ERROR ANALYSIS OF LOCAL REFINEMENTS OF POLYGONAL DOMAINS*[1)]

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Abstract

This paper gives a thorough analysis of the local refinement method on plane polygonal domains, with special attention to the treatment of reentrant corner. Convergence rates of the finite element method under various norms are derived via a systematic treatment of the interpolation theory in weighted Sobolev spaces. It is proved that by refining the mesh suitably, the finite element approximations for problems with singularities achieve the same convergence rates as those for smooth solutions.

§ 1. Introduction

To the authors knowledge, analysis of the local refinement method was initiated by A. Schatz and L. Wahlbin. Based on an asymptotic expression of the solution near a corner, which reveals the singularities of the solution, they obtained in [4] some error estimates on locally refined meshes. But from the point of view of approximation theory, their results are on functions of particular forms, rather than the norms of the error operator in suitable spaces. By introducing weighted Sobolev spaces, [1] gave a satisfactory answer to this problem. They also obtained the inverse estimates which show that their results cannot be improved. Unfortunately, the techniques used there cannot be extended readily to the case of high order elements. In this paper, by using a systematic treatment of the interpolation theory in weighted Sobolev spaces, we give a thorough analysis of the finite element method on locally refined meshes. Estimates obtained indicate the dependence of the approximation properties on the singularities of the solution, the order of the elements and the degree of refinement.

We use conventional notations in this paper, with C as a generic constant, which may assume different value in different places.

§ 2. Weighted Sobolev Spaces

Let Ω be a plane polygonal domain, with vertices $x_1, x_2, \cdots, x_M$ (see Fig. 1), ω_j is the inner angle of Ω at x_j, and $\alpha=\frac{\pi}{\omega_j}$,

$$V=\{x_1, x_2, \cdots, x_M\}.$$

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For $\beta=(\beta_1, \beta_2, \cdots, \beta_M)$, define

$$\Phi_\beta(x)=\prod_{i=1}^{M}|x-x_i|^{\beta_i},$$

where $|\cdot|$ is the Euclidean norm in R^2.

Let $\beta+j-2=(\beta_1+j-2, \cdots, \beta_M+j-2)$ and likewise $\beta j=(\beta_1 j, \cdots, \beta_M j)$.

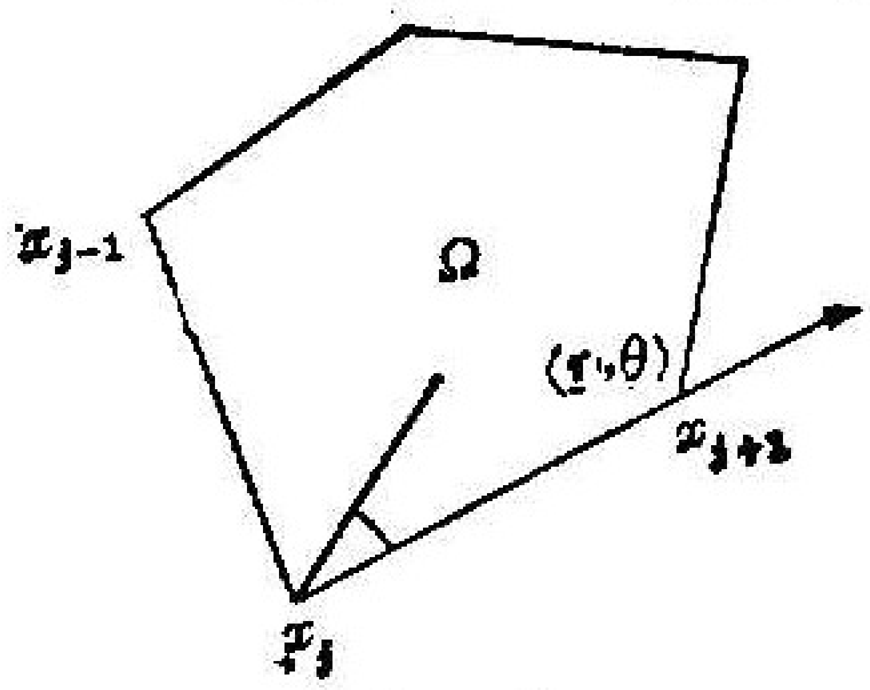

Fig. 1

For two vectors $\beta=(\beta_1, \cdots, \beta_M)$, $\gamma=(\gamma_1, \cdots, \gamma_M)$, we say $\beta>\gamma$, if $\beta_j>\gamma_j$, for all j, $1\leqslant j\leqslant M$. If, for instance, γ is real, then $\beta>\gamma$ is understood as $\beta_j>\gamma$, for all j, $1\leqslant j\leqslant M$.

Let $H_\beta^m(\Omega)$ be the complement of $C^\infty(\bar{\Omega})$ under the norm:

$$\|u\|_{m,\beta,\Omega}=\|u\|_{L^2(\Omega)}+|u|_{m,\beta,\Omega},$$

where

$$|D^j u|=\sum_{\alpha_1+\alpha_2=j}\left|\frac{\partial^j u}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\right|, \quad |u|_{m,\beta,\Omega}=\left(\int_\Omega \Phi_{\beta+m-2}^2|D^m u|^2\,dx\right)^{1/2}.$$

If $\beta<1$, then by using Lemma 4.9 of [3], we have

$$H_\beta^m(\Omega)\subsetneq H_\beta^1(\Omega)\subsetneq H^1(\Omega).$$

This important property will be used in proving Theorem 1.

§ 3. Interpolation Theory

Denote by d_T the diameter of a triangle T. For $\gamma=(\gamma_1, \cdots, \gamma_M)$, as in [1], we define a triangulation Δ to be of (h, θ_0, γ)-type, if

(1) Δ satisfies the conventional requirements on triangulations (see, e.g. [2], p. 51).

(2) (Zlamal's condition) For any $T\in\Delta$, if θ is an interior angle of T, then $\theta\geqslant\theta_0>0$.

(3) If $T\cap V=\emptyset$, then $C_0(\theta_0)\Phi_\gamma(x)h\leqslant d_T\leqslant C_1(\theta_0)\Phi_\gamma(x)h$, $\forall x\in T$.

(4) If $T\cap V\neq\emptyset$, then $C_0(\theta_0)\sup_{x\in T}\Phi_\gamma(x)h\leqslant d_T\leqslant C_1(\theta_0)\sup_{x\in T}\Phi_\gamma(x)h$.

Here, $C_0(\theta_0)$, $C_1(\theta_0)$ are constants depending only on θ_0.

If $\gamma_j>0$, the triangulation is no longer quasi-uniform. If we use d_j to denote the diameter of an element near the corner x_j, then

$$d_j\sim h^{1/(1-\gamma_j)}.$$

To define the interpolation, we need the following result, which can be easily proved by using the corresponding result of [1] and Lemma 4.9 of [3].

Lemma 1. *If* $0\leqslant\beta<1$, $m\geqslant2$, *then* $H_\beta^m(\Omega)\subsetneq C^0(\bar{\Omega})$.

Denote by $\varphi_{m-1}(\Delta)$ $(m\geqslant2)$ the space of continuous functions which are piecewise polynomials of degree $m-1$ or less on Δ. Because of Lemma 1, the Lagrangian interpolation operator

$$u\mapsto u_I$$

is well defined on $H_\beta^m(\Omega)$: $H_\beta^m(\Omega)\to\varphi_{m-1}(\Delta)$.

To estimate the interpolation error, we need the following theorem.

Theorem 1. *If $\beta<1$, then there exists a constant C, such that, for any $v\in H_\beta^m(\Omega)$,*

$$\inf_{\varphi\in P_{m-1}(\Omega)}\|v+\varphi\|_{m,\beta,\Omega}\leqslant C|v|_{m,\beta,\Omega}, \tag{3.1}$$

where $P_{m-1}(\Omega)$ is the space of polynomials of degree $m-1$ or less.

Proof. Let $N=\dim P_{m-1}(\Omega)$ and f_i $(1\leqslant i\leqslant N)$ be a basis of the dual space of $P_{m-1}(\Omega)$. By the Hahn–Banach extension theorem, there exist continuous linear functionals on $P_{m-1}(\Omega)$, again denoted by f_i $(1\leqslant i\leqslant N)$, such that for any $\varphi\in P_{m-1}(\Omega)$,

$$f_i(\varphi)=0, \text{ for every } 1\leqslant i\leqslant N \text{ iff } \varphi=0.$$

We show that there exists a constant C, such that for any $v\in H_\beta^m(\Omega)$,

$$\|v\|_{m,\beta,\Omega}\leqslant C\left\{|v|_{m,\beta,\Omega}+\sum_{j=1}^N|f_j(v)|\right\}. \tag{3.2}$$

As a direct consequence of (3.2), we get (3.1).

Assume (3.2) fails. Then there exists a sequence $\{v_l\}_{l=1}^\infty\subset H_\beta^m(\Omega)$, such that

$$\|v_l\|_{m,\beta,\Omega}=1, \tag{3.3}$$

$$\lim_{l\to\infty}\left(|v_l|_{m,\beta,\Omega}+\sum_{j=1}^N|f_j(v_l)|\right)=0. \tag{3.4}$$

Because $\beta<1$, $H_\beta^m(\Omega)\hookrightarrow H^1(\Omega)$, by the Rellich theorem we have

$$H_\beta^m(\Omega)\overset{C}{\hookrightarrow}L^2(\Omega).$$

Thus, by (3.3), there exists a subsequence of $\{v_l\}_{l=1}^\infty$, again denoted by $\{v_l\}_{l=1}^\infty$, which converges in $L^2(\Omega)$ to v:

$$\|v_l-v\|_{L^2(\Omega)}\to 0,\quad l\to\infty.$$

From (3.4), we see that $\{v_l\}$ is a Cauchy sequence in $H_\beta^m(\Omega)$. So it should converge in $H_\beta^m(\Omega)$ to a function v, and $|v|_{m,\beta,\Omega}=0$. Because Ω is connected, this implies that $v\in P_{m-1}(\Omega)$. Note further that $f_j(v)=\lim\limits_{l\to\infty}f_j(v_l)=0$ $(1\leqslant j\leqslant N)$. So $v=0$, a contradiction to (3.3). Hence we have (3.2). This completes the proof of (3.1).

Theorem 2. *Assume the triangulation Δ is of (h,θ,γ)-type, $0<\beta<1$, $\gamma m\geqslant\beta+m-2$. Then there exists a constant C, which is independent of h, such that for any $u\in H_\beta^m(\Omega)$,*

$$\|u-u_I\|_{L^2(\Omega)}\leqslant Ch^m|u|_{m,\beta,\Omega}.$$

Proof. Let T be an element of Δ. If $T\cap V=\emptyset$, then we may use the interpolation theory on (unweighted) Sobolev spaces (cf. [2])

$$\|u-u_I\|_{L^2(T)}^2\leqslant C(\theta_0)d_T^{2m}|u|_{m,T}^2\leqslant C(\theta_0)h^{2m}\int_T\Phi_\gamma^m|D^m u|^2dx$$

$$\leqslant C(\theta_0)h^{2m}\int_T\Phi_{\beta+m-2}^2|D^m u|^2dx. \tag{3.5}$$

If $T\cap V=\{x_i\}$, we use the scaling argument.

Let $\hat{T}$ be the reference element, and $\hat{u}$ denote the result of u (on T) transplanted on $\hat{T}$. Then we have

$$\|\hat{u}_I\|_{L^2(\hat{T})}\leqslant C\|\hat{u}_I\|_{C^0(\hat{T})}\leqslant C\|\hat{u}\|_{m,\beta,\hat{T}},$$

$$\|\hat{u}-\hat{u}_I\|_{L^2(\hat{T})}\leqslant C\|\hat{u}\|_{m,\beta,\hat{T}}.$$

Use $\hat{u}+\varphi$ $(\varphi\in P_{m-1}(\hat{T}))$ instead of v on the above inequality,

$$\|\hat{u}-\hat{u}_I\|_{L^2(\hat{T})}\leqslant C\inf_{\varphi\in P_{m-1}(\hat{T})}\|\hat{u}+\varphi\|_{m,\beta,\hat{T}}.$$

By Theorem 1,

$$\|\hat{u}-\hat{u}_I\|_{L^2(\hat{T})}\leqslant C|\hat{u}|_{m,\beta,\hat{T}}.$$

So on the original element T, we have

$$\|u-u_I\|^2_{L^2(T)}\leqslant C(\theta_0)d_T^{2(2-\beta_i)}|u|^2_{m,\beta,T}.$$

Because

$$d_T\leqslant Ch\sup_{x\in T}\Phi_\gamma(x)\leqslant Ch\,d_T^{\gamma_i},$$

we have

$$d_T\leqslant Ch^{\frac{1}{1-\gamma_i}},$$

$$\|u-u_I\|_{L^2(T)}\leqslant C(\theta_0)h^{\frac{2-\beta_i}{1-\gamma_i}}|u|_{m,\beta,T}\leqslant C(\theta_0)h^m|u|_{m,\beta,T}. \tag{3.6}$$

Combining (3.5) with (3.6), we complete the proof of the theorem.

Theorem 3. *Assume the triangulation Δ is of (h,θ_0,γ)-type, $0<\beta<1$, $(m-1)\gamma\geqslant\beta+m-2$. Then there exists a constant C, which is independent of h, such that for any $u\in H^m_\beta(\Omega)$,*

$$\|u-u_I\|_{1,\Omega}\leqslant C(\theta_0)h^{m-1}|u|_{m,\beta,\Omega}.$$

Proof. We proceed in a similar way as in the proof of Theorem 2. Let T be an element of Δ. If $T\cap V=\emptyset$, then

$$\|u-u_I\|^2_{1,T}\leqslant C(\theta_0)d_T^{2(m-1)}|u|^2_{m,T}\leqslant C(\theta_0)h^{2(m-1)}\int_T\Phi_\gamma^{2(m-1)}|D^m u|^2\,dx$$

$$\leqslant C(\theta_0)h^{2(m-1)}\int_T\Phi^2_{\beta+m-2}|D^m u|^2\,dx.$$

If $T\cap V=\{x_i\}$, by using a suitable affine mapping with the origin as the image of x_i, we may first consider the reference element $\hat{T}$. Using the same notation as above, we have

$$\|\hat{u}_I\|_{1,\hat{T}}\leqslant C\|\hat{u}\|_{m,\beta,\hat{T}}.$$

Because $H^m_\beta(\Omega)\subsetneq H^1(\Omega)$,

$$\|\hat{u}\|_{1,\hat{T}}\leqslant C\|\hat{u}\|_{m,\beta,\hat{T}},$$

$$\|\hat{u}-\hat{u}_I\|_{1,\hat{T}}\leqslant C\|\hat{u}\|_{m,\beta,\hat{T}},$$

$$\|\hat{u}-\hat{u}_I\|_{1,\hat{T}}\leqslant C|\hat{u}|_{m,\beta,\hat{T}}.$$

So on T, we have

$$\|u-u_I\|_{1,T}\leqslant Cd_T^{1-\beta_i}|u|_{m,\beta,T}\leqslant Ch^{\frac{1-\beta_i}{1-\gamma_i}}|u|_{m,\beta,T}\leqslant Ch^{m-1}|u|_{m,\beta,T}$$

and Theorem 3 is proved by combining these inequalities.

§ 4. Error Estimates in H^1 and L^2 Norms

Define a bilinear form $B(\cdot,\cdot)$: $H^1_0(\Omega)\times H^1_0(\Omega)\to R^1$

$$B(u, v)=\int_\Omega \nabla u\cdot\nabla v\,dx, \quad u, v\in H_0^1(\Omega).$$

Denote by u_h the finite element approximation of u in the space $\varphi_{m-1}(\Delta)$.

Theorem 4. *If the triangulation Δ is of (h, θ_0, γ)-type, $0<\beta<1$, $(1-\beta_i)/(1-\gamma_i)\geqslant m-1$ $(1\leqslant i\leqslant M)$, then there exists a constant C which is independent of h, such that*

$$\|u-u_h\|_{1,\Omega}\leqslant Ch^{m-1}\|u\|_{m,\beta,\Omega}.$$

Proof. This is a direct consequence of the Céa lemma and Theorem 3.

The above theorem indicates that, even if the solution of the differential equation has some singularities, by suitably refining the mesh, the finite element approximation can still achieve the best possible convergence rate.

It is proved in [1] that if $0\leqslant\gamma\leqslant1$, then $\dim\varphi_{m-1}(\Delta)=N(\Delta)\leqslant Ch^{-2}$, where C is independent of h. So we have

Corollary. Assume the conditions of Theorem 4. Then there is a constant C which is independent of h, such that

$$\|u-u_h\|_{1,\Omega}\leqslant CN(\Delta)^{-\frac{m-1}{2}}\|u\|_{m,\beta,\Omega}$$

for any $u\in H_\beta^m(\Omega)$.

Theorem 4 does not consider the under-refinement case $\min\limits_{1\leqslant i\leqslant M}\frac{1-\beta_i}{1-\gamma_i}<m-1$. To compensate for this, we introduce the following Besov spaces.

If $0<\theta<1$, let $X(1, m, \beta, \theta)=[H_0^1(\Omega), H_0^1\cap H_\beta^m(\Omega)]_{\theta,\infty}$. Here $[\cdot,\cdot]_{\theta,+\infty}$ denotes the interpolation space using K-interpolation method. To be more precise, let

$$K(u, t)=\inf_{\substack{u=v+w\\ v\in H_0^1,\\ w\in H_\beta^m\cap H_0^1}}\{\|v\|_{H_0^1(\Omega)}+t\|w\|_{m,\beta,\Omega}\},$$

$$\|u\|_{X(1,m,\beta,\theta)}=\sup_{t>0}\{t^{-\theta}K(u, t)\}.$$

The following lemma helps us to get an insight on how large the space $X(1, m, \beta, \theta)$ is. Its proof is much the same as that of Lemma 2.1 in [1]. We omit it here.

Lemma 2. *If ρ, φ are smooth functions, $0<\alpha<1-\beta$, $\theta=\min\frac{\alpha_i}{1-\beta_i}$, then the $H_0^1(\Omega)$ function $u=r^\alpha\varphi(\theta)\rho(r)\in X(1, m, \beta, \theta)$. Here the polar coordinate system is situated at some vertex x_i of Ω. See Fig. 1.*

Theorem 5. *If the triangulation Δ is of (h, θ_0, γ)-type and β satisfies $\min\limits_{1\leqslant i\leqslant M}\frac{1-\beta_i}{1-\gamma_i}\geqslant m-1$, then there exists a constant C which is independent of h, such that for any $u\in X(1, m, \beta, \theta)$*

$$\|u-u_h\|_{1,\Omega}\leqslant Ch^{\theta(m-1)}\|u\|_{X(1,m,\beta,\theta)}.$$

Proof. Define the operator E_h as $E_hu=u-u_h$. Then E_h as an operator from $H_0^1(\Omega)$ to $H_0^1(\Omega)$ has its norm bounded by 1, and as an operator on $H_\beta^m(\Omega)\cap H_0^1(\Omega)$ has its norm bounded by Ch^{m-1}. So using the interpolation inequality, we have that E_h as an operator from $X(1, m, \beta, \theta)$ to $H_0^1(\Omega)$ has its norm bounded by $Ch^{\theta(m-1)}$. This is what the theorem asserts.

Combining Theorems 4 and 5, with Lemma 2, we have arrived at the following.

Theorem 6. *If the mesh Δ is of (h, θ_0, γ)-type, the function $u - r^\alpha\varphi(\theta)\rho(r) \in H_0^1(\Omega)$, β satisfies $\frac{1-\beta_i}{1-\gamma_i} = m-1$ $(1 \leqslant i \leqslant M)$, then for any positive number ε, there exists a constant C, which does not depend on h, but may depend on ε, such that*

$$\|u-u_h\|_{1,\Omega} \leqslant Ch^\mu.$$

Here

$$\mu = \begin{cases} s & if \quad s < m-1, \\ m-1-\varepsilon & if \quad s = m-1, \\ m-1 & if \quad s > m-1, \end{cases}$$

$$s = \min_i \frac{\alpha_i}{1-\gamma_i}.$$

The above theorem reveals the dependence of the convergence rate on the singularities of the solution, the order of the finite element space, and the degree of refinement.

A result on L^2-norm estimates has been given in [1]. We quote it here. Define

$$\|v\|_{0,-\beta} = \left\{\int_\Omega \Phi_\beta^{-2} v^2 dx\right\}^{1/2}.$$

Theorem 7. *If $\min \frac{\alpha_i}{1-\gamma_i} > 1$, and Δ is of (h, θ_0, γ)-type, then there exists a constant C, which is independent of h, such that*

$$\|u-u_h\|_{0,-\gamma} \leqslant Ch\|u-u_h\|_1.$$

The theory we developed above was extended to the case with general $p>1$; see [7], where results on inverse estimates and maximum norm estimates were also given. The inverse theorem is much the same as the one given in [1], and indicates that our results on error estimates here are optimal.

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