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Discontinuous Galerkin methods for solving the Signorini problem

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We study several discontinuous Galerkin methods for solving the Signorini problem. A unified error analysis is provided for the methods. The error estimates are of optimal order for linear elements. A numerical example is reported to illustrate numerical convergence orders.

Keywords: discontinuous Galerkin method; Signorini problem; error analysis.

1. Introduction

In this paper we extend the ideas in Wang *et al.* (2010) in which the discontinuous Galerkin methods (DGMs) for variational inequalities were analysed to solve the well-known Signorini problem. The initial DGM was introduced by Reed & Hill (1973) for numerically solving the neutron transport equation. In the past two decades DGMs have been widely used for a variety of partial differential equations, such as hyperbolic equations, convection–diffusion equations, Navier–Stokes equations, Hamilton–Jacobi equations and so on. We refer to Cockburn *et al.* (2000) for a historical survey about DGMs.

DGMs provide discontinuous approximations by using the Galerkin method element-by-element and transfer information between two neighbouring elements through the use of numerical traces (numerical fluxes). The discontinuity property means that DGMs easily handle elements of arbitrary shapes and irregular meshes with hanging nodes and have the flexibility to construct local shape function spaces (hp-adaptivity). The increase of locality in discretization, which enhances the degree of parallelizability, is one of the main advantages. In addition, DGMs permit easy treatment of nonhomogeneous boundary conditions, which greatly increases the robustness and accuracy of any boundary condition implementation.

For elliptic problems there are two basic ways to construct DGMs. The first way is to replace the bilinear form of a weak formulation by a new bilinear form with a penalty term penalizing the inter-element discontinuity, see, e.g. Babuška & Zlámal (1973), Douglas & Dupont (1976), Rivière *et al.* (1999), Brezzi *et al.* (2000). The second one is to choose suitable numerical fluxes to make the DG schemes consistent, conservative and stable, see, e.g. Bassi & Rebay (1997), Cockburn & Shu (1998), Cockburn (2003). Arnold *et al.* (2002, 2005) provide a unified error analysis of DGMs for elliptic problems and succeed in building a bridge between these two families, establishing a framework to understand their properties, differences and the connections between them. In Wang *et al.* (2010) *a priori* error estimates were established for the DGMs for solving an obstacle problem and a simplified frictional problem, which reach optimal order for linear elements. We will extend the ideas therein to solve the Signorini problem with DGMs.

The paper is organized as follows: in Section 2 we introduce the Signorini problem and the DG formulations for solving it. Then we show the consistency of the DG schemes and the boundedness and stability of the bilinear forms in Section 3. In Section 4 we establish *a priori* error estimates for these DGMs. In the last section we present results from a numerical example, paying particular attention to numerical convergence orders.

2. Signorini problem and DG formulations

2.1 *Signorini problem and its weak formulation*

The Signorini problem is an elastostatics problem describing the contact of a deformable body with a rigid frictionless foundation. It is an example of an elliptic variational inequality of the first kind. Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be an open bounded connected domain with a Lipschitz boundary Γ that is divided into three parts $\overline{\Gamma_{\rm D}}$, $\overline{\Gamma_{\rm F}}$ and $\overline{\Gamma_{\rm C}}$ with $\Gamma_{\rm D}$, $\Gamma_{\rm F}$ and $\Gamma_{\rm C}$ relatively open and mutually disjoint such that $\text{meas}(\Gamma_{\rm D}) > 0$. The displacement $\boldsymbol{u}\colon \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$ is a vector-valued function. The linearized strain tensor is

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}\big(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\big).$$

Consider a homogeneous, isotropic, linearized elastic material. Then the stress tensor is

$$\boldsymbol{\sigma} = \lambda(\text{tr}\,\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}, \tag{2.1}$$

where $\lambda > 0$ and $\mu > 0$ are the Lamé coefficients. The linearized strain and stress tensors are second-order symmetric tensors, which take values in $\mathbb{S}^d$, the space of second-order symmetric tensors on $\mathbb{R}^d$ with the inner product $\boldsymbol{\sigma}\colon \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}$. Let $\boldsymbol{\nu}$ be the unit outward normal to Γ. For a vector $\boldsymbol{v}$ denote its normal component and tangential component by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$ on the boundary. Similarly, for a tensor-valued function $\boldsymbol{\sigma}\colon \Omega \to \mathbb{S}^d$, we define its normal component $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and tangential component $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$. We have the decomposition formula

$$(\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{v} = (\sigma_\nu \boldsymbol{\nu} + \boldsymbol{\sigma}_\tau) \cdot (v_\nu \boldsymbol{\nu} + \boldsymbol{v}_\tau) = \sigma_\nu v_\nu + \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau.$$

Given $\boldsymbol{f} \in [L^2(\Omega)]^d$, $\boldsymbol{g} \in [L^2(\Gamma_{\rm F})]^d$, the Signorini problem is to find a displacement field $\boldsymbol{u}\colon \Omega \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma}\colon \Omega \to \mathbb{S}^d$ such that (Kikuchi & Oden, 1988)

$$\frac{1}{2\mu}\boldsymbol{\sigma} - \frac{\lambda}{2\mu(d\lambda + 2\mu)}\text{tr}(\boldsymbol{\sigma})\boldsymbol{I} = \boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{2.2}$$

$$-\text{div}\,\boldsymbol{\sigma} = \boldsymbol{f} \quad \text{in } \Omega, \tag{2.3}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_{\rm D}, \tag{2.4}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{g} \quad \text{on } \Gamma_{\rm F}, \tag{2.5}$$

$$u_\nu \leqslant 0,\ \sigma_\nu \leqslant 0,\ \sigma_\nu u_\nu = 0,\ \boldsymbol{\sigma}_\tau = \boldsymbol{0} \quad \text{on } \Gamma_{\rm C}. \tag{2.6}$$

Here (2.2) follows from the constitutive relation of the elastic material, (2.3) is the equilibrium equation in which volume forces of density $\boldsymbol{f}$ act in Ω. Boundary condition (2.4) means that the body is clamped on Γ_D and so the displacement field vanishes there. Surface tractions of density $\boldsymbol{g}$ act on Γ_F in (2.5). The body is in frictionless contact with a rigid foundation on Γ_C.

For a tensor-valued function $\boldsymbol{\sigma}$ define its divergence by

$$\operatorname{div}\boldsymbol{\sigma} = (\partial_j \sigma_{ij})_{1\leqslant i\leqslant d}.$$

Then, for any symmetric tensor $\boldsymbol{\sigma}$ and any vector field $\boldsymbol{v}$, both being continuously differentiable over $\overline{\Omega}$, we have the following integration by parts formula:

$$\int_\Omega \operatorname{div}\boldsymbol{\sigma}\cdot\boldsymbol{v}\,\mathrm{d}x = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}\cdot\boldsymbol{v}\,\mathrm{d}s - \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x. \tag{2.7}$$

To give the weak formulation of the Signorini problem we define

$$Q = \left\{\boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega)^{d\times d} \mid \tau_{ij} = \tau_{ji},\ 1 \leqslant i, j \leqslant d\right\}, \tag{2.8}$$

$$V = \left\{\boldsymbol{v} \in \left[H^1(\Omega)\right]^d \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D\right\}, \tag{2.9}$$

$$K = \{\boldsymbol{v} \in V \mid v_\nu \leqslant 0 \text{ a.e. on } \Gamma_C\}. \tag{2.10}$$

The admissible set K is nonempty, closed and convex. Following a standard argument (Han & Sofonea, 2002) we proceed to derive a weak formulation of the problem (2.2)–(2.6). For an arbitrary smooth vector-valued function $\boldsymbol{v} \in K$, multiplying equation (2.3) by $(\boldsymbol{v}-\boldsymbol{u})$ and integrating over Ω, we obtain by (2.7)

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}x + \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}\cdot(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}s.$$

We use the boundary conditions (2.4) and (2.5) to deduce the equality

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}x + \int_{\Gamma_F} \boldsymbol{g}\cdot(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}s + \int_{\Gamma_C} \boldsymbol{\sigma}\boldsymbol{\nu}\cdot(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}s. \tag{2.11}$$

Over Γ_C, we have

$$\boldsymbol{\sigma}\boldsymbol{\nu}\cdot(\boldsymbol{v}-\boldsymbol{u}) = \sigma_\nu(v_\nu - u_\nu) + \boldsymbol{\sigma}_\tau\cdot(\boldsymbol{v}_\tau - \boldsymbol{u}_\tau) = \sigma_\nu v_\nu,$$

where the boundary conditions (2.6) are used. Note that over Γ_C, we also have $\sigma_\nu \leqslant 0$ and $v_\nu \leqslant 0$. Then

$$\boldsymbol{\sigma}\boldsymbol{\nu}\cdot(\boldsymbol{v}-\boldsymbol{u}) \geqslant 0 \quad \text{on } \Gamma_C.$$

Therefore, we derive from (2.11) that

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}x \geqslant \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}x + \int_{\Gamma_F} \boldsymbol{g}\cdot(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}s.$$

From the boundary conditions (2.4) and (2.6) we know $\boldsymbol{u} \in K$. By the above argument, the variational formulation of the Signorini problem (2.2)–(2.6) is: find a displacement field $\boldsymbol{u} \in K$ such that

$$a(\boldsymbol{\sigma}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})) \geqslant \ell(\boldsymbol{v}-\boldsymbol{u}) \quad \forall\, \boldsymbol{v} \in K, \tag{2.12}$$

where $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{u})$ is given by (2.1) and the bilinear form $a(\cdot,\cdot)$ and the linear form $\ell \in V'$ are

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\,\mathrm{d}x \quad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in Q, \tag{2.13}$$

$$\ell(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_F} \boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}s \quad \forall\, \boldsymbol{v} \in V. \tag{2.14}$$

This problem has a unique solution (Kikuchi & Oden, 1988).

2.2 *Notation and DG formulations*

For definiteness, in the following, we only consider the case $d = 2$, although the discussion can be adapted to the three-dimensional case. Given a bounded domain $D \subset \mathbb{R}^2$ and a positive integer m, $H^m(D)$ is the usual Sobolev space with the corresponding norm $\|\cdot\|_{m,D}$ and seminorm $|\cdot|_{m,D}$. Let $\boldsymbol{u} = (u_1, u_2)^T \in [H^m(D)]^2$ and define the corresponding norm and seminorm by $\|\boldsymbol{u}\|^2_{m,D} = \sum_{i=1}^2 \|u_i\|^2_{m,D}$ and $|\boldsymbol{u}|^2_{m,D} = \sum_{i=1}^2 |u_i|^2_{m,D}$. Similarly, $\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2}_s$ is a matrix-valued function with each component $\tau_{ij} \in L^2(\Omega)$ and $\tau_{12} = \tau_{21}$. We assume Ω is a polygonal domain and consider a regular family of triangulations of $\overline{\Omega}$ denoted by $\{\mathcal{T}_h\}_h$ that are compatible with the boundary splitting $\Gamma = \overline{\Gamma_{\mathrm{D}}} \cup \overline{\Gamma_{\mathrm{F}}} \cup \overline{\Gamma_{\mathrm{C}}}$, i.e., if an element edge has a nonempty intersection with one of the sets Γ_{D}, Γ_{F} and Γ_{C}, then the edge lies entirely in the corresponding closed set $\overline{\Gamma_{\mathrm{D}}}$, $\overline{\Gamma_{\mathrm{F}}}$ or $\overline{\Gamma_{\mathrm{C}}}$. Let $h_K = \mathrm{diam}(K)$ and $h = \max\{h_K : K \in \mathcal{T}_h\}$. Denote by $\mathcal{E}_h$ the union of the boundaries of the elements K of $\mathcal{T}_h$, $\mathcal{E}_h^{\mathrm{i}} = \mathcal{E}_h \backslash \Gamma$ the set of all interior edges, and $\mathcal{E}_h^0 = \mathcal{E}_h \backslash (\Gamma_{\mathrm{F}} \cup \Gamma_{\mathrm{C}})$.

We introduce the following finite element spaces:

$$V_h = \left\{\boldsymbol{v}_h \in \left[L^2(\Omega)\right]^2 : v_{hi}|_K \in P_1(K)\ \forall K \in \mathcal{T}_h,\ i = 1, 2\right\},$$
$$W_h = \left\{\boldsymbol{\tau}_h \in \left[L^2(\Omega)\right]^{2\times 2}_s :\ \tau_{hij}|_K \in P_l(K)\ \forall K \in \mathcal{T}_h,\ i, j = 1, 2\right\}, \quad l = 0 \text{ or } 1.$$

We define the finite element set K_h to approximate K as follows:

$$K_h = \left\{\boldsymbol{v}_h \in V_h : v_{h\nu}(\boldsymbol{x}) \leqslant 0\ \forall \text{ nodes } \boldsymbol{x} \in \overline{\Gamma_{\mathrm{C}}}\right\}.$$

Because $v_{h\nu} \in K_h$ is a linear finite element function, $v_{h\nu} \leqslant 0$ at all nodes on $\overline{\Gamma_{\mathrm{C}}}$ ensures $v_{h\nu} \leqslant 0$ on $\overline{\Gamma_{\mathrm{C}}}$. For all vector-valued functions $\boldsymbol{v}$ and matrix-valued functions $\boldsymbol{\tau}$ then $\boldsymbol{\varepsilon}_h(\boldsymbol{v})$ and $\mathrm{div}_h\boldsymbol{\tau}$ are defined by the relations $\boldsymbol{\varepsilon}_h(\boldsymbol{v}) = \boldsymbol{\varepsilon}(\boldsymbol{v})$ and $\mathrm{div}_h\boldsymbol{\tau} = \mathrm{div}\,\boldsymbol{\tau}$ on any element $K \in \mathcal{T}_h$.

Let e be an edge shared by two elements K^+ and K^-, and $\boldsymbol{n}^\pm = \boldsymbol{n}|_{\partial K^\pm}$ be the unit outward normal vector on $\partial K^\pm$. For a scalar function w, let $w^\pm = w|_{\partial K^\pm}$ and similarly, for a vector-valued function $\boldsymbol{v}$ and a matrix-valued function $\boldsymbol{\tau}$, let $\boldsymbol{v}^\pm = \boldsymbol{v}|_{\partial K^\pm}$, $\boldsymbol{\tau}^\pm = \boldsymbol{\tau}|_{\partial K^\pm}$. Then define the averages $\{\cdot\}$ and the jumps $[\cdot]$, $[\![\cdot]\!]$ on $e \in \mathcal{E}_h^i$ by

$$\{w\} = \frac{1}{2}\left(w^+ + w^-\right), \quad [w] = w^+\boldsymbol{n}^+ + w^-\boldsymbol{n}^-,$$
$$\{\boldsymbol{v}\} = \frac{1}{2}\left(\boldsymbol{v}^+ + \boldsymbol{v}^-\right), \quad [\![\boldsymbol{v}]\!] = \frac{1}{2}(\boldsymbol{v}^+ \otimes \boldsymbol{n}^+ + \boldsymbol{n}^+ \otimes \boldsymbol{v}^+ + \boldsymbol{v}^- \otimes \boldsymbol{n}^- + \boldsymbol{n}^- \otimes \boldsymbol{v}^-),$$
$$\{\boldsymbol{\tau}\} = \frac{1}{2}\left(\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-\right), \quad [\boldsymbol{\tau}] = \boldsymbol{\tau}^+\boldsymbol{n}^+ + \boldsymbol{\tau}^-\boldsymbol{n}^-.$$

If e lies on the boundary Γ, we set

$$\{w\} = w, \quad [w] = w\boldsymbol{\nu},$$
$$\{\boldsymbol{v}\} = \boldsymbol{v}, \quad [\![\boldsymbol{v}]\!] = \tfrac{1}{2}(\boldsymbol{v} \otimes \boldsymbol{\nu} + \boldsymbol{\nu} \otimes \boldsymbol{v}),$$
$$\{\boldsymbol{\tau}\} = \boldsymbol{\tau}, \quad [\boldsymbol{\tau}] = \boldsymbol{\tau}\boldsymbol{\nu}.$$

Here $\boldsymbol{u} \otimes \boldsymbol{v}$ is a matrix with $u_i v_j$ as its (i, j)th element.

For a vector-valued function $\boldsymbol{v}$ and a matrix-valued function $\boldsymbol{\tau}$, after direct manipulation, we have

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} (\boldsymbol{\tau}\boldsymbol{n}) \cdot \boldsymbol{v}\, \mathrm{d}s = \sum_{e\in\mathcal{E}_h^i} \int_e [\boldsymbol{\tau}] \cdot \{\boldsymbol{v}\}\, \mathrm{d}s + \sum_{e\in\mathcal{E}_h} \int_e \{\boldsymbol{\tau}\} : [\![\boldsymbol{v}]\!]\, \mathrm{d}s. \tag{2.15}$$

To give the DG formulations we need lifting operators $\boldsymbol{r}_0$: $\left(L^2(\mathcal{E}_h^0)\right)_s^{2\times 2} \to W_h$, $\boldsymbol{r}_{\mathrm{e}}$: $\left(L^2(e)\right)_s^{2\times 2} \to W_h$ defined by

$$\int_\Omega \boldsymbol{r}_0(\boldsymbol{\phi}) : \boldsymbol{\tau}\,\mathrm{d}x = -\int_{\mathcal{E}_h^0} \boldsymbol{\phi} : \{\boldsymbol{\tau}\}\,\mathrm{d}s \quad \forall\, \boldsymbol{\tau} \in W_h,\ \ \boldsymbol{\phi} \in \left(L^2(\mathcal{E}_h^0)\right)_s^{2\times 2}, \tag{2.16}$$

$$\int_\Omega \boldsymbol{r}_{\mathrm{e}}(\boldsymbol{\phi}) : \boldsymbol{\tau}\,\mathrm{d}x = -\int_{e} \boldsymbol{\phi} : \{\boldsymbol{\tau}\}\,\mathrm{d}s \quad \forall\, \boldsymbol{\tau} \in W_h,\ \ \boldsymbol{\phi} \in \left(L^2(e)\right)_s^{2\times 2}. \tag{2.17}$$

It is easy to check that the following identity holds:

$$\boldsymbol{r}_0(\boldsymbol{\phi}) = \sum_{e\in\mathcal{E}_h^0} \boldsymbol{r}_{\mathrm{e}}(\boldsymbol{\phi}|_e) \quad \forall\, \boldsymbol{\phi} \in \left(L^2(\mathcal{E}_h^0)\right)_s^{2\times 2},$$

so we have

$$\|\boldsymbol{r}_0(\boldsymbol{\phi})\|^2 = \Big\| \sum_{e\in\mathcal{E}_h^0} \boldsymbol{r}_{\mathrm{e}}(\boldsymbol{\phi}|_e)\Big\|^2 \leqslant 3 \sum_{e\in\mathcal{E}_h^0} \|\boldsymbol{r}_{\mathrm{e}}(\boldsymbol{\phi}|_e)\|^2. \tag{2.18}$$

We now present some DGMs for the Signorini problem (2.2)–(2.6). We multiply equations (2.2) and (2.3) by test functions $\boldsymbol{\tau}$ and $\boldsymbol{v}$, respectively, and integrate on a subset $D \subset \Omega$. By (2.7) We get

$$\int_D \left(\frac{1}{2\mu}\boldsymbol{\sigma} : \boldsymbol{\tau} - \frac{\lambda}{4\mu(\lambda+\mu)}\mathrm{tr}(\boldsymbol{\sigma})\mathrm{tr}(\boldsymbol{\tau})\right)\mathrm{d}x = -\int_D \boldsymbol{u}\cdot \mathrm{div}\,\boldsymbol{\tau}\,\mathrm{d}x + \int_{\partial D} \boldsymbol{u}\cdot(\boldsymbol{\tau}\boldsymbol{n})\,\mathrm{d}s, \tag{2.19}$$

$$\int_D \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x = \int_D \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x - \int_{\partial D} (\boldsymbol{\sigma}\boldsymbol{n})\cdot\boldsymbol{v}\,\mathrm{d}s. \tag{2.20}$$

In the equations above we append the subscript h to $\boldsymbol{\sigma}$, $\boldsymbol{u}$, div and $\boldsymbol{\varepsilon}$, add over all the elements, and use numerical traces $\widehat{\boldsymbol{u}_h}$ and $\widehat{\boldsymbol{\sigma}_h}$ to approximate $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ over element edges to obtain

$$\int_\Omega \left(\frac{1}{2\mu}\boldsymbol{\sigma}_h : \boldsymbol{\tau}_h - \frac{\lambda}{4\mu(\lambda+\mu)}\mathrm{tr}(\boldsymbol{\sigma}_h)\mathrm{tr}(\boldsymbol{\tau}_h)\right)\mathrm{d}x = -\int_\Omega \boldsymbol{u}_h\cdot \mathrm{div}_h\boldsymbol{\tau}_h\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\int_{\partial K} \widehat{\boldsymbol{u}_h}\cdot(\boldsymbol{\tau}_h\boldsymbol{n}_K)\,\mathrm{d}s, \tag{2.21}$$

$$\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}_h\,\mathrm{d}x = \int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)\,\mathrm{d}x - \sum_{K\in\mathcal{T}_h}\int_{\partial K} (\widehat{\boldsymbol{\sigma}_h}\boldsymbol{n}_K)\cdot\boldsymbol{v}_h\,\mathrm{d}s \tag{2.22}$$

for all $(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in W_h \times V_h$ and all $K \in \mathcal{T}_h$. The numerical traces $\widehat{\boldsymbol{\sigma}_h}$ and $\widehat{\boldsymbol{u}_h}$ will be selected to guarantee consistency and stability of the above scheme.

To derive a new formulation which does not rely on $\boldsymbol{\sigma}_h$ explicitly, using (2.7) and (2.15), we have from (2.21) and (2.22) that

$$\begin{aligned}\int_\Omega \left(\frac{1}{2\mu}\boldsymbol{\sigma}_h : \boldsymbol{\tau}_h - \frac{\lambda}{4\mu(\lambda+\mu)}\mathrm{tr}(\boldsymbol{\sigma}_h)\mathrm{tr}(\boldsymbol{\tau}_h)\right)\mathrm{d}x &= \int_\Omega \boldsymbol{\varepsilon}_h(\boldsymbol{u}_h) : \boldsymbol{\tau}_h\,\mathrm{d}x + \int_{\mathcal{E}_h^i} \{\widehat{\boldsymbol{u}_h} - \boldsymbol{u}_h\}\cdot[\boldsymbol{\tau}_h]\,\mathrm{d}s \\ &\quad + \int_{\mathcal{E}_h} [\![\widehat{\boldsymbol{u}_h} - \boldsymbol{u}_h]\!] : \{\boldsymbol{\tau}_h\}\,\mathrm{d}s,\end{aligned} \tag{2.23}$$

$$\begin{aligned}\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}_h\,\mathrm{d}x &= \int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)\,\mathrm{d}x - \int_{\mathcal{E}_h^i} [\widehat{\boldsymbol{\sigma}_h}]\cdot\{\boldsymbol{v}_h\}\,\mathrm{d}s \\ &\quad - \int_{\mathcal{E}_h} [\![\boldsymbol{v}_h]\!] : \{\widehat{\boldsymbol{\sigma}_h}\}\,\mathrm{d}s.\end{aligned} \tag{2.24}$$

Choosing $\boldsymbol{\tau}_h = 2\mu\, \boldsymbol{\varepsilon}_h(\boldsymbol{v}_h) + \lambda\, \mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h))\, \boldsymbol{I}$ in (2.23) we get

$$\begin{aligned}\int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)\, \mathrm{d}x = &\int_\Omega (2\mu\, \boldsymbol{\varepsilon}_h(\boldsymbol{u}_h) : \boldsymbol{\varepsilon}_h(\boldsymbol{v}_h) + \lambda\, \mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h))\mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)))\, \mathrm{d}x\\ &+ \int_{\mathcal{E}_h^i} \{\widehat{\boldsymbol{u}}_h - \boldsymbol{u}_h\} \cdot (2\mu[\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)] + \lambda[\mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h))])\, \mathrm{d}s\\ &+ \int_{\mathcal{E}_h} [\![\widehat{\boldsymbol{u}}_h - \boldsymbol{u}_h]\!] : (2\mu\{\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)\} + \lambda\, \mathrm{tr}(\{\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)\})\boldsymbol{I})\, \mathrm{d}s.\end{aligned}$$

The combination of the last equation and (2.24) yields

$$\begin{aligned}&\int_\Omega (2\mu\, \boldsymbol{\varepsilon}_h(\boldsymbol{u}_h) : \boldsymbol{\varepsilon}_h(\boldsymbol{v}_h) + \lambda\, \mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h))\mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)))\, \mathrm{d}x + \int_{\mathcal{E}_h^i} \{\widehat{\boldsymbol{u}}_h - \boldsymbol{u}_h\} \cdot (2\mu[\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)] + \lambda[\mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h))])\, \mathrm{d}s\\ &+ \int_{\mathcal{E}_h} [\![\widehat{\boldsymbol{u}}_h - \boldsymbol{u}_h]\!] : (2\mu\{\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)\} + \lambda\, \mathrm{tr}(\{\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)\})\boldsymbol{I})\, \mathrm{d}s\\ &- \int_{\mathcal{E}_h^i} [\widehat{\boldsymbol{\sigma}}_h] \cdot \{\boldsymbol{v}_h\}\, \mathrm{d}s - \int_{\mathcal{E}_h} [\![\boldsymbol{v}_h]\!] : \{\widehat{\boldsymbol{\sigma}}_h\}\, \mathrm{d}s = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h\, \mathrm{d}x. \qquad (2.25)\end{aligned}$$

We can get DGMs from (2.25) by correct choices of numerical traces $\widehat{\boldsymbol{\sigma}}_h$ and $\widehat{\boldsymbol{u}}_h$. There are three principles for choosing appropriate numerical traces. Conservation requires the numerical traces to be single valued over all edges; consistency of the numerical traces requires $\widehat{\boldsymbol{u}}_h(\boldsymbol{u}) = \boldsymbol{u}|_{\mathcal{E}_h}$ and $\widehat{\boldsymbol{\sigma}}_h(\boldsymbol{\sigma}) = \boldsymbol{\sigma}|_{\mathcal{E}_h}$; stability is not easily ensured and it is usual to add a suitable penalty term (stability term) to guarantee it. We will introduce five consistent and stable DGMs. For example, take

$$\begin{cases}\widehat{\boldsymbol{u}}_h = \{\boldsymbol{u}_h\} \quad \text{on } \mathcal{E}_h \backslash \Gamma_{\mathrm{D}}, \quad \widehat{\boldsymbol{u}}_h = \boldsymbol{0} \quad \text{on } \Gamma_{\mathrm{D}},\\ \widehat{\boldsymbol{\sigma}}_h = 2\mu\{\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h)\} + \lambda\, \mathrm{tr}(\{\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h)\})\boldsymbol{I} - \dfrac{\eta}{h_e}[\![\boldsymbol{u}_h]\!] \quad \text{on } \mathcal{E}_h^0,\\ \widehat{\boldsymbol{\sigma}}_h \boldsymbol{\nu} = \boldsymbol{g} \quad \text{on } \Gamma_{\mathrm{F}}, \quad \widehat{\boldsymbol{\sigma}}_{h\tau} = \boldsymbol{0}, \quad \widehat{\sigma}_{h\nu} \leqslant 0, \quad \widehat{\sigma}_{h\nu} u_{h\nu} = 0 \quad \text{on } \Gamma_{\mathrm{C}},\end{cases}$$

where the function η equals a constant η_e on each $e \in \mathcal{E}_h^0$, with $\{\eta_e\}_{e\in\mathcal{E}_h^0}$ having a uniform positive bound from above and below. We obtain from (2.25) that

$$B_{1,h}^{(1)}(\boldsymbol{u}_h, \boldsymbol{v}_h) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h\, \mathrm{d}x + \int_{\Gamma_{\mathrm{F}}} \boldsymbol{g} \cdot \boldsymbol{v}_h\, \mathrm{d}s + \int_{\Gamma_{\mathrm{C}}} \widehat{\boldsymbol{\sigma}}_h \boldsymbol{\nu} \cdot \boldsymbol{v}_h\, \mathrm{d}s, \qquad (2.26)$$

where

$$\begin{aligned}B_{1,h}^{(1)}(\boldsymbol{u}_h, \boldsymbol{v}_h) := &\int_\Omega (2\mu\, \boldsymbol{\varepsilon}_h(\boldsymbol{u}_h) : \boldsymbol{\varepsilon}_h(\boldsymbol{v}_h) + \lambda\, \mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h))\mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)))\, \mathrm{d}x\\ &- \int_{\mathcal{E}_h^0} [\![\boldsymbol{u}_h]\!] : (2\mu\{\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)\} + \lambda\, \mathrm{tr}(\{\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)\})\boldsymbol{I})\, \mathrm{d}s\\ &- \int_{\mathcal{E}_h^0} [\![\boldsymbol{v}_h]\!] : (2\mu\{\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h)\} + \lambda\, \mathrm{tr}(\{\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h)\})\boldsymbol{I})\, \mathrm{d}s + \int_{\mathcal{E}_h^0} \frac{\eta}{h_e} [\![\boldsymbol{u}_h]\!] : [\![\boldsymbol{v}_h]\!]\, \mathrm{d}s. \qquad (2.27)\end{aligned}$$

Let $\boldsymbol{v}_h = \boldsymbol{w}_h - \boldsymbol{u}_h$ with $\boldsymbol{w}_h \in K_h$. Since $\widehat{\boldsymbol{\sigma}}_{h\tau} = \boldsymbol{0}$, $\widehat{\sigma}_{h\nu} \leqslant 0$, $\widehat{\sigma}_{h\nu} u_{h\nu} = 0$ on Γ_{C}, equation (2.26) leads to the inequality

$$B_{1,h}^{(1)}(\boldsymbol{u}_h, \boldsymbol{w}_h - \boldsymbol{u}_h) \geqslant \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{w}_h - \boldsymbol{u}_h)\, \mathrm{d}x + \int_{\Gamma_{\mathrm{F}}} \boldsymbol{g} \cdot (\boldsymbol{w}_h - \boldsymbol{u}_h)\mathrm{d}s \quad \forall\, \boldsymbol{w}_h \in K_h.$$

The term $\int_{\mathcal{E}_h^0} \eta h_e^{-1} [\![\boldsymbol{u}_h]\!] : [\![\boldsymbol{v}_h]\!]\,\mathrm{d}s$ is the penalty term. This is the interior penalty (IP) formulation (Douglas & Dupont, 1976). With the lift operator $\boldsymbol{r}_0$, we can rewrite $B_{1,h}^{(1)}$ as

$$B_{2,h}^{(1)}(\boldsymbol{u}_h, \boldsymbol{v}_h) := \int_\Omega (2\mu\, \boldsymbol{\varepsilon}_h(\boldsymbol{u}_h) : \big(\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h) + \boldsymbol{r}_0([\![\boldsymbol{v}_h]\!])\big) + \lambda\, \mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h))\mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h) + \boldsymbol{r}_0([\![\boldsymbol{v}_h]\!])))\,\mathrm{d}x$$
$$+ \int_\Omega \boldsymbol{r}_0([\![\boldsymbol{u}_h]\!]) : (2\mu\, \boldsymbol{\varepsilon}_h(\boldsymbol{v}_h) + \lambda\, \mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h))\boldsymbol{I})\,\mathrm{d}x + \int_{\mathcal{E}_h^0} \frac{\eta}{h_e} [\![\boldsymbol{u}_h]\!] : [\![\boldsymbol{v}_h]\!]\,\mathrm{d}s. \tag{2.28}$$

Note that (2.27) and (2.28) are equivalent on V_h, implying that either one can be used to define the numerical solution u_h. In this paper we give an *a priori* error estimate for the first formula (2.27). Because (2.27) and (2.28) are equivalent on V_h we will prove stability for the second formula $B_{2,h}^{(1)}$ on V_h, which guarantees the stability of the first formulation $B_{1,h}^{(1)}$ on V_h. This comment is valid for the other DGMs to be introduced later.

By changing the sign of the second term in the bilinear form $B_{1,h}^{(1)}$, we can give a nonsymmetric interior penalty (NIPG) formulation (see Rivière *et al.*, 1999),

$$B_{1,h}^{(2)}(\boldsymbol{u}_h, \boldsymbol{v}_h) := \int_\Omega (2\mu\, \boldsymbol{\varepsilon}_h(\boldsymbol{u}_h) : \boldsymbol{\varepsilon}_h(\boldsymbol{v}_h) + \lambda\, \mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h))\mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)))\,\mathrm{d}x$$
$$+ \int_{\mathcal{E}_h^0} [\![\boldsymbol{u}_h]\!] : (2\mu\{\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)\} + \lambda\, \mathrm{tr}(\{\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)\})\boldsymbol{I})\,\mathrm{d}s$$
$$- \int_{\mathcal{E}_h^0} [\![\boldsymbol{v}_h]\!] : (2\mu\{\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h)\} + \lambda\, \mathrm{tr}(\{\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h)\})\boldsymbol{I})\,\mathrm{d}s + \int_{\mathcal{E}_h^0} \frac{\eta}{h_e} [\![\boldsymbol{u}_h]\!] : [\![\boldsymbol{v}_h]\!]\,\mathrm{d}s,$$

or equivalently,

$$B_{2,h}^{(2)}(\boldsymbol{u}_h, \boldsymbol{v}_h) := \int_\Omega \big(2\mu\, \boldsymbol{\varepsilon}_h(\boldsymbol{u}_h) : (\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h) + \boldsymbol{r}_0([\![\boldsymbol{v}_h]\!])) + \lambda\, \mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h))\mathrm{tr}\big(\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h) + \boldsymbol{r}_0([\![\boldsymbol{v}_h]\!])\big)\big)\,\mathrm{d}x$$
$$- \int_\Omega \boldsymbol{r}_0([\![\boldsymbol{u}_h]\!]) : (2\mu\, \boldsymbol{\varepsilon}_h(\boldsymbol{v}_h) + \lambda\, \mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h))\boldsymbol{I})\,\mathrm{d}x + \int_{\mathcal{E}_h^0} \frac{\eta}{h_e} [\![\boldsymbol{u}_h]\!] : [\![\boldsymbol{v}_h]\!]\,\mathrm{d}s.$$

Using the local lifting operator $\boldsymbol{r}_\mathrm{e}$, we can give the third example. Taking

$$\begin{cases} \widehat{\boldsymbol{u}_h} = \{\boldsymbol{u}_h\} \quad \text{on } \mathcal{E}_h \backslash \Gamma_\mathrm{D}, \quad \widehat{\boldsymbol{u}_h} = \boldsymbol{0} \quad \text{on } \Gamma_\mathrm{D}, \\ \widehat{\boldsymbol{\sigma}_h} = 2\mu\{\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h)\} + \lambda\, \mathrm{tr}(\{\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h)\})\, \boldsymbol{I} + 2\mu\big\{\boldsymbol{r}_0([\![\boldsymbol{u}_h]\!])\big\} + \lambda\big\{\mathrm{tr}\big(\boldsymbol{r}_0([\![\boldsymbol{u}_h]\!])\big)\, \boldsymbol{I}\big\} \\ \qquad + 2\mu\big\{\eta \boldsymbol{r}_\mathrm{e}([\![\boldsymbol{u}_h]\!])\big\} + \lambda\big\{\eta\, \mathrm{tr}\big(\boldsymbol{r}_\mathrm{e}([\![\boldsymbol{u}_h]\!])\big)\big\}\boldsymbol{I} \quad \text{on } \mathcal{E}_h^0, \\ \widehat{\boldsymbol{\sigma}_h}\boldsymbol{\nu} = \boldsymbol{g} \quad \text{on } \Gamma_\mathrm{F}, \quad \widehat{\boldsymbol{\sigma}_h}_\tau = \boldsymbol{0}, \quad \widehat{\sigma}_{h\nu} \leqslant 0, \quad \widehat{\sigma}_{h\nu} u_{h\nu} = 0 \quad \text{on } \Gamma_\mathrm{C}, \end{cases}$$

from (2.25), we get

$$B_{1,h}^{(3)}(\boldsymbol{u}_h, \boldsymbol{v}_h) := \int_\Omega (2\mu\, \boldsymbol{\varepsilon}_h(\boldsymbol{u}_h) : \boldsymbol{\varepsilon}_h(\boldsymbol{v}_h) + \lambda\, \mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h))\mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)))\,\mathrm{d}x$$
$$- \int_{\mathcal{E}_h^0} [\![\boldsymbol{u}_h]\!] : (2\mu\{\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)\} + \lambda\, \mathrm{tr}(\{\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)\})\boldsymbol{I})\,\mathrm{d}s$$
$$- \int_{\mathcal{E}_h^0} [\![\boldsymbol{v}_h]\!] : (2\mu\{\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h)\} + \lambda\, \mathrm{tr}(\{\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h)\})\boldsymbol{I})\,\mathrm{d}s$$

$$+\int_{\Omega} \boldsymbol{r}_0([\![\boldsymbol{v}_h]\!]) : \big(2\mu\, \boldsymbol{r}_0([\![\boldsymbol{u}_h]\!]) + \lambda\, \mathrm{tr}\big(\boldsymbol{r}_0([\![\boldsymbol{u}_h]\!])\big)\boldsymbol{I}\big)\, \mathrm{d}x$$
$$+\sum_{e\in\mathcal{E}_h^0}\int_{\Omega} \eta\big(2\mu\, \boldsymbol{r}_\mathrm{e}([\![\boldsymbol{u}_h]\!]) : \boldsymbol{r}_\mathrm{e}([\![\boldsymbol{v}_h]\!]) + \lambda\, \mathrm{tr}\big(\boldsymbol{r}_\mathrm{e}([\![\boldsymbol{u}_h]\!])\big)\mathrm{tr}\big(\boldsymbol{r}_\mathrm{e}([\![\boldsymbol{v}_h]\!])\big)\big)\, \mathrm{d}x,$$

or equivalently,

$$\begin{aligned}B_{2,h}^{(3)}(\boldsymbol{u}_h,\boldsymbol{v}_h) := &\int_{\Omega} 2\mu\big(\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h)+\boldsymbol{r}_0([\![\boldsymbol{u}_h]\!])\big) : \big(\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)+\boldsymbol{r}_0([\![\boldsymbol{v}_h]\!])\big)\, \mathrm{d}x\\ &+\int_{\Omega} \lambda\, \mathrm{tr}\big(\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h)+\boldsymbol{r}_0([\![\boldsymbol{u}_h]\!])\big)\mathrm{tr}\big(\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)+\boldsymbol{r}_0([\![\boldsymbol{v}_h]\!])\big)\, \mathrm{d}x\\ &+\sum_{e\in\mathcal{E}_h^0}\int_{\Omega} \eta\big(2\mu\, \boldsymbol{r}_\mathrm{e}([\![\boldsymbol{u}_h]\!]) : \boldsymbol{r}_\mathrm{e}([\![\boldsymbol{v}_h]\!]) + \lambda\, \mathrm{tr}\big(\boldsymbol{r}_\mathrm{e}([\![\boldsymbol{u}_h]\!])\big)\mathrm{tr}\big(\boldsymbol{r}_\mathrm{e}([\![\boldsymbol{v}_h]\!])\big)\big)\, \mathrm{d}x,\end{aligned}$$

which is an extension of the method of Brezzi *et al.* (1999).

With the choice

$$\begin{cases} \widehat{\boldsymbol{u}_h} = \{\boldsymbol{u}_h\} \quad \text{on } \mathcal{E}_h\backslash\Gamma_\mathrm{D}, \quad \widehat{\boldsymbol{u}_h} = \boldsymbol{0} \quad \text{on } \Gamma_\mathrm{D},\\ \widehat{\boldsymbol{\sigma}_h} = 2\mu\{\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h)\} + \lambda\, \mathrm{tr}(\{\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h)\})\,\boldsymbol{I} + 2\mu\big\{\eta\boldsymbol{r}_\mathrm{e}([\![\boldsymbol{u}_h]\!])\big\} + \lambda\big\{\eta\, \mathrm{tr}\big(\boldsymbol{r}_\mathrm{e}([\![\boldsymbol{u}_h]\!])\big)\big\}\boldsymbol{I} \quad \text{on } \mathcal{E}_h^0,\\ \widehat{\boldsymbol{\sigma}_h}\boldsymbol{\nu} = \boldsymbol{g} \quad \text{on } \Gamma_\mathrm{F}, \quad \widehat{\boldsymbol{\sigma}}_{h\tau} = \boldsymbol{0}, \quad \widehat{\sigma}_{h\nu} \leqslant 0, \quad \widehat{\sigma}_{h\nu}u_{h\nu} = 0 \quad \text{on } \Gamma_\mathrm{C}, \end{cases}$$

we obtain a DG formulation extended from the method of Bassi *et al.* (1997),

$$\begin{aligned}B_{1,h}^{(4)}(\boldsymbol{u}_h,\boldsymbol{v}_h) := &\int_{\Omega} (2\mu\, \boldsymbol{\varepsilon}_h(\boldsymbol{u}_h) : \boldsymbol{\varepsilon}_h(\boldsymbol{v}_h) + \lambda\, \mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h))\mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)))\, \mathrm{d}x\\ &-\int_{\mathcal{E}_h^0} [\![\boldsymbol{u}_h]\!] : (2\mu\{\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)\} + \lambda\, \mathrm{tr}(\{\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)\})\boldsymbol{I})\, \mathrm{d}s\\ &-\int_{\mathcal{E}_h^0} [\![\boldsymbol{v}_h]\!] : (2\mu\{\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h)\} + \lambda\, \mathrm{tr}(\{\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h)\})\boldsymbol{I})\, \mathrm{d}s\\ &+\sum_{e\in\mathcal{E}_h^0}\int_{\Omega} \eta\big(2\mu\, \boldsymbol{r}_\mathrm{e}([\![\boldsymbol{u}_h]\!]) : \boldsymbol{r}_\mathrm{e}([\![\boldsymbol{v}_h]\!]) + \lambda\, \mathrm{tr}\big(\boldsymbol{r}_\mathrm{e}([\![\boldsymbol{u}_h]\!])\big)\mathrm{tr}\big(\boldsymbol{r}_\mathrm{e}([\![\boldsymbol{v}_h]\!])\big)\big)\, \mathrm{d}x,\end{aligned}$$

or equivalently,

$$\begin{aligned}B_{2,h}^{(4)}(\boldsymbol{u}_h,\boldsymbol{v}_h) := &\int_{\Omega} \big(2\mu\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h) : \big(\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)+\boldsymbol{r}_0([\![\boldsymbol{v}_h]\!])\big) + \lambda\, \mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h))\mathrm{tr}\big(\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)+\boldsymbol{r}_0([\![\boldsymbol{v}_h]\!])\big)\big)\, \mathrm{d}x\\ &+\int_{\Omega} \boldsymbol{r}_0([\![\boldsymbol{u}_h]\!]) : (2\mu\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h) + \lambda\, \mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h))\boldsymbol{I})\, \mathrm{d}x\\ &+\sum_{e\in\mathcal{E}_h^0}\int_{\Omega} \eta\big(2\mu\boldsymbol{r}_\mathrm{e}([\![\boldsymbol{u}_h]\!]) : \boldsymbol{r}_\mathrm{e}([\![\boldsymbol{v}_h]\!]) + \lambda\, \mathrm{tr}\big(\boldsymbol{r}_\mathrm{e}([\![\boldsymbol{u}_h]\!])\big)\mathrm{tr}\big(\boldsymbol{r}_\mathrm{e}([\![\boldsymbol{v}_h]\!])\big)\big)\, \mathrm{d}x.\end{aligned}$$

If we choose

$$\begin{cases} \widehat{\boldsymbol{u}_h} = \{\boldsymbol{u}_h\} \quad \text{on } \mathcal{E}_h\backslash\Gamma_\mathrm{D}, \quad \widehat{\boldsymbol{u}_h} = \boldsymbol{0} \quad \text{on } \Gamma_\mathrm{D},\\ \widehat{\boldsymbol{\sigma}_h} = 2\mu\{\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h)\} + \lambda\, \mathrm{tr}(\{\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h)\})\boldsymbol{I} + 2\mu\big\{\boldsymbol{r}_0([\![\boldsymbol{u}_h]\!])\big\} + \lambda\big\{\boldsymbol{r}_0\big(\mathrm{tr}([\![\boldsymbol{u}_h]\!])\boldsymbol{I}\big)\big\} - \dfrac{\eta}{h_e}[\![\boldsymbol{u}_h]\!] \quad \text{on } \mathcal{E}_h^0,\\ \widehat{\boldsymbol{\sigma}_h}\boldsymbol{\nu} = \boldsymbol{g} \quad \text{on } \Gamma_\mathrm{F}, \quad \widehat{\boldsymbol{\sigma}}_{h\tau} = \boldsymbol{0}, \quad \widehat{\sigma}_{h\nu} \leqslant 0, \quad \widehat{\sigma}_{h\nu}u_{h\nu} = 0 \quad \text{on } \Gamma_\mathrm{C}, \end{cases}$$

then

$$
\begin{aligned}
B_{1,h}^{(5)}(\boldsymbol{u}_h, \boldsymbol{v}_h) := &\int_\Omega (2\mu\, \boldsymbol{\varepsilon}_h(\boldsymbol{u}_h) : \boldsymbol{\varepsilon}_h(\boldsymbol{v}_h) + \lambda\, \mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h))\mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)))\, \mathrm{d}x \\
&- \int_{\mathcal{E}_h^0} [\![\boldsymbol{u}_h]\!] : (2\mu\{\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)\} + \lambda\, \mathrm{tr}(\{\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h)\})\boldsymbol{I})\, \mathrm{d}s \\
&- \int_{\mathcal{E}_h^0} [\![\boldsymbol{v}_h]\!] : (2\mu\{\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h)\} + \lambda\, \mathrm{tr}(\{\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h)\})\boldsymbol{I})\, \mathrm{d}s \\
&+ \int_\Omega \boldsymbol{r}_0([\![\boldsymbol{v}_h]\!]) : \big(2\mu \boldsymbol{r}_0([\![\boldsymbol{u}_h]\!]) + \lambda\, \mathrm{tr}\big(\boldsymbol{r}_0([\![\boldsymbol{u}_h]\!])\big)\boldsymbol{I}\big)\, \mathrm{d}x + \int_{\mathcal{E}_h^0} \frac{\eta}{h_e} [\![\boldsymbol{u}_h]\!] : [\![\boldsymbol{v}_h]\!]\, \mathrm{d}s,
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
B_{2,h}^{(5)}(\boldsymbol{u}_h, \boldsymbol{v}_h) := &\int_\Omega 2\mu\big(\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h) + \boldsymbol{r}_0([\![\boldsymbol{u}_h]\!])\big) : \big(\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h) + \boldsymbol{r}_0([\![\boldsymbol{v}_h]\!])\big)\, \mathrm{d}x \\
&+ \int_\Omega \lambda\, \mathrm{tr}\big(\boldsymbol{\varepsilon}_h(\boldsymbol{u}_h) + \boldsymbol{r}_0([\![\boldsymbol{u}_h]\!])\big)\mathrm{tr}\big(\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h) + \boldsymbol{r}_0([\![\boldsymbol{v}_h]\!])\big)\, \mathrm{d}x + \int_{\mathcal{E}_h^0} \frac{\eta}{h_e} [\![\boldsymbol{u}_h]\!] : [\![\boldsymbol{v}_h]\!]\, \mathrm{d}s,
\end{aligned}
$$

which is an extension of the local discontinuous Galerkin (LDG) method of Cockburn & Shu (1998).

Let $B_h(\boldsymbol{u}_h, \boldsymbol{v}_h)$ be one of the bilinear forms $B_{1,h}^{(j)}(\boldsymbol{u}_h, \boldsymbol{v}_h)$ with $j = 1, \ldots, 5$. Then a DGM for the Signorini problem (2.12) is: find $\boldsymbol{u}_h \in K_h$ such that

$$
B_h(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) \geqslant \ell(\boldsymbol{v}_h - \boldsymbol{u}_h) \quad \forall\, \boldsymbol{v}_h \in K_h. \tag{2.29}
$$

3. Consistency, boundedness and stability

We note that if the solution of (2.12) has the regularity $\boldsymbol{u} \in [H^2(\Omega)]^2$, then $\boldsymbol{u}$ is the solution of (2.2)–(2.6), and on any interior edge e, $[\![\boldsymbol{u}]\!] = \boldsymbol{0}$, $\{\boldsymbol{u}\} = \boldsymbol{u}$, $\{\boldsymbol{\varepsilon}(\boldsymbol{u})\} = \boldsymbol{\varepsilon}(\boldsymbol{u})$, $[\boldsymbol{\sigma}] = \boldsymbol{0}$, $\{\boldsymbol{\sigma}\} = \boldsymbol{\sigma}$. For all DGMs introduced in the Section 2.2 we first show the consistency of the DG schemes.

LEMMA 3.1 (Consistency). Assume $\boldsymbol{u} \in [H^2(\Omega)]^2$ is the solution of (2.12). Then for the DGMs $B_h(\boldsymbol{w}, \boldsymbol{v}) = B_{1,h}^{(j)}(\boldsymbol{w}, \boldsymbol{v})$ with $j = 1, \ldots, 5$, we have

$$
B_h(\boldsymbol{u}, \boldsymbol{v}_h - \boldsymbol{u}) \geqslant \ell(\boldsymbol{v}_h - \boldsymbol{u}) \quad \forall\, \boldsymbol{v}_h \in K_h. \tag{3.1}
$$

Proof. Using (2.1), we obtain, for any $\boldsymbol{v}_h \in K_h$,

$$
\begin{aligned}
B_h(\boldsymbol{u}, \boldsymbol{v}_h - \boldsymbol{u}) = &\int_\Omega (2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}_h(\boldsymbol{v}_h - \boldsymbol{u}) + \lambda\, \mathrm{tr}(\boldsymbol{\varepsilon}(\boldsymbol{u}))\mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{v}_h - \boldsymbol{u})))\, \mathrm{d}x \\
&- \int_{\mathcal{E}_h^0} [\![\boldsymbol{v}_h - \boldsymbol{u}]\!] : (2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda\, \mathrm{tr}(\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{I})\, \mathrm{d}s \\
= &\sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{\sigma} : \boldsymbol{\varepsilon}_h(\boldsymbol{v}_h - \boldsymbol{u})\mathrm{d}x - \int_{\mathcal{E}_h^0} [\![\boldsymbol{v}_h - \boldsymbol{u}]\!] : \boldsymbol{\sigma}\, \mathrm{d}s.
\end{aligned}
$$

By (2.7), (2.15) and noting $[\boldsymbol{\sigma}] = \mathbf{0}$ on $\mathcal{E}_h^i$, we get

$$\begin{aligned}\sum_{K\in\mathcal{T}_h}\int_K \boldsymbol{\sigma} : \boldsymbol{\varepsilon}_h(\boldsymbol{v}_h-\boldsymbol{u})\,\mathrm{d}x &= \sum_{K\in\mathcal{T}_h}\int_K -\mathrm{div}\,\boldsymbol{\sigma}\cdot(\boldsymbol{v}_h-\boldsymbol{u})\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\int_{\partial K}(\boldsymbol{\sigma}\boldsymbol{n}_K)\cdot(\boldsymbol{v}_h-\boldsymbol{u})\,\mathrm{d}s\\ &= \sum_{K\in\mathcal{T}_h}\int_K -\mathrm{div}\,\boldsymbol{\sigma}\cdot(\boldsymbol{v}_h-\boldsymbol{u})\,\mathrm{d}x + \int_{\mathcal{E}_h}[\![\boldsymbol{v}_h-\boldsymbol{u}]\!] : \boldsymbol{\sigma}\,\mathrm{d}s.\end{aligned}$$

Then

$$\begin{aligned}B_h(\boldsymbol{u},\boldsymbol{v}_h-\boldsymbol{u}) &= \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}_h-\boldsymbol{u})\,\mathrm{d}x + \int_{\Gamma_\mathrm{F}}\boldsymbol{g}\cdot(\boldsymbol{v}_h-\boldsymbol{u})\,\mathrm{d}x + \int_{\Gamma_\mathrm{C}}(\boldsymbol{\sigma}\boldsymbol{\nu})\cdot(\boldsymbol{v}_h-\boldsymbol{u})\,\mathrm{d}x\\ &= \ell(\boldsymbol{v}_h-\boldsymbol{u}) + \int_{\Gamma_\mathrm{C}}\sigma_\nu v_{h\nu}\,\mathrm{d}s \geqslant \ell(\boldsymbol{v}_h-\boldsymbol{u}).\end{aligned}$$

The last inequality is obtained by (2.6) and $v_{h\nu}\leqslant 0$ for all $\boldsymbol{v}_h\in K_h$. Hence, (3.1) holds. □

To consider the boundedness and stability of the bilinear form B_h, as in Wang *et al.* (2010), let $V(h) = V_h + V\cap[H^2(\Omega)]^2$, and for $\boldsymbol{v}\in V(h)$ define seminorms as follows:

$$|\boldsymbol{v}|_K^2 := \int_K \boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x,\quad |\boldsymbol{v}|_h^2 := \sum_{K\in\mathcal{T}_h}|\boldsymbol{v}|_K^2,\quad |\boldsymbol{v}|_*^2 := \sum_{e\in\mathcal{E}_h^0} h_e^{-1}\|[\![\boldsymbol{v}]\!]\|_{0,e}^2,$$

where

$$\|[\![\boldsymbol{v}]\!]\|_{0,e}^2 = \int_e [\![\boldsymbol{v}]\!] : [\![\boldsymbol{v}]\!]\,\mathrm{d}s.$$

Then define norms by

$$|||\boldsymbol{v}|||_*^2 := |\boldsymbol{v}|_h^2 + |\boldsymbol{v}|_*^2,\quad |||\boldsymbol{v}|||^2 := |||\boldsymbol{v}|||_*^2 + \sum_{K\in\mathcal{T}_h} h_K^2|\boldsymbol{v}|_{2,K}^2. \tag{3.2}$$

The norm $|||\cdot|||_*$ defined in (3.2) is equivalent to the usual DG norm $(|\cdot|_{1,h}^2 + |\cdot|_*^2)^{1/2}$, thanks to Korn's inequality (see Brenner, 2004; see also Proposition 4.6 of Arnold *et al.*, 2005).

Before presenting the boundedness and stability of the bilinear forms, we give a useful estimate for the lifting operator $\boldsymbol{r}_\mathrm{e}$. The following lemma is a trivial extension to vectors $\boldsymbol{v}$ of Lemma 2 of Brezzi *et al.* (2000), also restated in Arnold *et al.* (2002).

LEMMA 3.2 For any $\boldsymbol{v}\in V(h)$ and $e\in\mathcal{E}_h^0$,

$$C_1 h_e^{-1}\|[\![\boldsymbol{v}]\!]\|_{0,e}^2 \leqslant \|\boldsymbol{r}_\mathrm{e}([\![\boldsymbol{v}]\!])\|_{0,h}^2 \leqslant C_2 h_e^{-1}\|[\![\boldsymbol{v}]\!]\|_{0,e}^2. \tag{3.3}$$

From (3.3) and (2.18) we have

$$\|\boldsymbol{r}_0([\![\boldsymbol{v}]\!])\|_{0,h}^2 = \Big\|\sum_{e\in\mathcal{E}_h^0}\boldsymbol{r}_\mathrm{e}([\![\boldsymbol{v}]\!])\Big\|_{0,h}^2 \leqslant 3C_2\sum_{e\in\mathcal{E}_h^0} h_e^{-1}\|[\![\boldsymbol{v}]\!]\|_{0,e}^2 = 3C_2|\boldsymbol{v}|_*^2.$$

To consider the boundedness of the primal forms B_h, noting $\|\mathrm{tr}(\boldsymbol{\tau})\| \leqslant \|\boldsymbol{\tau}\|$ for a matrix-valued function $\boldsymbol{\tau}$, we use the Cauchy–Schwarz inequality to bound them term-by-term,

$$\int_\Omega \boldsymbol{\varepsilon}_h(\boldsymbol{w}) : \boldsymbol{\varepsilon}_h(\boldsymbol{v})\,\mathrm{d}x \leqslant |\boldsymbol{w}|_h|\boldsymbol{v}|_h, \tag{3.4}$$

$$\int_\Omega \boldsymbol{r}_0([\![\boldsymbol{w}]\!]) : \boldsymbol{r}_0([\![\boldsymbol{v}]\!])\,\mathrm{d}x \lesssim |\boldsymbol{w}|_*|\boldsymbol{v}|_*, \tag{3.5}$$

$$\int_{\mathcal{E}_h^0} \eta h_e^{-1}[\![\boldsymbol{w}]\!] : [\![\boldsymbol{v}]\!]\,\mathrm{d}s \leqslant \sup_{e\in\mathcal{E}_h^0} \eta_e |\boldsymbol{w}|_*|\boldsymbol{v}|_*, \tag{3.6}$$

$$\sum_{e\in\mathcal{E}_h^0}\int_\Omega \eta\, \boldsymbol{r}_{\mathrm{e}}([\![\boldsymbol{w}]\!]) : \boldsymbol{r}_{\mathrm{e}}([\![\boldsymbol{v}]\!])\ \mathrm{d}x \lesssim \sup_{e\in\mathcal{E}_h^0} \eta_e |\boldsymbol{w}|_*|\boldsymbol{v}|_*. \tag{3.7}$$

Here '$\lesssim \ \ldots$' stands for '$\leqslant\ C\ldots$', where C is a positive generic constant independent of h and other parameters, which may take different values in different appearances. Using the trace inequality $\|\boldsymbol{\varepsilon}_h(\boldsymbol{v})\|_{0,e}^2 \lesssim h_e^{-1}|v|_{1,K}^2 + h_e|v|_{2,K}^2$, we have

$$\begin{aligned}\int_{\mathcal{E}_h^0}[\![\boldsymbol{w}]\!] : \{\boldsymbol{\varepsilon}_h(\boldsymbol{v})\}\,\mathrm{d}s &\leqslant \left(\sum_{e\in\mathcal{E}_h^0} h_e^{-1}\big\|[\![\boldsymbol{w}]\!]\big\|_{0,e}^2\right)^{1/2}\left(\sum_{e\in\mathcal{E}_h^0} h_e\|\{\boldsymbol{\varepsilon}_h(\boldsymbol{v})\}\|_{0,e}^2\right)^{1/2}\\ &\lesssim |\boldsymbol{w}|_*\left(\sum_{K\in\mathcal{T}_h}\left(|\boldsymbol{v}|_{1,K}^2 + h_e^2|\boldsymbol{v}|_{2,K}^2\right)\right)^{1/2} \leqslant |\boldsymbol{w}|_*\ |\!|\!|\ \boldsymbol{v}\ |\!|\!|\ .\end{aligned} \tag{3.8}$$

The inequalities (3.4) and (3.8) are needed by all bilinear forms. For the DGMs with the bilinear form $B_{1,h}^{(j)},\ j = 1,\ 2,\ 5$, inequality (3.6) is needed. Inequality (3.5) is needed by the formulas $B_{1,h}^{(j)}$ with $j = 3,\ 5$. For the methods with the bilinear forms $B_{1,h}^{(j)},\ j = 3,\ 4$, the inequality (3.7) is needed. So we have the following results about the boundedness of B_h.

LEMMA 3.3 (Boundedness). For $1 \leqslant j \leqslant 5$, $B_h = B_{1,h}^{(j)}$ satisfies

$$B_h(\boldsymbol{w},\boldsymbol{v}) \lesssim |\!|\!|\boldsymbol{w}\ |\!|\!|\ \ |\!|\!|\boldsymbol{v}\ |\!|\!| \quad \forall\, \boldsymbol{w},\boldsymbol{v} \in V(h). \tag{3.9}$$

For stability, note that $|\!|\!|\boldsymbol{v}|\!|\!| = |\!|\!|\boldsymbol{v}|\!|\!|_*$ for any $\boldsymbol{v}\in V_h$. Since $B_{1,h}^{(j)}$ and $B_{2,h}^{(j)}$ coincide on V_h, once we have proved the stability for $B_{2,h}^{(j)}$ on V_h, the stability of $B_{1,h}^{(j)}$ on V_h follows. We use the Cauchy–Schwarz inequality and Lemma 3.2 to get

$$\begin{aligned}B_{2,h}^{(1)}(\boldsymbol{v},\boldsymbol{v}) = 2\mu\int_\Omega \boldsymbol{\varepsilon}_h(\boldsymbol{v}) : \boldsymbol{\varepsilon}_h(\boldsymbol{v})\,\mathrm{d}x + \lambda\int_\Omega (\mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{v})))^2\,\mathrm{d}x + 4\mu\int_\Omega \boldsymbol{\varepsilon}_h(\boldsymbol{v}) : \boldsymbol{r}_0([\![\boldsymbol{v}]\!])\,\mathrm{d}x\\ + 2\lambda\int_\Omega \mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{v}))\mathrm{tr}\big(\boldsymbol{r}_0([\![\boldsymbol{v}]\!])\big)\,\mathrm{d}x + \int_{\mathcal{E}_h^0}\eta h_e^{-1}[\![\boldsymbol{v}]\!] : [\![\boldsymbol{v}]\!]\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&\geqslant 2\mu|\boldsymbol{v}|_h^2 + \lambda\|\mathrm{div}_h\boldsymbol{v}\|_{0,h}^2 - 2\mu\big(\epsilon|\boldsymbol{v}|_h^2 + \frac{1}{\epsilon}\big\|\boldsymbol{r}_0([\![\boldsymbol{v}]\!])\big\|_{0,h}^2\big) \\
&\quad - \lambda\big(\|\mathrm{div}_h\boldsymbol{v}\|_{0,h}^2 + \big\|\mathrm{tr}\big(\boldsymbol{r}_0([\![\boldsymbol{v}]\!])\big)\big\|_{0,h}^2\big) + \eta_0\sum_{e\in\mathcal{E}_h^0} h_e^{-1}\big\|[\![\boldsymbol{v}]\!]\big\|_{0,e}^2 \\
&\geqslant 2\mu(1-\epsilon)|\boldsymbol{v}|_h^2 + \left(\eta_0 - \frac{6\mu C_2}{\epsilon} - 6C_2\lambda\right)|\boldsymbol{v}|_*^2.
\end{aligned}
$$

Here $\|v\|_{0,h}^2 = \sum_{K\in\mathcal{T}_h}\|v\|_{0,K}^2$, $0 < \epsilon < 1$ is a constant and C_2 is the positive constant in (3.3). Therefore, stability is valid for the IP method when $\min_{e\in\mathcal{E}_h^0}\eta_e = \eta_0 > 6C_2(\mu+\lambda)$.

$$
\begin{aligned}
B_{2,h}^{(2)}(\boldsymbol{v},\boldsymbol{v}) &= 2\mu\int_\Omega \boldsymbol{\varepsilon}_h(\boldsymbol{v}) : \boldsymbol{\varepsilon}_h(\boldsymbol{v})\,\mathrm{d}x + \lambda\int_\Omega(\mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{v})))^2\,\mathrm{d}x + \int_{\mathcal{E}_h^0}\eta h_e^{-1}[\![\boldsymbol{v}]\!] : [\![\boldsymbol{v}]\!]\,\mathrm{d}s \\
&\geqslant 2\mu|\boldsymbol{v}|_h^2 + \eta_0|\boldsymbol{v}|_*^2.
\end{aligned}
$$

So stability holds for the NIPG method for any $\eta_0 > 0$.

$$
\begin{aligned}
B_{2,h}^{(3)}(\boldsymbol{v},\boldsymbol{v}) &\geqslant 2\mu\big\|\boldsymbol{\varepsilon}_h(\boldsymbol{v}) + \boldsymbol{r}_0([\![\boldsymbol{v}]\!])\big\|_{0,h}^2 + \eta_0\sum_{e\in\mathcal{E}_h^0}\big(2\mu\big\|\boldsymbol{r}_{\mathrm{e}}([\![\boldsymbol{v}]\!])\big\|_{0,h}^2 + \lambda\big\|\mathrm{tr}\big(\boldsymbol{r}_{\mathrm{e}}([\![\boldsymbol{v}]\!])\big)\big\|_{0,h}^2\big) \\
&\geqslant 2\mu\left(|\boldsymbol{v}|_h^2 + \big\|\boldsymbol{r}_0([\![\boldsymbol{v}]\!])\big\|_{0,h}^2 + 2\sum_{K\in\mathcal{T}_h}\int_K \boldsymbol{\varepsilon}_h(\boldsymbol{v}) : \boldsymbol{r}_0([\![\boldsymbol{v}]\!])\,\mathrm{d}x\right) + 2\mu C_1\eta_0|\boldsymbol{v}|_*^2 \\
&\geqslant 2\mu\left((1-\epsilon)|\boldsymbol{v}|_h^2 + \left(1-\frac{1}{\epsilon}\right)\big\|\boldsymbol{r}_0([\![\boldsymbol{v}]\!])\big\|_{0,h}^2\right) + 2\mu C_1\eta_0|\boldsymbol{v}|_*^2 \\
&\geqslant 2\mu(1-\epsilon)|\boldsymbol{v}|_h^2 + 2\mu\left(3C_2\left(1-\frac{1}{\epsilon}\right) + C_1\eta_0\right)|\boldsymbol{v}|_*^2.
\end{aligned}
$$

Stability holds for $\eta_0 > 0$.

$$
\begin{aligned}
B_{2,h}^{(4)}(\boldsymbol{v},\boldsymbol{v}) &\geqslant 2\mu|\boldsymbol{v}|_h^2 + \lambda\|\mathrm{div}_h\boldsymbol{v}\|_{0,h}^2 + 4\mu\int_\Omega\boldsymbol{\varepsilon}_h(\boldsymbol{v}) : \boldsymbol{r}_0([\![\boldsymbol{v}]\!])\,\mathrm{d}x + 2\lambda\int_\Omega \mathrm{div}_h\boldsymbol{v}\,\mathrm{tr}\big(\boldsymbol{r}_0([\![\boldsymbol{v}]\!])\big)\,\mathrm{d}x \\
&\quad + \eta_0\sum_{e\in\mathcal{E}_h^0}\big(2\mu\big\|\boldsymbol{r}_{\mathrm{e}}([\![\boldsymbol{v}]\!])\big\|_{0,h}^2 + \lambda\big\|\mathrm{tr}\big(\boldsymbol{r}_{\mathrm{e}}([\![\boldsymbol{v}]\!])\big)\big\|_{0,h}^2\big) \\
&\geqslant 2\mu|\boldsymbol{v}|_h^2 + \lambda\|\mathrm{div}_h\boldsymbol{v}\|_{0,h}^2 - 2\mu\left(\epsilon|\boldsymbol{v}|_h^2 + \frac{1}{\epsilon}\big\|\boldsymbol{r}_0([\![\boldsymbol{v}]\!])\big\|_{0,h}^2\right) - \lambda\|\mathrm{div}_h\boldsymbol{v}\|_{0,h}^2 \\
&\quad - \lambda\big\|\mathrm{tr}\big(\boldsymbol{r}_0([\![\boldsymbol{v}]\!])\big)\big\|_{0,h}^2 + \eta_0 C_1 2\mu\sum_{e\in\mathcal{E}_h^0} h_e^{-1}\big\|[\![\boldsymbol{v}]\!]\big\|_{0,e}^2 + \eta_0\lambda\sum_{e\in\mathcal{E}_h^0}\big\|\mathrm{tr}\big(\boldsymbol{r}_{\mathrm{e}}([\![\boldsymbol{v}]\!])\big)\big\|_{0,h}^2 \\
&\geqslant 2(1-\epsilon)\mu|\boldsymbol{v}|_h^2 + 2\mu\left(\eta_0 C_1 - \frac{3C_2}{\epsilon}\right)|\boldsymbol{v}|_*^2 + \lambda\,(\eta_0 - 3)\sum_{e\in\mathcal{E}_h^0}\big\|\mathrm{tr}\big(\boldsymbol{r}_{\mathrm{e}}([\![\boldsymbol{v}]\!])\big)\big\|_{0,h}^2.
\end{aligned}
$$

Since $C_2 > C_1$, $\eta_0 > 3$ is guaranteed from $\eta_0 > 3C_2/C_1$. So stability is valid for this DG formulation when $\eta_0 > 3C_2/C_1$.

$$
\begin{aligned}
B_{2,h}^{(5)}(v,v) &\geqslant 2\mu \left\| \boldsymbol{\varepsilon}_h(\boldsymbol{v}) + \boldsymbol{r}_0([\![\boldsymbol{v}]\!]) \right\|_{0,h}^2 + \int_{\mathcal{E}_h^0} \frac{\eta_e}{h_e} [\![\boldsymbol{v}]\!] : [\![\boldsymbol{v}]\!] \, \mathrm{d}s \\
&\geqslant 2\mu \left(|\boldsymbol{v}|_h^2 + \left\| \boldsymbol{r}_0([\![\boldsymbol{v}]\!]) \right\|_{0,h}^2 + 2 \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{\varepsilon}_h(\boldsymbol{v}) : \boldsymbol{r}_0([\![\boldsymbol{v}]\!]) \, \mathrm{d}x \right) + \eta_0 |\boldsymbol{v}|_*^2 \\
&\geqslant 2\mu \left((1-\epsilon)|\boldsymbol{v}|_h^2 + \left(1 - \frac{1}{\epsilon}\right) \left\| \boldsymbol{r}_0([\![\boldsymbol{v}]\!]) \right\|_{0,h}^2 \right) + \eta_0 |\boldsymbol{v}|_*^2 \\
&\geqslant 2\mu(1-\epsilon)|\boldsymbol{v}|_h^2 + \left(\eta_0 + 6\mu C_2 - \frac{6\mu C_2}{\epsilon} \right) |\boldsymbol{v}|_*^2.
\end{aligned}
$$

It is clear that stability holds for the LDG method when $\eta_0 > 0$. Summarizing the above argument we have the next result.

LEMMA 3.4 (Stability). For $1 \leqslant j \leqslant 5$, $B_h = B_{1,h}^{(j)}$ and $B_{2,h}^{(j)}$ satisfy

$$
B_h(\boldsymbol{v}, \boldsymbol{v}) \gtrsim |\!|\!| \boldsymbol{v} |\!|\!|_*^2 \quad \forall\, \boldsymbol{v} \in V_h \tag{3.10}
$$

if $\eta_0 = \min_{e \in \mathcal{E}_h^0} \eta_e > 0$ for the methods with $j = 2, 3, 5$, and η_0 is large enough for the methods with $j = 1, 4$.

4. Approximation and error estimates

Considering the error estimation for the DG methods, we first write the error as

$$
\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}_h = (\boldsymbol{u} - \boldsymbol{u}_I) + (\boldsymbol{u}_I - \boldsymbol{u}_h),
$$

where $\boldsymbol{u}_I \in V_h$ is the usual continuous piecewise linear interpolant of the exact solution $\boldsymbol{u}$. Then $[\![\boldsymbol{u} - \boldsymbol{u}_I]\!] = \boldsymbol{0}$ on the interelement boundaries. By the definition of norm (3.2) we have the approximation property, assuming $\boldsymbol{u} \in [H^2(\Omega)]^2$,

$$
|\!|\!| \boldsymbol{u} - \boldsymbol{u}_I |\!|\!|^2 = |\boldsymbol{u} - \boldsymbol{u}_I|_h^2 + \sum_{K \in \mathcal{T}_h} h_K^2 |\boldsymbol{u} - \boldsymbol{u}_I|_{2,K}^2 \lesssim h^2 |\boldsymbol{u}|_{2,\Omega}^2. \tag{4.1}
$$

In the next result we need some additional solution regularity assumption. Assume that both tangential and normal derivatives of $\boldsymbol{u}$ on Γ_{C} are piecewise in $[L^\infty]^2$, i.e., on each line segment piece of Γ_{C}, both tangential and normal derivatives of $\boldsymbol{u}$ belong to the space $[L^\infty]^2$ (a similar assumption is made in Brezzi *et al.*, 1977 in the proof of Lemma 6.1 there). As in Brezzi *et al.* (1977) we also assume that the number of changes from $u_\nu < 0$ to $u_\nu = 0$ on Γ_{C} is finite.

THEOREM 4.1 Let $\boldsymbol{u}$ and $\boldsymbol{u}_h$ be the solutions of (2.12) and (2.29), respectively. Assume $\boldsymbol{u} \in [H^2(\Omega)]^2$, both tangential and normal derivatives of $\boldsymbol{u}$ on Γ_{C} are piecewise in $[L^\infty]^2$, and the number of changes from $u_\nu < 0$ to $u_\nu = 0$ on Γ_{C} is finite. Then for the DGMs with $j = 1, \ldots, 5$, we have the error bound

$$
|\!|\!| \boldsymbol{u} - \boldsymbol{u}_h |\!|\!| \lesssim h. \tag{4.2}
$$

Proof. By the stability of the bilinear form B_h we have

$$
|\!|\!| \boldsymbol{u}_I - \boldsymbol{u}_h |\!|\!|^2 \lesssim B_h(\boldsymbol{u}_I - \boldsymbol{u}_h, \boldsymbol{u}_I - \boldsymbol{u}_h) \equiv T_1 + T_2, \tag{4.3}
$$

where

$$T_1 = B_h(\boldsymbol{u}_I - \boldsymbol{u}, \boldsymbol{u}_I - \boldsymbol{u}_h),$$
$$T_2 = B_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_I - \boldsymbol{u}_h).$$

We bound T_1 as follows, using the boundedness of B_h,

$$T_1 \lesssim |||\boldsymbol{u}_I - \boldsymbol{u}|||\ |||\boldsymbol{u}_I - \boldsymbol{u}_h||| \lesssim \epsilon\ |||\ \boldsymbol{u}_I - \boldsymbol{u}_h\ |||^2 + \frac{1}{4\epsilon}\ |||\ \boldsymbol{u}_I - \boldsymbol{u}|||^2, \tag{4.4}$$

where $\epsilon > 0$ is an arbitrarily small number.

Then we bound T_2. Note that on an interior edge, $[\![\boldsymbol{u}]\!] = \boldsymbol{0}$, $\{\boldsymbol{u}\} = \boldsymbol{u}$, $\{\boldsymbol{\sigma}\} = \boldsymbol{\sigma}$, and on Γ_{D}, $[\![\boldsymbol{u}]\!] = \boldsymbol{0}$. Then

$$\begin{aligned} B_h(\boldsymbol{u}, \boldsymbol{u}_I - \boldsymbol{u}_h) &= \int_\Omega (2\mu\, \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}_h(\boldsymbol{u}_I - \boldsymbol{u}_h) + \lambda\ \mathrm{tr}(\boldsymbol{\varepsilon}(\boldsymbol{u}))\mathrm{tr}(\boldsymbol{\varepsilon}_h(\boldsymbol{u}_I - \boldsymbol{u}_h)))\ \mathrm{d}x \\ &\quad - \int_{\mathcal{E}_h^0} [\![\boldsymbol{u}_I - \boldsymbol{u}_h]\!] : (2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda\ \mathrm{tr}(\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{I})\ \mathrm{d}s \\ &= \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{\sigma} : \boldsymbol{\varepsilon}_h(\boldsymbol{u}_I - \boldsymbol{u}_h)\ \mathrm{d}x - \int_{\mathcal{E}_h^0} [\![\boldsymbol{u}_I - \boldsymbol{u}_h]\!] : \boldsymbol{\sigma}\ \mathrm{d}s. \end{aligned}$$

Since $[\boldsymbol{\sigma}] = \boldsymbol{0}$ on an interior edge and remembering (2.3) we have

$$\begin{aligned} \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{\sigma} : \boldsymbol{\varepsilon}_h(\boldsymbol{u}_I - \boldsymbol{u}_h)\ \mathrm{d}x &= \sum_{K\in\mathcal{T}_h} \int_K -\mathrm{div}\, \boldsymbol{\sigma} \cdot (\boldsymbol{u}_I - \boldsymbol{u}_h)\ \mathrm{d}x + \sum_{K\in\mathcal{T}_h} \int_{\partial K} (\boldsymbol{\sigma}\boldsymbol{n}_K) \cdot (\boldsymbol{u}_I - \boldsymbol{u}_h)\ \mathrm{d}s \\ &= \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{f} \cdot (\boldsymbol{u}_I - \boldsymbol{u}_h)\ \mathrm{d}x + \int_{\mathcal{E}_h} [\![\boldsymbol{u}_I - \boldsymbol{u}_h]\!] : \boldsymbol{\sigma}\ \mathrm{d}s. \end{aligned}$$

Then

$$B_h(\boldsymbol{u}, \boldsymbol{u}_I - \boldsymbol{u}_h) = \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{u}_I - \boldsymbol{u}_h)\ \mathrm{d}x + \int_{\Gamma_{\mathrm{F}}} \boldsymbol{g} \cdot (\boldsymbol{u}_I - \boldsymbol{u}_h)\ \mathrm{d}s + \int_{\Gamma_{\mathrm{C}}} (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot (\boldsymbol{u}_I - \boldsymbol{u}_h)\ \mathrm{d}s. \tag{4.5}$$

Choosing $\boldsymbol{v}_h = \boldsymbol{u}_I$ in (2.29) we have

$$B_h(\boldsymbol{u}_h, \boldsymbol{u}_I - \boldsymbol{u}_h) \geqslant \ell(\boldsymbol{u}_I - \boldsymbol{u}_h). \tag{4.6}$$

Let Γ_{T} and Γ_{N} denote the sets of edges $\subset \Gamma_{\mathrm{C}}$ where $u_\nu = 0$ and $u_\nu < 0$, respectively. Combining (4.6) and (4.5) we obtain

$$\begin{aligned} T_2 = B_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_I - \boldsymbol{u}_h) &\leqslant \int_{\Gamma_{\mathrm{C}}} (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot (\boldsymbol{u}_I - \boldsymbol{u}_h)\ \mathrm{d}s \\ &= \int_{\Gamma_{\mathrm{C}}} \sigma_\nu (u_{I\nu} - u_{h\nu})\ \mathrm{d}s \leqslant \int_{\Gamma_{\mathrm{C}}} \sigma_\nu u_{I\nu}\ \mathrm{d}s = T_3 + T_4 + T_5, \end{aligned} \tag{4.7}$$

where

$$T_3 = \int_{e\subset\Gamma_{\mathrm{T}}} \sigma_\nu u_{I\nu}\ \mathrm{d}s, \quad T_4 = \int_{e\subset\Gamma_{\mathrm{N}}} \sigma_\nu u_{I\nu}\ \mathrm{d}s, \quad T_5 = \int_{e\subset\Gamma_{\mathrm{C}}\backslash(\Gamma_{\mathrm{T}}\cup\Gamma_{\mathrm{N}})} \sigma_\nu u_{I\nu}\ \mathrm{d}s.$$

From $\sigma_\nu u_\nu = 0$ on Γ_C it is easy to know that $T_3 = 0$ and $T_4 = 0$. Consider the term T_5. If $e \subset \Gamma_C \backslash (\Gamma_T \cup \Gamma_N)$, then there exists a point $P \in e$ satisfying $u_\nu(P) = 0$. By the regularity assumption over Γ_C, we have $u_{I\nu} = \mathcal{O}(h)$ on $e \subset \Gamma_C \backslash (\Gamma_T \cup \Gamma_N)$ and $\sigma_\nu \in L^\infty(\Gamma_C)$. Hence,

$$T_5 = \int_{e \subset \Gamma_C \backslash (\Gamma_T \cup \Gamma_N)} \sigma_\nu u_{I\nu} \,\mathrm{d}s \lesssim h^2.$$

Thus, under the stated regularity assumptions, $T_2 \lesssim h^2$ and the proof is completed. □

REMARK 4.2 For the scalar unilateral variational inequality

$$u \in K, \quad \int_\Omega [\nabla u \cdot \nabla (v-u) + u(v-u)] \,\mathrm{d}x \geqslant \int_\Omega f(v-u) \,\mathrm{d}x \quad \forall v \in K,$$

where $\Omega \subset \mathbb{R}^2$ is a bounded, convex polygonal domain, and

$$K = \left\{ v \in H^1(\Omega) \colon v \geqslant 0 \text{ a.e. on } \partial\Omega \right\},$$

the optimal linear convergence for the linear element solution is proved in Dobrowolski & Staib (1992) without the additional assumption that the number of switches between $u > 0$ and $u = 0$ on Γ_C is finite. For the Signorini problem that we are studying in this paper, it does not appear possible to adapt the arguments in Dobrowolski & Staib (1992) to show the optimal error bound (4.2) without the additional assumption that the number of changes from $u_\nu < 0$ to $u_\nu = 0$ on Γ_C is finite.

5. Numerical example

We report some numerical results on a two-dimensional test problem solved by the LDG method. The physical setting is given in Fig. 1. The domain $\Omega = (0,4) \times (0,4)$ can be viewed as the cross section of a linearized elastic body. On the boundary $\Gamma_D = \{4\} \times (0,4)$, the body is clamped and hence the displacement field vanishes there. The traction acts on the boundary $\{0\} \times (0,4)$ and the boundary of $(0,4) \times \{4\}$ is traction free. Therefore, $\Gamma_F = \{0\} \times (0,4) \cup (0,4) \times \{4\}$. On the boundary $\Gamma_C = (0,4) \times \{0\}$, the body contacts with a frictionless rigid foundation. No volume force is assumed to act on the body Ω. Let E be Young's modulus and κ be the Poisson ratio of the material. Then the Lamé coefficients are

$$\lambda = \frac{E\kappa}{(1+\kappa)(1-2\kappa)}, \qquad \mu = \frac{E}{2(1+\kappa)}.$$

In this example we use the following data:

$$E = 200 \text{ daN/mm}^2, \quad \kappa = 0.3, \quad \boldsymbol{f} = \boldsymbol{0} \text{ daN/mm}^2,$$
$$\boldsymbol{g}(x_1, x_2) = (0.02(5 - x_2), -0.01) \text{ daN/mm}^2,$$

where the unit daN/mm^2 denotes decanewtons per square milimeter. We solve the discretized problem on uniform triangular meshes, see Fig. 2.

To consider the convergence order we solve the problem on a family of uniform meshes. We start with $h = 2\sqrt{2}$, which decreases by half, and adopt the numerical solution on the mesh $h = \sqrt{2}/16$ as the 'exact' solution computing errors of the numerical solution on other meshes. In Fig. 3 we show the deformed mesh (amplified by 200) for $h = \sqrt{2}/16$. We observe from Fig. 4 that the numerical convergence orders in both norms $|||\cdot|||$ and $|\cdot|_h$ are $\sim$1, matching the theoretical prediction well.

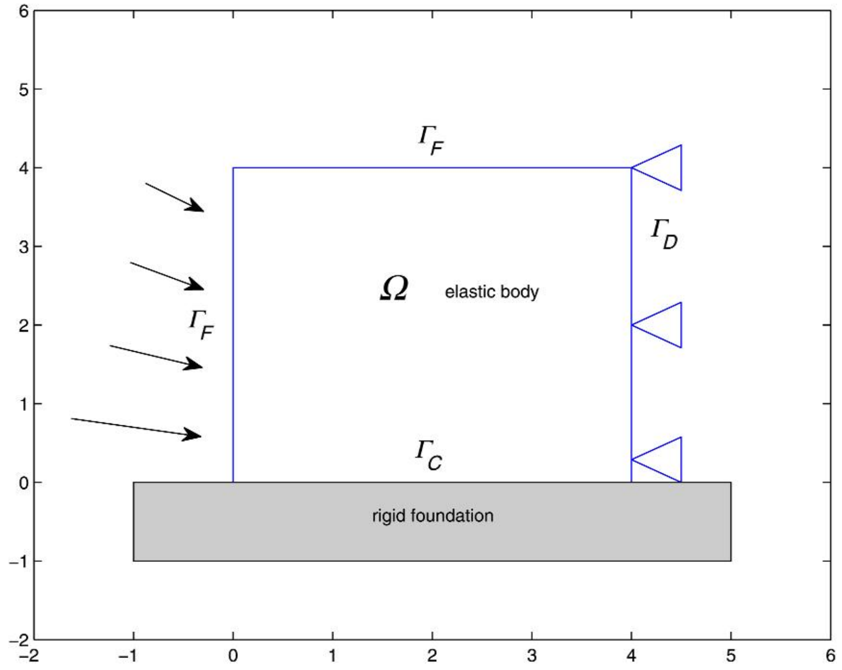

FIG. 1. An elastic body on a frictionless rigid foundation.

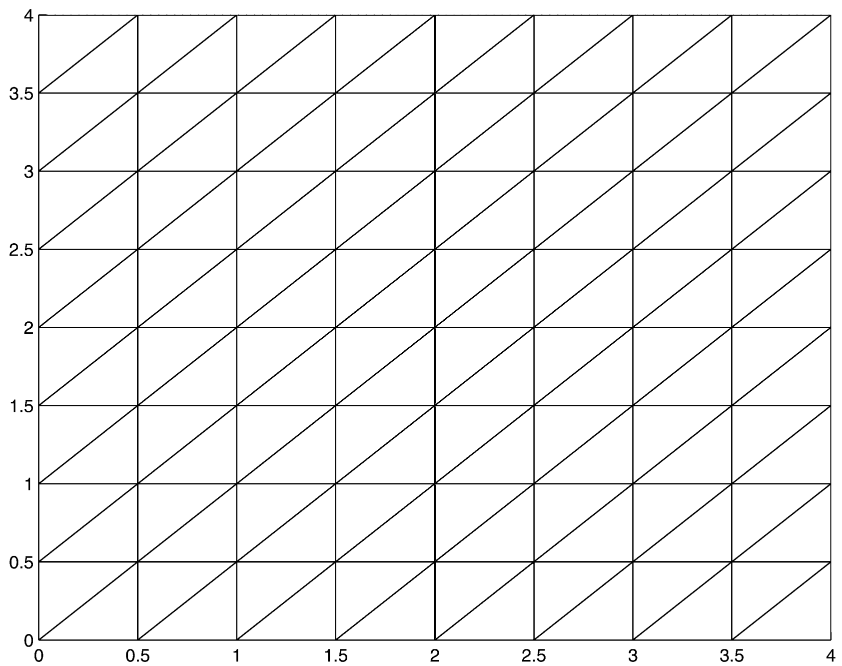

FIG. 2. A uniform triangular mesh of the domain.

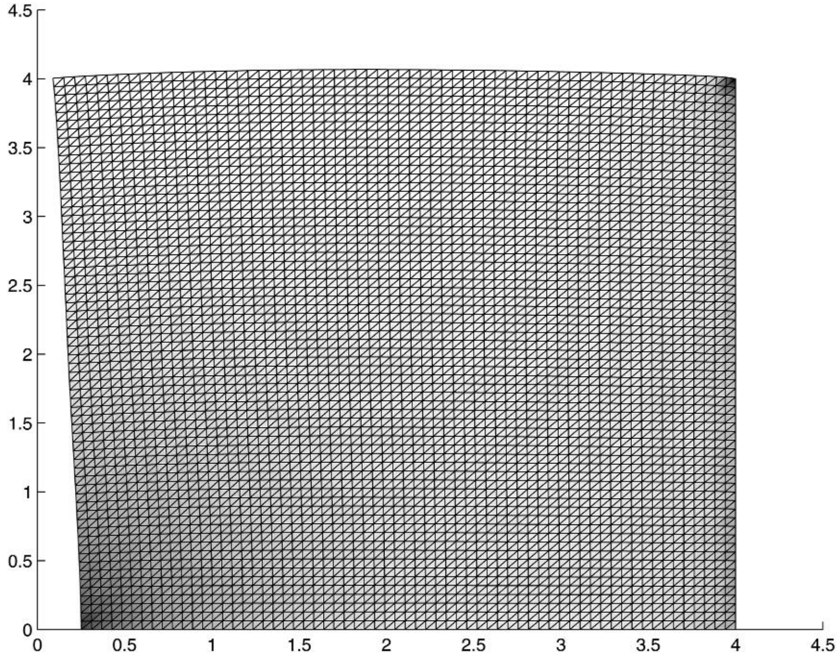

FIG. 3. Deformed mesh (amplified by 200) for $h = \sqrt{2}/16$.

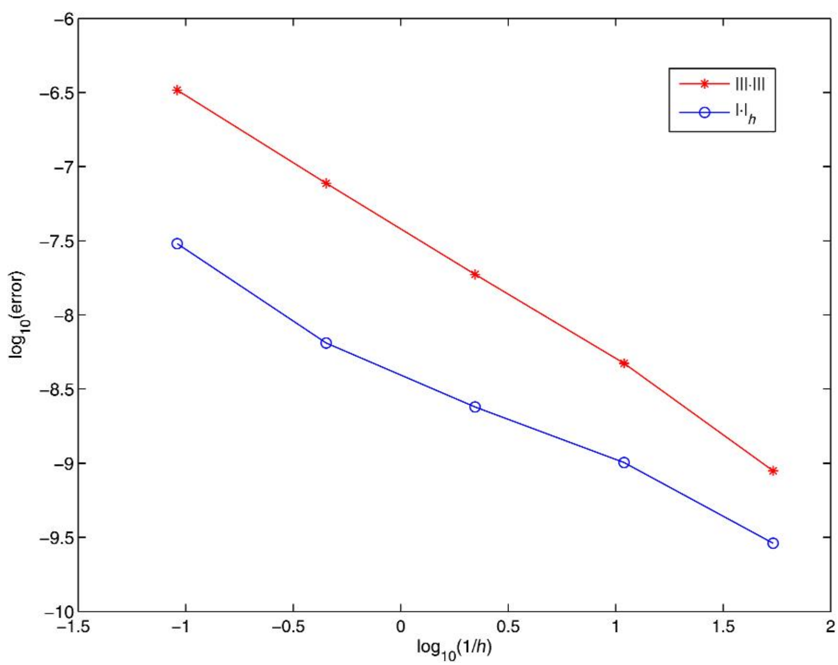

FIG. 4. Numerical errors.

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