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Discontinuous Galerkin methods for solving a hyperbolic inequality

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In this paper, we study spatially semi-discrete and fully discrete schemes to numerically solve a hyperbolic variational inequality, with discontinuous Galerkin (DG) discretization in space and finite difference discretization in time. Under appropriate regularity assumptions on the solution, a unified error analysis is established for four DG schemes, which reaches the optimal convergence order for linear elements. A numerical example is presented, and the numerical results confirm the theoretical error estimates.

KEYWORDS
discontinuous Galerkin methods, hyperbolic variational inequality, optimal order error estimate

1 | INTRODUCTION

In physical and engineering sciences, many problems are modeled by partial differential equations with proper boundary and/or initial conditions. However, various more complex physical processes are described by variational inequalities (VIs), which form an important and very useful class of nonlinear problems arising in diverse application areas of physical, engineering, financial, and management sciences, such as elastoplasticity and contact mechanics [1–4], heat control problem [1], pricing of options [5], and Nash-equilibria [6]. Various numerical methods, such as finite element method [7–10], finite difference method [11], finite volume method [12], and spectral element method [13], have been applied to discretize variational inequalities.

In the past four decades, due to their flexibility in constructing feasible local shape function spaces and their capability to capture nonsmooth or oscillatory solutions effectively, discontinuous Galerkin (DG) methods have been developed to solve a variety of equations, such as convection-diffusion equations [14, 15], hyperbolic equations [16–19], Navier–Stokes equations [20, 21], Hamilton-Jacobi equations [22, 23], the radiative transfer equation [24] and so on. A historical account of the methods can be found in [25]. A unified analysis of DG methods for elliptic problems was presented in [26].

The DG methods discretize differential equations in an element-by-element fashion, and glue neighboring elements together through numerical traces, which makes the methods locally conservative. A penalty term is added in the bilinear form of DG method to force the continuity of the primal variable, and this built-in stabilization mechanism does not degrade the high-order accuracy. Since no inter-element continuity is required in the function spaces, DG methods allow general meshes with hanging nodes and elements of different shapes, so they are very suitable for the implementation of *hp*-adaptive algorithm. Moreover, locality of the discretization makes the DG methods ideally suited for parallel computing (see [25, 26] and the references therein). Recently, DG methods have been applied for solving VIs, such as gradient plasticity problem [27, 28], obstacle problems [29, 30], Signorini problem [31, 32], quasistatic contact problems [33], plate contact problem [34–36], two membranes problem [37] and Stokes or Navier–Stokes flows with slip boundary condition [38, 39]. A posteriori error analysis of DG methods for VIs was also considered in [40–44].

However, to our best knowledge, there is no literature studying DG methods for hyperbolic type variational inequalities. In this paper, we study some DG methods to solve a hyperbolic variational inequality problem from ([21], Chapter 6, Section 8.2). Given an open bounded connected domain $\Omega \subset \mathbb{R}^d (d=2,3)$ with a Lipschitz boundary Γ, let us consider a hyperbolic type variational inequality [1, 9, 45]: Find $u \in L^2(0,T;V)$ with $\dot{u} \in L^2(0,T;V)$, $\ddot{u} \in L^2(0,T;V')$ s.t. for a.e. $t \in [0,T]$,

$$\dot{u}(t) \in K, \tag{1.1}$$

$$(\ddot{u}(t), v - \dot{u}(t)) + a(u(t), v - \dot{u}(t)) \geq (f(t), v - \dot{u}(t)) \qquad \forall v \in K, \tag{1.2}$$

and

$$u(0) = u_0, \quad \dot{u}(0) = v_0, \tag{1.3}$$

where

$$\begin{aligned} V &= H_0^1(\Omega), \\ K &= \{v \in V : v \geq 0 \text{ a.e. in } \Omega\}, \end{aligned} \tag{1.4}$$

and the bilinear forms

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx, \qquad (u,v) = \int_\Omega uv dx.$$

The VI (1.1)–(1.3) can be regarded as a scalar version of a moderated mechanical system problem where the velocity components are non-negative and when a velocity component is positive, then the motion equation in the corresponding coordinate is enforced with an adjusted external force.

For the well-posedness of the problem (1.1)–(1.3), we have the following theorem.

Theorem 1.1 *([21, 39], page 478) Assume*

$$f, \dot{f} \in L^2(0,T;L^2(\Omega)), -\Delta u_0 \in L^2(\Omega), \qquad v_0 \in K.$$

Then the problem (1.1)–(1.3) has a unique solution $u \in L^\infty(0,T;V)$, and $\dot{u} \in L^\infty(0,T;V)$, $\ddot{u} \in L^\infty(0,T;L^2(\Omega))$.

We observe that the solution has the continuity properties $u \in C([0,T];V)$ and $\dot{u} \in C([0,T];L^2(\Omega))$. The classical formulation of VI (1.1)–(1.3) is

$$\ddot{u} - \Delta u - f \geq 0, \qquad \dot{u} \geq 0, \qquad \dot{u}(\ddot{u} - \Delta u - f) = 0 \qquad \text{a.e. in } \Omega \times [0,T], \tag{1.5}$$

$$u = 0 \quad \text{a.e. on } \Gamma \quad \text{and} \quad u(0) = u_0, \quad \dot{u}(0) = v_0. \tag{1.6}$$

In general, the solution regularity for VIs is limited no matter how smooth the problem data are. A sample regularity result for the elliptic obstacle problem was proved by Brezies, compare [46, 47]. For the hyperbolic variational inequality (1.1)–(1.3), it appears that no higher solution regularity is available in the literature.

In this paper, we study four DG methods for solving this hyperbolic variational inequality, and provide a unified error analysis for these DG schemes. We show optimal order error estimates for linear elements. The paper is organized as follows: in Section 2 we introduce spatially semi-discrete and fully discrete schemes with DG discretization in space and finite difference discretization in time. Next, we derive a priori error estimates for the spatially semi-discrete schemes of these DG methods in Section 3, and for fully discrete scheme in Section 4. Then in Section 5, we report simulation results on a numerical example to show the numerical convergence orders that match the theoretical predictions.

2 | DG SCHEMES FOR THE HYPERBOLIC VI

2.1 | Notation

For definiteness, we only consider the case $d = 2$ in the rest of the paper, even though the discussion can be extended to the three-dimensional case. Given a bounded domain $D \subset \mathbb{R}^2$ and an integer $m \geq 0$, $W^{m,p}(D)$ is the Sobolev space with the corresponding usual norm $\|\cdot\|_{m,p,D}$ and semi-norm $|\cdot|_{m,p,D}$. We abbreviate them by $\|\cdot\|_{m,p}$ and $|\cdot|_{m,p}$, respectively when D is chosen as Ω. When $p = 2$, $W^{m,2}(D)$ is written as $H^m(D)$ for convenience, and the associated norm and semi-norm are denoted by $\|\cdot\|_{m,D}$ and $|\cdot|_{m,D}$, respectively. In addition, $\|\cdot\|_D$ is the norm of Lebesgue space $L^2(D)$. Furthermore, for the time dependent functions, we introduce the space

$$W^{m,p}(0,T;V) = \left\{ v \in L^p(0,T;V) \,:\, \|\partial_t^l v\|_{L^p(0,T;V)} < \infty \quad \forall l \leq m \right\}$$

with the norm

$$\|v\|_{W^{m,p}(0,T;V)} = \begin{cases} \left(\int_0^T \sum\limits_{0\leq l\leq m} \|\partial_t^l v\|_V^p dt \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max_{0\leq l\leq m} \operatorname{esssup}_{0\leq t\leq T} \|\partial_t^l v\|_V & \text{if } p = \infty; \end{cases}$$

and the space

$$C^m([0,T];V) = \left\{ v \in C([0,T];V) \,:\, \partial_t^l v \in C([0,T];V) \ \forall l \leq m \right\}$$

with the norm

$$\|v\|_{C^m([0T];V)} = \sum_{l=0}^{m} \max_{t\in[0,T]} \|\partial_t^l v\|_V.$$

We assume Ω is a polygonal domain and consider a regular family of triangulations of $\overline{\Omega}$ denoted by $\{\mathcal{T}_h\}_h$ such that the minimal angle condition is satisfied. Let $h_K = \text{diam}(K)$ and $h = \max\{h_K \,:\, K \in \mathcal{T}_h\}$. Denote by $\mathcal{E}_h$ the collection of all the edges of $\mathcal{T}_h$, $\mathcal{E}_h^i$ the set of all interior edges, and $\mathcal{E}_h^\partial = \mathcal{E}_h \backslash \mathcal{E}_h^i$. Let e be an edge shared by two elements K^+ and K^-, and $\boldsymbol{n}^\pm = \boldsymbol{n}|_{\partial K^\pm}$ be the unit outward normal vector on $\partial K^\pm$. For a piecewise smooth scalar-valued function v, let $v^\pm = v|_{\partial K^\pm}$, and define the average $\{v\}$ and the jump $[\![v]\!]$ on $\mathcal{E}_h^i$ as follows:

$$\{v\} = \frac{1}{2}(v^+ + v^-), \qquad [\![v]\!] = v^+\boldsymbol{n}^+ + v^-\boldsymbol{n}^- \qquad \text{on } e \in \mathcal{E}_h^i.$$

For a piecewise smooth vector-valued function $\boldsymbol{w}$, we denote $\boldsymbol{w}^{\pm} = \boldsymbol{w}|_{\partial K^{\pm}}$ and set the average $\{\boldsymbol{w}\}$ and the jump $[\boldsymbol{w}]$ on $\mathcal{E}_h^i$ as follows:

$$\{\boldsymbol{w}\} = \frac{1}{2}(\boldsymbol{w}^+ + \boldsymbol{w}^-), \qquad [\boldsymbol{w}] = \boldsymbol{w}^+ \cdot \boldsymbol{n}^+ + \boldsymbol{w}^- \cdot \boldsymbol{n}^- \text{ on } e \in \mathcal{E}_h^i.$$

If $e \in \mathcal{E}_h^\partial$, the set of boundary edges, we let

$$[\![v]\!] = v\boldsymbol{n}, \qquad \{\boldsymbol{w}\} = \boldsymbol{w} \text{ on } e \in \mathcal{E}_h^\partial,$$

where $\boldsymbol{n}$ is the unit outward normal on Γ.

With the above definitions of average and jumps, after direct manipulation, we have

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} v\boldsymbol{w} \cdot \boldsymbol{n}_K ds = \int_{\mathcal{E}_h} [\![v]\!] \cdot \{\boldsymbol{w}\} ds + \int_{\mathcal{E}_h^i} \{v\}[\boldsymbol{w}] ds, \tag{2.1}$$

where v is a scalar-valued function and $\boldsymbol{w}$ is a vector-valued function.

Let us introduce the following discontinous finite element spaces:

$$V^h = \{v^h \in L^2(\Omega) : \ v^h|_K \in P_1(K) \ \forall \mathrm{K} \in \mathcal{T}_h\},$$
$$\boldsymbol{W}^h = \{\boldsymbol{w}^h \in [L^2(\Omega)]^2 : \ \boldsymbol{w}^h|_K \in [P_1(K)]^2 \ \forall \mathrm{K} \in \mathcal{T}_h\},$$

where $P_1(K)$ denotes the polynomial space of degree 1. We use the following subset of the finite element space V^h to approximate the adimissible set K defined in (1.4):

$$K^h = \{v^h \in V^h : \ v_h(x) \geq 0 \text{ at all nodes of } \mathcal{T}_h\}.$$

2.2 | Spatially semi-discrete DG approximation

Before presenting the DG schemes, we define lifting operators $r : [L^2(\mathcal{E}_h)]^2 \to \boldsymbol{W}^h$, $r_\partial : [L^2(\mathcal{E}_h^\partial)]^2 \to \boldsymbol{W}^h$, and $r_e : [L^2(e)]^2 \to \boldsymbol{W}^h$ by

$$\int_\Omega r(\boldsymbol{q}) \cdot \boldsymbol{w}^h dx = -\int_{\mathcal{E}_h} \boldsymbol{q} \cdot \{\boldsymbol{w}^h\} ds, \qquad \int_\Omega r_\partial(\boldsymbol{q}) \cdot \boldsymbol{w}^h dx = -\int_{\mathcal{E}_h^\partial} \boldsymbol{q} \cdot \{\boldsymbol{w}^h\} ds, \tag{2.2}$$

$$\int_\Omega r_e(\boldsymbol{q}) \cdot \boldsymbol{w}^h dx = -\int_e \boldsymbol{q} \cdot \{\boldsymbol{w}^h\} ds, \qquad \forall \boldsymbol{w}^h \in \boldsymbol{W}^h. \tag{2.3}$$

Spatially semi-discrete DG formulation for the VI (1.1)–(1.3) is: Find $u^h : [0, T] \to V^h$ such that $\dot{u}^h \in K^h$ and

$$(\ddot{u}^h, v^h - \dot{u}^h) + B_h(u^h, v^h - \dot{u}^h) \geq (f, v^h - \dot{u}^h) \qquad \forall v^h \in K^h, \tag{2.4}$$

$$u^h(0) = P_B^h u_0, \tag{2.5}$$

$$\dot{u}^h(0) = P_B^h v_0, \tag{2.6}$$

where P_B^h is the Galerkin projection from V to V^h defined by

$$B_h(P_B^h v - v, v^h) = 0 \qquad \forall v^h \in V^h.$$

We introduce four choices of the bilinear form $B_h = B_h^{(j)}$ ($j = 1$–4) in the following. The bilinear form of interior penalty (IP) method ([48–50]) is

$$B_h^{(1)}(u, v) = \int_\Omega \nabla_h u \cdot \nabla_h v dx - \int_{\mathcal{E}_h} [\![u]\!] \cdot \{\nabla_h v\} ds - \int_{\mathcal{E}_h} \{\nabla_h u\} \cdot [\![v]\!] ds + \int_{\mathcal{E}_h} \eta \ [\![u]\!] \cdot [\![v]\!] \, ds,$$

where the penalty weighting function $\eta : \mathcal{E}_h \to \mathbb{R}$ is given by $\eta_e h_e^{-1}$ on each $e \in \mathcal{E}_h$ with η_e being a positive number. Here, the broken gradient operator ∇_h is defined by the relation $\nabla_h v = \nabla v$ on any element $K \in T_h$. For the method of Bassi et al. [51], the bilinear form is

$$B_h^{(2)}(u,v) = \int_\Omega \nabla_h u \cdot \nabla_h v dx - \int_{\mathcal{E}_h} [\![u]\!] \cdot \{\nabla_h v\} ds - \int_{\mathcal{E}_h} \{\nabla_h u\} \cdot [\![v]\!] ds + \sum_{e\in\mathcal{E}_h} \int_\Omega \eta_e r_e([\![u]\!]) \cdot r_e([\![v]\!]) dx.$$

The bilinear form of Brezzi et al. method [52] is given by

$$B_h^{(3)}(u,v) = \int_\Omega \nabla_h u \cdot \nabla_h v dx - \int_{\mathcal{E}_h} [\![u]\!] \cdot \{\nabla_h v\} ds - \int_{\mathcal{E}_h} \{\nabla_h u\} \cdot [\![v]\!] ds + \int_\Omega r([\![u]\!]) \cdot r([\![v]\!]) dx + \sum_{e\in\mathcal{E}_h} \int_\Omega \eta_e r_e([\![u]\!]) \cdot r_e([\![v]\!]) dx.$$

The last one is the simplified local DG (LDG) method [53], the bilinear form is

$$B_h^{(4)}(u,v) = \int_\Omega \nabla_h u \cdot \nabla_h v dx - \int_{\mathcal{E}_h} [\![u]\!] \cdot \{\nabla_h v\} ds - \int_{\mathcal{E}_h} \{\nabla_h u\} \cdot [\![v]\!] ds + \int_\Omega r([\![u]\!]) \cdot r([\![v]\!]) dx + \int_{\mathcal{E}_h} \eta\, [\![u]\!] \cdot [\![v]\!] ds.$$

Remark 2.1 For the general boundary condition $u = g$ on Γ, the DG scheme needs an extra linear form on the right hand side of (2.4). For the DG methods with $j = 1, \ldots, 4$, the associated linear forms are

$$F^{(1)}(v) = \int_{\mathcal{E}_h^\partial} g(\eta v - \nabla_h v \cdot \boldsymbol{n}) ds,$$

$$F^{(2)}(v) = \sum_{e\in\mathcal{E}_h^\partial} \int_\Omega \eta_e r_e(g\boldsymbol{n}) \cdot r_e(v\boldsymbol{n}) dx - \int_{\mathcal{E}_h^\partial} g\nabla_h v \cdot \boldsymbol{n} ds,$$

$$F^{(3)}(v) = \int_\Omega r_\partial(g\boldsymbol{n}) \cdot r([\![v]\!]) dx + \sum_{e\in\mathcal{E}_h^\partial} \int_\Omega \eta_e r_e(g\boldsymbol{n}) \cdot r_e(v\boldsymbol{n}) dx - \int_{\mathcal{E}_h^\partial} g\nabla_h v \cdot \boldsymbol{n} ds,$$

$$F^{(4)}(v) = \int_\Omega r_\partial(g\boldsymbol{n}) \cdot r([\![v]\!]) dx + \int_{\mathcal{E}_h^\partial} g(\eta v - \nabla_h v \cdot \boldsymbol{n}) ds.$$

2.3 | Fully discrete approximation scheme

We need a partition of the time interval

$$[0,T] = \bigcup_{n=1}^{N} [t_{n-1}, t_n], \quad 0 = t_0 < t_1 < \cdots < t_N = T.$$

For simplicity in notation, we use evenly spaced nodes $t_n = nk$, $0 \le n \le N$, with a uniform time step $k = T/N$. For a continuous function v, we use the notation $v_n = v(t_n)$. We define

$$\gamma_k v_n = \frac{v_{n+1} + v_{n-1}}{2}, \qquad \delta_k v_n = \frac{v_{n+1} - v_{n-1}}{2k} \qquad \text{and} \qquad d_k v_n = \frac{v_{n+1} - 2v_n + v_{n-1}}{k^2}.$$

Let $B_h(\cdot,\cdot)$ be one of the bilinear forms $B_h^{(j)}(\cdot,\cdot)$ with $j = 1, \ldots, 4$, and F_n be the associated linear form with the boundary condition $g_n = g(t_n)$. Then a fully discrete approximation of (1.1)–(1.3) is:

Find $\{u_n^{hk}\}_{n=0}^N \subset V^h$ such that

$$\delta_k u_n^{hk} \in K^h, \tag{2.7}$$

for $1 \le n \le N-1$,

$$(d_k u_n^{hk}, v^h - \delta_k u_n^{hk}) + B_h(\gamma_k u_n^{hk}, v^h - \delta_k u_n^{hk}) \ge (f_n, v^h - \delta_k u_n^{hk}) \qquad \forall v^h \in K^h, \tag{2.8}$$

and

$$u_0^{hk} = P_B^h u_0, \tag{2.9}$$

$$u_1^{hk} = u_0^{hk} + kP_B^h v_0. \tag{2.10}$$

Remark 2.2 In the above fully discrete scheme, the temporal discretization in (2.8) is 2nd-order. However, as proved in Section 4, it only achieves linear convergence order in time due to the limitation on the accuracy provided by (2.10). In [18], for the wave equation, a second order scheme was given to approximate the initial condition (1.3), so that the fully discrete approximation therein achieves 2nd order in time. Unfortunately, due to the inequality feature, the same ideas cannot be applied to the hyperbolic variational inequality problem (1.1)–(1.3). It can be seen from the proof of Theorem 4.2 that the fully discrete scheme can achieve 2nd order convergence in time if a higher order approximation can be constructed for the initial condition (1.3).

2.4 | Properties of DG schemes

As a preparation for error analysis, we first show the consistency of the DG schemes, and then give the boundedness and stability of the bilinear forms under DG norms.

Lemma 2.3 *(Consistency) Assume $u \in L^2(0, T; H^2(\Omega))$ is the solution of the VI (1.1)–(1.3). Then for all DG methods $B_h(w, v) = B_h^{(j)}(w, v)$ with $j = 1, \ldots, 4$, we have for almost everywhere $t \in [0, T]$,*

$$(\ddot{u}, v^h - \dot{u}) + B_h(u, v^h - \dot{u}) \ge (f, v^h - \dot{u}) \qquad \forall v^h \in K^h. \tag{2.11}$$

Proof. Notice that $u(t) \in H^2(\Omega)$ for almost everywhere $t \in [0, T]$, so $[\![u]\!] = 0$, $\{u\} = u$, $[\nabla u] = 0$, and $\{\nabla u\} = \nabla u$ on any interior edge. For any $v^h \in K^h$, using integration by parts formula, we get

$$\begin{aligned} B_h(u, v^h - \dot{u}) &= \int_\Omega \nabla_h u \cdot \nabla_h (v^h - \dot{u}) dx - \int_{\mathcal{E}_h} \{\nabla_h u\} \cdot [\![v^h - \dot{u}]\!] ds \\ &= \int_\Omega -\Delta u (v^h - \dot{u}) dx + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \nabla u \cdot \boldsymbol{n}_K (v^h - \dot{u}) ds - \int_{\mathcal{E}_h} \{\nabla_h u\} \cdot [\![v^h - \dot{u}]\!] ds \\ &= \int_\Omega -\Delta u (v^h - \dot{u}) dx. \end{aligned}$$

Then we use the relation (1.5) to obtain

$$\begin{aligned} (\ddot{u}, v^h - \dot{u}) + B_h(u, v^h - \dot{u}) &= \int_\Omega (\ddot{u} - \Delta u)(v^h - \dot{u}) dx \\ &= \int_\Omega (\ddot{u} - \Delta u) v^h dx - \int_\Omega (\ddot{u} - \Delta u - f) \dot{u} dx - \int_\Omega f \dot{u} dx \end{aligned}$$

$$\geq \int_\Omega f v^h dx - \int_\Omega f \dot{u} dx$$
$$= \int_\Omega f(v^h - \dot{u}) dx,$$

that is, (2.11) holds. ■

To consider the boundedness and stability of the bilinear form B_h, as in [30], let $V(h) = V_h + H^2(\Omega) \cap H_0^1(\Omega)$, and define seminorms and norms for $v \in V[h]$ by the following relations:

$$|v|_{1,h}^2 = \sum_{K\in\mathcal{T}_h} |v|_{1,K}^2, \qquad |v|_{1,*}^2 = \sum_{e\in\mathcal{E}_h} h_e^{-1} \| [\![v]\!] \|_e^2, \qquad |||v|||^2 = |v|_{1,h}^2 + \sum_{K\in\mathcal{T}_h} h_K^2 |v|_{2,K}^2 + |v|_{1,*}^2. \tag{2.12}$$

We have the following inequalities ([[3], Lemma 2.1])

$$\|v\| \lesssim (|v|_{1,h}^2 + |v|_{1,*}^2)^{1/2} \lesssim |||v||| \qquad \forall v \in V(h). \tag{2.13}$$

Here "$\lesssim\cdots$" stands for "$\leq C\cdots$", where C is a positive generic constant independent of h, k and T, which may take on different values at different places. In the analysis, we shall use space $L^p(0, T; V(h))$ with the norm

$$|||v|||_{L^p(0,T;V(h))} = \begin{cases} \left(\int_0^T |||v|||^p dt\right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \text{esssup}_{0\leq t\leq T} |||v|||, & \text{if } p = \infty. \end{cases}$$

The boundedness and stability of the bilinear form $B_h(u, v)$ was given in [26, 30]. Here, we state them as lemmas.

Lemma 2.4 *(Boundedness) For $B_h = B_h^{(j)}$, $1 \leq j \leq 4$,*

$$B_h(u, v) \lesssim |||u||| \; |||v||| \qquad \forall u, v \in V(h). \tag{2.14}$$

Lemma 2.5 *(Stability) For $B_h = B_h^{(j)}$, $1 \leq j \leq 4$,*

$$B_h(v, v) \gtrsim |||v|||^2 \qquad \forall v \in V_h, \tag{2.15}$$

if $\eta_0 = \inf_e \eta_e$ is large enough for IP method ($j = 1$), $\eta_0 > 3$ for the method with $j = 2$, and $\eta_0 > 0$ for the methods with $j = 3, 4$.

3 | ERROR ESTIMATES FOR THE SPATIALLY SEMI-DISCRETE SCHEMES

3.1 | Interpolation errors

If $u \in L^2(0, T; H^2(\Omega))$, let $\Pi^h u \in V^h$ be the usual continuous piecewise linear polynomial interpolant, then the jumps of $u - \Pi^h u$ will be zero at the interelement boundaries. It is easy to see that [26, 30] for almost everywhere $t \in [0, T]$

$$|||u(t) - \Pi^h u(t)||| \lesssim h|u(t)|_2. \tag{3.1}$$

To extend the analysis to nonconforming meshes, it is convenient to use an interpolant $\Pi^h u$ which is discontinuous across the interelement boundaries. As in [26], we just require the local approximation property

$$|u(t) - \Pi^h u(t)|_{1,K} \lesssim h|u(t)|_{2,K};$$

then for the global approximation error, we have for almost everywhere $t \in [0, T]$,

$$|||u - \Pi^h u||| \lesssim h|u(t)|_2. \tag{3.2}$$

Similarly, if $\dot{u} \in L^2(0,T;H^2(\Omega))$ and $\ddot{u} \in L^2(0,T;H^2(\Omega))$, we have for almost everywhere $t \in [0, T]$,

$$|||\dot{u} - \Pi^h \dot{u}||| \lesssim h|\dot{u}(t)|_2, \qquad |||\ddot{u} - \Pi^h \ddot{u}||| \lesssim h|\ddot{u}(t)|_2. \tag{3.3}$$

Define $u^I(t) \in V^h$ by

$$B_h(u^I(t) - u(t), v^h) = 0 \qquad \forall v^h \in V^h. \tag{3.4}$$

Then we have the following approximation property (see [18, 33]).

Lemma 3.1 *Assume $u \in H^2(0, T;H^2(\Omega))$, we have*

$$|||\partial_t^i(u^I - u)||| \lesssim h\|\partial_t^i u\|_2, \qquad \|\partial_t^i(u^I - u)\| \lesssim h^2\|\partial_t^i u\|_2, \qquad i = 0, 1, 2. \tag{3.5}$$

3.2 | A priori error estimates

Theorem 3.2 *Let u and u_h be the solutions of (1.1)–(1.3) and (2.4)–(2.6), respectively. Assume $u \in H^2(0, T;H^2(\Omega))$, then for the DG methods with $j = 1, \ldots, 4$, we have*

$$\|\dot{u}(t) - \dot{u}^h(t)\| + |||u(t) - u^h(t)||| \le Ch, \qquad \textit{for } \text{ a.e. } t \in [0, T]. \tag{3.6}$$

Here, the constant C depends on $\|u\|_{H^2(0,T;H^2(\Omega))}$, and $\|f\|_{L^2(0,T;L^2(\Omega))}$.

Proof. Note that $u^I(0) = P_B^h u_0$ and $\dot{u}^I(0) = P_B^h v_0$. Thus

$$u^I(0) = u^h(0), \qquad \dot{u}^I(0) = \dot{u}^h(0). \tag{3.7}$$

Now we write the error as

$$e = u - u^h = (u - u^I) + (u^I - u^h) := e^I + e^h.$$

Let $v^h = \dot{u}^h$ in (2.11) to get

$$(\ddot{u}, \dot{u}^h - \dot{u}) + B_h(u, \dot{u}^h - \dot{u}) \ge (f, \dot{u}^h - \dot{u}). \tag{3.8}$$

Combining with (2.4), we obtain for all $v^h \in K^h$,

$$-B_h(u^h, v^h - \dot{u}^h) \le B_h(u, \dot{u}^h - \dot{u}) + (\ddot{u}^h, v^h - \dot{u}^h) + (\ddot{u}, \dot{u}^h - \dot{u}) - (f, v^h - \dot{u}). \tag{3.9}$$

Using symmetry of B_h and orthogonality (3.4), from (3.9), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}[\|\dot{e}^h\|^2 + B_h(e^h, e^h)] &= (\ddot{e}^h, \dot{e}^h) + B_h(e^h, \dot{e}^h) \\ &= (\ddot{e}^h, \dot{e}^h) + B_h(u^I - u^h, \dot{u}^I - \dot{u}) + B_h(u^I - u^h, \dot{u} - v^h) \\ &\quad + B_h(u^I, v^h - \dot{u}^h) - B_h(u^h, v^h - \dot{u}^h) \\ &\le T_1 + T_2 + T_3 + T_4,\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}T_1 &= B_h(u^I(t) - u^h(t), \dot{u}^I(t) - \dot{u}(t)), \\ T_2 &= B_h(u^I(t) - u^h(t), \dot{u}(t) - v^h),\end{aligned}$$

$$T_3 = (\ddot{u}, v^h - \dot{u}) + B_h(u(t), v^h - \dot{u}(t)) - (f, v^h - \dot{u}(t)),$$
$$T_4 = (\ddot{e}^h, \dot{e}^h) + (\ddot{u}^h - \ddot{u}, v^h - \dot{u}^h) = (\ddot{e}^h, \dot{u}^I - v^h) + (\ddot{u}^I - \ddot{u}, v^h - \dot{u}^h).$$

In the above argument, we used the fact that $B_h(u^I - u, v^h - \dot{u}^h) = 0$ due to (3.4). By the boundedness of the bilinear form B_h, we have

$$T_1 \lesssim |||u^I - u^h||| \ \ |||\dot{u}^I - \dot{u}||| \lesssim |||u^I - u^h|||^2 + |||\dot{u}^I - \dot{u}|||^2, \tag{3.11}$$

$$T_2 \lesssim |||u^I - u^h||| \ \ |||\dot{u} - v^h||| \lesssim |||u^I - u^h|||^2 + |||\dot{u} - v^h|||^2. \tag{3.12}$$

We turn to bound T_3. Note that on an interior edge, $[\![u]\!] = 0$, $\{u\} = u$, $\{\nabla u\} = \nabla u$, and on Γ, $[\![u]\!] = g\boldsymbol{n}$. Then

$$B_h(u, v^h - \dot{u}) = \int_\Omega \nabla_h u \cdot \nabla_h(v^h - \dot{u}) dx - \int_{\mathcal{E}_h} \nabla u \cdot [\![v^h - \dot{u}]\!] ds.$$

Since $[\nabla u] = 0$ on an interior edge and remembering (2.1), we have

$$\sum_{K\in\mathcal{T}_h} \int_K \nabla_h u \cdot \nabla_h(v^h - \dot{u}) dx = \sum_{K\in\mathcal{T}_h} \int_K -\Delta u(v^h - \dot{u}) dx + \sum_{K\in\mathcal{T}_h} \int_{\partial K} (\nabla u \cdot \boldsymbol{n}_K)(v^h - \dot{u}) ds$$
$$= \sum_{K\in\mathcal{T}_h} \int_K -\Delta u(v^h - \dot{u}) dx + \int_{\mathcal{E}_h} \nabla u \cdot [\![v^h - \dot{u}]\!] ds.$$

Then

$$B_h(u, v^h - \dot{u}) = \sum_{K\in\mathcal{T}_h} \int_K -\Delta u(v^h - \dot{u}) dx.$$

Hence, we get

$$T_3 = (\ddot{u} - \Delta u - f, v^h - \dot{u}) \le \|\ddot{u} - \Delta u - f\| \|v^h - \dot{u}\|. \tag{3.13}$$

For T_4, we have

$$\begin{aligned} T_4 &= (\ddot{e}^h, \dot{u}^I - v^h) + (\ddot{u}^I - \ddot{u}, v^h - \dot{u}^I) + (\ddot{u}^I - \ddot{u}, \dot{u}^I - \dot{u}^h) \\ &\le (\ddot{e}^h, \dot{u}^I - v^h) + \|\ddot{u}^I - \ddot{u}\| \|v^h - \dot{u}^I\| + \|\ddot{u}^I - \ddot{u}\| \|\dot{u}^I - \dot{u}^h\| \\ &\le (\ddot{e}^h, \dot{u}^I - v^h) + \|\ddot{u}^I - \ddot{u}\|^2 + \frac{1}{2}\|v^h - \dot{u}^I\|^2 + \frac{1}{2}\|\dot{e}^h\|^2. \end{aligned} \tag{3.14}$$

We apply (3.11), (3.12), (3.13) and (3.14) in the inequality (3.10), and then integrate this inequality over the time interval $(0, s)$ for a fixed $s \in I$. This yields

$$\begin{aligned} \frac{1}{2}\|\dot{e}^h(s)\|^2 + \frac{1}{2}B_h(e^h(s), e^h(s)) \lesssim\ & \frac{1}{2}\|\dot{e}^h(0)\|^2 + \frac{1}{2}B_h(e^h(0), e^h(0)) + 2\int_0^s |||e^h|||^2 dt \\ &+ \int_0^s |||\dot{u}^I - \dot{u}|||^2 dt + \int_0^s |||\dot{u} - v^h|||^2 dt \\ &+ \int_0^s \|\ddot{u} - \Delta u - f\| \|v^h - \dot{u}\| dt \\ &+ \int_0^s \|\ddot{u}^I - \ddot{u}\|^2 dt + \frac{1}{2}\int_0^s \|v^h - \dot{u}^I\|^2 dt \\ &+ \frac{1}{2}\int_0^s \|\dot{e}^h\|^2 dt + \int_0^s (\ddot{e}^h, \dot{u}^I - v^h) dt. \end{aligned} \tag{3.15}$$

Let $v^h = \Pi^h \dot{u}$ in the above inequality. Applying integration by parts to the last term on the right-hand side, we get

$$\begin{aligned}\int_0^s (\ddot{e}^h, \dot{u}^I - \Pi^h \dot{u})dt &= -\int_0^s (\dot{e}^h, \partial_t(\dot{u}^I - \Pi^h \dot{u}))dt + (\dot{e}^h, \dot{u}^I - \Pi^h \dot{u})|_{t=0}^{t=s} \\ &\leq \frac{1}{2}\int_0^s \|\dot{e}^h\|^2 dt + \frac{1}{2}\int_0^s \|\ddot{u}^I - \Pi^h \ddot{u}\|^2 dt + \frac{\varepsilon}{2}\|\dot{e}^h(s)\|^2 \\ &\quad + \frac{1}{2\varepsilon}\|\dot{u}^I(s) - \Pi^h \dot{u}(s)\|^2 + \frac{1}{2}\|\dot{e}^h(0)\|^2 + \frac{1}{2}\|\dot{u}^I(0) - \Pi^h \dot{u}(0)\|^2.\end{aligned}$$

Here, in the last inequality, the geometric–arithmetic mean inequality $| ab | \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2$ is used with any arbitrary small constant $\varepsilon > 0$.

Next, by the boundedness and stability of the bilinear forms, we make some algebra manipulation and apply Gronwall's inequality to the inequality (3.15). Then, we obtain

$$\begin{aligned}\|\dot{e}^h(s)\| + |||e^h(s)||| \lesssim & \|\dot{e}^h(0)\| + |||e^h(0)||| + ||| \dot{u}^I - \dot{u}|||_{L^2(0,T;V(h))} \\ & + |||\dot{u} - \Pi^h \dot{u}|||_{L^2(0,T;V(h))} + \|\Pi^h \dot{u} - \dot{u}\|^{1/2}_{L^2(0,T;L^2(\Omega))} \\ & + \|\ddot{u}^I - \ddot{u}\|_{L^2(0,T;L^2(\Omega))} + \|\Pi^h \dot{u} - \dot{u}^I\|_{L^2(0,T;L^2(\Omega))} \\ & + \|\ddot{u}^I - \Pi^h \ddot{u}\|_{L^2(0,T;L^2(\Omega))} + \|\dot{u}^I(s) - \Pi^h \dot{u}(s)\| + \|\dot{u}^I(0) - \Pi^h \dot{u}(0)\|.\end{aligned}$$

By the relation (3.7), we know that

$$e^h(0) = u^I(0) - u^h(0) = 0, \qquad \dot{e}^h(0) = \dot{u}^I(0) - \dot{u}^h(0) = 0.$$

Then by (3.2), (3.3) and (3.5), we get

$$\|\dot{e}^h(s)\| + |||e^h(s)||| \lesssim h.$$

Finally, by the triangle inequality, we complete the proof of (3.6). ■

4 | ERROR ESTIMATES FOR THE FULLY DISCRETE SCHEME

In this section, we analyze the fully discrete scheme. First, we show the well-posedness of problem (2.7)–(2.10).

Theorem 4.1 *The problem (2.7)–(2.10) admits a unique solution u^{hk}, which is stable in the sense that for given $u_{1,0}$, $u_{2,0} \in V$, and $f_1, f_2 \in W^{1,\infty}(0,T;V)$, the corresponding solutions $u^{hk}_{1,n}$ and $u^{hk}_{2,n}$, $0 \leq n \leq N$, satisfy the inequality*

$$\max_{0\leq n\leq N}(k^{-1}\|e_{n+1} - e_n\| + |||e_n|||) \lesssim |||P^h_B(u_{1,0} - u_{2,0})||| + \|P^h_B(v_{1,0} - v_{2,0})\| + \|f_1 - f_2\|_{W^{1,\infty}(0,T;V)}. \quad (4.1)$$

Here, $e_n = u^{hk}_{1,n} - u^{hk}_{2,n}$.

Proof. The inequality (2.8) can be rewritten as

$$\begin{aligned}\frac{2}{k}(\delta_k u^{hk}_n, v^h - \delta_k u^{hk}_n) + kB_h(\delta_k u^{hk}_n, v^h - \delta_k u^{hk}_n) \geq & (f_n, v^h - \delta_k u^{hk}_n) + \frac{2}{k^2}(u^{hk}_n - u^{hk}_{n-1}, v^h - \delta_k u^{hk}_n) \\ & - B_h(u^{hk}_{n-1}, v^h - \delta_k u^{hk}_n) \qquad \forall v^h \in K^h.\end{aligned}$$

This inequality problem admits a unique solution $\delta_k u^{hk}_n \in K^h$ by the boundedness and stability of the bilinear form B_h.

Then we turn to deduce the inequality (4.1). With $n = 1, 2, \ldots, N-1$, for all $v^h \in K^h$, we have

$$(d_k u_{1,n}^{hk}, v^h - \delta_k u_{1,n}^{hk}) + B_h(\gamma_k u_{1,n}^{hk}, v^h - \delta_k u_{1,n}^{hk}) \geq (f_{1,n}, v^h - \delta_k u_{1,n}^{hk}), \tag{4.2}$$

$$(d_k u_{2,n}^{hk}, v^h - \delta_k u_{2,n}^{hk}) + B_h(\gamma_k u_{2,n}^{hk}, v^h - \delta_k u_{2,n}^{hk}) \geq (f_{2,n}, v^h - \delta_k u_{2,n}^{hk}). \tag{4.3}$$

Taking $v^h = \delta_k u_{2,n}^{hk}$ in (4.2) and $v^h = \delta_k u_{1,n}^{hk}$ in (4.3), and adding the two inequalities, we obtain

$$A_n := (d_k e_n, \delta_k e_n) + B_h(\gamma_k e_n, \delta_k e_n) \leq (f_{1,n} - f_{2,n}, \delta_k e_n).$$

We can get the lower bound as

$$A_n \gtrsim \frac{1}{k^3}(\|e_{n+1} - e_n\|^2 - \|e_n - e_{n-1}\|^2) + \frac{1}{k}(|||e_{n+1}|||_{B_h}^2 - |||e_{n-1}|||_{B_h}^2). \tag{4.4}$$

Here, the norm $|||\cdot|||_{B_h}^2 := B_h(\cdot,\cdot)$ is equivalent with $|||\cdot|||^2$ due to continuity and coercivity of the bilinear form B_h. Then for $1 \leq n \leq N-1$, we obtain

$$\frac{1}{k^2}(\|e_{n+1} - e_n\|^2 - \|e_n - e_{n-1}\|^2) + |||e_{n+1}|||_{B_h}^2 - |||e_{n-1}|||_{B_h}^2 \lesssim (f_{1,n} - f_{2,n}, e_{n+1} - e_{n-1}).$$

A simple induction yields

$$\begin{aligned}
\frac{1}{k^2}(\|e_{n+1} - e_n\|^2 - \|e_1 - e_0\|^2) + |||e_{n+1}|||_{B_h}^2 - |||e_0|||_{B_h}^2 &\lesssim \sum_{j=1}^{n}(f_{1,j} - f_{2,j}, e_{j+1} - e_{j-1}) \\
= (f_{1,n-1} - f_{2,n-1}, e_n) + (f_{1,n-2} - f_{2,n-2}, e_{n-1}) &- (f_{1,1} - f_{2,1}, e_0) - (f_{1,2} - f_{2,2}, e_1) \\
+ \sum_{j=1}^{n-3}((f_{1,j} - f_{1,j+2}) - (f_{2,j} - f_{2,j+2}), e_{j+1}).
\end{aligned}$$

Recall (2.9)–(2.10), we have

$$e_1 = u_{1,1}^{hk} - u_{2,1}^{hk} = u_{1,0}^{hk} + kP_B^h v_{1,0} - u_{2,0}^{hk} - kP_B^h v_{2,0} = e_0 + kP_B^h(v_{1,0} - v_{2,0}),$$

which implies

$$\frac{1}{k^2}\|e_1 - e_0\|^2 \leq \|P_B^h(v_{1,0} - v_{2,0})\|.$$

Let $M = \max_{1 \leq n \leq N}(k^{-1}\|e_{n+1} - e_n\|^2 + |||e_n|||)$, we obtain

$$\begin{aligned}
M^2 &\lesssim \|P_B^h(v_{1,0} - v_{2,0})\|^2 + |||P_B^h(u_{1,0} - u_{2,0})|||^2 + \Bigg(\|f_{1,n-1} - f_{2,n-1}\| + \|f_{1,n-2} - f_{2,n-2}\| \\
&\quad + \|f_{1,1} - f_{2,1}\| + \|f_{1,2} - f_{2,2}\| + \sum_{j=1}^{n-3}\|(f_{1,j} - f_{1,j+2}) - (f_{2,j} - f_{2,j+2})\|\Bigg)M \\
&\lesssim \|P_B^h(v_{1,0} - v_{2,0})\|^2 + |||P_B^h(u_{1,0} - u_{2,0})|||^2 + M\|f_1 - f_2\|_{W^{1,\infty}(0,T;V)}.
\end{aligned}$$

Applying the following inequality

$$x, a, b \geq 0 \ \text{ and } \ x^2 \leq ax + b \Rightarrow x \lesssim a + b^{1/2}, \tag{4.5}$$

we then obtain the stability inequality (4.1). ■

Now we give error estimates for the fully discrete scheme in the following theorem.

Theorem 4.2 *Let u and u^{hk} be the solutions of (1.1)–(1.3) and (2.4)–(2.6), respectively. Assume $u \in C^2([0,T];H^2(\Omega))$, $\partial_t^3 u \in L^\infty(0,T;H^2(\Omega))$, $\partial_t^4 u \in L^\infty(0,T;L^2(\Omega))$. Then the* *following error bound holds*

$$\max_j(k^{-1}\|(u_{j+1}-u^{hk}_{j+1})-(u_j-u^{hk}_j)\|+|||u_j-u^{hk}_j|||)\le C(h+k), \tag{4.6}$$

where the constant C depends on $\|u\|_{C^2([0T];H^2(\Omega))}$, $\|\partial_t^3 u\|_{L^\infty(0,T;H^2(\Omega))}$, $\|\partial_t^4 u\|_{L^\infty(0,T;L^2(\Omega))}$, *and* $\|f\|_{L^2(0,T;L^2(\Omega))}$.

Proof. Define $e_n=u_n-u^{hk}_n$ for $n=1, 2, \ldots, N$. We have

$$e_n=(u_n-u^I_n)+(u^I_n-u^{hk}_n):=e^I_n+e^h_n,$$

where $u^I_n=u^I(t_n)$. Denote

$$A^h_n=(d_ke^h_n,\delta_ke^h_n)+B_h(\gamma_ke^h_n,\delta_ke^h_n).$$

As (4.4), we have

$$A^h_n\gtrsim\frac{1}{k^3}(\|e^h_{n+1}-e^h_n\|^2-\|e^h_n-e^h_{n-1}\|^2)+\frac{1}{k}(|||e^h_{n+1}|||^2_{B_h}-|||e^h_{n-1}|||^2_{B_h}). \tag{4.7}$$

For an upper bound of A^h_n, write

$$\begin{aligned}A^h_n=&(d_ku^I_n-\ddot u_n,\delta_ku^I_n-\delta_ku^{hk}_n)+B_h(\gamma_ku^I_n-u_n,\delta_ku^I_n-\delta_ku^{hk}_n)\\&+(\ddot u_n,\delta_ku^I_n-\delta_ku^{hk}_n)+B_h(u_n,\delta_ku^I_n-\delta_ku^{hk}_n)\\&-(d_ku^{hk}_n,\delta_ku^I_n-v^h_n)-B_h(\gamma_ku^{hk}_n,\delta_ku^I_n-v^h_n)\\&-(d_ku^{hk}_n,v^h_n-\delta_ku^{hk}_n)-B_h(\gamma_ku^{hk}_n,v^h_n-\delta_ku^{hk}_n),\end{aligned} \tag{4.8}$$

where $v^h_n\in K^h$ is arbitrary. We take $v^h=\delta_ku^{hk}_n\in K^h$ in (2.11) at $t=t_n$ to get

$$(\ddot u_n,\delta_ku^{hk}_n-\dot u_n)+B_h(u_n,\delta_ku^{hk}_n-\dot u_n)\ge(f_n,\delta_ku^{hk}_n-\dot u_n).$$

Combining the above inequality with (2.8), we have,

$$\begin{aligned}-(d_ku^{hk}_n,v^h_n-\delta_ku^{hk}_n)-B_h(\gamma_ku^{hk}_n,v^h_n-\delta_ku^{hk}_n)\le&(\ddot u_n,\delta_ku^{hk}_n-\dot u_n)+B_h(u_n,\delta_ku^{hk}_n-\dot u_n)\\&-(f_n,v^h_n-\dot u_n).\end{aligned} \tag{4.9}$$

In Equation (4.8), inserting

$$(d_ku^I_n,\delta_ku^I_n-v^h_n)-(d_ku^I_n,\delta_ku^I_n-v^h_n)+B_h(u^I_n,\delta_ku^I_n-v^h_n)-B_h(u^I_n,\delta_ku^I_n-v^h_n)$$

and applying (4.9), we get

$$A^h_n\le T^1_n+T^2_n+T^3_n+T^4_n, \tag{4.10}$$

where

$$\begin{aligned}T^1_n&=(d_ku^I_n-\ddot u_n,\delta_ku^I_n-\delta_ku^{hk}_n)+B_h(\gamma_ku^I_n-u_n,\delta_ku^I_n-\delta_ku^{hk}_n),\\T^2_n&=(d_ku^I_n-\ddot u_n,v^h_n-\delta_ku^I_n)+B_h(e^I_n,\delta_ku^I_n-v^h_n),\\T^3_n&=(d_ke^h_n,\delta_ku^I_n-v^h_n)+B_h(u^I_n-\gamma_ku^{hk}_n,\delta_ku^I_n-v^h_n),\\T^4_n&=(\ddot u_n,v^h_n-\dot u_n)+B_h(u_n,v^h_n-\dot u_n)-(f_n,v^h_n-\dot u_n).\end{aligned}$$

From the lower bound (4.7) and the inequality (4.10), we obtain

$$\frac{1}{k^3}(\|e^h_{n+1}-e^h_n\|^2-\|e^h_n-e^h_{n-1}\|^2)+\frac{1}{k}(|||e^h_{n+1}|||^2_{B_h}-|||e^h_{n-1}|||^2_{B_h})\lesssim T^1_n+T^2_n+T^3_n+T^4_n.$$

By an induction, we get

$$\frac{1}{k^3}(\|e_{n+1}^h - e_n^h\|^2 - \|e_1^h - e_0^h\|^2) + \frac{1}{k}(|||e_{n+1}^h|||_{B_h}^2 - |||e_0^h|||_{B_h}^2) \lesssim \sum_{j=1}^{n}(T_j^1 + T_j^2 + T_j^3 + T_j^4),$$

which implies

$$\frac{1}{k^2}\|e_{n+1}^h - e_n^h\|^2 + |||e_{n+1}^h|||_{B_h}^2 \lesssim k\sum_{j=1}^{n}(T_j^1 + T_j^2 + T_j^3 + T_j^4) + \frac{1}{k^2}\|e_1^h - e_0^h\|^2 + |||e_0^h|||_{B_h}^2 . \tag{4.11}$$

Set $M = \max_j |||\, e_j^h \,|||$, $\xi_j = d_k u_j^I - \ddot{u}_j$, $\zeta_j = \gamma_k u_j^I - u_j$ and $\theta_j = u_j^I - \gamma_k u_j^{hk}$, we get

$$\begin{aligned}
\sum_{j=1}^{n} 2kT_j^1 &= \sum_{j=1}^{n}(\xi_j, e_{j+1}^h - e_{j-1}^h) + \sum_{j=1}^{n} B_h(\zeta_j, e_{j+1}^h - e_{j-1}^h)\\
&= \sum_{j=1}^{n}(\xi_j, e_{j+1}^h - e_j^h) + \sum_{j=1}^{n}(\xi_j, e_j^h - e_{j-1}^h) + \sum_{j=1}^{n-2} B_h(\zeta_j - \zeta_{j+2}, e_{j+1}^h)\\
&\quad + B_h(\zeta_n, e_{n+1}^h) + B_h(\zeta_{n-1}, e_n^h) - B_h(\zeta_2, e_1^h) - B_h(\zeta_1, e_0^h)\\
&\lesssim \sum_{j=1}^{n} k\|\xi_j\|^2 + \sum_{j=0}^{n}\frac{1}{k}\|e_{j+1}^h - e_j^h\|^2 + \left(\sum_{j=1}^{n-2} |||\zeta_j - \zeta_{j+2}||| + \max_j |||\zeta_j|||\right)M,
\end{aligned} \tag{4.12}$$

$$\sum_{j=1}^{n} kT_j^2 \lesssim \max_j(\|\xi_j\| + |||e_j^I|||)\sum_{j=1}^{n} k\,|||v_j^h - \delta_k u_j^I|||, \tag{4.13}$$

and

$$\begin{aligned}
\sum_{j=1}^{n} kT_j^3 &= \frac{1}{k}\left[\sum_{j=1}^{n}(e_{j+1}^h - e_j^h \delta_k u_j^I - v_j^h) - \sum_{j=1}^{n}(e_j^h - e_{j-1}^h, \delta_k u_j^I - v_j^h)\right] + \sum_{j=1}^{n} kB_h(\theta_j, \delta_k u_j^I - v_j^h)\\
&= \frac{1}{k}\left[\sum_{j=1}^{n-1}(e_{j+1}^h - e_j^h, \delta_k(u_j^I - u_{j+1}^I) - (v_j^h - v_{j+1}^h)) + (e_{n+1}^h - e_n^h, \delta_k u_n^I - v_n^h)\right.\\
&\quad \left. - (e_1^h - e_0^h, \delta_k u_1^I - v_1^h)\right] + \sum_{j=1}^{n} kB_h(\theta_j, \delta_k u_j^I - v_j^h)\\
&\lesssim \frac{1}{k}\sum_{j=1}^{n-1}\|e_{j+1}^h - e_j^h\|^2 + \frac{1}{k}\sum_{j=1}^{n-1}\|\delta_k(u_j^I - u_{j+1}^I) - (v_j^h - v_{j+1}^h)\|^2 + \frac{\varepsilon}{k^2}\|e_{n+1}^h - e_n^h\|^2\\
&\quad + \frac{1}{4\varepsilon}\|\delta_k u_n^I - v_n^h\|^2 + \frac{1}{k}\|e_1^h - e_0^h\|\|\delta_k u_1^I - v_1^h\| + \max_j |||\theta_j||| \sum_{j=1}^{n} k\,|||v_j^h - \delta_k u_j^I|||.
\end{aligned} \tag{4.14}$$

Then, by Gronwall's inequality, we obtain from (4.11) to (4.14)

$$\begin{aligned}
\frac{1}{k^2}\|e_{n+1}^h - e_n^h\|^2 + M^2 &\lesssim \sum_{j=1}^{n} k\|\xi_j\|^2 + \left(\sum_{j=1}^{n-2} |||\zeta_j - \zeta_{j+2}||| + \max_j |||\zeta_j|||\right)M\\
&\quad + \max_j(\|\xi_j\| + |||e_j^I||| + |||\theta_j|||)\sum_{j=1}^{n} k\,|||v_j^h - \delta_k u_j^I|||
\end{aligned}$$

$$+\frac{1}{k}\sum_{j=1}^{n-1}\|\delta_k(u_j^I-u_{j+1}^I)-(v_j^h-v_{j+1}^h)\|^2+\|\delta_k u_n^I-v_n^h\|^2$$
$$+\|\delta_k u_1^I-v_1^h\|^2+\sum_{j=1}^{n}k\mid T_j^4\mid+\frac{1}{k^2}\|e_1^h-e_0^h\|^2+|||e_0^h|||^2. \tag{4.15}$$

Note that $|||\theta_j|||=|||u_j^I-\gamma_k u_j^I+\gamma_k e_j^h|||\le u_j^I-\gamma_k u_j^I\,|||+M$, by the relation (4.5), we obtain

$$\max_j\left(\frac{1}{k}\|e_{j+1}^h-e_j^h\|+|||e_j^h|||\right)$$
$$\lesssim\left(\sum_{j=1}^{n}k\|\xi_j\|^2\right)^{1/2}+\sum_{j=1}^{n-2}|||\zeta_j-\zeta_{j+2}|||+\max_j|||\zeta_j|||+\sum_{j=1}^{n}k\;|||v_j^h-\delta_k u_j^I|||$$
$$+\left(\max_j(\|\xi_j\|+\|e_j^I\|+\|u_j^I-\gamma_k u_j^I\|)\sum_{j=1}^{n}k\;\;|||v_j^h-\delta_k u_j^I|||\right)^{1/2}$$
$$+\left(\frac{1}{k}\sum_{j=1}^{n-1}\|\delta_k(u_j^I-u_{j+1}^I)-(v_j^h-v_{j+1}^h)\|^2\right)^{1/2}+\|\delta_k u_n^I-v_n^h\|$$
$$+\|\delta_k u_1^I-v_1^h\|+\left(\sum_{j=1}^{n}k|T_j^4|\right)^{1/2}+\frac{1}{k}\|e_1^h-e_0^h\|+|||e_0^h|||\,. \tag{4.16}$$

Now, let us estimate the above inequality term by term. By Taylor's theorem, we have

$$v_{j+1}=v_j+k\dot v_j+\frac{k^2}{2}\ddot v_j+\cdots+\frac{k^m}{m!}\partial_t^m v_j+\int_{t_j}^{t_{j+1}}\partial_t^{m+1}v(t)\frac{(t_{j+1}-t)^m}{m!}dt,$$
$$v_{j-1}=v_j-k\dot v_j+\frac{k^2}{2}\ddot v_j+\cdots+\frac{(-k)^m}{m!}\partial_t^m v_j+\int_{t_j}^{t_{j-1}}\partial_t^{m+1}v(t)\frac{(t_{j-1}-t)^m}{m!}dt,$$

then

$$d_k v_j=\frac{v_{j+1}-2v_j+v_{j-1}}{k^2}=\frac{1}{k^2}\int_{t_{j-1}}^{t_{j+1}}\ddot v(t)(k-|t-t_j|)dt$$
$$=\ddot v_j+\frac{1}{6k^2}\int_{t_{j-1}}^{t_{j+1}}\partial_t^4 v(t)(k-|t-t_j|)^3dt,$$

$$\delta_k v_j=\frac{v_{j+1}-v_{j-1}}{2k}=\dot v_j+\frac{1}{4k}\int_{t_j}^{t_{j+1}}\partial_t^3 v(t)(t_{j+1}-t)^2dt-\frac{1}{4k}\int_{t_j}^{t_{j-1}}\partial_t^3 v(t)(t_{j-1}-t)^2dt,$$

and

$$\gamma_k v_j=\frac{v_{j+1}+v_{j-1}}{2}=v_j+\frac{1}{2}\int_{t_{j-1}}^{t_{j+1}}\ddot v(t)(k-|t-t_j|)dt.$$

Note that $k-|t-t_j|\le k$. We obtain

$$\|\xi_j\|=\|d_k u_j^I-\ddot u_j\|\le\|d_k(u_j^I-u_j)\|+\|d_k u_j-\ddot u_j\|$$
$$\le\frac{1}{k}\int_{t_{j-1}}^{t_{j+1}}\|\ddot u^I-\ddot u\|\,dt+\frac{k}{6}\int_{t_{j-1}}^{t_{j+1}}\|\partial_t^4 u\|\,dt$$

$$\lesssim \frac{h^2}{k}\int_{t_{j-1}}^{t_{j+1}} \|\ddot{u}\|_2\,dt + k\int_{t_{j-1}}^{t_{j+1}} \|\partial_t^4 u\|\,dt$$
$$\lesssim \frac{h^2}{k^{1/2}}\left(\int_{t_{j-1}}^{t_{j+1}} \|\ddot{u}\|_2^2\,dt\right)^{1/2} + k^{3/2}\left(\int_{t_{j-1}}^{t_{j+1}} \|\partial_t^4 u\|^2\,dt\right)^{1/2}.$$

Therefore,

$$\left(\sum_{j=1}^{n} k\|\xi_j\|^2\right)^{1/2} \lesssim h^2\|\ddot{u}\|_{L^2(0,T;H^2(\Omega))} + k^2\|\partial_t^4 u\|_{L^2(0,T;L^2(\Omega))}. \tag{4.17}$$

Similarly, we have

$$\begin{aligned}
|||\zeta_j||| &= |||\gamma_k u_j^I - u_j||| \le |||\gamma_k(u_j^I - u_j)||| + |||\gamma_k u_j - u_j|||\\
&\le |||u_j^I - u_j||| + \frac{k}{2}\int_{t_{j-1}}^{t_{j+1}} |||\ddot{u}^I - \ddot{u}|||\,dt + \frac{k}{2}\int_{t_{j-1}}^{t_{j+1}} |||\ddot{u}|||\,dt\\
&\lesssim h|u_j|_2 + kh\int_{t_{j-1}}^{t_{j+1}} |\ddot{u}|_2 dt + k\int_{t_{j-1}}^{t_{j+1}} |||\ddot{u}|||\,dt\\
&\lesssim (h+k^2)\|u\|_{C^2([0,T];H^2(\Omega))},
\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}
\sum_{j=1}^{n-2} |||\zeta_j - \zeta_{j+2}||| &= \sum_{j=1}^{n-2} |||\int_{t_j}^{t_{j+2}} \dot{\zeta}(t)dt|||\\
&\lesssim \sum_{j=1}^{n-2}\int_{t_j}^{t_{j+2}}\left(h|\dot{u}_j(t)|_2 + kh\int_{t-k}^{t+k} |\partial_\tau^3 u|_2 d\tau + k\int_{t-k}^{t+k} |||\partial_\tau^3 u|||\,d\tau\right)dt\\
&\lesssim h\|\dot{u}\|_{L^\infty(0,T;H^2(\Omega))} + k^2\|\partial_t^3 u\|_{L^\infty(0,T;H^2(\Omega))}.
\end{aligned} \tag{4.19}$$

Choose $v_j^h = \Pi^h \dot{u}_j$, we have

$$\begin{aligned}
\sum_{j=1}^{n} k\,|||v_j^h - \delta_k u_j^I||| &= \sum_{j=1}^{n} k\,|||\Pi^h \dot{u}_j - \delta_k u_j^I|||\\
&\le \sum_{j=1}^{n} k\left(|||\Pi^h \dot{u}_j - \dot{u}_j^I||| + k\int_{t_{j-1}}^{t_{j+1}} |||\partial_t^3 u^I|||\,dt\right)\\
&\lesssim h\|\dot{u}\|_{L^\infty(0,T;H^2(\Omega))} + k^2\|\partial_t^3 u\|_{L^1(0,T;H^2(\Omega))}.
\end{aligned} \tag{4.20}$$

In addition,

$$\begin{aligned}
&\|\xi_j\| + |||e_j^I||| + |||u_j^I - \gamma_k u_j^I|||\\
&\quad\lesssim \frac{h^2}{k}\int_{t_{j-1}}^{t_{j+1}} \|\ddot{u}\|_2 dt + \frac{k}{6}\int_{t_{j-1}}^{t_{j+1}} \|\partial_t^4 u\|ds + h\|u\|_{L^\infty(0,T;H^2(\Omega))} + k\int_{t_{j-1}}^{t_{j+1}} \|\ddot{u}^I\|_2 ds\\
&\quad\lesssim (h+k^2)\|u\|_{C^2([0,T];H^2(\Omega))} + k^2\|\partial_t^4 u\|_{L^\infty(0,T;L^2(\Omega))}.
\end{aligned}$$

Therefore,

$$\left(\max_j(\|\xi_j\| + \|e_j^I\| + \|u_j^I - \gamma_k u_j^I\|)\sum_{j=1}^{n} k\,|||v_j^h - \delta_k u_j^I|||\right)^{1/2} \lesssim h + k^2. \tag{4.21}$$

TABLE 1 Numerical convergence orders when $t = 1$ for fixed $k = 0.01$

h	2^0	2^{-1}	2^{-2}	2^{-3}	2^{-4}
H^1 errors	7.7073e−001	3.4212e−001	1.2933e−001	7.0273e−002	3.0137e−002
Order	–	1.1717	1.4034	0.8801	1.2214

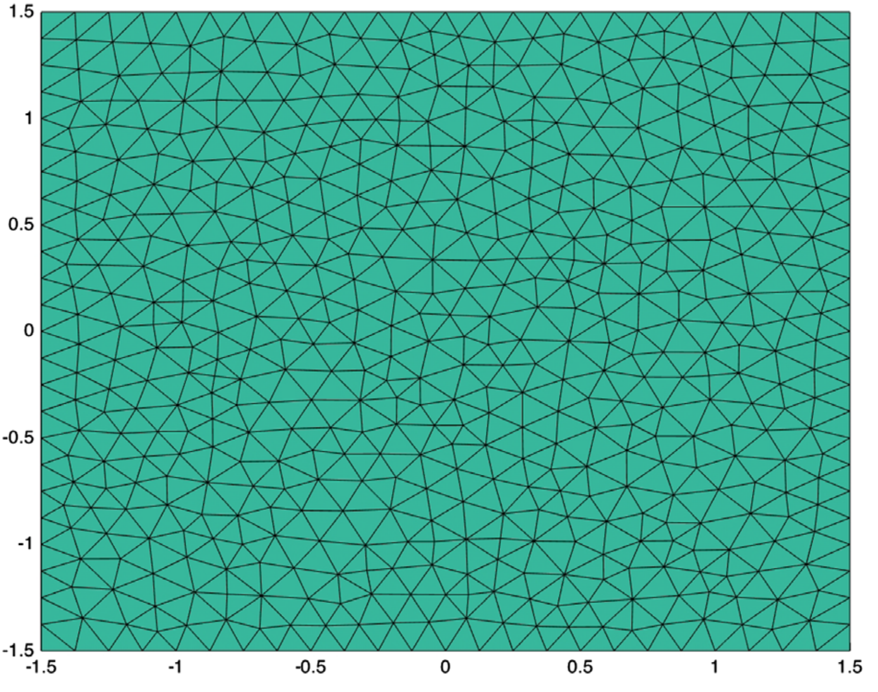

FIGURE 1 Quasi-uniform triangulation with $h = 0.125$ [Color figure can be viewed at wileyonlinelibrary.com]

TABLE 2 Numerical convergence orders when $t = 1$ for fixed $h = 0.03$

k	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}
H^1 errors	9.8526e−001	1.7645e−001	1.3772e−001	6.5695e−002	26168e−0.02
Order	--	2.4812	0.3576	1.0678	1.3280

Next, we estimate

$$
\begin{aligned}
&\|\delta_k(u_j^I - u_{j+1}^I) - (\Pi^h \dot{u}_j - \Pi^h \dot{u}_{j+1})\| \\
&\quad \le \|(\delta_k u_j^I - \dot{u}_j^I) - (\delta_k u_{j+1}^I - \dot{u}_{j+1}^I)\| + \|(\dot{u}_j^I - \dot{u}_{j+1}^I) - (\Pi^h \dot{u}_j - \Pi^h \dot{u}_{j+1})\| \\
&\quad = \frac{1}{4k}\left(\int_{t_j}^{t_{j+1}} \partial_t^3 u^I(t)(t_{j+1}-t)^2 dt + \int_{t_{j-1}}^{t_j} \partial_t^3 u^I(t)(t_{j-1}-t)^2 dt\right. \\
&\qquad \left. - \int_{t_{j+1}}^{t_{j+2}} \partial_t^3 u^I(t)(t_{j+2}-t)^2 dt - \int_{t_j}^{t_{j+1}} \partial_t^3 u^I(t)(t_j - t)^2 dt\right) \\
&\qquad + \left\|\int_{t_j}^{t_{j+1}} \ddot{u}^I\, dt - \int_{t_j}^{t_{j+1}} \Pi^h \ddot{u}\, dt\right\|
\end{aligned}
$$

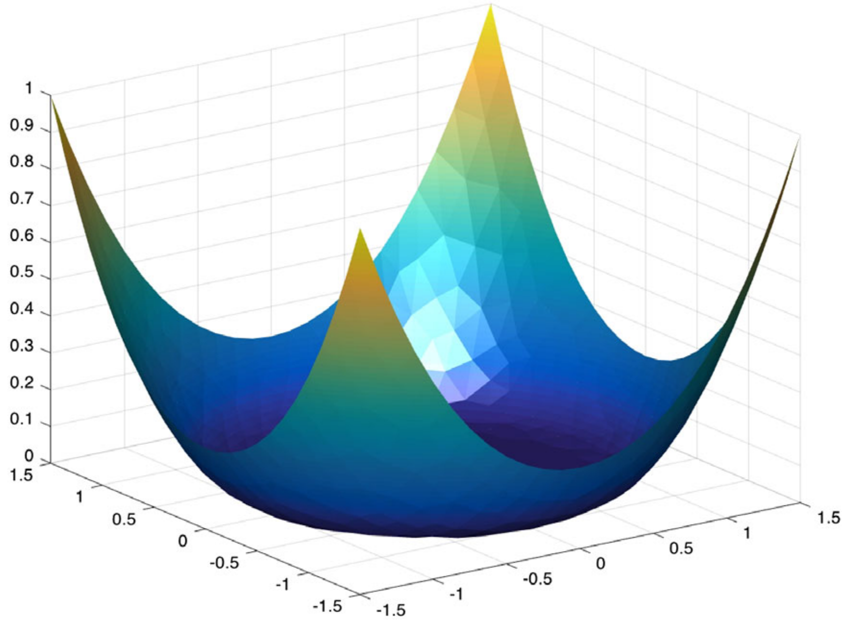

FIGURE 2 Numerical solution on mesh with $h = 0.125$ when $t = 1$ [Color figure can be viewed at wileyonlinelibrary.com]

$$\lesssim k^2 \int_{t_{j-1}}^{t_{j+2}} \|\partial_t^4 u^I\|\, dt + h^2 \int_{t_j}^{t_{j+1}} \|\partial_t^2 u\|_2\, dt$$

$$\lesssim k^{5/2} \left(\int_{t_{j-1}}^{t_{j+2}} \|\partial_t^4 u^I\|^2\, dt \right)^{1/2} + h^2 k^{1/2} \left(\int_{t_j}^{t_{j+1}} \|\partial_t^2 u\|_2^2\, dt \right)^{1/2}.$$

In the third inequality, we use the results in Lemma 3.1 and the fact that

$$\partial_t^3 u^I(t) = \partial_t^3 u^I(t_j) + \int_{t_j}^{t} \partial_\tau^4 u^I(\tau) d\tau.$$

Hence, we have

$$\left(\frac{1}{k} \sum_{j=1}^{n-1} \|\delta_k(u_j^I - u_{j+1}^I) - (\Pi^h \dot{u}_j - \Pi^h \dot{u}_{j+1})\|^2 \right)^{1/2} \lesssim k^2 \|\partial_t^4 u\|_{L^2(0,T;L^2(\Omega))} + h^2 \|\ddot{u}\|_{L^2(0,T;H^2(\Omega))}. \quad (4.22)$$

Similarly,

$$\begin{aligned} \|\delta_k u_j^I - \Pi^h \dot{u}_j\| &\le \|\delta_k u_j^I - \dot{u}_j^I\| + \|\dot{u}_j^I - \Pi^h \dot{u}_j\| \\ &= \frac{1}{4k} \left(\int_{t_j}^{t_{j+1}} \partial_t^3 u^I(t)(t_{j+1} - t)^2 dt - \int_{t_j}^{t_{j-1}} \partial_t^3 u^I(t)(t_{j-1} - t)^2 dt \right) + \|\dot{u}_j^I - \Pi^h \dot{u}_j\| \\ &\lesssim k^2 \int_{t_{j-1}}^{t_{j+1}} \|\partial_t^4 u^I\| dt + h^2 \|\dot{u}_j\|_2. \end{aligned}$$

Therefore,

$$\|\delta_k u_n^I - \Pi^h \dot{u}_n\| + \|\delta_k u_1^I - \Pi^h \dot{u}_1\| \lesssim k^2 \|\partial_t^4 u\|_{L^1(0,T;L^2(\Omega))} + h^2 \|\dot{u}\|_{L^\infty(0,T;H^2(\Omega))}. \quad (4.23)$$

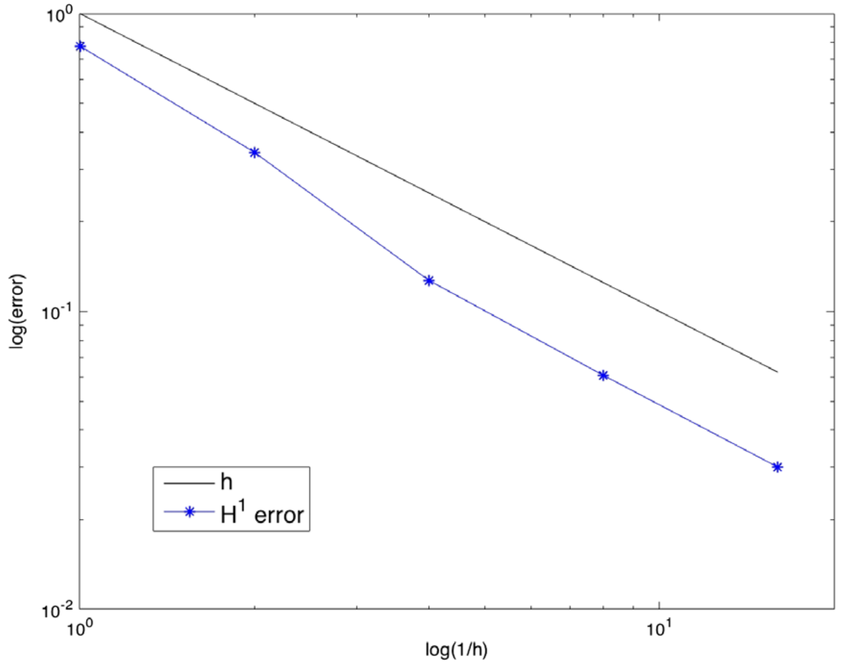

FIGURE 3 Numerical errors for IPDG method when $t = 1$ for fixed $k = 0.01$ [Color figure can be viewed at wileyonlinelibrary.com]

To estimate the term $| T_j^4 |$, doing similar argument for deriving (3.13), we obtain

$$| T_j^4 | \leq \|\ddot{u}_j - \Delta u_j - f_j\| \|v_j^h - \dot{u}_j\| \lesssim h^2.$$

Hence,

$$\left(\sum_{j=1}^{n} k|T_j^4| \right)^{1/2} \lesssim h \tag{4.24}$$

Because $u_0^{hk} = P_B^h u_0 = u_0^I$, we have $e_0^h = 0$, and

$$e_1^h - e_0^h = u_1^I - u_1^{hk} = u_1^I - u_0^{hk} - kP_B^h v_0 = u_1^I - u_0^I - k\dot{u}_0^I = k^2 \ddot{u}^I(\alpha),$$

we get

$$\frac{1}{k}\|e_1^h - e_0^h\| + |||e_0^h||| \lesssim k. \tag{4.25}$$

Summarizing the results (4.17)–(4.25), we obtain

$$\max_j \left(\frac{1}{k}\|e_{j+1}^h - e_j^h\| + |||e_j^h||| \right) \lesssim h + k.$$

Finally, we apply the triangle inequality to finish the proof

$$k^{-1}\|e_{j+1} - e_j\| + |||e_j||| \leq k^{-1}\|e_{j+1}^I - e_j^I\| + |||e_j^I||| + k^{-1}\|e_{j+1}^h - e_j^h\| + |||e_j^h||| \lesssim h + k.$$

Here, we use the fact that

$$\|e_{j+1}^I - e_j^I\| = \|k\dot{e}_j^I + \int_{t_j}^{t_{j+1}} \ddot{e}^I(t)\frac{t_{j+1} - t}{2} dt\| \lesssim kh^2\|\dot{u}\|_2 + kh^2 \int_{t_j}^{t_{j+1}} \|\ddot{u}\|_2 \, dt.$$

■

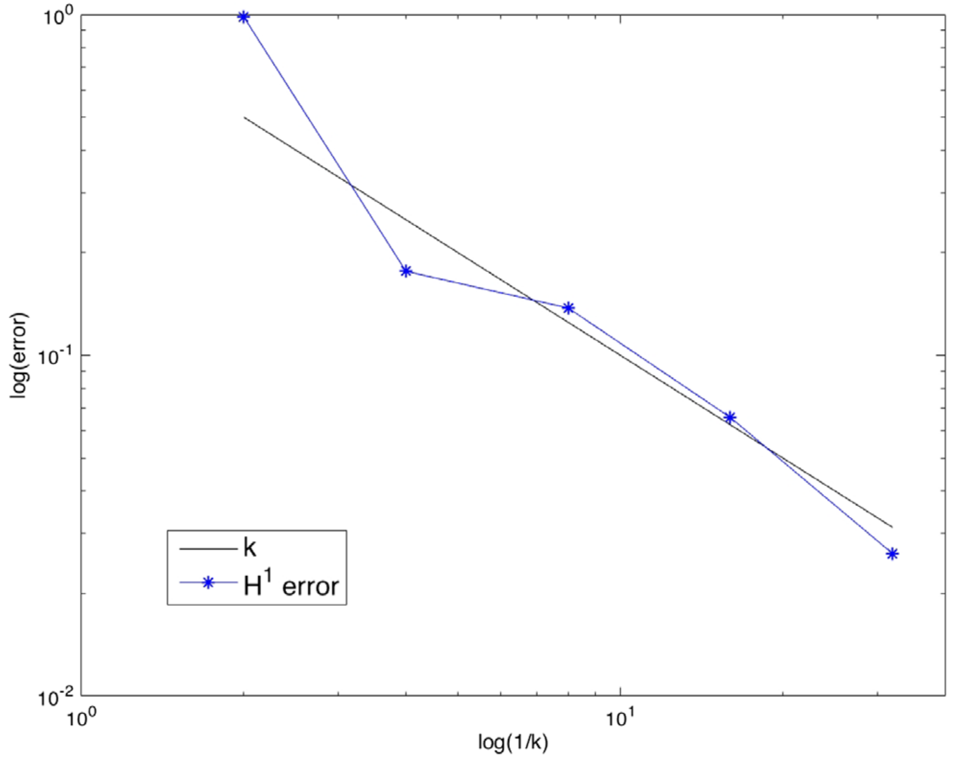

FIGURE 4 Numerical errors for IPDG method when $t = 1$ for fixed $h = 0.03$ [Color figure can be viewed at wileyonlinelibrary.com]

5 | NUMERICAL EXAMPLE

In this section, we present a numerical example on convergence orders. The hyperbolic variational inequality problem (1.1)–(1.3) is discretized by the IPDG scheme in space and finite difference scheme in time as stated in Section 2.3. In each time step, the discretized problem is solved by primal-dual active set method [54].

Example. Let the domain $\Omega := (-1.5, 1.5)^2$ and denote $r = (x^2 + y^2)^{1/2}$. Given a function

$$\psi(x,y) = \begin{cases} \frac{r^2}{2} - \ln(r) - \frac{1}{2}, & \text{if } r \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

we set the right side function $f(t, x, y) = 2\,\psi(x, y) - 2\,t^2$ and define the Dirichlet boundary condition as the trace of the exact solution $u(t, x, y) = t^2\psi(x, y)$.

We use quasi-uniform triangulations $\mathcal{T}_h$, as shown in Figure 1 for $h = 0.125$. Figure 2 shows the numerical solution when $t = 1$ on the mesh with $h = 0.125$.

To observe how the numerical errors depend on the mesh size h, we fix time step $k = 0.01$, and let $h = 2^0, 2^{-1}, \ldots, 2^{-4}$. The numerical errors and convergence orders for $t = 1$ are summarized in Table 1 and shown in Figure 3. We see that the numerical convergence order for H^1 error is around 1, which matches well the theoretical prediction.

Then we fix mesh size $h = 0.03$, and observe how the numerical errors depend on the time step size k, see Table 2 and Figure 4. We see that the convergence order is linear with respect to k.

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