inequality over $\boldsymbol{V}$ takes the form

$$\|\boldsymbol{v}\|_{0,\Gamma_S} \le \mu_1^{-1/2}|\boldsymbol{v}|_{1,\Omega} \quad \forall\,\boldsymbol{v}\in\boldsymbol{V}, \tag{2.3}$$

where $\mu_1 > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\begin{aligned} -\Delta\boldsymbol{u} &= \boldsymbol{0} && \text{in } \Omega,\\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \Gamma_D,\\ u_n &= 0 && \text{on } \Gamma_S,\\ \partial\boldsymbol{u}_\tau/\partial\boldsymbol{n} &= \mu\,\boldsymbol{u}_\tau && \text{on } \Gamma_S. \end{aligned}$$

Denote

$$\boldsymbol{V}_0 = \{\boldsymbol{v}\in\boldsymbol{V} : \mathrm{div}\,\boldsymbol{v} = 0 \text{ in } \Omega\}.$$

By restricting the test function $\boldsymbol{v}$ to the subspace $\boldsymbol{V}_0$, it is easy to see that the problem (2.1) is reduced to the following problem: Find $\boldsymbol{u}\in\boldsymbol{V}_0$ such that

$$a(\boldsymbol{u},\boldsymbol{v}) + \int_{\Gamma_S} j^0(\boldsymbol{u}_\tau;\boldsymbol{v}_\tau)\, ds \ge (\boldsymbol{f},\boldsymbol{v}) \quad \forall\,\boldsymbol{v}\in\boldsymbol{V}_0. \tag{2.4}$$

Then, we have the following existence, uniqueness and equivalence results; the proof is similar to that of [14, Theorems 3.4 and 3.5], and is hence omitted.

Theorem 2.2 *Under the assumption* (2.2) *and the smallness condition* $m_\tau \mu_1^{-1} < \nu$, *for any* $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, *the problem* (2.1) *has a unique solution* $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$, *and* $\boldsymbol{u} \in \boldsymbol{V}_0$ *is also the unique solution of the hemivariational inequality* (2.4).

3 VEM for the Hemivariational Inequality

We focus on the study of the virtual element solution of the hemivariational inequality (2.1) for the two-dimensional case, and comment that the discussion can be extended to the three-dimensional case.

Let Ω be a bounded polygonal domain. We express $\overline{\Gamma_S}$ as the union of closed flat components with disjoint interior: $\overline{\Gamma_S} = \cup_{l=1}^{l_S} \Gamma_{S,l}$. Let $\mathcal{T}_h$ be a decomposition of $\overline{\Omega}$ into general polygonal elements denoted by K. Let $h_K = \text{diam}(K)$, $h = \max\{h_K : K \in \mathcal{T}_h\}$, and $h_e = \text{length}(e)$. Following [2,3], for each h and every $K \in \mathcal{T}_h$, we assume that there exists a constant $\gamma > 0$ such that

A1 K is star-shaped with respect to a ball of radius $\geq \gamma h_K$;

A2 the distance between any two vertices of K is $\geq \gamma h_K$.

For each polygon $K \in \mathcal{T}_h$, let us consider the triangulation $\mathcal{T}_h^K$ of K obtained by connecting each vertex of K with the center of the ball with respect to which K is star-shaped. Let $\mathcal{E}_h$ stand for the union of the boundaries of all the elements in $\mathcal{T}_h$, $\mathcal{E}_h^i$ the set of all interior edges, $\mathcal{E}_h^S$ the set of all the edges on Γ_S, and $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \mathcal{E}_h^S$. Let K^+ and K^- be two neighboring elements with a common edge e. For a scalar function q, we denote by $q^\pm$ the trace of $q|_{K^\pm}$ on e from within $K^\pm$ and by $\boldsymbol{n}^\pm$ the unit normal on e in the outward direction with respect to $K^\pm$. Then, the jump of q across e is defined as $[q] = q^+\boldsymbol{n}^+ + q^-\boldsymbol{n}^-$. Similarly, the jump of a vector-valued function $\boldsymbol{v}$ on the interior edge e is given by

$$[\boldsymbol{v}] = \boldsymbol{v}^+ \cdot \boldsymbol{n}^+ + \boldsymbol{v}^- \cdot \boldsymbol{n}^-, \quad [\![\boldsymbol{v}]\!] = \boldsymbol{v}^+ - \boldsymbol{v}^-.$$

On an element edge $e \subset \Gamma$, we set

$$[q] = q\boldsymbol{n}, \quad [\boldsymbol{v}] = \boldsymbol{v} \cdot \boldsymbol{n}, \quad [\![\boldsymbol{v}]\!] = \boldsymbol{v}.$$

Let $k \geq 1$ be an integer. We use P_k to denote the space of the polynomials of degree $\leq k$, and use $\boldsymbol{P}_k = (P_k)^2$ for the corresponding vector-valued polynomial space. On each element $K \in \mathcal{T}_h$, we define the local virtual element space

$$\boldsymbol{V}_h^K = \Big\{ \boldsymbol{v} \in \boldsymbol{H}^1(K) : \Delta \boldsymbol{v} \in \boldsymbol{P}_{k-2}(K),\ \nabla \boldsymbol{v} \boldsymbol{n}_e \in \boldsymbol{P}_{k-1}(e) \quad \forall e \in \partial K \cap \mathcal{E}_h^0,$$
$$\boldsymbol{v} \in C^0(\partial K \cap \Gamma_S),\ \boldsymbol{v}|_e \in \boldsymbol{P}_k(e) \quad \forall e \in \partial K \cap \mathcal{E}_h^S \Big\}$$

and

$$Q_h^K = P_{k-1}(K).$$

It is easy to check that $\boldsymbol{P}_k(K) \subseteq \boldsymbol{V}_h^K$. For $\boldsymbol{v} \in \boldsymbol{V}_h^K$, we can choose the degrees of freedom $\boldsymbol{D}_{\boldsymbol{V}}^K(\boldsymbol{v}) = \{\boldsymbol{D}_{\boldsymbol{V}}^{K,1}(\boldsymbol{v}), \boldsymbol{D}_{\boldsymbol{V}}^{K,2}(\boldsymbol{v}), \boldsymbol{D}_{\boldsymbol{V}}^{K,3}(\boldsymbol{v})\}$ with

- $\boldsymbol{D}_{\boldsymbol{V}}^{K,1}(\boldsymbol{v})$: the values of $\boldsymbol{v}$ at $k+1$ Gauss–Lobatto points of edge $e \in \partial K \cap \mathcal{E}_h^S$,
- $\boldsymbol{D}_{\boldsymbol{V}}^{K,2}(\boldsymbol{v})$: the moments $\frac{1}{h_e}\int_e \boldsymbol{v} \cdot \boldsymbol{q}_{k-1}\, ds$ for any $\boldsymbol{q}_{k-1} \in \boldsymbol{P}_{k-1}(e)$, $e \in \partial K \cap \mathcal{E}_h^0$,
- $\boldsymbol{D}_{\boldsymbol{V}}^{K,3}(\boldsymbol{v})$: the moments $\frac{1}{|K|}\int_K \boldsymbol{v} \cdot \boldsymbol{q}_{k-2}\, dx$ for any $\boldsymbol{q}_{k-2} \in \boldsymbol{P}_{k-2}(K)$.

Furthermore, given $q \in Q_h^K$, we consider the following degrees of freedom:

- $\boldsymbol{D}_Q^K(q)$: the moments $\int_K q\, p_{k-1}\, dx$ for any $p_{k-1} \in P_{k-1}(K)$.

Lemma 3.1 *The degrees of freedom $\boldsymbol{D}_{\boldsymbol{V}}^K(\boldsymbol{v})$ uniquely determine $\boldsymbol{v} \in \boldsymbol{V}_h^K$.*

Proof We only need to show that if the degrees of freedom $\boldsymbol{D}_{\boldsymbol{V}}^K(\boldsymbol{v})$ are all zero for a given $\boldsymbol{v} \in \boldsymbol{V}_h^K$, then $\boldsymbol{v} = \boldsymbol{0}$. By integration by parts,

$$\int_K \nabla \boldsymbol{v} : \nabla \boldsymbol{v}\, dx = \sum_{e \in \partial K \cap \mathcal{E}_h^S} \int_e \nabla \boldsymbol{v} \boldsymbol{n} \cdot \boldsymbol{v}\, ds + \sum_{e \in \partial K \cap \mathcal{E}_h^0} \int_e \nabla \boldsymbol{v} \boldsymbol{n} \cdot \boldsymbol{v}\, ds - \int_K \Delta \boldsymbol{v} \cdot \boldsymbol{v}\, dx.$$

Each of the three terms on the right side of the above relation is zero since all the degrees of freedom for $\boldsymbol{v}$ are zero. Thus, $\nabla \boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{v}$ is a constant vector in K. If $\partial K \cap \mathcal{E}_h^S \neq \emptyset$, then $\boldsymbol{v} = \boldsymbol{0}$ on $\partial K \cap \mathcal{E}_h^S$. Otherwise, $\int_{\partial K} \boldsymbol{v}\, ds = \boldsymbol{0}$. In either case, we deduce that $\boldsymbol{v} = \boldsymbol{0}$. □

For $K \in \mathcal{T}_h$, let $\Pi_k^{0,K} : \boldsymbol{L}^2(K) \to \boldsymbol{P}_k(K)$ be the $\boldsymbol{L}^2(K)$ projection operator defined by

$$\int_K \Pi_k^{0,K} \boldsymbol{v} \cdot \boldsymbol{q}_k\, dx = \int_K \boldsymbol{v} \cdot \boldsymbol{q}_k\, dx \quad \forall \boldsymbol{q}_k \in \boldsymbol{P}_k(K). \tag{3.1}$$

For an interior edge e shared by $K^+, K^- \in \mathcal{T}_h$, we define the $\boldsymbol{L}^2(e)$ projection operator $\Pi_k^{0,e} : \boldsymbol{L}^2(e) \to \boldsymbol{P}_k(e)$ similarly. We recall the following approximation results [8].

Lemma 3.2 *Under the assumptions* **A1** *and* **A2** *on the decomposition $\mathcal{T}_h$, the following statements are valid.*

(a) *For any $\boldsymbol{u} \in \boldsymbol{H}^{k+1}(K)$,*

$$\|\boldsymbol{u} - \Pi_k^{0,K} \boldsymbol{u}\|_{0,K} + h_K |\boldsymbol{u} - \Pi_k^{0,K} \boldsymbol{u}|_{1,K} \lesssim h_K^{k+1} |\boldsymbol{u}|_{k+1,K}. \tag{3.2}$$

(b) *For any $\boldsymbol{u} \in \boldsymbol{H}^{k+1}(K^+ \cup K^-)$,*

$$\|\boldsymbol{u} - \Pi_k^{0,e} \boldsymbol{u}\|_{0,e} + h_e |\boldsymbol{u} - \Pi_k^{0,e} \boldsymbol{u}|_{1,e} \lesssim h_e^{k+1/2} |\boldsymbol{u}|_{k+1,K^+ \cup K^-}. \tag{3.3}$$

Remark 3.3 We note that the degrees of freedom $\boldsymbol{D}_{\boldsymbol{V}}^{K,3}(\boldsymbol{v})$ allow us to compute exactly the $\boldsymbol{L}^2(K)$ projection $\Pi_{k-2}^{0,K} \boldsymbol{v}$.

We now define the global virtual element spaces

$$\boldsymbol{V}_h = \Big\{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{v}|_K \in \boldsymbol{V}_h^K\ \forall K \in \mathcal{T}_h,\ \int_e [\![\boldsymbol{v}]\!] \cdot \boldsymbol{q}_{k-1}\, ds = 0\ \forall \boldsymbol{q}_{k-1} \in \boldsymbol{P}_{k-1}(e)\ \forall e \in \mathcal{E}_h^0,$$
$$\boldsymbol{v} \in C^0(\Gamma_S),\ \ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_S \Big\}$$

and

$$Q_h = \{ q \in Q : q|_K \in Q_h^K\ \forall K \in \mathcal{T}_h \}.$$

According to the discrete Poincaré-Friedrichs inequality for the piecewise $\boldsymbol{H}^1$ functions in $\boldsymbol{V}_h$ with $k \geq 1$ [7], the broken $\boldsymbol{H}^1$-seminorm

$$|\boldsymbol{v}_h|_{1,h} = \left(\sum_{K \in \mathcal{T}_h} |\boldsymbol{v}_h|_{1,K}^2 \right)^{\frac{1}{2}}$$

is a norm on $\boldsymbol{V}_h$, and it will be chosen as the norm on $\boldsymbol{V}_h$ in the rest of the paper.

In the following analysis, we extend the definition of the continuous bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ to $\boldsymbol{V}_h$ by

$$a(\boldsymbol{u}, \boldsymbol{v}_h) = \sum_{K\in\mathcal{T}_h} a^K(\boldsymbol{u}, \boldsymbol{v}_h) \quad \forall \boldsymbol{u} \in \boldsymbol{V}, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,$$
$$b(\boldsymbol{v}_h, q) = \sum_{K\in\mathcal{T}_h} b^K(\boldsymbol{v}_h, q) \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \quad \forall q \in Q,$$

where $a^K(\cdot,\cdot)$ and $b^K(\cdot,\cdot)$ are the restrictions of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ on the element K, respectively.

We now define the approximate bilinear forms $a_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ for the VEM. Let

$$\begin{aligned} b_h(\boldsymbol{v}_h, q_h) &= \sum_{K\in\mathcal{T}_h} b_h^K(\boldsymbol{v}_h, q_h) = -\sum_{K\in\mathcal{T}_h} \int_K q_h \operatorname{div} \boldsymbol{v}_h \, dx \\ &= -\sum_{K\in\mathcal{T}_h} \int_{\partial K} \boldsymbol{v}_h \cdot \boldsymbol{n}\, q_h \, ds + \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{v}_h \cdot \nabla q_h \, dx \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \quad \forall q_h \in Q_h. \end{aligned}$$

We notice that the right-hand side is computable using only the degrees of freedom of $\boldsymbol{v}_h$, since q_h is a polynomial of degree $\leq k-1$ in each element $K \in \mathcal{T}_h$. For any $K \in \mathcal{T}_h$, we define the projection $\Pi^K : \boldsymbol{V}_h^K \to \boldsymbol{P}_k(K)$ by

$$\begin{cases} a^K(\boldsymbol{q}, \boldsymbol{v}_h - \Pi^K \boldsymbol{v}_h) = 0 & \forall \boldsymbol{q} \in \boldsymbol{P}_k(K), \\ \int_{\partial K} (\boldsymbol{v}_h - \Pi^K \boldsymbol{v}_h) \, ds = 0 & \text{for} \quad k = 1, \\ \int_K (\boldsymbol{v}_h - \Pi^K \boldsymbol{v}_h) \, dx = 0 & \text{for} \quad k \geq 2. \end{cases} \tag{3.4}$$

In fact, $\Pi^K \boldsymbol{v}_h$ is computable in terms of the degrees of freedom $\boldsymbol{D}_V^K(\boldsymbol{v}_h)$: for all $\boldsymbol{q} \in \boldsymbol{P}_k(K)$ we have

$$a^K(\boldsymbol{q}, \boldsymbol{v}_h) = \nu \int_K \nabla \boldsymbol{q} : \nabla \boldsymbol{v}_h \, dx = \nu \int_{\partial K} \nabla \boldsymbol{q}\boldsymbol{n} \cdot \boldsymbol{v}_h \, ds - \nu \int_K \Delta \boldsymbol{q} \cdot \boldsymbol{v}_h \, d\boldsymbol{x}, \tag{3.5}$$

and the right-hand side is directly computable from $\boldsymbol{D}_V^K(\boldsymbol{v}_h)$. Clearly, $\Pi^K \boldsymbol{q}_k = \boldsymbol{q}_k$ for all $\boldsymbol{q}_k \in \boldsymbol{P}_k(K)$. Then, we can set [2]

$$a_h^K(\boldsymbol{u}_h, \boldsymbol{v}_h) = a^K(\Pi^K \boldsymbol{u}_h, \Pi^K \boldsymbol{v}_h) + S^K((I - \Pi^K)\boldsymbol{u}_h, (I - \Pi^K)\boldsymbol{v}_h) \quad \forall \boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{V}_h^K, \tag{3.6}$$

with

$$S^K(\boldsymbol{u}_h, \boldsymbol{v}_h) = \sum_{i=1}^{N^K} \mathrm{dof}_i(\boldsymbol{u}_h)\mathrm{dof}_i(\boldsymbol{v}_h), \tag{3.7}$$

where N^K is the dimension of $\boldsymbol{V}_h^K$, and $\mathrm{dof}_i(\boldsymbol{v}_h)$ denotes the ith-degree of freedom of $\boldsymbol{v}_h$. For a_h^K thus defined, the following properties hold true [2].

- *k-consistency* for all $\boldsymbol{q} \in \boldsymbol{P}_k(K)$ and $\boldsymbol{v}_h \in \boldsymbol{V}_h^K$,

$$a_h^K(\boldsymbol{q}, \boldsymbol{v}_h) = a^K(\boldsymbol{q}, \boldsymbol{v}_h); \tag{3.8}$$

- *stability* there exist two positive constants α_* and α^* independent of h and K such that

$$\alpha_* a^K(\boldsymbol{v}_h, \boldsymbol{v}_h) \leq a_h^K(\boldsymbol{v}_h, \boldsymbol{v}_h) \leq \alpha^* a^K(\boldsymbol{v}_h, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^K. \tag{3.9}$$

Then we define the bilinear form $a_h(\cdot,\cdot) : \boldsymbol{V}_h \times \boldsymbol{V}_h \to \mathbb{R}$ by the formula

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \sum_{K \in \mathcal{T}_h} a_h^K(\boldsymbol{u}_h, \boldsymbol{v}_h) \quad \forall\, \boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{3.10}$$

For a computable approximation of the right-hand side $(\boldsymbol{f}, \boldsymbol{v})$ in (2.1), we define

$$\boldsymbol{f}_h|_K = \begin{cases} \Pi_0^{0,K} \boldsymbol{f} & \text{for} \quad k = 1, \\ \Pi_{k-2}^{0,K} \boldsymbol{f} & \text{for} \quad k \geq 2. \end{cases} \tag{3.11}$$

Denote by $\overline{\boldsymbol{v}_h}$ the piecewise average of $\boldsymbol{v}_h$, i.e.,

$$\overline{\boldsymbol{v}_h}|_K = \frac{1}{|\partial K|} \int_{\partial K} \boldsymbol{v}_h \, ds \quad \text{for any element } K.$$

Then the discrete right-hand side is defined by

$$\langle \boldsymbol{f}_h, \boldsymbol{v}_h \rangle = \begin{cases} \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{f}_h \cdot \overline{\boldsymbol{v}_h} \, dx & \text{for} \quad k = 1, \\ \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{f}_h \cdot \boldsymbol{v}_h \, dx & \text{for} \quad k \geq 2. \end{cases} \tag{3.12}$$

The following error bounds hold [2]

$$|\langle \boldsymbol{f}_h, \boldsymbol{v}_h \rangle - (\boldsymbol{f}, \boldsymbol{v}_h)| \lesssim h^k \|\boldsymbol{f}\|_{k-1,\Omega} |\boldsymbol{v}_h|_{1,h}. \tag{3.13}$$

The virtual element method for solving the inequality problem (2.1) is to find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ such that

$$\begin{cases} a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) + \int_{\Gamma_S} j^0(\boldsymbol{u}_{h\tau}; \boldsymbol{v}_{h\tau}) \, ds \geq \langle \boldsymbol{f}_h, \boldsymbol{v}_h \rangle & \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ b_h(\boldsymbol{u}_h, q_h) = 0 & \forall\, q_h \in Q_h. \end{cases} \tag{3.14}$$

It can be verified that $a_h(\cdot,\cdot)$ is continuous and coercive over $\boldsymbol{V}_h$. By the broken trace theorem [9,21], there exists a constant λ such that

$$\|\boldsymbol{v}_h\|_{0,\Gamma_S} \leq \lambda^{-1/2} |\boldsymbol{v}_h|_{1,h} \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{3.15}$$

For an error analysis of the virtual element scheme (3.14), we assume

$$m_\tau \lambda^{-1} < \nu \alpha_* \tag{3.16}$$

from now on. Therefore, the well-posedness of the discrete problem (3.14) will follow if a suitable inf-sup condition is fulfilled, which is the topic of Sect. 4.1.

4 A Priori Error Analysis

First we quote the following result from the classical Scott–Dupont theory [8].

Lemma 4.1 *Assume* **A1**. *Let $K \in \mathcal{T}_h$. Then for any $\boldsymbol{u} \in \boldsymbol{H}^{k+1}(K)$, there exists a polynomial $\boldsymbol{u}_\pi \in \boldsymbol{P}_k(K)$ such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_\pi\|_{0,K} + h_K |\boldsymbol{u} - \boldsymbol{u}_\pi|_{1,K} \lesssim h_K^{k+1} |\boldsymbol{u}|_{k+1,K}. \tag{4.1}$$

For any $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, define $\boldsymbol{u}_I \in \boldsymbol{V}_h^K$ by the degrees of freedom as follows:

$$\boldsymbol{u}_I = \boldsymbol{u}_c \quad \text{on } \partial K \cap \mathcal{E}_h^S, \tag{4.2}$$

$$\int_e \boldsymbol{u}_I \cdot \boldsymbol{q}_{k-1} \, ds = \int_e \boldsymbol{u} \cdot \boldsymbol{q}_{k-1} \, ds \quad \forall\, \boldsymbol{q}_{k-1} \in \boldsymbol{P}_{k-1}(e), \quad \forall e \in \partial K \cap \mathcal{E}_h^0, \tag{4.3}$$

$$\int_K \boldsymbol{u}_I \cdot \boldsymbol{q}_{k-2}\, dx = \int_K \boldsymbol{u} \cdot \boldsymbol{q}_{k-2}\, dx \quad \forall \boldsymbol{q}_{k-2} \in \boldsymbol{P}_{k-2}(K), \tag{4.4}$$

where $\boldsymbol{u}_c \in \boldsymbol{H}^1(\Omega)$, whose restriction on K is the standard nodal interpolant of $\boldsymbol{u}$ in the continuous finite element space of piecewise polynomials corresponding to the local triangulation $\mathcal{T}_h^K$ of K. Let $\boldsymbol{u} \in \boldsymbol{V}$ be the solution of the problem (2.1). Observe that the global function $\boldsymbol{u}_I \in \boldsymbol{V}_h$. Then, for any $q_h \in Q_h$, we have

$$\begin{aligned} b_h(\boldsymbol{u}_I, q_h) &= -\sum_{K\in\mathcal{T}_h} \sum_{e\in\partial K\cap\mathcal{E}_h^0} \int_e \boldsymbol{u}_I \cdot \boldsymbol{n}\, q_h\, ds + \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{u}_I \cdot \nabla q_h\, dx \\ &= -\sum_{K\in\mathcal{T}_h} \int_{\partial K} \boldsymbol{u} \cdot \boldsymbol{n}\, q_h\, ds + \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{u} \cdot \nabla q_h\, dx \\ &= b(\boldsymbol{u}, q_h), \end{aligned} \tag{4.5}$$

which gives $b_h(\boldsymbol{u}_I, q_h) = 0$. Moreover, we have the following approximation result, which can be proven as in [2,25].

Lemma 4.2 *Let $\boldsymbol{u} \in \boldsymbol{V} \cap \boldsymbol{H}^{k+1}(K)$. Under the assumptions* **A1** *and* **A2** *on the decomposition $\mathcal{T}_h$, it holds*

$$\|\boldsymbol{u} - \boldsymbol{u}_I\|_{0,K} + h_K |\boldsymbol{u} - \boldsymbol{u}_I|_{1,K} \lesssim h_K^{k+1} |\boldsymbol{u}|_{k+1,K}. \tag{4.6}$$

For the pressure $p \in H^k(\Omega)$, from classic polynomial approximation theory [8], it holds

$$\|p - p_I\|_{0,\Omega} \lesssim h^k \|p\|_{k,\Omega}, \tag{4.7}$$

where p_I is the L^2 projection of p to the space Q_h.

4.1 The inf-sup condition

The aim of this section is to prove that the following inf-sup condition holds.

Lemma 4.3 *There exists a positive constant β independent of h such that*

$$\sup_{\boldsymbol{v}_h \in \widetilde{\boldsymbol{V}}_h} \frac{b_h(\boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_{1,h}} \geq \beta \|q_h\|_{0,\Omega} \quad \forall q_h \in Q_h, \tag{4.8}$$

where $\widetilde{\boldsymbol{V}}_h = \{\boldsymbol{v}_h \in \boldsymbol{V}_h : \boldsymbol{v}_{h\tau} = \boldsymbol{0} \text{ on } \Gamma_S\}$.

Proof It is well known that there exists a positive constant β independent of h such that for any $q_h \in Q_h$ there exists a function $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$ satisfying

$$\frac{b(\boldsymbol{v}, q_h)}{|\boldsymbol{v}|_{1,\Omega}} \geq \beta \|q_h\|_{0,\Omega}. \tag{4.9}$$

From (4.5) we know that $b_h(\boldsymbol{v}_I, q_h) = b(\boldsymbol{v}, q_h)$. By noting $\boldsymbol{v}_I \in \widetilde{\boldsymbol{V}}_h$, we have

$$\begin{aligned} |\boldsymbol{v}_I|_{1,K}^2 &= \int_K \nabla \boldsymbol{v}_I : \nabla \boldsymbol{v}_I\, dx \\ &= \sum_{e\in\partial K\cap\mathcal{E}_h^0} \int_e \nabla \boldsymbol{v}_I \boldsymbol{n} \cdot \boldsymbol{v}_I\, ds - \int_K \Delta \boldsymbol{v}_I \cdot \boldsymbol{v}_I\, dx \end{aligned}$$

$$= \int_{\partial K} \nabla \boldsymbol{v}_I \boldsymbol{n} \cdot \boldsymbol{v}\, ds - \int_K \Delta \boldsymbol{v}_I \cdot \boldsymbol{v}\, dx$$
$$= \int_K \nabla \boldsymbol{v}_I : \nabla \boldsymbol{v}\, dx \le |\boldsymbol{v}_I|_{1,K} |\boldsymbol{v}|_{1,K}.$$

The last inequality implies that $|\boldsymbol{v}_I|_{1,h} \le |\boldsymbol{v}|_{1,\Omega}$. Then

$$\beta \|q_h\|_{0,\Omega} \le \frac{b(\boldsymbol{v}, q_h)}{|\boldsymbol{v}|_{1,\Omega}} \le \frac{b_h(\boldsymbol{v}_I, q_h)}{|\boldsymbol{v}_I|_{1,h}}. \tag{4.10}$$

Thus, we finish the proof of (4.8). □

Denote $\boldsymbol{V}_{h,0} = \{\boldsymbol{v}_h \in \boldsymbol{V}_h : b_h(\boldsymbol{v}_h, q_h) = 0\ \forall q_h \in Q_h\}$. Using the discrete inf-sup condition (4.8), the problem (3.14) is equivalent to the following problem: Find $\boldsymbol{u}_h \in \boldsymbol{V}_{h,0}$ such that

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + \int_{\Gamma_S} j^0(\boldsymbol{u}_{h\tau}; \boldsymbol{v}_{h\tau})\, ds \ge \langle \boldsymbol{f}_h, \boldsymbol{v}_h \rangle \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_{h,0}. \tag{4.11}$$

Similar to Theorem 2.2, we have the following result.

Theorem 4.4 *Under the assumptions* (2.2) *and* (3.16)*, there is a unique function* $\boldsymbol{u}_h \in \boldsymbol{V}_{h,0}$ *satisfying* (4.11).

Let us show that the solution $\boldsymbol{u}_h$ is uniformly bounded.

Lemma 4.5 *The solution* $\boldsymbol{u}_h$ *of the problem* (4.11) *is uniformly bounded independent of* h.

Proof We let $\boldsymbol{v}_h = -\boldsymbol{u}_h$ in (4.11) to get

$$a_h(\boldsymbol{u}_h, \boldsymbol{u}_h) \le \int_{\Gamma_S} j^0(\boldsymbol{u}_{h\tau}; -\boldsymbol{u}_{h\tau})\, ds + \langle \boldsymbol{f}_h, \boldsymbol{u}_h \rangle. \tag{4.12}$$

From (2.2) and (3.15), we have

$$\begin{aligned} \int_{\Gamma_S} j^0(\boldsymbol{u}_{h\tau}; -\boldsymbol{u}_{h\tau})\, ds &\le \int_{\Gamma_S} m_\tau |\boldsymbol{u}_{h\tau}|^2\, ds - \int_{\Gamma_S} j^0(0; \boldsymbol{u}_{h\tau})\, ds \\ &\le m_\tau \lambda^{-1} |\boldsymbol{u}_h|_{1,h}^2 + \int_{\Gamma_S} c_0 |\boldsymbol{u}_{h\tau}|\, ds \\ &\le m_\tau \lambda^{-1} |\boldsymbol{u}_h|_{1,h}^2 + C |\boldsymbol{u}_h|_{1,h}. \end{aligned} \tag{4.13}$$

Apply (3.9) and (4.13) in (4.12),

$$(\nu \alpha_* - m_\tau \lambda^{-1}) |\boldsymbol{u}_h|_{1,h}^2 \le C\, |\boldsymbol{u}_h|_{1,h}. \tag{4.14}$$

Since $m_\tau \lambda^{-1} < \nu \alpha_*$, the above inequality implies the uniform boundedness of $|\boldsymbol{u}_h|_{1,h}$ with respect to h. □

4.2 Error estimation

We now bound the error for the VEM of an arbitrary order $k \ge 1$.

Theorem 4.6 *Let $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ be the solution of the problem (2.1) and $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ be the solution of the problem (3.14). Assume $\boldsymbol{u} \in \boldsymbol{H}^{k+1}(\Omega)$, $\boldsymbol{u}|_{\Gamma_{S,l}} \in \boldsymbol{H}^{k+1}(\Gamma_{S,l})$, $1 \le l \le l_S$, and $p \in H^k(\Omega)$. Then*

$$|\boldsymbol{u} - \boldsymbol{u}_h|_{1,h} + \|p - p_h\|_{0,\Omega} \lesssim h^{\frac{k+1}{2}} \left(\|\boldsymbol{u}\|_{k+1,\Omega} + \|p\|_{k,\Omega} + \|\boldsymbol{f}\|_{k-1,\Omega} + \left(\sum_{l=1}^{l_S} \|\boldsymbol{u}\|^2_{k+1,\Gamma_{S,l}} \right)^{\frac{1}{4}} \right). \tag{4.15}$$

Proof First, we derive from the inequality in (2.1) that

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega).$$

Thus,

$$-\nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in the sense of distribution in } \Omega.$$

Under the stated solution regularity, we further have

$$-\nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{a.e. in } \Omega. \tag{4.16}$$

We multiply both sides of the equation (4.16) by an arbitrary $\boldsymbol{v} \in \boldsymbol{V}$ and integrate over Ω:

$$-\nu \int_\Omega \Delta \boldsymbol{u} \cdot \boldsymbol{v}\, dx + \int_\Omega \nabla p \cdot \boldsymbol{v}\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx.$$

Then integrate by parts and make use of the homogeneous boundary conditions on $\boldsymbol{v} \in \boldsymbol{V}$ to obtain

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + \int_{\Gamma_S} (-\boldsymbol{\sigma}_\tau) \cdot \boldsymbol{v}_\tau\, ds = (\boldsymbol{f}, \boldsymbol{v}).$$

Comparing this equality with (2.1), we find

$$\int_{\Gamma_S} \left[-\boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau - j^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau) \right] ds \le 0 \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}.$$

It can be derived from the above inequality that

$$-\boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau \le j^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau) \quad \text{a.e. on } \Gamma_S.$$

Since the vector $\boldsymbol{v}_\tau$ is arbitrary, we see that

$$-\boldsymbol{\sigma}_\tau \in \partial j(\boldsymbol{u}_\tau) \quad \text{a.e. on } \Gamma_S. \tag{4.17}$$

Denote $\boldsymbol{e}_h = \boldsymbol{u}_I - \boldsymbol{u}_h$. Then

$$\begin{aligned} \nu \alpha_* |\boldsymbol{e}_h|^2_{1,h} &\le a_h(\boldsymbol{e}_h, \boldsymbol{e}_h) = a_h(\boldsymbol{u}_I, \boldsymbol{e}_h) - a_h(\boldsymbol{u}_h, \boldsymbol{e}_h) \\ &\le \sum_{K \in \mathcal{T}_h} (a_h^K(\boldsymbol{u}_I - \boldsymbol{u}_\pi, \boldsymbol{e}_h) + a^K(\boldsymbol{u}_\pi - \boldsymbol{u}, \boldsymbol{e}_h)) + a(\boldsymbol{u}, \boldsymbol{e}_h) \\ &\quad + b_h(\boldsymbol{e}_h, p_h) + \int_{\Gamma_S} j^0(\boldsymbol{u}_{h\tau}; \boldsymbol{e}_{h\tau})\, ds - \langle \boldsymbol{f}_h, \boldsymbol{e}_h \rangle. \end{aligned}$$

By integration by parts,

$$a(\boldsymbol{u}, \boldsymbol{e}_h) = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \nu \nabla \boldsymbol{u} \boldsymbol{n} \cdot \boldsymbol{e}_h\, ds - \sum_{K \in \mathcal{T}_h} \int_K \nu \Delta \boldsymbol{u} \cdot \boldsymbol{e}_h\, dx.$$

Making use of the pointwise relation (4.16), we have

$$a(\boldsymbol{u},\boldsymbol{e}_h)=\sum_{K\in\mathcal{T}_h}\int_{\partial K}\nu\nabla\boldsymbol{u}\boldsymbol{n}\cdot\boldsymbol{e}_h\,ds-\sum_{K\in\mathcal{T}_h}\int_{\partial K}p\boldsymbol{n}\cdot\boldsymbol{e}_h\,ds-b(\boldsymbol{e}_h,p)+(\boldsymbol{f},\boldsymbol{e}_h).$$

Using the definition of the jump operators, we can write

$$\begin{aligned}a(\boldsymbol{u},\boldsymbol{e}_h)=\nu\sum_{e\in\mathcal{E}_h^0}\int_e\nabla\boldsymbol{u}\boldsymbol{n}\cdot[\![\boldsymbol{e}_h]\!]\,ds-\sum_{e\in\mathcal{E}_h^0}\int_e p\,[\boldsymbol{e}_h]\,ds+\int_{\Gamma_S}\boldsymbol{\sigma}_\tau\cdot\boldsymbol{e}_{h\tau}\,ds\\-b(\boldsymbol{e}_h,p)+(\boldsymbol{f},\boldsymbol{e}_h).\end{aligned}\tag{4.18}$$

Then apply (4.17),

$$\begin{aligned}a(\boldsymbol{u},\boldsymbol{e}_h)\le\nu\sum_{e\in\mathcal{E}_h^0}\int_e\nabla\boldsymbol{u}\boldsymbol{n}\cdot[\![\boldsymbol{e}_h]\!]\,ds-\sum_{e\in\mathcal{E}_h^0}\int_e p\,[\boldsymbol{e}_h]\,ds+\int_{\Gamma_S}j^0(\boldsymbol{u}_\tau;\boldsymbol{u}_{h\tau}-\boldsymbol{u}_{I\tau})\,ds\\-b(\boldsymbol{e}_h,p)+(\boldsymbol{f},\boldsymbol{e}_h).\end{aligned}$$

Hence,

$$\nu\alpha_*|\boldsymbol{e}_h|_{1,h}^2\le I_1+I_2+I_3+I_4+I_5,\tag{4.19}$$

where

$$I_1=\sum_{K\in\mathcal{T}_h}(a_h^K(\boldsymbol{u}_I-\boldsymbol{u}_\pi,\boldsymbol{e}_h)+a^K(\boldsymbol{u}_\pi-\boldsymbol{u},\boldsymbol{e}_h)),\tag{4.20}$$

$$I_2=(\boldsymbol{f},\boldsymbol{e}_h)-\langle\boldsymbol{f}_h,\boldsymbol{e}_h\rangle,\tag{4.21}$$

$$I_3=b_h(\boldsymbol{e}_h,p_h)-b(\boldsymbol{e}_h,p),\tag{4.22}$$

$$I_4=\nu\sum_{e\in\mathcal{E}_h^0}\int_e\nabla\boldsymbol{u}\boldsymbol{n}\cdot[\![\boldsymbol{e}_h]\!]\,ds-\sum_{e\in\mathcal{E}_h^0}\int_e p\,[\boldsymbol{e}_h]\,ds,\tag{4.23}$$

$$I_5=\int_{\Gamma_S}j^0(\boldsymbol{u}_\tau;\boldsymbol{u}_{h\tau}-\boldsymbol{u}_{I\tau})\,ds+\int_{\Gamma_S}j^0(\boldsymbol{u}_{h\tau};\boldsymbol{u}_{I\tau}-\boldsymbol{u}_{h\tau})\,ds.\tag{4.24}$$

Let us bound each of the terms I_1, I_2, I_3, I_4 and I_5. By the boundedness of the bilinear forms and the modified Young's inequality with an arbitrarily small $\varepsilon>0$, we have

$$\begin{aligned}I_1&\le\nu(\alpha^*|\boldsymbol{u}_I-\boldsymbol{u}_\pi|_{1,h}+|\boldsymbol{u}-\boldsymbol{u}_\pi|_{1,h})|\boldsymbol{e}_h|_{1,h}\\&\le\frac{\varepsilon}{4}|\boldsymbol{e}_h|_{1,h}^2+C(|\boldsymbol{u}-\boldsymbol{u}_I|_{1,h}^2+|\boldsymbol{u}-\boldsymbol{u}_\pi|_{1,h}^2),\end{aligned}\tag{4.25}$$

$$I_2\le\|\boldsymbol{f}-\boldsymbol{f}_h\|_{\boldsymbol{V}_h^*}|\boldsymbol{e}_h|_{1,h}\le\frac{\varepsilon}{4}|\boldsymbol{e}_h|_{1,h}^2+C\|\boldsymbol{f}-\boldsymbol{f}_h\|_{\boldsymbol{V}_h^*}^2.\tag{4.26}$$

By using the fact that $b_h(\boldsymbol{u}_I,q_h)=0$ for any $q_h\in Q_h$, and (3.14), we obtain

$$I_3=b_h(\boldsymbol{e}_h,p_I)-b(\boldsymbol{e}_h,p)\lesssim\|p-p_I\|_{0,\Omega}|\boldsymbol{e}_h|_{1,h}\le\frac{\varepsilon}{4}|\boldsymbol{e}_h|_{1,h}^2+C\|p-p_I\|_{0,\Omega}^2.\tag{4.27}$$

Using the definition of $\boldsymbol{V}_h$, we have

$$\int_e[\![\boldsymbol{e}_h]\!]\cdot\boldsymbol{q}_{k-1}\,ds=0\quad\forall\,\boldsymbol{q}_{k-1}\in\boldsymbol{P}_{k-1}(e),\quad\forall\,e\in\mathcal{E}_h^0.$$

Thus, it holds that

$$\int_e[\boldsymbol{e}_h]\,q_{k-1}\,ds=0\quad\forall\,q_{k-1}\in P_{k-1}(e),\quad\forall\,e\in\mathcal{E}_h^0.$$

Then we obtain

$$\nu \sum_{e\in\mathcal{E}_h^0} \int_e \nabla \boldsymbol{u}\boldsymbol{n} \cdot [\![\boldsymbol{e}_h]\!]\, ds = \nu \sum_{e\in\mathcal{E}_h^0} \int_e (\nabla \boldsymbol{u}\boldsymbol{n} - \Pi_{k-1}^{0,e}(\nabla \boldsymbol{u}\boldsymbol{n})) \cdot ([\![\boldsymbol{e}_h]\!] - \Pi_0^{0,e}[\![\boldsymbol{e}_h]\!])\, ds$$

for the velocity and

$$\sum_{e\in\mathcal{E}_h^0} \int_e p\, [\boldsymbol{e}_h]\, ds = \sum_{e\in\mathcal{E}_h^0} \int_e (p - \Pi_{k-1}^{0,e} p)\, ([\boldsymbol{e}_h] - \Pi_0^{0,e}[\boldsymbol{e}_h])\, ds$$

for the pressure. Using the Cauchy–Schwartz inequality and then applying the approximation estimates of Lemma 3.2, we get

$$\begin{aligned} I_4 &\lesssim h^k \sum_{e\in\mathcal{E}_h^0} (\|\boldsymbol{u}\|_{k+1,K^+\cup K^-} + \|p\|_{k,K^+\cup K^-})|\boldsymbol{e}_h|_{1,K^+\cup K^-} \\ &\lesssim h^k(\|\boldsymbol{u}\|_{k+1,\Omega} + \|p\|_{k,\Omega})|\boldsymbol{e}_h|_{1,h} \\ &\le \frac{\varepsilon}{4}|\boldsymbol{e}_h|_{1,h}^2 + Ch^{2k}(\|\boldsymbol{u}\|_{k+1,\Omega}^2 + \|p\|_{k,\Omega}^2). \end{aligned} \tag{4.28}$$

By the subadditivity of the generalized directional derivative, we have

$$\begin{aligned} I_5 \le& \int_{\Gamma_S} j^0(\boldsymbol{u}_\tau; \boldsymbol{u}_{h\tau} - \boldsymbol{u}_\tau)\, ds + \int_{\Gamma_S} j^0(\boldsymbol{u}_{h\tau}; \boldsymbol{u}_\tau - \boldsymbol{u}_{h\tau})\, ds \\ &+ \int_{\Gamma_S} j^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau - \boldsymbol{u}_{I\tau})\, ds + \int_{\Gamma_S} j^0(\boldsymbol{u}_{h\tau}; \boldsymbol{u}_{I\tau} - \boldsymbol{u}_\tau)\, ds. \end{aligned}$$

By (2.2) (d) and (3.15),

$$\begin{aligned} &\int_{\Gamma_S} j^0(\boldsymbol{u}_\tau; \boldsymbol{u}_{h\tau} - \boldsymbol{u}_\tau)\, ds + \int_{\Gamma_S} j^0(\boldsymbol{u}_{h\tau}; \boldsymbol{u}_\tau - \boldsymbol{u}_{h\tau})\, ds \le \int_{\Gamma_S} m_\tau |\boldsymbol{u}_\tau - \boldsymbol{u}_{h\tau}|^2\, ds \\ &\quad\le m_\tau \lambda^{-1}|\boldsymbol{u} - \boldsymbol{u}_h|_{1,h}^2 \le (m_\tau\lambda^{-1} + \varepsilon)|\boldsymbol{e}_h|_{1,h}^2 + C|\boldsymbol{u} - \boldsymbol{u}_I|_{1,h}^2. \end{aligned}$$

From (2.2) (c),

$$\begin{aligned} \int_{\Gamma_S} j^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau - \boldsymbol{u}_{I\tau})\, ds &\le \int_{\Gamma_S} (c_0 + c_1|\boldsymbol{u}_\tau|)|\boldsymbol{u}_\tau - \boldsymbol{u}_{I\tau}|\, ds, \\ \int_{\Gamma_S} j^0(\boldsymbol{u}_{h\tau}; \boldsymbol{u}_{I\tau} - \boldsymbol{u}_\tau)\, ds &\le \int_{\Gamma_S} (c_0 + c_1|\boldsymbol{u}_{h\tau}|)|\boldsymbol{u}_\tau - \boldsymbol{u}_{I\tau}|\, ds. \end{aligned}$$

Note that $\|\boldsymbol{u}_h\|_{0,\Gamma_S}$ is bounded by a constant independent of h. Thus,

$$I_5 \le (m_\tau\lambda^{-1} + \varepsilon)|\boldsymbol{e}_h|_{1,h}^2 + C(|\boldsymbol{u} - \boldsymbol{u}_I|_{1,h}^2 + \|\boldsymbol{u} - \boldsymbol{u}_I\|_{0,\Gamma_S}). \tag{4.29}$$

With the solution regularity assumption $\boldsymbol{u}|_{\Gamma_{S,l}} \in \boldsymbol{H}^{k+1}(\Gamma_{S,l})$, $1 \le l \le l_S$, we obtain

$$\|\boldsymbol{u} - \boldsymbol{u}_I\|_{0,\Gamma_S} = \|\boldsymbol{u} - \boldsymbol{u}_c\|_{0,\Gamma_S} \lesssim h^{k+1}\left(\sum_{l=1}^{l_S} \|\boldsymbol{u}\|_{k+1,\Gamma_{S,l}}^2\right)^{\frac{1}{2}}. \tag{4.30}$$

Then, combining (4.25)–(4.30) in (4.19), we get

$$(\nu\alpha_* - m_\tau\lambda^{-1} - 2\varepsilon)|\boldsymbol{e}_h|_{1,h}^2 \lesssim h^{2k}(\|\boldsymbol{u}\|_{k+1,\Omega}^2 + \|p\|_{k,\Omega}^2 + \|\boldsymbol{f}\|_{k-1,\Omega}^2) + h^{k+1}\left(\sum_{l=1}^{l_S}\|\boldsymbol{u}\|_{k+1,\Gamma_{S,l}}^2\right)^{\frac{1}{2}}. \tag{4.31}$$

Since $m_\tau\lambda^{-1} < \nu\alpha_*$, we can choose $\varepsilon = (\nu\alpha_* - m_\tau\lambda^{-1})/4 > 0$ to get

$$|\boldsymbol{e}_h|_{1,h} \lesssim h^{\frac{k+1}{2}}\left(\|\boldsymbol{u}\|_{k+1,\Omega} + \|p\|_{k,\Omega} + \|\boldsymbol{f}\|_{k-1,\Omega} + \left(\sum_{l=1}^{l_S}\|\boldsymbol{u}\|_{k+1,\Gamma_{S,l}}^2\right)^{\frac{1}{4}}\right). \tag{4.32}$$

By the triangle inequality, we finish the estimate for $|\boldsymbol{u} - \boldsymbol{u}_h|_{1,h}$.

We next estimate the error for the pressure. From the discrete inf-sup condition (4.8), we have

$$\beta\|p_h - p_I\|_{0,\Omega} \le \sup_{\boldsymbol{v}_h \in \widetilde{\boldsymbol{V}}_h} \frac{b_h(\boldsymbol{v}_h, p_h - p_I)}{|\boldsymbol{v}_h|_{1,h}}. \tag{4.33}$$

Take $\boldsymbol{v}_h \in \widetilde{\boldsymbol{V}}_h$ (defined in Lemma 4.3) as a test function in the first relation of (3.14) to obtain

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) = \langle \boldsymbol{f}_h, \boldsymbol{v}_h\rangle \quad \forall\, \boldsymbol{v}_h \in \widetilde{\boldsymbol{V}}_h. \tag{4.34}$$

Then,

$$\begin{aligned} b_h(\boldsymbol{v}_h, p_h - p_I) &= b_h(\boldsymbol{v}_h, p_h) - b_h(\boldsymbol{v}_h, p_I) \\ &= -a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + \langle \boldsymbol{f}_h, \boldsymbol{v}_h\rangle - b_h(\boldsymbol{v}_h, p_I) \\ &= a(\boldsymbol{u}, \boldsymbol{v}_h) - a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + \langle \boldsymbol{f}_h, \boldsymbol{v}_h\rangle - (\boldsymbol{f}, \boldsymbol{v}_h) \\ &\quad + (\boldsymbol{f}, \boldsymbol{v}_h) - a(\boldsymbol{u}, \boldsymbol{v}_h) - b_h(\boldsymbol{v}_h, p_I). \end{aligned}$$

By integration by parts and using the pointwise relation (4.16), we obtain

$$(\boldsymbol{f}, \boldsymbol{v}_h) - a(\boldsymbol{u}, \boldsymbol{v}_h) = -\sum_{K\in\mathcal{T}_h}\int_{\partial K} \nu\nabla\boldsymbol{u}\boldsymbol{n}\cdot\boldsymbol{v}_h\, ds + \sum_{K\in\mathcal{T}_h}\int_{\partial K} p\boldsymbol{n}\cdot\boldsymbol{v}_h\, ds + b(\boldsymbol{v}_h, p).$$

Using the definition of the jump operators for $\boldsymbol{v}_h \in \widetilde{\boldsymbol{V}}_h$, we can write

$$(\boldsymbol{f}, \boldsymbol{v}_h) - a(\boldsymbol{u}, \boldsymbol{v}_h) = -\nu\sum_{e\in\mathcal{E}_h^0}\int_e \nabla\boldsymbol{u}\boldsymbol{n}\cdot[\![\boldsymbol{v}_h]\!]\, ds + \sum_{e\in\mathcal{E}_h^0}\int_e p\,[\boldsymbol{v}_h]\, ds + b(\boldsymbol{v}_h, p).$$

Then we obtain

$$\begin{aligned} b_h(\boldsymbol{v}_h, p_h - p_I) = &\sum_{K\in\mathcal{T}_h}(a^K(\boldsymbol{u} - \boldsymbol{u}_\pi, \boldsymbol{v}_h) + a_h^K(\boldsymbol{u}_\pi - \boldsymbol{u}_h, \boldsymbol{v}_h)) + \langle \boldsymbol{f}_h, \boldsymbol{v}_h\rangle - (\boldsymbol{f}, \boldsymbol{v}_h) \\ &- \nu\sum_{e\in\mathcal{E}_h^0}\int_e \nabla\boldsymbol{u}\boldsymbol{n}\cdot[\![\boldsymbol{v}_h]\!]\, ds + \sum_{e\in\mathcal{E}_h^0}\int_e p\,[\boldsymbol{v}_h]\, ds + b(\boldsymbol{v}_h, p) - b_h(\boldsymbol{v}_h, p_I). \end{aligned} \tag{4.35}$$

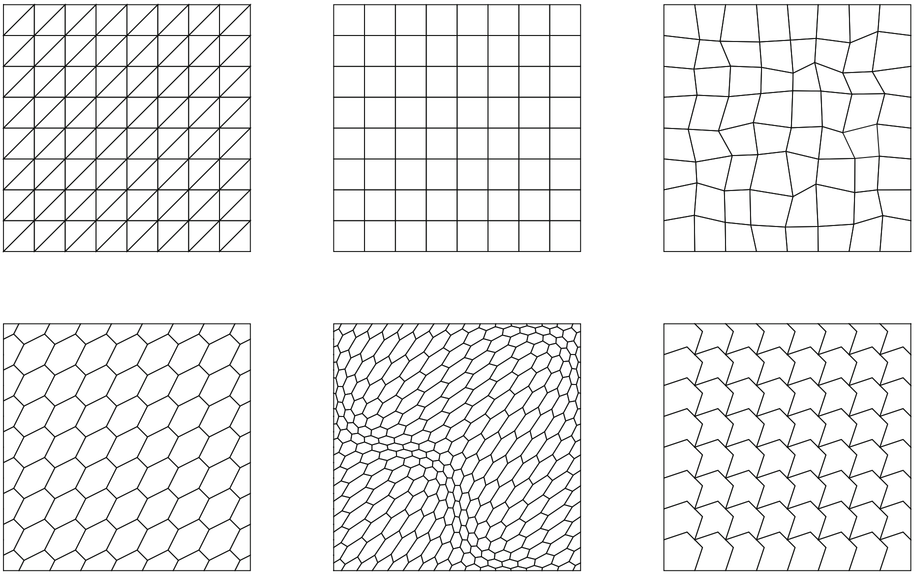

Fig. 1 Sample meshes: $\mathcal{T}_h^1$, $\mathcal{T}_h^2$, $\mathcal{T}_h^3$, $\mathcal{T}_h^4$, $\mathcal{T}_h^5$, and $\mathcal{T}_h^6$

Then, combining (3.13), (4.1), (4.7) and (4.28) in (4.35), we get

$$\begin{aligned} b_h(\boldsymbol{v}_h, p_h - p_I) &\lesssim |\boldsymbol{u} - \boldsymbol{u}_h|_{1,h}|\boldsymbol{v}_h|_{1,h} + h^k(\|\boldsymbol{u}\|_{k+1,\Omega} + \|p\|_{k,\Omega} + \|\boldsymbol{f}\|_{k-1,\Omega})|\boldsymbol{v}_h|_{1,h} \\ &\lesssim h^{\frac{k+1}{2}}\left(\|\boldsymbol{u}\|_{k+1,\Omega} + \|p\|_{k,\Omega} + \|\boldsymbol{f}\|_{k-1,\Omega} + \left(\sum_{l=1}^{l_S}\|\boldsymbol{u}\|^2_{k+1,\Gamma_{S,l}}\right)^{\frac{1}{4}}\right)|\boldsymbol{v}_h|_{1,h}, \end{aligned}$$

which gives

$$\|p_h - p_I\|_{0,\Omega} \lesssim h^{\frac{k+1}{2}}\left(\|\boldsymbol{u}\|_{k+1,\Omega} + \|p\|_{k,\Omega} + \|\boldsymbol{f}\|_{k-1,\Omega} + \left(\sum_{l=1}^{l_S}\|\boldsymbol{u}\|^2_{k+1,\Gamma_{S,l}}\right)^{\frac{1}{4}}\right). \tag{4.36}$$

By the triangle inequality, $\|p - p_h\|_{0,\Omega} \le \|p - p_I\|_{0,\Omega} + \|p_h - p_I\|_{0,\Omega}$; thus,

$$\|p - p_h\|_{0,\Omega} \lesssim h^{\frac{k+1}{2}}\left(\|\boldsymbol{u}\|_{k+1,\Omega} + \|p\|_{k,\Omega} + \|\boldsymbol{f}\|_{k-1,\Omega} + \left(\sum_{l=1}^{l_S}\|\boldsymbol{u}\|^2_{k+1,\Gamma_{S,l}}\right)^{\frac{1}{4}}\right). \tag{4.37}$$

we finish the proof of (4.15), which is optimal when $k = 1$. □

5 Numerical Example

In this section, we report some numerical results of an example to show the performance of the nonconforming virtual element method.

Table 1 Numerical errors of the lowest-order VEM on $\mathcal{T}_h^1$

h	$\Vert \boldsymbol{e_u} \Vert_{0,\Omega}$	Order	$\vert \boldsymbol{e_u} \vert_{1,h}$	Order	$\Vert e_p \Vert_{0,\Omega}$	Order
2^{-2}	3.2624e−01	–	4.7435	–	2.6099	–
2^{-3}	9.2558e−02	1.8175	2.4808	0.9351	1.2485	1.0638
2^{-4}	2.4607e−02	1.9113	1.2625	0.9746	6.0706e−01	1.0403
2^{-5}	6.2594e−03	1.9749	6.3464e−01	0.9922	2.9921e−01	1.0207
2^{-6}	1.5439e−03	2.0194	3.2109e−01	0.9829	1.4737e−01	1.0217

Table 2 Numerical errors of the lowest-order VEM on $\mathcal{T}_h^2$

h	$\Vert \boldsymbol{e_u} \Vert_{0,\Omega}$	Order	$\vert \boldsymbol{e_u} \vert_{1,h}$	Order	$\Vert e_p \Vert_{0,\Omega}$	Order
2^{-2}	5.5854e−01	–	5.7929	–	3.7521	–
2^{-3}	1.9279e−01	1.5347	2.8255	1.0358	1.9142	0.9709
2^{-4}	5.7292e−02	1.7506	1.3308	1.0862	8.9025e−01	1.1044
2^{-5}	1.5328e−02	1.9022	6.3676e−01	1.0635	4.1127e−01	1.1141
2^{-6}	3.7637e−03	2.0259	3.0604e−01	1.0569	1.9365e−01	1.0866

Table 3 Numerical errors of the lowest-order VEM on $\mathcal{T}_h^3$

h	$\Vert \boldsymbol{e_u} \Vert_{0,\Omega}$	Order	$\vert \boldsymbol{e_u} \vert_{1,h}$	Order	$\Vert e_p \Vert_{0,\Omega}$	Order
2^{-2}	5.4736e−01	–	5.8751	–	3.8700	–
2^{-3}	1.8643e−01	1.5538	2.9281	1.0047	1.8914	1.0329
2^{-4}	5.5458e−02	1.7492	1.4191	1.0449	9.1009e−01	1.0554
2^{-5}	1.4536e−02	1.9317	6.9011e−01	1.0401	4.2671e−01	1.0927
2^{-6}	3.5693e−03	2.0259	3.4286e−01	1.0092	2.0741e−01	1.0408

Table 4 Numerical errors of the lowest-order VEM on $\mathcal{T}_h^4$

h	$\Vert \boldsymbol{e_u} \Vert_{0,\Omega}$	Order	$\vert \boldsymbol{e_u} \vert_{1,h}$	Order	$\Vert e_p \Vert_{0,\Omega}$	Order
2^{-2}	4.1482e−01	–	4.7781	–	3.5181	–
2^{-3}	1.5843e−01	1.3887	2.5472	0.9075	1.8706	0.9113
2^{-4}	4.8624e−02	1.7041	1.2702	1.0038	8.7314e−01	1.0992
2^{-5}	1.3141e−02	1.8875	6.3708e−01	0.9955	4.1569e−01	1.0707
2^{-6}	3.3091e−03	1.9896	3.3172e−01	0.9415	2.1213e−01	0.9705

Let $\Omega = (0, 1) \times (0, 1)$, $\Gamma_S = (0, 1) \times \{0\}$ and $\Gamma_D = \Gamma \setminus \Gamma_S$. We choose $\nu = 1$, and let

$$\boldsymbol{f}(x, y) = \begin{bmatrix} 4\pi^2 (\sin(2\pi x) + \sin(2\pi y) - 2\sin(2\pi y)\cos(2\pi x)) \\ -4\pi^2 (\sin(2\pi x) + \sin(2\pi y) - 2\sin(2\pi x)\cos(2\pi y)) \end{bmatrix}.$$

Table 5 Numerical errors of the lowest-order VEM on $\mathcal{T}_h^5$

h	$\|\boldsymbol{e_u}\|_{0,\Omega}$	Order	$\|\boldsymbol{e_u}\|_{1,h}$	Order	$\|e_p\|_{0,\Omega}$	Order
2^{-2}	4.3661e−01	–	4.9079	–	3.6378	–
2^{-3}	2.2891e−01	0.9316	2.9144	0.7519	2.3561	0.6266
2^{-4}	9.6538e−02	1.2456	1.5395	0.9207	1.1978	0.9760
2^{-5}	3.1138e−02	1.6324	7.4261e−01	1.0518	5.2324e−01	1.1948
2^{-6}	8.3124e−03	1.9053	3.5959e−01	1.0462	2.2562e−01	1.2136

Table 6 Numerical errors of the lowest-order VEM on $\mathcal{T}_h^6$

h	$\|\boldsymbol{e_u}\|_{0,\Omega}$	Order	$\|\boldsymbol{e_u}\|_{1,h}$	Order	$\|e_p\|_{0,\Omega}$	Order
2^{-2}	4.7488e−01	–	5.3179	–	3.9472	–
2^{-3}	1.7041e−01	1.4785	2.7234	0.9655	1.9847	0.9919
2^{-4}	5.1707e−02	1.7206	1.3095	1.0564	9.0027e−01	1.1405
2^{-5}	1.3778e−02	1.9079	6.3585e−01	1.0422	4.1084e−01	1.1318
2^{-6}	3.4239e−03	2.0087	3.2048e−01	0.9884	2.0617e−01	0.9947

for positive parameters $a > b$ and α, we let

$$\mu(t) = (a-b)e^{-\alpha t} + b, \quad j(\boldsymbol{u}_\tau) = \int_0^{|\boldsymbol{u}_\tau|} \mu(t)\, dt.$$

Then the slip boundary condition $-\boldsymbol{\sigma}_\tau \in \partial j(\boldsymbol{u}_\tau)$ from (1.3) is equivalent to

$$|\boldsymbol{\sigma}_\tau| \le \mu(0) \quad \text{if } \boldsymbol{u}_\tau = \boldsymbol{0}, \quad \boldsymbol{\sigma}_\tau = -\mu(|\boldsymbol{u}_\tau|)\, \frac{\boldsymbol{u}_\tau}{|\boldsymbol{u}_\tau|} \quad \text{if } \boldsymbol{u}_\tau \ne \boldsymbol{0}, \quad \text{on } \Gamma_S. \tag{5.1}$$

It can be verified that for this choice of j, (2.2)(d) is satisfied with $m_\tau = \alpha(a-b)$. We take $a = 9.01$, $b = 9.0$, and $\alpha = 10$ for the function j in the numerical tests.

Because the true solution is unknown, the numerical solution on a sufficiently fine mesh ($h = 2^{-8}$) is used as the reference solution $(\boldsymbol{u}^*, p^*)$. Then we compare the numerical solutions $(\boldsymbol{u}_h, p_h)$ on the coarser meshes ($h = 2^{-n}$, $n = 2, 3, \ldots, 6$) with $(\boldsymbol{u}^*, p^*)$. We compute the numerical solution errors element-wise on the fine mesh ($h = 2^{-8}$). For each element of the fine mesh, we calculate its centroid and identify the element in the coarse mesh that contains the centroid. The general polygonal meshes are not nested when the mesh is refined, which gives rise to an additional error in computing the numerical solution errors. However, the additional error is expected to be of higher-order compared to the numerical solution errors because the mesh-size of the fine mesh ($h = 2^{-8}$) is much smaller than that of the coarse meshes ($h = 2^{-n}$ for $n \le 6$). Denote $\boldsymbol{e_u} = \Pi\boldsymbol{u}^* - \Pi\boldsymbol{u}_h$ and $e_p = p^* - p_h$. We compute the errors $\|\boldsymbol{e_u}\|_{0,\Omega}$, $|\boldsymbol{e_u}|_{1,h}$, and $\|e_p\|_{0,\Omega}$ for the lowest order method ($k = 1$) on six types of meshes: uniform triangulation $\mathcal{T}_h^1$, uniform rectangle mesh $\mathcal{T}_h^2$, quadrilateral mesh $\mathcal{T}_h^3$ by perturbing the interior nodes of $\mathcal{T}_h^2$ with a parameter 0.25, polygonal mesh $\mathcal{T}_h^4$ generated by the dual of the triangle mesh $\mathcal{T}_h^1$, distorted polygonal mesh $\mathcal{T}_h^5$, non-convex mesh $\mathcal{T}_h^6$, respectively (see Fig. 1). In Tables 1, 2, 3, 4, 5 and 6, we report the numerical solution errors of the velocity and pressure, respectively. We observe that the broken $\boldsymbol{H}^1$

error for the velocity and L^2 error for the pressure are of the order $O(h)$ for all types of the meshes, which matches the theoretical result in Theorem 4.6 with $k = 1$.

Data availability The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

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