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Radiative transfer with delta-Eddington-type phase functions

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ABSTRACT

The radiative transfer equation (RTE) arises in a wide variety of applications, in particular, in biomedical imaging applications associated with the propagation of light through the biological tissue. However, highly forward-peaked scattering feature in a biological medium makes it very challenging to numerically solve the RTE problem accurately. One idea to overcome the difficulty associated with the highly forward-peaked scattering is through the use of a delta-Eddington phase function. This paper is devoted to an RTE framework with a family of delta-Eddington-type phase functions. Significance in biomedical imaging applications of the RTE with delta-Eddington-type phase functions are explained. Mathematical studies of the problems include solution existence, uniqueness, and continuous dependence on the problem data: the inflow boundary value, the source function, the absorption coefficient, and the scattering coefficient. Numerical results are presented to show that employing a delta-Eddington-type phase function with properly chosen parameters provides accurate simulation results for light propagation within highly forward-peaked scattering media.

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1. Introduction

The radiative transfer equation (RTE) arises in a wide variety of applications, such as astrophysics [22], atmosphere and ocean [26,31], heat transfer [20], neutron transport [7,9], optical molecular imaging [21,28], and so on. Recently, there is much interest in analysis and numerical simulation of the RTE and its related inverse problems, motivated by applications in biomedical optics [2,3,5,11–15,23,25].

Photon propagation in biological or engineered tissues can be well described by the radiative transport equation (RTE). However, the direct solution of the RTE is computationally expensive because of the dimensionality of the equation and the complexity of the phase function. It is rather common in practice that the diffusion approximation is based upon to enable optical molecular tomographic techniques that reveal optically labeled molecular and cellular activities in vivo. A majority of such studies target small animal models of human diseases and 3D tissue engineering constructs of regenerative functionalities. Photon propagation in these media is strongly affected by scattering. When samples are not large, characteristic forward scattering is observable and responsible for substantial components in the measurement. Inspired by this observation, delta-Eddington-type phase functions were proposed to model the underlying physics, simplify the solution of the RTE, and successfully applied in multiple applications. Therefore, it is desirable and timely to generalize this approach and establish its theoretical foundation.

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The RTE for the complicated process of absorption and scattering of light within the biological medium is

$$\omega\cdot\nabla u+\mu_t u-\mu_s Su=f \quad \text{in } X\times\Omega. \tag{1.1}$$

Here, X is a domain in $\mathbb{R}^3$ occupied by the biological medium and Ω is the unit sphere in $\mathbb{R}^3$ for the directions of the photon propagation. A generic point in X is denoted by $\boldsymbol{x}$ whereas a generic point in Ω is denoted by $\boldsymbol{\omega}$. The symbol ∇ stands for the gradient with respect to the spatial variable $\boldsymbol{x}$. The unknown function $u(\boldsymbol{x}, \boldsymbol{\omega})$ is the angular flux at the point $\boldsymbol{x}$ in the direction $\boldsymbol{\omega}$. The RTE (1.1) contains two medium parameters, the total cross-section $\mu_t(\boldsymbol{x})$ and the scattering cross-section $\mu_s(\boldsymbol{x})$, that are related by $\mu_t=\mu_a+\mu_s$ with μ_a being the absorption cross-section. The integral operator S is given by the formula

$$Su(x,\omega)=\int_\Omega p(\hat\omega\cdot\omega)\,u(x,\hat\omega)\,d\hat\omega, \tag{1.2}$$

where the phase function $p(\hat\omega\cdot\omega)$ is non-negative and is normalized:

$$\int_\Omega p(\hat\omega\cdot\omega)\,d\hat\omega=1,$$

or equivalently,

$$\int_{-1}^{1} p(t)\,dt=\frac{1}{2\pi}.$$

The function $f(\boldsymbol{x}, \boldsymbol{\omega})$ represents a source density.

The phase function p describes the scattering property of the biological medium. The precise form of the phase function is usually unknown for applications, and a benchmark choice is the Henyey–Greenstein phase function (cf. [16]):

$$p_{HG,g}(t)=\frac{1-g^2}{4\pi\left(1+g^2-2gt\right)^{3/2}},\quad t:=\hat\omega\cdot\omega\in[-1,1],$$

where the parameter $g\in(-1,1)$ is the anisotropy factor of the scattering medium. For isotropic scattering, $g=0$; for forward scattering, $g>0$; and for backward scattering, $g<0$. For applications in biomedical imaging, the value of g is typically between 0.9 and 0.95. For this range of the value of g, the corresponding integral operator (1.2) presents numerical singularity, bringing in additional difficulty in numerically solving the RTE problem. The biological tissue scatters light strongly in the forward direction, and so it is natural to approximately model the effect of the strongly forward scattering through the inclusion of a delta function in the phase function. In this paper, we consider the RTE problem with a general delta-Eddington-type phase function of the following form

$$p(t)=\frac{1}{4\pi}[(1-p_0)\,r(t)+2\,p_0\delta(1-t)], \tag{1.3}$$

where $p_0\in[-1,1]$ is the weighting factor measuring the anisotropy of the photon scattering, δ is the Dirac delta function, and $r(t)$ represents a remainder part of the phase function which is smooth and slowly varying. For strongly forward peaked media, p_0 is less than but close to 1:

$$1-\varepsilon<p_0<1,$$

where $\varepsilon>0$ is a small number. Physical considerations dictate that the remainder function $r(t)$ satisfies the following condition:

$$r(t)\ge 0,\quad \frac{1}{2}\int_{-1}^{1} r(t)\,dt=1. \tag{1.4}$$

The formula (1.3) includes as particular cases several phase functions proposed in the literature. We list some of them in the following.

The transport approximation [10] corresponds to the choice $r(t)=1$, i.e., the phase function is the sum of a forward delta function and an isotropic scattering function.

The delta-Eddington phase function [17]

$$p_{dE}(t)=\frac{1}{4\pi}\left[(1-p_0)\left(1+3g't\right)+2\,p_0\delta(1-t)\right], \tag{1.5}$$

corresponds to the choice

$$r(t)=1+3g't, \tag{1.6}$$

where g' is an asymmetry factor of the phase function used to modulate the weakly anisotropic scattering. The phase function (1.5) is a linear combination of a forward delta function and a weakly anisotropic scattering function. Formally, the transport approximation is a special case of the delta-Eddington phase function with $g'=0$. Note that the condition (1.4) reduces to

$$-\frac{1}{3}\le g'\le\frac{1}{3}.$$

Since the anisotropic scattering is weak, the asymmetry factor g' of the phase function is a small positive number. Thus, for the phase function (1.5), we have the following characteristics:

$$1-\varepsilon < p_0 < 1, \qquad 0 < g' \le \frac{1}{3}. \tag{1.7}$$

In [8], for a reflection boundary condition, the boundary value problem of the RTE (1.1) with the phase function (1.5) is transformed to a system of two integral equations, which provides convenience to numerical treatment in some applications. Some other related references include [6,18,24].

The delta-M method [29] is an extension of the delta-Eddington approximation and has the following form for the remainder function:

$$r(t) = \sum_{n=0}^{N} (2n+1) a_n P_n(t),$$

where N is a non-negative integer and P_n is the Legendre polynomial of degree n. The coefficients $\{a_n\}_{n=0}^N$ are chosen so that the delta-M phase function has certain number of correct moments.

For the delta-Henyey–Greenstein phase function, the remainder function is [19]

$$r(t) = p_{HG,g_0}(t),$$

where $g_0 \in [0, 1)$ is chosen so that the delta-Henyey–Greenstein phase function has the correct values of the first couple of moments. The usefulness of this approach is that due to the presence of the delta function term in the phase function, the parameter g_0 is expected to be much smaller than the anisotropy factor g of the scattering medium, and then the numerical treatment of the RTE with the delta-Henyey–Greenstein phase function can be conducted much more efficiently.

In this paper, we explore the solution existence, uniqueness and continuous dependence properties for the RTE (2.1) with the general delta-Eddington-type phase function (1.3), and provide numerical evidence that properly chosen values of the parameters in the phase function leads to accurate numerical simulation results. The rest of the paper is organized as follows. In Section 2, we state the boundary value problem of the RTE and introduce some function spaces needed in the mathematical study of the problem. In Section 3, we show rigorously the well-posedness of the problem. In Section 4, we present results from numerical experiments. The paper ends with a section of concluding remarks.

2. Preliminaries

We will assume that X is a domain in $\mathbb{R}^3$ with a Lipschitz boundary ∂X. For each fixed direction $\boldsymbol{\omega} \in \Omega$, introduce a new Cartesian coordinate system (z_1, z_2, s) by the relations $x = z + s\omega$, $z = z_1\omega_1 + z_2\omega_2$, where $(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega})$ is an orthonormal basis of $\mathbb{R}^3$, $z_1, z_2, s \in \mathbb{R}$. With respect to this new coordinate system, we denote by $X_{\boldsymbol{\omega}}$ the projection of X on the plane $s = 0$ in $\mathbb{R}^3$, and by $X_{\boldsymbol{\omega},\boldsymbol{z}}$ ($\boldsymbol{z} \in X_{\boldsymbol{\omega}}$) the intersection of the straight line $\{z + s\omega \mid s \in \mathbb{R}\}$ with X. We assume that the domain X is such that for any $(\boldsymbol{\omega}, \boldsymbol{z})$ with $\boldsymbol{z} \in X_{\boldsymbol{\omega}}$, $X_{\boldsymbol{\omega},\boldsymbol{z}}$ is the union of a finite number of line segments:

$$X_{\boldsymbol{\omega},\boldsymbol{z}} = \cup_{i=1}^{N(\boldsymbol{\omega},\boldsymbol{z})} \{z + s\omega \mid s \in (s_{i,-}, s_{i,+})\}.$$

Here $s_{i,\pm} = s_{i,\pm}(\omega, z)$ depend on $\boldsymbol{\omega}$ and $\boldsymbol{z}$, and $x_{i,\pm} := z + s_{i,\pm}\omega$ are the intersection points of the line $\{z + s\omega \mid s \in \mathbb{R}\}$ with ∂X. We assume $\sup_{\boldsymbol{\omega},\boldsymbol{z}} N(\omega, z) < \infty$; this assumption is known as a generalized convexity condition in the literature [27]. A convex domain X satisfies the generalized convexity condition, since $\sup_{\boldsymbol{\omega},\boldsymbol{z}} N(\omega, z) = 1$. Consider the following subsets of ∂X:

$$\partial X_{\boldsymbol{\omega},\pm} = \{z + s_{i,\pm}\omega \mid 1 \le i \le N(\omega, z),\ z \in X_{\boldsymbol{\omega}}\},$$

and introduce the following incoming and outgoing boundaries as subsets of $\Gamma = \partial X \times \Omega$:

$$\Gamma_- = \{(x, \omega) \mid x \in \partial X_{\boldsymbol{\omega},-},\ \omega \in \Omega\}, \qquad \Gamma_+ = \{(x, \omega) \mid x \in \partial X_{\boldsymbol{\omega},+},\ \omega \in \Omega\}.$$

Consider a Dirichlet boundary value problem for the RTE (1.1), i.e. the following boundary value problem:

$$\omega\cdot\nabla u + \mu_t u - \mu_s S u = f \quad \text{in } X \times \Omega, \tag{2.1}$$

$$u = u_{\text{in}} \quad \text{on } \Gamma_-. \tag{2.2}$$

We need some function spaces in studying the boundary value problem (2.1) and (2.2). Let

$$Q := L^2(X \times \Omega), \tag{2.3}$$

be the Hilbert space of measurable functions on $X \times \Omega$ with the inner product

$$(u, v)_Q := \int_{X\times\Omega} u(x, \omega)\, v(x, \omega)\, dx\, d\omega$$

and the norm $\|v\|_Q := (v, v)_Q^{1/2}$. The solution u of the boundary value problem (2.1) and (2.2) will be sought from the space

$$V := \{v \in Q \mid \omega\cdot\nabla v \in Q\}. \tag{2.4}$$

This space is denoted as $H^{1,\,2}(X \times \Omega)$ in [1], and is a Hilbert space with the inner product

$$(u, v)_V := \int_{X\times\Omega} [\omega\cdot\nabla u(x, \omega)\, \omega\cdot\nabla v(x, \omega) + u(x, \omega)\, v(x, \omega)]dx\, d\omega,$$

and the norm $\|v\|_V := (v, v)_V^{1/2}$. We also need function spaces $L^2(\Gamma_\pm)$ on $\Gamma_\pm$. They are Hilbert spaces of functions v on $\Gamma_\pm$ with inner products

$$(u, v)_{L^2(\Gamma_\pm)} := \int_\Omega d\omega \int_{X_\omega} \sum_{i=1}^{N(\omega,z)} u(z + s_{i,\pm}\omega, \omega)\, v(z + s_{i,\pm}\omega, \omega)\, dz,$$

and corresponding norms $\|v\|_{L^2(\Gamma_\pm)}$. We have the following statement for the trace of functions in V [1]. If $v \in V$ and $v|_{\Gamma_-} \in L^2(\Gamma_-)$, then $v|_{\Gamma_+} \in L^2(\Gamma_+)$ and for some constant c depending only on X,

$$\|v\|_{L^2(\Gamma_+)} \le c\big[\|v\|_V + \|v\|_{L^2(\Gamma_-)}\big]. \tag{2.5}$$

The statement remains valid by switching Γ_+ and Γ_-.

3. Well-posedness of the problem

This section is devoted to a well-posedness analysis of the boundary value problem (2.1) and (2.2) with the phase function (1.3). Without loss of generality, we prove the well-posedness result for the case where X is a convex domain in $\mathbb{R}^3$. This allows us to simplify various expressions and to focus on the essentials in the argument. All the discussions in the rest of the paper can be extended to the general case where the domain X satisfies the generalized convexity condition. Under the convexity assumption on X, for each $\boldsymbol{\omega} \in \Omega$ and $\boldsymbol{z} \in X_{\boldsymbol{\omega}}$, $X_{\boldsymbol{\omega},\boldsymbol{z}} := \{z + s\omega \mid s \in (s_-, s_+)\}$ is a line segment, where $s_\pm = s_\pm(\omega, z)$ depend on $\boldsymbol{\omega}$ and $\boldsymbol{z}$, and $x_\pm := z + s_\pm\omega$ are the intersection points of the line $\{z + s\omega \mid s \in \mathbb{R}\}$ with ∂X.

Theorem 3.1. *Assume $0 < p_0 < 1$, (1.4), μ_a, $\mu_s \in L^\infty(X)$, μ_a, $\mu_s \ge 0$ a.e. in X, and*

$$\kappa := \sup_{\boldsymbol{x}\in X} \frac{(1 - p_0)\mu_s(x)}{\mu_a(x) + (1 - p_0)\mu_s(x)} < 1. \tag{3.1}$$

Then, given

$$f \in Q, \qquad u_{\rm in} \in L^2(\Gamma_-), \tag{3.2}$$

the problem (2.1) and (2.2) with the phase function (1.3) has a unique solution $u \in V$, and

$$\|u\|_V \le c\big[\|u_{\rm in}\|_{L^2(\Gamma_-)} + \|f\|_Q\big]. \tag{3.3}$$

Moreover, the solution depends continuously on all the data.

Proof. We define modified scattering parameter $\tilde\mu_s$ and the modified total attenuation parameter $\tilde\mu_t$ as follows:

$$\tilde\mu_s = (1 - p_0)\mu_s, \qquad \tilde\mu_t = \mu_a + \tilde\mu_s.$$

Then with the phase function (1.3), we can rewrite the RTE (2.1) as

$$\omega\cdot\nabla u(x, \omega) + \tilde\mu_t(x)\, u(x, \omega) = \tilde\mu_s S_1 u(x, \omega) + f(x, \omega), \tag{3.4}$$

where

$$S_1 u(x, \omega) = \frac{1}{4\pi} \int_\Omega r(\hat\omega\cdot\omega)\, u(x, \hat\omega)\, d\hat\omega. \tag{3.5}$$

Let us convert the boundary value problem (3.4) and (2.2) to a fixed-point formulation. In the following, we write $s_\pm$ instead of $s_\pm(\boldsymbol{\omega}, \boldsymbol{z})$ wherever there is no danger for confusion. We write the Eq. (3.4) as

$$\frac{\partial}{\partial s} u(z + s\omega, \omega) + \tilde\mu_t(z + s\omega)\, u(z + s\omega, \omega) = \tilde\mu_s(z + s\omega)\, S_1 u(z + s\omega, \omega) + f(z + s\omega, \omega) \tag{3.6}$$

and multiply it by an integrating factor $\lambda(s) := \exp\big(\int_{s_-}^{s} \tilde\mu_t(z + s\omega)\, ds\big)$ to obtain

$$\frac{\partial}{\partial s}[\lambda(s)\, u(z + s\omega, \omega)] = \lambda(s)[\tilde\mu_s(z + s\omega)\, S_1 u(z + s\omega, \omega) + f(z + s\omega, \omega)].$$

Integrate this equation from s_- to s:

$$\lambda(s)\, u(z + s\omega, \omega) - u_{\rm in}(z + s_-\omega, \omega) = \int_{s_-}^{s} \lambda(t)[\tilde\mu_s(z + t\omega)\, S_1 u(z + t\omega, \omega) + f(z + t\omega, \omega)]dt.$$

Thus,

$$u = Au + F, \tag{3.7}$$

where

$$Au(z+s\omega,\omega) = \lambda(s)^{-1}\int_{s_-}^{s} \lambda(t)\,\tilde{\mu}_s(z+t\omega)\,S_1u(z+t\omega,\omega)\,dt,$$

$$F(z+s\omega,\omega) = \lambda(s)^{-1}\left[u_{\rm in}(z+s_-\omega,\omega) + \int_{s_-}^{s} \lambda(t)\,f(z+t\omega,\omega)\,dt\right].$$

Reversing the above procedure, we can derive (3.6), and then (2.1) and (2.2), from (3.7).

Let $Q_{\tilde{\mu}_t}$ be the space of $L^2(X\times\Omega)$ functions with the weighted norm

$$\|v\|_{Q_{\tilde{\mu}_t}} := \left[\int_{X\times\Omega} \tilde{\mu}_t(x)\,v(x,\omega)^2 dx\,d\omega\right]^{1/2}.$$

This weighted norm is equivalent to the norm $\|v\|_Q$ for $v\in Q$. Hence, $Q_{\tilde{\mu}_t}$ is also a Hilbert space. We will show the operator $A : Q_{\tilde{\mu}_t} \to Q_{\tilde{\mu}_t}$ is a contraction. We start with the inequality

$$\int_{s_-}^{s_+} \tilde{\mu}_t(z+s\omega)|Au(z+s\omega,\omega)|^2 ds \le \int_{s_-}^{s_+} \frac{\tilde{\mu}_t(z+s\omega)}{\lambda(s)^2}\left[\int_{s_-}^{s} \lambda(t)\,\tilde{\mu}_s(z+t\omega)\,dt\right] \cdot \left[\int_{s_-}^{s} \lambda(t)\,\tilde{\mu}_s(z+t\omega)|S_1u(z+t\omega,\omega)|^2 dt\right] ds.$$

The assumption (3.1) implies

$$\kappa = \sup_{x\in X} \frac{\tilde{\mu}_s(x)}{\tilde{\mu}_t(x)} < 1.$$

Since

$$\int_{s_-}^{s} \lambda(t)\,\tilde{\mu}_s(z+t\omega)\,dt \le \kappa \int_{s_-}^{s} \lambda(t)\,\tilde{\mu}_t(z+t\omega)\,dt = \kappa\,(\lambda(s)-1) < \kappa\,\lambda(s),$$

we have

$$\begin{aligned}
&\int_{s_-}^{s_+} \tilde{\mu}_t(z+s\omega)|Au(z+s\omega,\omega)|^2 ds \\
&\quad\le \kappa \int_{s_-}^{s_+} \lambda(s)^{-1}\tilde{\mu}_t(z+s\omega)\int_{s_-}^{s} \lambda(t)\,\tilde{\mu}_s(z+t\omega)|S_1u(z+t\omega,\omega)|^2 dt\,ds \\
&\quad= \kappa \int_{s_-}^{s_+} \lambda(t)\,\tilde{\mu}_s(z+t\omega)|S_1u(z+t\omega,\omega)|^2\left[\int_{t}^{s_+} \lambda(s)^{-1}\tilde{\mu}_t(z+s\omega)\,ds\right]dt.
\end{aligned}$$

Now

$$\int_{t}^{s_+} \lambda(s)^{-1}\tilde{\mu}_t(z+s\omega)\,ds = \lambda(t)^{-1} - \lambda(s_+)^{-1} < \lambda(t)^{-1},$$

we obtain

$$\begin{aligned}
\int_{s_-}^{s_+} \tilde{\mu}_t(z+s\omega)|Au(z+s\omega,\omega)|^2 ds &\le \kappa \int_{s_-}^{s_+} \tilde{\mu}_s(z+t\omega)|S_1u(z+t\omega,\omega)|^2 dt \\
&\le \kappa^2 \int_{s_-}^{s_+} \tilde{\mu}_t(z+t\omega)|S_1u(z+t\omega,\omega)|^2 dt.
\end{aligned}$$

Integrating the above inequality first with respect to $\boldsymbol{z}\in X_{\boldsymbol{\omega}}$ and then with respect to $\boldsymbol{\omega}\in\Omega$, we have thus proved the inequality

$$\|Au\|_{Q_{\tilde{\mu}_t}} \le \kappa\,\|S_1u\|_{Q_{\tilde{\mu}_t}}. \tag{3.8}$$

Furthermore, we have

$$\begin{aligned}
\|S_1u\|^2_{Q_{\tilde{\mu}_t}} &= \int_{X\times\Omega} \tilde{\mu}_t(x)\,|S_1u(x,\omega)|^2 dx\,d\omega \\
&= \frac{1}{(4\pi)^2}\int_{X\times\Omega} \tilde{\mu}_t(x)\left|\int_{\Omega} r(\hat{\omega}\cdot\omega)\,u(x,\hat{\omega})\,d\hat{\omega}\right|^2 dx\,d\omega.
\end{aligned}$$

Note that

$$\left|\int_\Omega r(\hat\omega\cdot\omega)\,u(x,\hat\omega)\,d\hat\omega\right|^2 \le \int_\Omega r(\hat\omega\cdot\omega)\,d\hat\omega \int_\Omega r(\hat\omega\cdot\omega)\,u(x,\hat\omega)^2 d\hat\omega$$
$$= 4\pi \int_\Omega r(\hat\omega\cdot\omega)\,u(x,\hat\omega)^2 d\hat\omega,$$

and

$$\int_{X\times\Omega} \tilde\mu_t(x) \int_\Omega r(\hat\omega\cdot\omega)\,u(x,\hat\omega)^2 d\hat\omega\,dx\,d\omega = 4\pi \int_{X\times\Omega} \tilde\mu_t(x) u(x,\hat\omega)^2 dx\,d\hat\omega.$$

Thus,

$$\|S_1 u\|^2_{Q_{\tilde\mu_t}} \le \|u\|^2_{Q_{\tilde\mu_t}},$$

i.e.,

$$\|S_1 u\|_{Q_{\tilde\mu_t}} \le \|u\|_{Q_{\tilde\mu_t}}. \tag{3.9}$$

Combining (3.8) and (3.9), we see that the operator $A : Q_{\tilde\mu_t} \to Q_{\tilde\mu_t}$ is contractive and

$$\|Au\|_{Q_{\tilde\mu_t}} \le \kappa\, \|u\|_{Q_{\tilde\mu_t}}. \tag{3.10}$$

By an application of the Banach fixed-point theorem (cf. [4] or [32]), we conclude that the Eq. (3.7) has a unique solution $u \in Q_{\tilde\mu_t}$, i.e., $u \in Q$. We further deduce from the Eq. (3.4) that $\boldsymbol{\omega}\cdot\nabla u(\boldsymbol{x},\boldsymbol{\omega}) \in Q$. Therefore, the solution $u \in V$. Using (3.10),

$$\|u\|_{Q_{\tilde\mu_t}} \le \|Au\|_{Q_{\tilde\mu_t}} + \|\tilde\mu_t^{1/2} F\|_{L^2(X\times\Omega)} \le \kappa\, \|u\|_{Q_{\tilde\mu_t}} + c\big[\|u_{\rm in}\|_{L^2(\Gamma_-)} + \|f\|_{L^2(X\times\Omega)}\big].$$

Thus,

$$\|u\|_Q \le c\big[\|u_{\rm in}\|_{L^2(\Gamma_-)} + \|f\|_Q\big],$$

and furthermore, by (3.4) we have the bound (3.3).

The Lipschitz continuity of the solution with respect to the source function $f \in Q$ and the inflow value function $u_{\rm in} \in L^2(\Gamma_-)$ follows from (3.3).

Finally consider continuous dependence of the solution on the optical parameters μ_t and μ_s. Let $(\mu_{t,1}, \mu_{s,1})$ and $(\mu_{t,2}, \mu_{s,2})$ be two sets of the parameters satisfying the stated assumptions, and let $u_1 \in V$ and $u_2 \in V$ be the solutions of the corresponding boundary value problems. The difference $e := u_1 - u_2$ satisfies the equation

$$\omega\cdot\nabla e + \mu_{t,2} e - \mu_{s,2} S e = -(\mu_{t,1} - \mu_{t,2}) u_1 + (\mu_{s,1} - \mu_{s,2}) S u_1 \quad \text{in } X\times\Omega,$$

together with the homogeneous boundary condition on Γ_-. Applying (3.3), we have

$$\|e\|_V \le c\big[\|(\mu_{t,1} - \mu_{t,2}) u_1\|_Q + \|(\mu_{s,1} - \mu_{s,2}) S u_1\|_Q\big]$$
$$\le c\big[\|\mu_{t,1} - \mu_{t,2}\|_{L^2(X)} \|u_1\|_Q + \|\mu_{s,1} - \mu_{s,2}\|_{L^2(X)} \|S u_1\|_Q\big].$$

So we have local Lipschitz continuous dependence of the solution with respect to the optical parameters in the $L^2(X)$ norm. □

Note that the condition (3.1) is trivially satisfied as long as the absorption effect cannot be ignored.

We now provide a positivity property for the model (3.4) and (2.2). This property is required for the model to be physically meaningful.

Proposition 3.2. *Keep the assumptions stated in Theorem 3.1. If $f \ge 0$ a.e. in $X \times \Omega$ and $u_{\rm in} \ge 0$ a.e. on Γ_-, then for the solution of the problem (3.4) and (2.2), $u \ge 0$ a.e. in $X \times \Omega$.*

Proof. From (3.7),

$$u = (I - A)^{-1} F = \sum_{j=0}^{\infty} A^j F.$$

By the assumptions, $F \ge 0$ a.e. in $X \times \Omega$. Let us prove that this implies $AF \ge 0$ a.e. in $X \times \Omega$. By definition of A,

$$AF(z + s\omega, \omega) = \lambda(s)^{-1} \int_{s_-}^{s} \lambda(t)\, \tilde\mu_s(z + t\omega)\, S_1 F(z + t\omega, \omega)\, dt.$$

Thus, $AF \ge 0$ a.e. in $X \times \Omega$, and then for any positive integer j, $A^j F \ge 0$ a.e. in $X \times \Omega$. Therefore, $u \ge 0$ a.e. in $X \times \Omega$. □

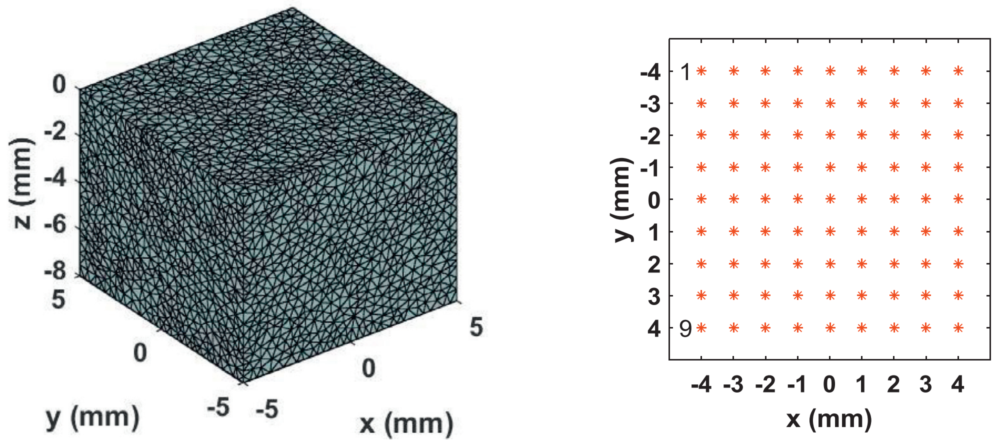

Fig. 1. 3D cube with $\mu_a = 0.1\,\text{mm}^{-1}$, $\mu_s = 15\,\text{mm}^{-1}$ and $g = 0.9$. For delta-Eddington phase function, $p_0 = 0.85$, $g' = 0.12$. The internal source is located at $(0, 0, -1)$. (a) The mesh of 3D cube; (b) The detector distribution at the bottom surface of the phantom.

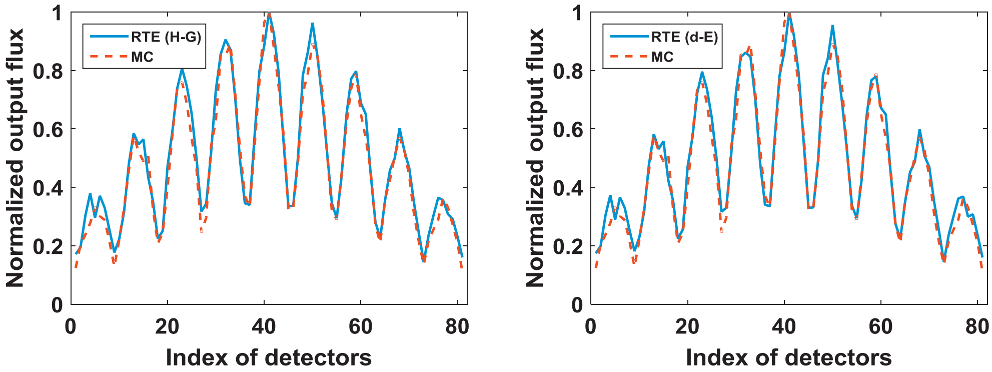

Fig. 2. Comparison between RTE and Monte Carlo (MC). (a) Comparison of normalized fluence obtained from RTE with Henyey–Greenstein phase function and MC simulation; (b) Comparison of normalized fluence obtained from RTE with delta-Eddington function and MC simulation.

4. Numerical results

We report numerical simulation results on an example. For this example, we use a homogeneous cubic phantom with the geometric dimensions $10 \times 10 \times 8\,\text{mm}^3$, with coordinates $[-5, 5] \times [-5, 5] \times [-8, 0]$, and assign values of the optical parameters for the phantom as follows: $\mu_a = 0.1\,\text{mm}^{-1}$, $\mu_s = 15\,\text{mm}^{-1}$, the anisotropy factor $g = 0.9$ and refractive index 1.0. The detectors are distributed at the bottom surface of the phantom, as shown in Fig. 1. A Gaussian beam whose center located at the position (0, 0, 0) is used to simulate photon propagation inside media. The Gaussian beam has intensity distribution

$$g(x, y; \sigma) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/(2\sigma^2)},$$

where σ controls the beam width and is set to be 0.5.

Using the Henyey–Greenstein (H–G) scattering phase function, radiative transport equation (RTE) simulator and Monte Carlo simulation are implemented respectively to simulate photons propagation in the object and the photon fluence is recorded by virtual detectors on the surface of the object. In our example, a mesh-based Monte Carlo (MC) with wide-filed sources simulation [30] was employed with 10^8 photons. Meanwhile, the delta-Eddington scattering phase function (1.5) is used to simulate photon propagation. According to our theoretical analysis, the parameters of the delta-Eddington (D-E) phase function stay in the range (1.7). The coefficients of the corresponding Legendre polynomial series for the remainder function $r(t)$ are $a_0 = 1$ and $a_1 = g'$. Here, we choose the anisotropy factor $g = 0.9$, and take the parameters as $p_0 = 0.85$ and $g' = 0.12$ in the delta-Eddington phase function. Based on the delta-Eddington scattering phase function, RTE is performed to acquire the photon fluence on the surface of the object. Fig. 2 gives the comparison of the photon fluences between the two scattering phase functions. The photon fluence on the surface of the phantom is in good agreement with the counterpart obtained from the Monte Carlo simulation with the Henyey–Greenstein scattering phase function.

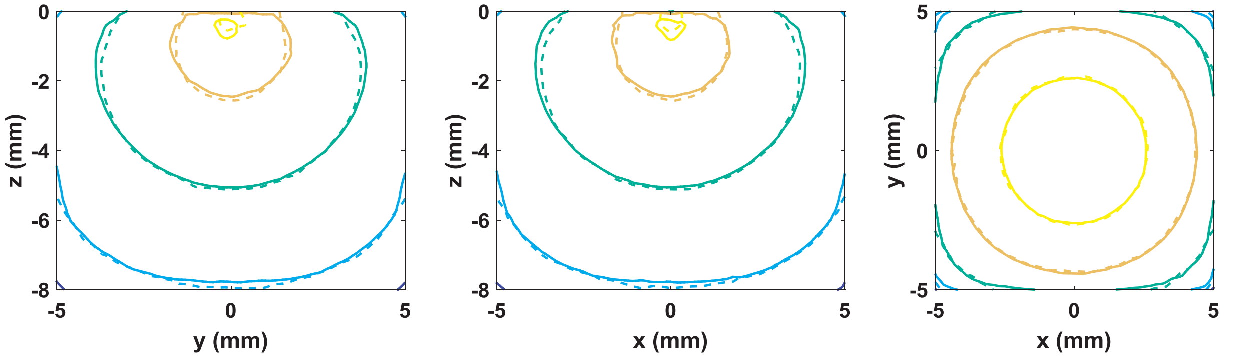

Fig. 3. Contours of logarithm of photon density acquired by solving RTE with the delta-Eddington phase function. The solid curves represent the solution to RTE and the dashed curves represent the Monte Carlo (MC) results. (a) *Oyz*-plane. The values from innermost curve to the outer curve are −1, −2, −3, and −4, respectively. (b) *Ozx*-plane. The values form innermost curve to the outer curve are −1, −2, −3, and −4, respectively. (c) The plane $z = -4$ mm. The values from innermost curve to the outer curve are −3, −3.5, and −4, respectively.

To illustrate the accuracy of the proposed method, the comparison of the light field propagated in the object is also performed. Contours of logarithm of photon density distribution in the phantom acquired from the solutions of RTE with the delta-Eddington phase function are shown in Fig. 3. The photon fluence in the phantom is also in excellent agreement with the counterpart obtained from the Monte Carlo simulation with the Henyey–Greenstein scattering phase function.

5. Concluding remarks

In this paper, we introduce a family of delta-Eddington-type phase functions for the radiative transfer equation (RTE). A proper selection of the delta-Eddington phase function can lead to much simplification in solving the RTE, and is thus of fundamental importance for applications in biomedical optics. A rigorous mathematical theory is developed for the RTE with a general delta-Eddington-type phase function. Numerical experiments show that with a proper choice of the parameter, the RTE with a generalized delta-Eddington phase function is able to provide accurate simulation results for light propagation within highly forward-peaked scattering media.

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References

[1] A. Agoshkov, Boundary Value Problems for Transport Equations, Birkhäuser, Boston, 1998.
[2] S.R. Arridge, Optical tomography in medical imaging, Inverse Probl. 15 (1999) R41–R93.
[3] S.R. Arridge, J.C. Schotland, Optical tomography: forward and inverse problems, Inverse Probl. 25 (2009) 59. 123010
[4] K. Atkinson, W. Han, Theoretical Numerical Analysis: A Functional Analysis Framework, third ed., Springer-Verlag, New York, 2009.
[5] G. Bal, Inverse transport theory and applications, Inverse Probl. 25 (2009) 48. 053001
[6] P. Boulet, A. Collin, J.-L. Consalvi, On the finite volume method and the discrete ordinates method regarding radiative heat transfer in acute forward anisotropic scattering media, J. Quant. Spectrosc. Radiat. Transf. 104 (2007) 460–473.
[7] K.M. Case, P.F. Zweifel, Linear Transport Theory, Addison-Wesley, Reading, MA, 1967.
[8] W.X. Cong, H.O. Shen, A. Cong, G. Wang, Integral equations of the photon fluence rate and flux based on a generalized delta-Eddington phase function, J. Biomed. Opt. 13 (2) (2008) 6. 024016
[9] J.J. Duderstadt, W.R. Martin, Transport Theory, John Wiley, New York, 1978.
[10] B. Davison, Neutron Transport Theory, Oxford University Press, Oxford, 1957.
[11] H. Gao, H. Zhao, Analysis of a numerical solver for radiative transport equation, Math. Comp. 82 (2013) 153–172.
[12] W. Han, J. Eichholz, X.L. Cheng, G. Wang, A theoretical framework of x-ray dark-field tomography, SIAM J. Appl. Math. 71 (2011a) 1557–1577.
[13] W. Han, J. Eichholz, J. Huang, J. Lu, RTE based bioluminescence tomography: a theoretical study, Inverse Probl. Sci. Eng. 19 (2011b) 435–459.
[14] W. Han, J. Eichholz, G. Wang, On a family of differential approximations of the radiative transfer equation, J. Math. Chem. 50 (2012) 689–702.
[15] W. Han, J. Huang, J. Eichholz, Discrete-ordinate discontinuous Galerkin methods for solving the radiative transfer equation, SIAM J. Sci. Comput. 32 (2010) 477–497.
[16] L. Henyey, J. Greenstein, Diffuse radiation in the galaxy, Astrophys. J. 93 (1941) 70–83.
[17] J.H. Joseph, W.J. Wiscombe, J.A. Weinman, The delta-Eddington approximation for radiative flux transfer, J. Atmos. Sci. 33 (1976) 2452–2459.
[18] S. Karanjai, M. Karanjai, Inverse radiative transfer problem with delta-Eddington phase function in a symmetric model, Astrophys. Space Sci. 117 (1985) 151–164.
[19] B.H.J. McKellar, M.A. Box, The scaling group of the radiative transfer equation, J. Atmos. Sci. 38 (1981) 1063–1068.
[20] M.F. Modest, Radiative Heat Transfer, second ed., Academic Press, Oxford, 2003.
[21] F. Natterer, F. Wübbeling, Mathematical Methods in Image Reconstruction, SIAM, Philadelphia, 2001.
[22] A. Peraiah, An Introduction to Radiative Transfer: Methods and applications in astrophysics, Cambridge University Press, Cambridge, 2004.
[23] K. Ren, Recent development in numerical techniques for transport-based medical imaging methods, Commun. Comput. Phys. 8 (2010) 1–50.

[24] B. Subasilar, Analytical approaches to the delta-Eddington model of the radiative transfer through vertically inhomogeneous optical depths, Appl. Math. Model. 32 (2008) 514–534.
[25] J. Tang, W. Han, B. Han, A theoretical study for RTE-based parameter identification problems, Inverse Probl. 29 (2013) 18. 095002
[26] G.E. Thomas, K. Stamnes, Radiative Transfer in the Atmosphere and Ocean, Cambridge University Press, Cambridge, 1999.
[27] V.S. Vladimirov, Mathematical Problems of One-Velocity Transport Theory, Trudy MIAN, 61, 1961.
[28] L.V. Wang, H.-i. Wu, Biomedical Optics: Principles and Imaging, Wiley, New York, 2007.
[29] W.J. Wiscombe, The delta-*m* method: rapid yet accurate radiative flux calculations for strongly asymmetric phase functions, J. Atmos. Sci. 34 (1977) 1408–1422.
[30] R. Yao, X. Intes, Q. Fang, Generalized mesh-based montecarlo for wide-field sources and detectors via mesh retesselation, Biomed. Opt. Express. 7 (2016) 171–184.
[31] W. Zdunkowski, T. Trautmann, A. Bott, Radiation in the Atmosphere: A Course in Theoretical Meteorology, Cambridge University Press, Cambridge, 2007.
[32] E. Zeidler, Nonlinear Functional Analysis and its Applications. I: Fixed-point Theorems, Springer-Verlag, New York, 1985.