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Numerical analysis of an evolutionary variational–hemivariational inequality with application in contact mechanics[☆]

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Highlights

- First paper on numerical analysis of evolutionary variational–hemivariational inequality problems.
- A Céa type inequality for fully discrete numerical solutions.
- An optimal order error estimate for linear element solutions.
- Numerical evidence on the theoretically predicted optimal order error estimate.

Abstract

Variational–hemivariational inequalities are useful in applications in science and engineering. This paper is devoted to numerical analysis for an evolutionary variational–hemivariational inequality. We introduce a fully discrete scheme for the inequality, using a finite element approach for the spatial approximation and a backward finite difference to approximate the time derivative. We present a Céa type inequality which is the starting point for error estimation. Then we apply the results in the numerical solution of a problem arising in contact mechanics, and derive an optimal order error estimate when the linear elements are used. Finally, we report numerical simulation results on solving a model contact problem, and provide numerical evidence on the theoretically predicted optimal order error estimate.

Keywords: Evolutionary variational–hemivariational inequality; Finite element method; Error estimation; Contact mechanics

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1. Introduction

Variational and hemivariational inequalities are mathematical models arising naturally in both qualitative studies and numerical analysis of various complicated and challenging problems in sciences and engineering. Variational inequalities are inequality problems with convex structure [1,2], whereas hemivariational inequalities are inequality problems involving non-convex terms [3–7].

The numerical analysis of variational inequalities is a well developed area, as illustrated in [8–10] and the references therein. In contrast, there are still relatively few publications devoted to numerical methods for hemivariational inequalities and, in particular, to evolutionary hemivariational inequalities. The basic reference in the area is the book [11]. However, while this book covers convergence of numerical methods for solving hemivariational inequalities, it does not provide error estimates for the numerical solutions. Recently a number of papers have been published on error estimation for numerical methods for hemivariational inequalities. In particular, in [12], an optimal order error estimate is derived for the linear element solution of a static variational–hemivariational inequality; the variational–hemivariational inequality reduces to a static hemivariational inequality in a special case. In [13], an optimal order error estimate with respect to both the spatial mesh-size and the temporal step-size is derived for the linear finite element solution of a hyperbolic hemivariational inequality. In [14], for a general family of elliptic hemivariational inequalities with or without convex constraints, the Galerkin method is shown to converge, and for various elliptic hemivariational inequalities arising in contact mechanics, optimal order error estimates are proved for the linear finite element solutions.

Variational–hemivariational inequalities involve both convex and nonconvex functions. Interest in their study is motivated by problems in mechanics [5]. Recent results in their study have been obtained in [12,15,16]. The inequalities studied in [12] are elliptic, whereas those in [15] are history-dependent ones. Numerical approximations of history-dependent variational–hemivariational inequalities are considered in [16]. First order evolutionary variational–hemivariational inequalities are studied in [17], and the purpose of this paper is to provide numerical analysis of such variational–hemivariational inequalities.

We will need the notion of the subdifferential in the sense of Clarke. All spaces used in this paper are real. For a normed space X, we denote its norm by $\|\cdot\|_X$, its topological dual by X^*, and the duality pairing of X and X^* by $\langle\cdot,\cdot\rangle_{X^*\times X}$. For a locally Lipschitz function $\varphi : X \to \mathbb{R}$, its Clarke generalized directional derivative at a point $x \in X$ in a direction $v \in X$ is defined by

$$\varphi^0(x; v) = \limsup_{y\to x,\lambda\downarrow 0} \frac{\varphi(y+\lambda v) - \varphi(y)}{\lambda}.$$

The Clarke subdifferential of φ at x is a subset of X^* given by

$$\partial_{Cl}\varphi(x) = \left\{\zeta \in X^* : \varphi^0(x; v) \ge \langle\zeta, v\rangle_{X^*\times X}\ \forall\, v \in X\right\}.$$

Discussions of the subdifferential in the sense of Clarke can be found in the books [4,5,18,19].

The rest of the paper is organized as follows. In Section 2 we introduce the class of evolutionary variational–hemivariational inequalities studied in [17] and comment on the solution existence and uniqueness. In Section 3, we introduce and analyse a fully discrete scheme for the inequalities, and present a Céa type inequality for error estimation. Then in Section 4, we apply the results of Section 3 to derive an error estimate for the evolutionary variational–hemivariational inequality that describes a frictionless contact problem with a Kelvin–Voigt viscoelastic material where the contact conditions are with normal compliance and unilateral constraints. The error estimate is of optimal order with respect to both time step-size and space mesh-size when linear finite elements are used in space discretization. In Section 5, we report computer simulation results that provide numerical evidence of the predicted optimal convergence order for numerical solutions of a model contact problem.

2. The evolutionary variational–hemivariational inequality

In this section, we introduce the evolutionary variational–hemivariational inequality studied in [17]. For this purpose, we need some function spaces.

Let V be a strictly convex, reflexive separable Banach space. We denote by V^* the dual of V, and by $\langle\cdot,\cdot\rangle$ the duality pairing between V^* and V. Let U be a reflexive Banach space with the dual U^*, and denote by $\langle\cdot,\cdot\rangle_{U^*\times U}$ the duality pairing between U^* and U. Let $\iota : V \to U$ be a linear operator and $\iota^* : U^* \to V^*$ be its adjoint operator.

For a positive number T, we introduce several function spaces defined on the interval $[0, T]$ with values in a Banach space: $\mathcal{V} = L^2(0, T; V)$, $\mathcal{V}^* = L^2(0, T; V^*)$, $\mathcal{U} = L^2(0, T; U)$, $\mathcal{U}^* = L^2(0, T; U^*)$, and $\mathcal{W} = \{v \in \mathcal{V} \mid \dot{v} \in \mathcal{V}^*\}$. Note that hereafter the dot above the name of a function represents its time derivative. We then define a space $M^{2,2}(0, T; V, V^*)$. Let π be a finite partition of the interval $(0, T)$ by a family of disjoint subintervals $\sigma_i = (a_i, b_i)$ such that $[0, T] = \cup_{i=1}^n \bar{\sigma}_i$. Let $\mathcal{F}$ denote the family of all such partitions. Then, we define the seminorm of a function $v : [0, T] \to V$ by

$$\|v\|^2_{BV^2(0,T;V)} = \sup_{\pi\in\mathcal{F}} \Big\{ \sum_{\sigma_i\in\pi} \|v(b_i) - v(a_i)\|^2_V \Big\}$$

and the space

$$BV^2(0, T; V) = \left\{v : [0, T] \to V : \|v\|_{BV^2(0;T;V)} < \infty\right\}.$$

We further define

$$M^{2,2}(0, T; V, V^*) = L^2(0, T; V) \cap BV^2(0, T; V^*).$$

It is well-known that $M^{2,2}(0, T; V, V^*)$ is a Banach space with the norm $\|\cdot\|_{L^2(0,T;V)} + \|\cdot\|_{BV^2(0,T;V^*)}$.

Let there be given operators $A, B : V \to V^*$, functionals $J : U \to \mathbb{R}$ and $\Phi : V \to \mathbb{R} \cup \{+\infty\}$, and a function $f : [0, T] \to V^*$. The pointwise formulation of the problem is the following.

PROBLEM (P). Find $u \in \mathcal{W}$ such that $u(0) = u_0$ and for a.e. $t \in (0, T)$,

$$\langle A\dot{u}(t) + Bu(t) + \iota^*\xi(t), v - u(t)\rangle + \Phi(v) - \Phi(u(t)) \geq \langle f(t), v - u(t)\rangle \quad \forall\, v \in V, \tag{2.1}$$

$$\xi(t) \in \partial_{Cl} J(\iota u(t)). \tag{2.2}$$

The corresponding integral form of the problem is studied in [17].

PROBLEM (P′). Find $u \in \mathcal{W}$ such that $u(0) = u_0$,

$$\int_0^T \left[\langle A\dot{u}(t) + Bu(t) + \iota^*\xi(t) - f(t), v(t) - u(t)\rangle + \Phi(v(t)) - \Phi(u(t))\right] dt \geq 0 \quad \forall\, v \in \mathcal{V},$$

and for a.e. $t \in (0, T)$,

$$\xi(t) \in \partial_{Cl} J(\iota u(t)).$$

As in [17], we make the following assumptions on the data.

$H(A)$. The operator $A : V \to V^*$ is linear, bounded, coercive and symmetric, i.e.

(i) $A \in \mathcal{L}(V, V^*)$.
(ii) $\langle Av, v\rangle \geq \alpha\|v\|^2_V\ \forall\, v \in V$ with $\alpha > 0$.
(iii) $\langle Av, w\rangle = \langle Aw, v\rangle\ \forall\, v, w \in V$.

$H(B)$. The operator $B : V \to V^*$ is linear, bounded and coercive, i.e.

(i) $B \in \mathcal{L}(V, V^*)$.
(ii) $\langle Bv, v\rangle \geq \beta\|v\|^2_V\ \forall\, v \in V$ with $\beta > 0$.

$H(J)$. The functional $J : U \to \mathbb{R}$ is such that

(i) J is locally Lipschitz.
(ii) There exists $c_J > 0$ such that $\|\xi\|_{U^*} \leq c_J(1 + \|u\|_U)\ \forall\, u \in U, \xi \in \partial_{Cl} J(u)$.
(iii) There exists $m > 0$, such that

$$\langle \xi - \eta, u - v\rangle_{U^*\times U} \geq -m\|u - v\|^2_U \quad \forall\, u, v \in U,\ \xi \in \partial_{Cl} J(u),\ \eta \in \partial_{Cl} J(v).$$

$H(\Phi)$. The functional $\Phi : V \to \mathbb{R} \cup \{+\infty\}$ is convex, proper and lower semicontinuous.

$H(\iota)$. The operator $\iota : V \to U$ is linear, continuous and compact. Moreover, the associated Nemytskii operator $\bar{\iota} : M^{2,2}(0, T; V, V^*) \to \mathcal{U}$ defined by $(\bar{\iota}v)(t) = \iota(v(t))$ for all $t \in [0, T]$ is also compact.

$H(0)$. $f \in H^1(0, T; V^*)$, $u_0 \in dom(\Phi)$ and the following compatibility condition holds: there exist $\xi_0 \in \partial_{Cl} J(\iota u_0)$ and $\eta_0 \in \partial_{Conv}\Phi(u_0)$ such that

$$Bu_0 + \iota^*\xi_0 + \eta_0 - f(0) \in V.$$

$H(s)$. Inequality $\beta > m\,\|\iota\|^2$ holds.

Remark 2.1. In order to guarantee the compactness of the Nemytskii operator $\bar{\iota}$, that is required in the assumption $H(\iota)$, it is enough to provide the following condition.

$H_1(\iota)$. There exists a space Z and a linear operator $\pi : Z \to U$ such that $V \subset Z$ compactly and $\iota = \pi \circ i$, where $i : V \to Z$ denotes the (compact) identity mapping.

This argumentation was justified and used in [17] to show the validity of the assumption $H(\iota)$ in particular problems of contact mechanics. In Section 4 we use the same idea, namely, we ensure that condition $H_1(\iota)$ is met in our case.

We note that the symbols $dom(\Phi)$ and $\partial_{Conv}\Phi$ in $H(0)$ denote the effective domain of Φ and its subdifferential in the sense of convex analysis, respectively.

It is proved in [17] that under the assumptions $H(A)$, $H(B)$, $H(J)$, $H(\Phi)$, $H(\iota)$, $H(0)$ and $H(s)$, Problem (P′) has a unique solution $u \in H^1(0, T; V)$. Under the same assumptions, it is straightforward to show a solution of Problem (P) is unique. Throughout the paper, we keep the assumptions $H(A)$, $H(B)$, $H(J)$, $H(\Phi)$, $H(\iota)$, $H(0)$ and $H(s)$, and assume that Problem (P), like Problem (P′), has a unique solution $u \in H^1(0, T; V)$.

3. A fully discrete approximation

We introduce and analyse a fully discrete numerical method to solve Problem (P).

3.1. The scheme

Let V^h be a finite dimensional subspace of V and $u_0^h \in V^h$ be an approximation of u_0. Let N be a positive integer, and denote by $k = T/N$ the time stepsize. We use notation $t_n = kn$, $n = 0, \ldots, N$. For any continuous function g defined on the interval $[0, T]$ we denote $g_n = g(t_n)$, $n = 0, \ldots, N$. For any sequence $\{z_n\}_{n=0}^N$ we introduce the notation

$$\delta z_n = \frac{1}{k}(z_n - z_{n-1}), \quad n = 1, \ldots, N.$$

Then a fully discrete approximation method for Problem (P) is the following.

Problem (P^{hk}). Find $u^{hk} = \{u_n^{hk}\}_{n=0}^N \subset V^h$ such that $u_0^{hk} = u_0^h$ and for $n = 1, 2, \ldots, N$,

$$\left\langle A\delta u_n^{hk} + Bu_n^{hk} + \iota^*\xi_n^{hk}, v^h - u_n^{hk}\right\rangle + \Phi(v^h) - \Phi(u_n^{hk}) \ge \langle f_n, v^h - u_n^{hk}\rangle \quad \forall\, v^h \in V^h, \tag{3.1}$$

$$\xi_n^{hk} \in \partial_{Cl} J(\iota u_n^{hk}). \tag{3.2}$$

In the well-posedness study of Problem (P^{hk}), we rewrite (3.1) as

$$\begin{aligned}&\left\langle Au_n^{hk} + k\, Bu_n^{hk} + k\, \iota^*\xi_n^{hk}, v^h - u_n^{hk}\right\rangle + k\, \Phi(v^h) - k\, \Phi(u_n^{hk})\\ &\quad \ge \langle Au_{n-1}^{hk} + k\, f_n, v^h - u_n^{hk}\rangle \quad \forall\, v^h \in V^h.\end{aligned} \tag{3.3}$$

This is a static variational–hemivariational inequality and we can apply the existence and uniqueness result of [20] to see that the problem has a unique solution $u_n^{hk} \in V^h$.

Next, we prove a boundedness property for the numerical solution, which will be useful in error analysis. We will need the identity

$$\langle A(u-v), u\rangle = \frac{1}{2}\langle Au, u\rangle - \frac{1}{2}\langle Av, v\rangle + \frac{1}{2}\langle A(u-v), u-v\rangle. \tag{3.4}$$

Also, it will be convenient to introduce the $\|\cdot\|_A$ norm through

$$\|v\|_A^2 = \langle Av, v\rangle.$$

Note that under the assumption $H(A)$, the norms $\|\cdot\|_A$ and $\|\cdot\|_V$ are equivalent.

For simplicity, assume $u_0 \in V^h$; this is the case, e.g., if $u_0 = 0$. Then by letting

$$u_{-1}^{hk} := u_0 + k\, A^{-1}\left(Bu_0 + \iota^*\xi_0 + \eta_0 - f(0)\right),$$

the inequality (3.1) holds also for $n = 0$. Recall that ξ_0 and η_0 denote the elements introduced in the assumption $H(0)$.

Proposition 3.1. *Under the stated assumptions, there is a constant $c > 0$ such that*

$$\max_{1\le n\le N} \left\|\delta u_n^{hk}\right\|_V^2 + k\sum_{n=1}^N \left\|\delta u_n^{hk}\right\|_V^2 \le c. \tag{3.5}$$

Proof. We take $v^h = u^{hk}_{n-1}$ in (3.1) to get

$$\left\langle A\delta u^{hk}_n + Bu^{hk}_n + \iota^*\xi^{hk}_n, -k\,\delta u^{hk}_n\right\rangle + \Phi(u^{hk}_{n-1}) - \Phi(u^{hk}_n) \geq -k\,\langle f_n, \delta u^{hk}_n\rangle.$$

Then, we write (3.1) with index $n-1$ instead of n, and take $v^h = u^{hk}_n$ to get

$$\left\langle A\delta u^{hk}_{n-1} + Bu^{hk}_{n-1} + \iota^*\xi^{hk}_{n-1}, k\,\delta u^{hk}_n\right\rangle + \Phi(u^{hk}_n) - \Phi(u^{hk}_{n-1}) \geq k\,\langle f_{n-1}, \delta u^{hk}_n\rangle.$$

Add the two inequalities,

$$\left\langle A\left(\delta u^{hk}_n - \delta u^{hk}_{n-1}\right) + k\,B\delta u^{hk}_n + k\,\iota^*\delta\xi^{hk}_n, \delta u^{hk}_n\right\rangle \leq k\,\langle \delta f_n, \delta u^{hk}_n\rangle.$$

Applying the identity (3.4) and using the assumptions $H(B)$ and $H(J)$, we derive from the above inequality that

$$\frac{1}{2}\left\|\delta u^{hk}_n\right\|^2_A - \frac{1}{2}\left\|\delta u^{hk}_{n-1}\right\|^2_A + k\left(\beta - m\,\|\iota\|^2\right)\left\|\delta u^{hk}_n\right\|^2_A \leq k\,\langle \delta f_n, \delta u^{hk}_n\rangle.$$

For any $\varepsilon > 0$,

$$\langle \delta f_n, \delta u^{hk}_n\rangle \leq \varepsilon\left\|\delta u^{hk}_n\right\|^2_V + \frac{1}{4\,\varepsilon}\|\delta f_n\|^2_{V^*}.$$

Thus,

$$\frac{1}{2}\left\|\delta u^{hk}_n\right\|^2_A - \frac{1}{2}\left\|\delta u^{hk}_{n-1}\right\|^2_A + k\left(\beta - m\,\|\iota\|^2 - \varepsilon\right)\left\|\delta u^{hk}_n\right\|^2_A \leq \frac{k}{4\,\varepsilon}\|\delta f_n\|^2_{V^*}.$$

Replacing n by j in the above inequality and making a summation from $j = 1$ to n, we obtain

$$\frac{1}{2}\left\|\delta u^{hk}_n\right\|^2_A + k\left(\beta - m\,\|\iota\|^2 - \varepsilon\right)\sum_{j=1}^{n}\left\|\delta u^{hk}_j\right\|^2_A \leq \frac{1}{2}\left\|\delta u^{hk}_0\right\|^2_A + \frac{k}{4\,\varepsilon}\sum_{j=1}^{n}\left\|\delta f_j\right\|^2_{V^*}. \tag{3.6}$$

Now,

$$\delta f_j = \frac{1}{k}\left(f_j - f_{j-1}\right) = \frac{1}{k}\int_{t_{j-1}}^{t_j}\dot{f}(s)\,ds.$$

Hence,

$$k\sum_{j=1}^{n}\left\|\delta f_j\right\|^2_{V^*} \leq \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|\dot{f}(s)\|^2_{V^*}ds \leq \|\dot{f}\|^2_{\mathcal{V}^*}.$$

When f_j is defined as the average

$$f_j := \frac{1}{k}\int_{t_{j-1}}^{t_j} f(s)\,ds,$$

we also have the above bound. Note that

$$\left\|\delta u^{hk}_0\right\|^2_A = \frac{1}{k^2}\left\|u_0 - u^{hk}_{-1}\right\|^2_A = \left\|Bu_0 + \iota^*\xi_0 + \eta_0 - f(0)\right\|^2_{V^*}.$$

Therefore, from (3.6), we conclude the boundedness (3.5). ∎

For $n = 1, 2, \ldots, N$, we can write

$$u^{hk}_n = u_0 + k\sum_{j=1}^{n}\delta u^{hk}_j.$$

Then we have the following corollary from Proposition 3.1.

Corollary 3.2. *Under the stated assumptions, there is a constant $c > 0$ such that*

$$\max_{1\leq n\leq N}\left\|u^{hk}_n\right\|_V \leq c. \tag{3.7}$$

3.2. Error analysis

We perform an error analysis of the fully discrete numerical method. Throughout this subsection we denote by c a positive constant that may differ from line to line. To simplify the notation, define

$$\delta_n := \delta u_n - \dot{u}_n.$$

From (2.1)–(2.2), we have, for $n = 1, 2, \ldots, N$,

$$\langle A\dot{u}_n + Bu_n + \iota^*\xi_n, v - u_n\rangle + \Phi(v) - \Phi(u_n) \ge \langle f_n, v - u_n\rangle \quad \forall\, v \in V, \tag{3.8}$$

$$\xi_n \in \partial_{Cl} J(\iota u_n). \tag{3.9}$$

From (3.8), we have

$$\left\langle A\delta u_n + Bu_n + \iota^*\xi_n, v - u_n\right\rangle + \Phi(v) - \Phi(u_n) \ge \langle f_n, v - u_n\rangle_{V^*\times V} + \langle A\delta_n, v - u_n\rangle \quad \forall\, v \in V.$$

Then,

$$\left\langle A\delta u_n + Bu_n + \iota^*\xi_n, u_n^{hk} - u_n\right\rangle + \Phi(u_n^{hk}) - \Phi(u_n) \ge \langle f_n, u_n^{hk} - u_n\rangle + \left\langle A\delta_n, u_n^{hk} - u_n\right\rangle. \tag{3.10}$$

Add (3.10) and (3.1),

$$\begin{aligned}&\left\langle A\delta u_n + Bu_n + \iota^*\xi_n, u_n^{hk} - u_n\right\rangle + \left\langle A\delta u_n^{hk} + Bu_n^{hk} + \iota^*\xi_n^{hk}, v^h - u_n^{hk}\right\rangle + \Phi(v^h) - \Phi(u_n)\\ &\quad\ge \langle f_n, v^h - u_n\rangle + \left\langle A\delta_n, u_n^{hk} - u_n\right\rangle,\end{aligned}$$

which is rewritten as

$$\begin{aligned}&\left\langle A\delta(u_n - u_n^{hk}) + B(u_n - u_n^{hk}) + \iota^*(\xi_n - \xi_n^{hk}), u_n^{hk} - u_n\right\rangle + \left\langle A\delta u_n^{hk} + Bu_n^{hk} + \iota^*\xi_n^{hk}, v^h - u_n\right\rangle + \Phi(v^h) - \Phi(u_n)\\ &\quad\ge \langle f_n, v^h - u_n\rangle + \left\langle A\delta_n, u_n^{hk} - u_n\right\rangle.\end{aligned}$$

Introduce the notation

$$e_n := u_n - u_n^{hk}.$$

Then,

$$\begin{aligned}&\frac{1}{2\,k}\left(\|e_n\|_A^2 - \|e_{n-1}\|_A^2\right) + \beta\,\|e_n\|_V^2 - m\,\|\iota e_n\|_U^2\\ &\quad\le \left\langle A\delta u_n^{hk} + Bu_n^{hk} + \iota^*\xi_n^{hk} - f_n, v^h - u_n\right\rangle + \Phi(v^h) - \Phi(u_n) - \left\langle A\delta_n, u_n^{hk} - u_n\right\rangle.\end{aligned}$$

Define a residual type quantity

$$R_n(v) := \langle A\dot{u}_n + Bu_n + \iota^*\xi_n - f_n, v - u_n\rangle + \Phi(v) - \Phi(u_n). \tag{3.11}$$

Then we have

$$\begin{aligned}&\frac{1}{2\,k}\left(\|e_n\|_A^2 - \|e_{n-1}\|_A^2\right) + \beta\,\|e_n\|_V^2 - m\,\|\iota e_n\|_U^2\\ &\quad\le \frac{1}{k}\left\langle Ae_n, u_n - v_n^h\right\rangle - \frac{1}{k}\left\langle Ae_{n-1}, u_n - v_n^h\right\rangle + \langle Be_n, u_n - v_n^h\rangle + \langle \xi_n - \xi_n^{hk}, \iota(u_n - v_n^h)\rangle\\ &\qquad + R_n(v_n^h) + \langle A\delta_n, v_n^h - u_n\rangle + \langle A\delta_n, e_n\rangle.\end{aligned} \tag{3.12}$$

Recall the assumption $H(J)$(ii). Corresponding to the solution $u \in H^1(0, T; V)$, $\{\|\xi_n\|_{U^*}\}_n$ is uniformly bounded. By Corollary 3.2, $\{\|\xi_n^{hk}\|_{U^*}\}$ is also uniformly bounded. Then,

$$\begin{aligned}&\|e_n\|_A^2 - \|e_{n-1}\|_A^2 + 2\beta k\,\|e_n\|_V^2 - 2mk\,\|\iota e_n\|_U^2\\ &\quad\le 2\left\langle Ae_n, u_n - v_n^h\right\rangle - 2\left\langle Ae_{n-1}, u_n - v_n^h\right\rangle + \varepsilon\,k\,\|e_n\|_V^2 + ck\,\|u_n - v_n^h\|_V^2\\ &\qquad + ck\,\|\iota(u_n - v_n^h)\|_U + 2k\,|R_n(v_n^h)| + ck\,\|A\delta_n\|_{V^*}^2.\end{aligned}$$

Choose $\varepsilon > 0$ sufficiently small and denote

$$c_0 := 2\left(\beta - m\,\|\iota\|^2\right) - \varepsilon > 0.$$

Then,

$$\|e_n\|_A^2 - \|e_{n-1}\|_A^2 + c_0 k\,\|e_n\|_V^2 \le 2\left\langle Ae_n, u_n - v_n^h\right\rangle - 2\left\langle Ae_{n-1}, u_n - v_n^h\right\rangle + ck\,\|u_n - v_n^h\|_V^2 + ck\,\|\iota(u_n - v_n^h)\|_U + 2k\,|R_n(v_n^h)| + ck\,\|A\delta_n\|_{V^*}^2. \tag{3.13}$$

Replace n by j in (3.13) and make a summation over $j = 1, 2, \ldots, n$:

$$\begin{aligned}\|e_n\|_A^2 + c_0 k \sum_{j=1}^{n} \|e_j\|_V^2 &\le \|e_0\|_A^2 + 2\left\langle Ae_n, u_n - v_n^h\right\rangle - 2\left\langle Ae_0, u_1 - v_1^h\right\rangle + 2\sum_{j=1}^{n-1}\langle Ae_j, (u_j - v_j^h) - (u_{j+1} - v_{j+1}^h)\rangle \\ &\quad + ck \sum_{j=1}^{n}\left(\|u_j - v_j^h\|_V^2 + \|\iota(u_j - v_j^h)\|_U + |R_j(v_j^h)| + \|A\delta_j\|_{V^*}^2\right).\end{aligned} \tag{3.14}$$

We now bound the terms on the right side of (3.14) involving the errors e_j, $0 \le j \le n$. We have

$$2\left\langle Ae_n, u_n - v_n^h\right\rangle \le \frac{1}{2}\,\|e_n\|_A^2 + c\,\|u_n - v_n^h\|_V^2,$$
$$-2\left\langle Ae_0, u_1 - v_1^h\right\rangle \le \|e_0\|_A^2 + c\,\|u_1 - v_1^h\|_V^2,$$

and

$$\begin{aligned}2\sum_{j=1}^{n-1}\langle Ae_j, (u_j - v_j^h) - (u_{j+1} - v_{j+1}^h)\rangle &\le 2k\,\|A\| \sum_{j=1}^{n-1} \|e_j\|_V \|\delta(u_{j+1} - v_{j+1}^h)\|_V \\ &\le \frac{1}{2}\,c_0 k \sum_{j=1}^{n-1}\|e_j\|_V^2 + c\,k \sum_{j=2}^{n}\|\delta(u_j - v_j^h)\|_V^2.\end{aligned}$$

Using these inequalities in (3.14), we have

$$\begin{aligned}\|e_n\|_A^2 + k\sum_{j=1}^{n}\|e_j\|_V^2 &\le c\Bigg[\|e_0\|_V^2 + \|u_n - v_n^h\|_V^2 + \|u_1 - v_1^h\|_V^2 \\ &\quad + k\sum_{j=1}^{n}\left(\|\delta(u_j - v_j^h)\|_V^2 + \|u_j - v_j^h\|_V^2 + \|\iota(u_j - v_j^h)\|_U + |R_j(v_j^h)| + \|A\delta_j\|_{V^*}^2\right)\Bigg].\end{aligned}$$

Therefore, we have the following Céa type inequality:

$$\begin{aligned}\max_{0\le n\le N}\|e_n\|_V^2 + k\sum_{n=1}^{N}\|e_n\|_V^2 &\le c\Bigg[\|e_0\|_V^2 + \max_{0\le n\le N}\|u_n - v_n^h\|_V^2 \\ &\quad + k\sum_{n=1}^{N}\left(\|\delta(u_n - v_n^h)\|_V^2 + \|\iota(u_n - v_n^h)\|_U + |R_n(v_n^h)| + \|\delta_n\|_V^2\right)\Bigg],\end{aligned} \tag{3.15}$$

for any $v_n^h \in V^h$, $0 \le n \le N$. This inequality is the starting point in deriving error estimates, as is shown in the next section.

4. Numerical analysis of a frictionless contact problem

In this section, we apply the result from the previous section to derive an optimal order error estimate for the fully discrete solution of a frictionless contact problem using linear finite elements. Modelling, analysis, and numerical methods for contact problems are topics of numerous publications, and some comprehensive references are [4,21–24].

We now introduce the contact problem for a viscoelastic body. Let Ω be a Lipschitz domain in $\mathbb{R}^d$ ($d = 2$ or 3) occupied by the body. Its boundary $\partial\Omega$ consists of three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that the measures of Γ_1 and Γ_3 are positive, and Γ_2 is allowed to be empty. The body is clamped on Γ_1, is subjected to the action of a volume force of density $\boldsymbol{f}_0$ in Ω and the action of a surface traction of density $\boldsymbol{f}_2$ on Γ_2. The functions $\boldsymbol{f}_0$ and $\boldsymbol{f}_2$ are allowed to be time-dependent. The body is in frictionless contact on Γ_3 with an obstacle, the so-called foundation. The foundation is made of a perfectly rigid material, covered by layer of deformable material of thickness $g > 0$. Therefore, the contact is modelled with a normal compliance unilateral condition. The process is quasistatic and the time interval of interest is $[0, T]$ for some constant $T > 0$.

Let $\mathbb{S}^d$ denote the space of second order symmetric tensors on $\mathbb{R}^d$. The canonical inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i, \quad \|\boldsymbol{v}\|_{\mathbb{R}^d} = (\boldsymbol{v}\cdot\boldsymbol{v})^{1/2} \quad \forall\,\boldsymbol{u} = (u_i),\ \boldsymbol{v} = (v_i) \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\tau}\|_{\mathbb{S}^d} = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2} \quad \forall\,\boldsymbol{\sigma} = (\sigma_{ij}),\ \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d,$$

respectively. Here, the indices i and j run between 1 and d and the summation convention over repeated indices is used. We use the notation $\boldsymbol{u} = (u_i) \in \mathbb{R}^d$, $\boldsymbol{\sigma} = (\sigma_{ij}) \in \mathbb{S}^d$ and $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})) \in \mathbb{S}^d$ for the displacement vector, the stress tensor, and the linearized strain tensor, respectively. Note that $\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$ where an index following a comma indicates a partial derivative with respect to the corresponding component of the spatial variable $\boldsymbol{x} = (x_i)$. The unit outward normal vector exists a.e. on $\partial\Omega$ and is denoted by $\boldsymbol{\nu} = (\nu_i) \in \mathbb{R}^d$. For a vector field $\boldsymbol{v}$ on $\partial\Omega$, the normal and tangential components of $\boldsymbol{v}$ are $v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}$. Similarly, for a stress field $\boldsymbol{\sigma}$, the normal and tangential components are $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$. We denote by $\boldsymbol{0}$ the zero element of $\mathbb{R}^d$.

Then the classical formulation of the contact problem is the following.

Problem ($\mathcal{P}_\mathcal{M}$). *Find a displacement field* $\boldsymbol{u} : \Omega\times[0, T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega\times[0, T] \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}) \qquad \text{in } \Omega\times(0, T), \tag{4.1}$$
$$\mathrm{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \qquad \text{in } \Omega\times(0, T), \tag{4.2}$$
$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma_1\times(0, T), \tag{4.3}$$
$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \qquad \text{on } \Gamma_2\times(0, T), \tag{4.4}$$
$$\left.\begin{array}{l} \sigma_\nu = \sigma_\nu^1 + \sigma_\nu^2, \\ -\sigma_\nu^1 \in \partial_{Cl} j(u_\nu), \\ u_\nu \le g, \quad \sigma_\nu^2 \le 0, \quad \sigma_\nu^2(u_\nu - g) = 0 \end{array}\right\} \qquad \text{on } \Gamma_3\times(0, T) \tag{4.5}$$
$$\boldsymbol{\sigma}_\tau = \boldsymbol{0} \qquad \text{on } \Gamma_3\times(0, T), \tag{4.6}$$
$$\boldsymbol{u}(0) = \boldsymbol{0} \qquad \text{in } \Omega. \tag{4.7}$$

We comment that Eq. (4.1) is known as the Kelvin–Voigt viscoelastic constitutive law, commonly used to model the deformation behaviour of certain metals, rubbers and polymers. Eq. (4.2) is the equilibrium equation for the quasistatic process. Conditions (4.3) and (4.4) are the ordinary displacement and traction boundary condition. The contact is frictionless and it is represented by (4.6). The initial condition is (4.7). In the contact condition (4.5), the normal stress σ_ν on the contact surface is split into two parts, σ_ν^1 and σ_ν^2. The first part σ_ν^1 describes the deformability of the obstacle with a normal compliance condition, governed by the subdifferential of a nonconvex potential j. The second part σ_ν^2 describes the rigidity of the obstacle with the Signorini unilateral contact condition. The condition (4.5) is used to model the contact of the body with a foundation made of a rigid body covered by a layer of elastic material. Note that penetration is allowed but is restricted by the relation $u_\nu \le g$, where g represents the thickness of the elastic layer. When there is penetration, as long as the normal displacement does not reach the bound g, the contact is described with a nonmonotone normal compliance condition $-\sigma_\nu \in \partial_{Cl} j(u_\nu)$. Due to the nonmonotonicity of $\partial_{Cl} j$, the condition can be used to describe the hardening or the softening phenomena of the foundation. Examples and mechanical interpretation associated with the nonmonotone normal compliance condition can be found in [4].

In the study of Problem ($\mathcal{P}_\mathcal{M}$), we need some standard Lebesgue and Sobolev spaces. For $\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d)$, we denote by $\gamma\boldsymbol{v}$ its trace of $\boldsymbol{v}$ on $\partial\Omega$. We introduce spaces

$$V = \{\boldsymbol{v} = (v_i) \in H^1(\Omega;\mathbb{R}^d) \mid \gamma\boldsymbol{v} = \boldsymbol{0} \ \text{ a.e. on } \Gamma_1\},$$
$$\mathcal{H} = \{\boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega;\mathbb{R}^{d\times d}) \mid \tau_{ij} = \tau_{ji},\ 1 \le i, j \le d\}.$$

The space $\mathcal{H}$ is a real Hilbert space with the canonical inner product given by

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_{\Omega} \sigma_{ij}(\boldsymbol{x})\, \tau_{ij}(\boldsymbol{x})\, dx \quad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H}$$

and the associated norm $\|\cdot\|_{\mathcal{H}}$. Since Γ_1 has a positive measure, it is known that V is a real Hilbert space with the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V \tag{4.8}$$

and the associated norm $\|\cdot\|_V$. The duality pairing between V and V^* is denoted by $\langle \cdot, \cdot \rangle$. Let $Z = H^{1-\varepsilon}(\Omega; \mathbb{R}^d)$ for some $\varepsilon \in (0, \frac{1}{2})$. Define the embedding $i_1 : V \to Z$, the trace operator $\gamma_1 : Z \to H^{\frac{1}{2}-\varepsilon}(\Gamma_3; \mathbb{R}^d)$, and the embedding $i_2 : H^{\frac{1}{2}-\varepsilon}(\Gamma_3; \mathbb{R}^d) \to L^2(\Gamma_3; \mathbb{R}^d)$. Consider the trace operator $\gamma = i_2 \circ \gamma_1 \circ i_1 : V \to L^2(\Gamma_3; \mathbb{R}^d)$. By the Sobolev trace theorem,

$$\|\gamma \boldsymbol{v}\|_{L^2(\Gamma_3;\mathbb{R}^d)} \le c_0 \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V \tag{4.9}$$

for a constant c_0 depending only on the domain Ω, Γ_1 and Γ_3. Let $U = L^2(\Gamma_3)$ and define operators $\nu : L^2(\Gamma_3; \mathbb{R}^d) \to U$, $\nu \boldsymbol{v} = v_\nu$ for $\boldsymbol{v} \in L^2(\Gamma_3; \mathbb{R}^d)$, and $\iota = \nu \circ \gamma : V \to U$. The spaces $\mathcal{V}, \mathcal{U}$ and $\mathcal{W}$ are as defined at the beginning of Section 2, with the spaces V and U introduced here.

On the data of Problem $\mathcal{P}_{\mathcal{M}}$, we assume the following.

$H(\mathcal{C})$. For the viscosity tensor $\mathcal{C} = (\mathcal{C}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$,

$$\begin{aligned}
&\mathcal{C}_{ijkl} \in L^\infty(\Omega), \quad 1 \le i, j, k, l \le d;\\
&\mathcal{C}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathcal{C}\boldsymbol{\tau} \quad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. in } \Omega;\\
&\mathcal{C}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \ge \alpha \|\boldsymbol{\tau}\|^2_{\mathbb{S}^d} \quad \forall\, \boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. in } \Omega, \text{ with } \alpha > 0.
\end{aligned}$$

$H(\mathcal{G})$. For the elasticity tensor $\mathcal{G} = (\mathcal{G}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$,

$$\begin{aligned}
&\mathcal{G}_{ijkl} \in L^\infty(\Omega), \quad 1 \le i, j, k, l \le d;\\
&\mathcal{G}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathcal{G}\boldsymbol{\tau} \quad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. in } \Omega;\\
&\mathcal{G}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \ge \beta \|\boldsymbol{\tau}\|^2_{\mathbb{S}^d} \quad \forall\, \boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. in } \Omega, \text{ with } \beta > 0.
\end{aligned}$$

$H(j)$. For the normal compliance function $j : \mathbb{R} \to \mathbb{R}$,

$$\begin{aligned}
&j \text{ is locally Lipschitz;}\\
&|\eta| \le c_1(1 + |s|)\, \forall\, \eta \in \partial_{Cl} j(s),\ s \in \mathbb{R} \text{ with } d > 0;\\
&(\eta_1 - \eta_2)(s_1 - s_2) \ge -c_2 |s_1 - s_2|^2\, \forall\, \eta_i \in \partial_{Cl} j(s_i),\ s_i \in \mathbb{R},\ i = 1, 2, \text{ with } c_2 > 0.
\end{aligned}$$

$H(f)$. For the densities of forces and traction,

$$\begin{aligned}
&\boldsymbol{f}_0 \in H^1(0, T; L^2(\Omega; \mathbb{R}^d)), \quad \boldsymbol{f}_2 \in H^1(0, T; L^2(\Gamma_2; \mathbb{R}^d));\\
&\boldsymbol{f}_0(0) \in V, \quad \boldsymbol{f}_2(0) = \boldsymbol{0}.
\end{aligned}$$

We define operators $A, B : V \to V^*$ by

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \quad \langle B\boldsymbol{u}, \boldsymbol{v}\rangle = (\mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V,$$

and functions $J : L^2(\Gamma_3) \to \mathbb{R}$ and $\boldsymbol{f} : (0, T) \to V^*$ by

$$\begin{aligned}
&J(w) = \int_{\Gamma_3} j(w)\, d\Gamma, \quad \forall\, w \in L^2(\Gamma_3),\\
&\langle \boldsymbol{f}(t), \boldsymbol{v}\rangle = \int_{\Omega} \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\, d\Gamma \quad \forall\, \boldsymbol{v} \in V,\ t \in [0, T].
\end{aligned}$$

Let $\Phi : V \to \mathbb{R} \cup \{+\infty\}$ be the indicator function of the set

$$K = \{\boldsymbol{v} \in V \mid v_\nu \le g \text{ a.e. on } \Gamma_3\}.$$

Then the weak formulation of Problem $\mathcal{P}_{\mathcal{M}}$ in terms of the displacement is the following.

PROBLEM ($\mathcal{P}_{\mathcal{M}}{}^{V}$). *Find a displacement field* $\boldsymbol{u} \in \mathcal{W}$ *such that* $\boldsymbol{u}(0) = \boldsymbol{0}$ *and for a.e.* $t \in (0, T)$,

$$\langle A\dot{\boldsymbol{u}}(t) + B\boldsymbol{u}(t) + \iota^*\xi(t) - \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle + \Phi(\boldsymbol{v}) - \Phi(\boldsymbol{u}(t)) \geq 0 \quad \forall\, \boldsymbol{v} \in V,$$
$$\xi(t) \in \partial_{Cl} J(\iota \boldsymbol{u}(t)).$$

It follows from the definition of function Φ that Problem $\mathcal{P}_{\mathcal{M}}{}^{V}$ reduces to the following.
PROBLEM ($\mathcal{P}_{\mathcal{M}}{}^{V,*}$). *Find a displacement field* $\boldsymbol{u} \in \mathcal{W}$ *such that* $\boldsymbol{u}(0) = \boldsymbol{0}$ *and for a.e.* $t \in (0, T)$, $\boldsymbol{u}(t) \in K$ *and*

$$\langle A\dot{\boldsymbol{u}}(t) + B\boldsymbol{u}(t) + \iota^*\xi(t) - \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle \geq 0 \quad \forall\, \boldsymbol{v} \in K,$$
$$\xi(t) \in \partial_{Cl} J(\iota \boldsymbol{u}(t)).$$

Problem ($\mathcal{P}_{\mathcal{M}}{}^{V}$) is in the form of Problem (P) discussed in Section 2. It can be verified without difficulty that under the assumptions $H(\mathcal{C})$, $H(\mathcal{G})$, $H(j)$, $H(f)$, and $\beta > c_2\|\iota\|^2$, the conditions $H(A)$, $H(B)$, $H(J)$, $H(\Phi)$, $H(\iota)$, and $H(s)$ of Section 2 are valid (see also Remark 2.1). Therefore, results on Problem (P) and its integral form can be applied. Thus, we assume Problem ($\mathcal{P}_{\mathcal{M}}{}^{V}$) admits a unique solution $\boldsymbol{u} \in H^1(0, T; V)$ and focus on numerical analysis of the contact problem in this section.

As in Section 3, for a positive integer N, let $k = N/T$ be the time stepsize. For simplicity, we assume Ω is a polygonal/polyhedral domain. Then

$$\overline{\Gamma_j} = \cup_{i=1}^{i_j} \Gamma_{j,i}, \quad 1 \leq j \leq 3$$

where $\Gamma_{j,i}$, $1 \leq i \leq i_j$, $1 \leq j \leq 3$, are closed flat components of the boundary and have pairwise disjoint interiors. Consider a regular family of meshes $\{\mathcal{T}^h\}$ that partition $\overline{\Omega}$ into triangles/tetrahedrons, compatible with the splitting of the boundary $\partial\Omega$ into $\Gamma_{j,i}$, $1 \leq i \leq i_j$, $1 \leq j \leq 3$, in the sense that if the intersection of one side/face of an element with one set $\Gamma_{j,i}$ has a positive measure with respect to $\Gamma_{j,i}$, then the side/face lies entirely in $\Gamma_{j,i}$. Corresponding to $\mathcal{T}^h$, we define the linear element space

$$V^h = \left\{ \boldsymbol{v}^h \in C(\overline{\Omega})^d | \boldsymbol{v}^h|_T \in \mathbb{P}_1(T)^d,\ T \in \mathcal{T}^h,\ \boldsymbol{v}^h = \boldsymbol{0} \text{ on } \Gamma_1 \right\}.$$

Then a fully discrete approximation method for Problem $\mathcal{P}_{\mathcal{M}}{}^{V}$ is the following.
PROBLEM ($\mathcal{P}_{\mathcal{M}}{}^{V,hk}$). Find $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset V^h$ such that $\boldsymbol{u}_0^{hk} = \boldsymbol{0}$ and for $n = 1, 2, \ldots, N$,

$$\left\langle A\delta\boldsymbol{u}_n^{hk} + B\boldsymbol{u}_n^{hk} + \iota^*\xi_n^{hk}, \boldsymbol{v}^h - \boldsymbol{u}_n^{hk}\right\rangle + \Phi(\boldsymbol{v}^h) - \Phi(\boldsymbol{u}_n^{hk}) \geq \langle \boldsymbol{f}_n, \boldsymbol{v}^h - \boldsymbol{u}_n^{hk}\rangle \quad \forall\, \boldsymbol{v}^h \in V^h, \tag{4.10}$$
$$\xi_n^{hk} \in \partial_{Cl} J(\iota \boldsymbol{u}_n^{hk}). \tag{4.11}$$

In the following, we assume g is concave. We can eliminate the explicit appearance of the function Φ in the formulation by introducing the finite element set

$$U^h = \left\{ \boldsymbol{v}^h \in V^h \mid v_\nu^h \leq g \text{ at the nodes on } \Gamma_3 \right\}.$$

Since g is a concave function, we have the inclusion

$$U^h \subset K.$$

Thus, an equivalent form of Problem $\mathcal{P}_{\mathcal{M}}{}^{V,hk}$ is:
PROBLEM ($\mathcal{P}_{\mathcal{M}}{}^{V,hk,*}$). Find $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset U^h$ such that $\boldsymbol{u}_0^{hk} = \boldsymbol{u}_0^h$ and for $n = 1, 2, \ldots, N$,

$$\left\langle A\delta\boldsymbol{u}_n^{hk} + B\boldsymbol{u}_n^{hk} + \iota^*\xi_n^{hk}, \boldsymbol{v}^h - \boldsymbol{u}_n^{hk}\right\rangle \geq \langle \boldsymbol{f}_n, \boldsymbol{v}^h - \boldsymbol{u}_n^{hk}\rangle \quad \forall\, \boldsymbol{v}^h \in U^h, \tag{4.12}$$
$$\xi_n^{hk} \in \partial_{Cl} J(\iota \boldsymbol{u}_n^{hk}). \tag{4.13}$$

For an error analysis, assume the solution regularity

$$\boldsymbol{u} \in H^1(0, T; H^2(\Omega)), \quad \ddot{\boldsymbol{u}} \in L^2(0, T; V), \tag{4.14}$$
$$u_\nu|_{\Gamma_{3,i}} \in C(0, T; H^2(\Gamma_{3,i})),\ \sigma_\nu|_{\Gamma_{3,i}} \in C(0, T; L^2(\Gamma_{3,i})),\ 1 \leq i \leq i_3. \tag{4.15}$$

We will apply standard finite element interpolation error estimates (cf. [25–27]).

The regularity (4.14) implies that $\boldsymbol{u}(t,\boldsymbol{x})$ is a continuous function of t and $\boldsymbol{x}$. Thus, the pointwise values of $\boldsymbol{u}$ are well defined. Choose $\boldsymbol{v}_n^h = \Pi^h \boldsymbol{u}_n \in V^h$ to be the finite element interpolant of $\boldsymbol{u}_n(\boldsymbol{x}) := \boldsymbol{u}(t_n,\boldsymbol{x})$. We apply (3.15) and

$$\begin{aligned}
&\max_{0\le n\le N} \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 + k\sum_{n=1}^N \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 \\
&\quad \le c\Bigg[\max_{0\le n\le N} \|\boldsymbol{u}_n - \Pi^h\boldsymbol{u}_n\|_V^2 \\
&\qquad + k\sum_{n=1}^N \Big(\|\delta(\boldsymbol{u}_n - \Pi^h\boldsymbol{u}_n)\|_V^2 + \|u_{n,\nu} - \Pi^h u_{n,\nu}\|_{L^2(\Gamma_3)} + |R_n(\Pi^h\boldsymbol{u}_n)| + \|\boldsymbol{\delta}_n\|_V^2\Big)\Bigg], \qquad (4.16)
\end{aligned}$$

where

$$\boldsymbol{\delta}_n := \delta\boldsymbol{u}_n - \dot{\boldsymbol{u}}_n, \qquad (4.17)$$

$$R_n(\boldsymbol{v}) := \langle A\dot{\boldsymbol{u}}_n + B\boldsymbol{u}_n + \iota^*\xi_n - \boldsymbol{f}_n, \boldsymbol{v} - \boldsymbol{u}_n\rangle + \Phi(\boldsymbol{v}) - \Phi(\boldsymbol{u}_n), \qquad (4.18)$$

$$\xi_n \in \partial_{Cl} J(\iota\boldsymbol{u}_n). \qquad (4.19)$$

From [28, Lemma 11.5], we have

$$\|\boldsymbol{\delta}_n\|_V \le \|\ddot{\boldsymbol{u}}\|_{L^1(t_{n-1},t_n;V)}.$$

Thus,

$$\|\boldsymbol{\delta}_n\|_V^2 \le k\,\|\ddot{\boldsymbol{u}}\|_{L^2(t_{n-1},t_n;V)}^2$$

and

$$k\sum_{n=1}^N \|\boldsymbol{\delta}_n\|_V^2 \le k^2 \|\ddot{\boldsymbol{u}}\|_{L^2(0,T;V)}^2. \qquad (4.20)$$

Write

$$\delta(\boldsymbol{u}_n - \Pi^h\boldsymbol{u}_n) = \frac{1}{k}\int_{t_{n-1}}^{t_n} \big(\dot{\boldsymbol{u}}(t) - \Pi^h\dot{\boldsymbol{u}}(t)\big)\,dt.$$

Then

$$\|\delta(\boldsymbol{u}_n - \Pi^h\boldsymbol{u}_n)\|_V^2 \le \frac{1}{k}\int_{t_{n-1}}^{t_n} \big\|\dot{\boldsymbol{u}}(t) - \Pi^h\dot{\boldsymbol{u}}(t)\big\|^2\,dt,$$

and

$$k\sum_{n=1}^N \|\delta(\boldsymbol{u}_n - \Pi^h\boldsymbol{u}_n)\|_V^2 \le \int_0^T \big\|\dot{\boldsymbol{u}}(t) - \Pi^h\dot{\boldsymbol{u}}(t)\big\|^2\,dt.$$

Therefore,

$$k\sum_{n=1}^N \|\delta(\boldsymbol{u}_n - \Pi^h\boldsymbol{u}_n)\|_V^2 \le c\,h^2 \|\dot{\boldsymbol{u}}\|_{L^2(0,T;H^2(\Omega))}^2. \qquad (4.21)$$

Note that on each component $\Gamma_{3,i}$, $\Pi^h u_{n,\nu}$ is the finite element interpolant of $u_{n,\nu}$. By the last part of the regularity assumption (4.14), we have

$$k\sum_{n=1}^N \|u_{n,\nu} - \Pi^h u_{n,\nu}\|_{L^2(\Gamma_3)} \le c\,h^2 \sum_{i=1}^{i_3} \|u_\nu\|_{L^\infty(0,T;H^2(\Gamma_{3,i}))}. \qquad (4.22)$$

We now bound $|R_n(\Pi^h\boldsymbol{u}_n)|$. Using the relation

$$\langle A\dot{\boldsymbol{u}}(t) + B\boldsymbol{u}(t) - \boldsymbol{f}(t), \boldsymbol{v}\rangle = \int_{\Gamma_3} \sigma_\nu(t) v_\nu\,d\Gamma \quad \forall\,\boldsymbol{v} \in V,$$

we have

$$R_n(\Pi^h \boldsymbol{u}_n) = \int_{\Gamma_3} \left(\sigma_{n,\nu} + \xi_n\right) \left(\Pi^h u_{n,\nu} - u_{n,\nu}\right) d\Gamma,$$

where $\sigma_{n,\nu} := \sigma_\nu(t_n)$. Then,

$$|R_n(\Pi^h \boldsymbol{u}_n)| \le c \, \|\Pi^h u_{n,\nu} - u_{n,\nu}\|_{L^2(\Gamma_3)},$$

and

$$k \sum_{n=1}^{N} |R_n(\Pi^h \boldsymbol{u}_n)| \le c\, h^2 \sum_{i=1}^{i_3} \|u_\nu\|_{L^\infty(0,T;H^2(\Gamma_{3,i}))}. \tag{4.23}$$

Using (4.20)–(4.23) in (4.16), we obtain the following result concerning optimal error estimate for the fully discrete scheme.

Corollary 4.1. *Let* $\boldsymbol{u}$ *and* $\boldsymbol{u}^{hk}$ *be solutions of Problem* $(\mathcal{P}_{\mathcal{M}}{}^V)$ *and Problem* $(\mathcal{P}_{\mathcal{M}}{}^{V,hk})$ *respectively. Assume the solution regularity* (4.14)– (4.15)*. Then*

$$\max_{0\le n\le N} \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 + k \sum_{n=1}^{N} \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 \le c\left(k^2 + h^2\right), \tag{4.24}$$

with a positive constant c *independent of* k *and* h*.*

5. Numerical simulations

This section provides computer simulation results on the contact Problem $(\mathcal{P}_{\mathcal{M}}{}^{V,hk,*})$, including numerical evidence of the theoretical error estimates obtained in Section 4 for the fully discrete approximation.

The solution of Problem $(\mathcal{P}_{\mathcal{M}}{}^{V,hk,*})$ is based on numerical methods described in [29,30]. Note that hemivariational inequalities arising in contact mechanics are related to the solution of non-convex problems. A numerical technique to solve this kind of problems is to use a “convexification” iterative procedure which leads to a sequence of convex programming problems. Then, the resulting nonsmooth convex iterative problems are solved by classical numerical methods that can be found for instance in [23,24].

Numerical example. The physical setting used for Problem $(\mathcal{P}_{\mathcal{M}}{}^{V,hk,*})$ is depicted in Fig. 1. The deformable body is a rectangle, $\Omega = (0, 2) \times (0, 1) \subset \mathbb{R}^2$, and its boundary Γ is split as follows:

$$\Gamma_1 = (\{0\} \times [0.5, 1]) \cup (\{2\} \times [0.5, 1]),$$
$$\Gamma_2 = ((0, 2) \times \{1\}) \cup (\{0\} \times (0, 0.5)) \cup (\{2\} \times (0, 0.5)),$$
$$\Gamma_3 = [0, 2] \times \{0\}.$$

The domain Ω represents the cross section of a three-dimensional linearly viscoelastic body subjected to the action of tractions in such a way that a plane stress hypothesis is assumed. On the part Γ_1 the body is clamped and, therefore, the displacement field vanishes there. Vertical compressions act on the part $(0, 2) \times \{1\}$ of the boundary Γ_2 and the part $(\{0\} \times (0, 0.5)) \cup (\{2\} \times (0, 0.5))$ is traction free. Constant vertical body forces are assumed to act on the viscoelastic body. The body is in frictionless contact with an obstacle on the part Γ_3 of the boundary.

The compressible material’s behaviour of the domain Ω is governed by a Kelvin–Voigt viscoelastic linear constitutive law of the form (4.1). In addition, we assume that the material is homogeneous and isotropic; then, the elasticity tensor $\mathcal{G}$ and the viscosity tensor $\mathcal{C}$ have the following forms

$$(\mathcal{G}\boldsymbol{\tau})_{ij} = \frac{E\kappa}{(1+\kappa)(1-2\kappa)}(\tau_{ii})\delta_{ij} + \frac{E}{1+\kappa}\tau_{ij}, \quad 1 \le i, j \le 2, \ \forall\, \boldsymbol{\tau} \in \mathbb{S}^2,$$
$$(\mathcal{C}\boldsymbol{\tau})_{ij} = \alpha(\tau_{ii})\delta_{ij} + \beta\tau_{ij}, \quad 1 \le i, j \le 2, \ \forall\, \boldsymbol{\tau} \in \mathbb{S}^2,$$

where the coefficients E and κ are the Young’s modulus and the Poisson’s ratio of the material, respectively, and α and β are the viscosity parameters. δ_{ij} denotes the Kronecker symbol.

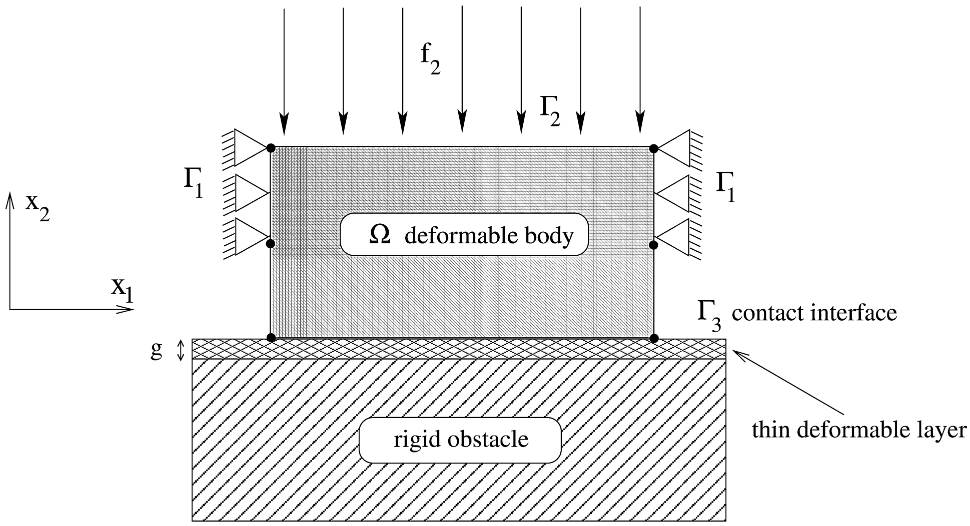

Fig. 1. Reference configuration of the two-dimensional example.

For the numerical simulation of Problem $(\mathcal{P}_{\mathcal{M}}{}^{V,hk,*})$, the following data are used:

$$E = 1000\ \mathrm{N/m^2},\quad \kappa = 0.4,\ \alpha = 100\ \mathrm{N/m^2},\ \beta = 50\ \mathrm{N/m^2},$$
$$\boldsymbol{f}_0 = (0, -0.1\times 10^{-3})\ \mathrm{N/m^2},$$
$$\boldsymbol{f}_2 = (-1.5\times 10^{-3}, 0)\ \mathrm{N/m}\ \text{on}\ (0,2)\times\{1\},$$
$$T = 1\ \mathrm{s},\ h = 1/128,\ k = T/N\ \text{with}\ N = 128.$$

The numerical results are presented in Figs. 2–5 and are described in what follows.

Numerical solution of Problem $(\mathcal{P}_{\mathcal{M}}{}^{V,hk,*})$. In Figs. 3–4, the deformed configurations as well as the interface forces on Γ_3 are plotted. In this case, the contact boundary conditions on Γ_3 are characterized by a frictionless multivalued normal compliance contact in which the response follows a nonmonotone law with respect to the normal displacement u_ν and for which the maximal penetration is restricted by a unilateral constraint as follows:

$$-\sigma_\nu^1 = \begin{cases} 0 & \text{if } u_\nu \le 0,\\ c_\nu^1 u_\nu & \text{if } u_\nu \in (0, u_\nu^1],\\ c_\nu^1 u_\nu^1 + c_\nu^2(u_\nu - u_\nu^1) & \text{if } u_\nu \in (u_\nu^1, u_\nu^2),\\ c_\nu^1 u_\nu^1 + c_\nu^2(u_\nu^2 - u_\nu^1) + c_\nu^3(u_\nu - u_\nu^2) & \text{if } u_\nu \ge u_\nu^2, \end{cases} \tag{5.1}$$

$$u_\nu \le g,\ \sigma_\nu^2 \le 0,\ (u_\nu - g)\sigma_\nu^2 = 0, \tag{5.2}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0} \tag{5.3}$$

with $c_\nu^1 = 200\ \mathrm{N/m^2}$, $c_\nu^2 = -100\ \mathrm{N/m^2}$, $c_\nu^3 = 300\ \mathrm{N/m^2}$, $u_\nu^1 = 0.05$ m and $u_\nu^2 = 0.075$ m and $g = -0.1$ m. Note that in the conditions (5.2), g represents the maximum value of the allowed penetration. When this value of penetration is reached, the contact follows a unilateral condition without any additional penetration. For the conditions (5.1), we use a multivalued normal compliance response in which the non-monotonic behaviour of $-\sigma_\nu^1$ is characterized, respectively, by an increasing, a decreasing and again an increasing with respect to the normal displacement u_ν. In order to better appreciate the non-monotonic character of the normal response, we show in Fig. 2 the dependence of $-\sigma_\nu$ as a function of the normal displacement u_ν related to the relations (5.1) and (5.2).

In Fig. 3, we plotted the deformed mesh as well as the interface forces on Γ_3. On the extremities of the boundary Γ_3, we can see the non-monotonic behaviour of the normal compliance response $-\sigma_\nu$ with respect to the penetration. On the centre of Γ_3, the nodes are in unilateral contact status since the penetration reached the maximum value g.

In Fig. 4, the deformed mesh as well as the interface forces on Γ_3 is plotted for various values of the maximal penetration g. It is obvious to observe that the number of nodes in unilateral contact status increases with the reduction of the thickness of deformable layer of value g.

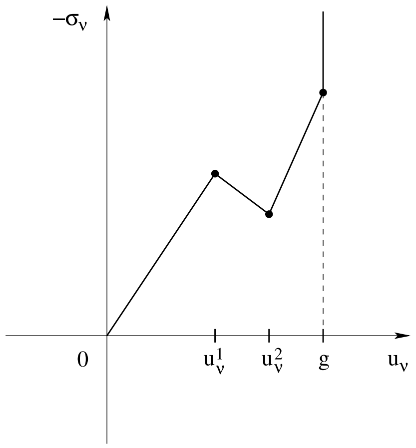

Fig. 2. Dependence of $-\sigma_\nu$ on u_ν.

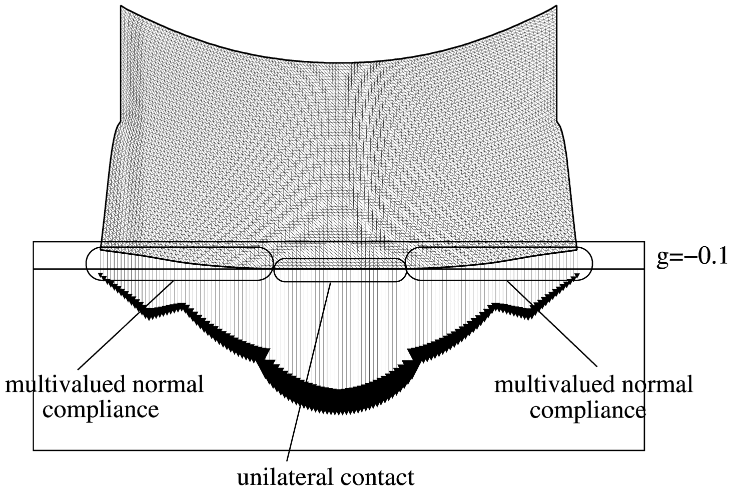

Fig. 3. Deformed mesh and interface force on Γ_3.

The details concerning the computation for the numerical simulation related to the solution of Problem $(\mathcal{P}_{\mathcal{M}}{}^{V,hk,*})$ with $g = -0.1$ m are the following. For instance, in Fig. 3, the problem was discretized in 16 513 finite elements with 128 contact elements ($h = 1/128$) and 128 time steps ($k = 1/128$); the total number of degrees of freedom was equal to 17 028. For information, the average iterations number of the "convexification" procedure for the solution of Problem $(\mathcal{P}_{\mathcal{M}}{}^{V,hk,*})$ was equal to 4 and the simulation runs in 665 (expressed in seconds) CPU time on a IBM computer equipped with Intel Dual core processors (Model 5148, 2.33 GHz).

Errors and numerical convergence orders. The aim of this part is to illustrate the convergence of the discrete scheme and to provide numerical evidence of the optimal error estimate obtained in Section 4. To this end, we computed a sequence of numerical solutions by using uniform discretization of Problem $(\mathcal{P}_{\mathcal{M}}{}^{V,hk,*})$ according to the spatial discretization parameter h and the time step k, respectively. For instance, for $h = 1/128$ and $k = 1/128$, we obtained the deformed configuration and the interface forces plotted in Fig. 3.

The numerical estimations of $\|\boldsymbol{u} - \boldsymbol{u}^{hk}\|_V$ are computed by using the energy norm $\|\cdot\|_E$ for several discretization parameters of h and k. The energy norm is defined by the formula

$$\|\boldsymbol{v}^{hk}\|_E := \frac{1}{\sqrt{2}}(\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{v}^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{v}^{hk}))_{\mathcal{H}}^{1/2}.$$

Since it is not possible to calculate the exact solution $\boldsymbol{u}$ analytically, we consider a "reference" solution $\boldsymbol{u}_{\text{ref}}$ corresponding to a fine approximation of Problem $(\mathcal{P}_{\mathcal{M}}{}^{V,hk,*})$. For this procedure, the boundary Γ of Ω is divided into

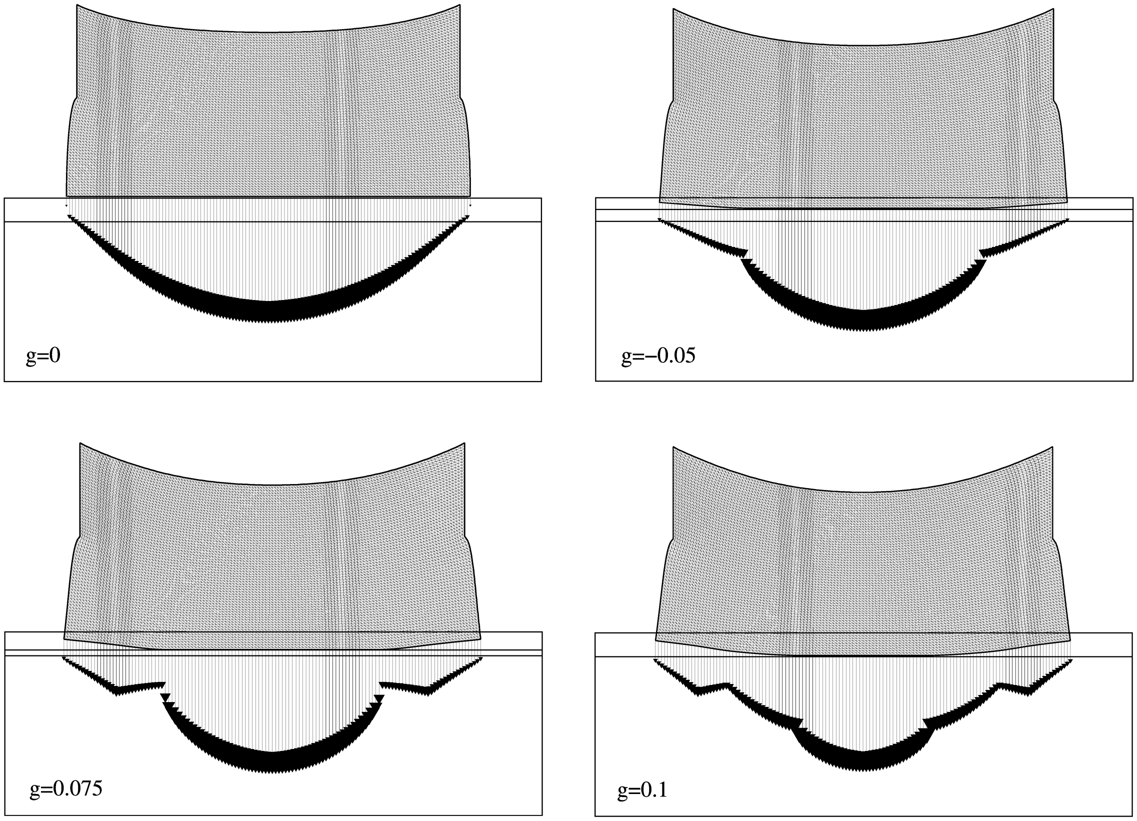

Fig. 4. Deformed meshes and interface forces on Γ_3 for various values of g.

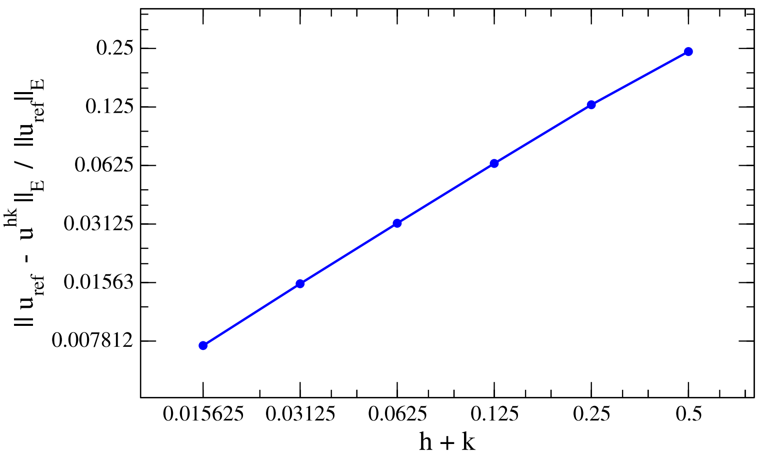

Fig. 5. Numerical errors.

$1/h$ equal parts and the time interval $[0, T]$ is divided into $1/k$ time steps. We start with $h = 1/4$ and $k = 1/4$ which are successively halved. The numerical solution $\boldsymbol{u}_{ref}$ corresponding to $h = 1/512$ and $k = 1/512$ was taken as the "reference" solution. This fine discretization corresponds to a problem with 264 508 degrees of freedom and 262 657 finite elements; the simulation runs in 93 878 (expressed in seconds) CPU time. The numerical results are presented in Fig. 5 and in Table 1 where the dependence of the relative error estimates $\|\boldsymbol{u}_{\text{ref}} - \boldsymbol{u}^h\|_E / \|\boldsymbol{u}_{\text{ref}}\|_E$ with respect to h

Table 1
Relative errors in energy norm.

$h+k$	1/2	1/4	1/8	1/16	1/32	1/64
Error	24.099%	12.835%	6.407%	3.153%	1.540%	0.741%

and k are plotted. Note that these results provide a good numerical evidence of the theoretically predicted first order convergence of the numerical solution measured in the energy norm.

References

[1] G. Duvaut, J.-L. Lions, Inequalities in Mechanics and Physics, Springer-Verlag, Berlin, 1976.
[2] J.-L. Lions, G. Stampacchia, Variational inequalities, Commun. Pure Appl. Anal. 20 (1967) 493–519.
[3] W. Han, S. Miǵorski, M. Sofonea (Eds.), Advances in Variational and Hemivariational Inequalities. Theory, Numerical Analysis and Applications, in: Advances in Mechanics and Mathematics, vol. 33, Springer, New York, 2015.
[4] S. Migórski, A. Ochal, M. Sofonea, Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems, in: Advances in Mechanics and Mathematics, vol. 26, Springer, New York, 2013.
[5] Z. Naniewicz, P.D. Panagiotopoulos, Mathematical Theory of Hemivariational Inequalities and Applications, Dekker, New York, 1995.
[6] P.D. Panagiotopoulos, Nonconvex energy functions, hemivariational inequalities and substationary principles, Acta Mech. 42 (1983) 160–183.
[7] P.D. Panagiotopoulos, Hemivariational Inequalities, Applications in Mechanics and Engineering, SpringerVerlag, Berlin, 1993.
[8] R. Glowinski, Numerical Methods for Nonlinear Variational Problems, Springer-Verlag, New York, 1984.
[9] R. Glowinski, J.-L. Lions, R. Trémolières, Numerical Analysis of Variational Inequalities, North-Holland, Amsterdam, 1981.
[10] I. Hlaváček, J. Haslinger, J. Necǎs, J. Lovíšek, Solution of Variational Inequalities in Mechanics, Springer-Verlag, New York, 1988.
[11] J. Haslinger, M. Miettinen, P.D. Panagiotopoulos, Finite Element Method for Hemivariational Inequalities: Theory, Methods and Applications, Kluwer Academic Publishers, Dordrecht, Boston, London, 1999.
[12] W. Han, S. Migórski, M. Sofonea, A class of variational-hemivariational inequalities with applications to frictional contact problems, SIAM J. Math. Anal. 46 (2014) 3891–3912.
[13] M. Barboteu, K. Bartosz, W. Han, T. Janiczko, Numerical analysis of a hyperbolic hemivariational inequality arising in dynamic contact, SIAM J. Numer. Anal. 53 (2015) 527–550.
[14] W. Han, M. Sofonea, M. Barboteu, Numerical analysis of elliptic hemivariational inequalities, SIAM J. Numer. Anal. (2017) in press.
[15] S. Migórski, A. Ochal, M. Sofonea, History-dependent variational-hemivariational inequalities in contact mechanics, Nonlinear Anal. RWA 22 (2015) 604–618.
[16] M. Sofonea, W. Han, S. Migorski, Numerical analysis of history-dependent variational-hemivariational inequalities with applications to contact problems, European J. Appl. Math. 26 (2015) 427–452.
[17] K. Bartosz, M. Sofonea, The Rothe method for variational-hemivariational inequalities with applications to contact mechanics, SIAM J. Math. Anal. 48 (2016) 861–883.
[18] F.H. Clarke, Optimization and Nonsmooth Analysis, Wiley, Interscience, New York, 1983.
[19] Z. Denkowski, S. Migórski, N.S. Papageorgiou, An Introduction to Nonlinear Analysis: Theory, Kluwer Academic, Plenum Publishers, Boston, Dordrecht, London, New York, 2003.
[20] S. Migórski, A. Ochal, M. Sofonea, A class of variational-hemivariational inequalities in reflexive Banach spaces, J. Elasticity (2017). http://dx.doi.org/10.1007/s10659-016-9600-7. in press.
[21] W. Han, M. Sofonea, Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity, American Mathematical Society, Providence, 2002 RI—Intl. Press, Sommerville, MA.
[22] N. Kikuchi, J.T. Oden, Contact Problems in Elasticity: A Study of Variational Inequalities Finite Element Methods, SIAM, Philadelphia, 1988.
[23] T. Laursen, Computational Contact and Impact Mechanics, Springer, 2002.
[24] P. Wriggers, Computational Contact Mechanics, second ed., Springer, Berlin, 2006.
[25] K. Atkinson, W. Han, Theoretical Numerical Analysis: A Functional Analysis Framework, third ed., Springer, New York, 2009.
[26] S.C. Brenner, L.R. Scott, The Mathematical Theory of Finite Element Methods, third ed., Springer-Verlag, New York, 2008.
[27] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.
[28] W. Han, B.D. Reddy, Plasticity: Mathematical Theory Numerical Analysis, second ed., Springer-Verlag, 2013.
[29] M. Barboteu, K. Bartosz, P. Kalita, An analytical and numerical approach to a bilateral contact problem with nonmonotone friction, Int. J. Appl. Math. Comput. Sci. 23 (2013) 263–276.
[30] M. Barboteu, K. Bartosz, P. Kalita, A. Ramadan, Analysis of a contact problem with normal compliance, finite penetration and nonmonotone slip dependent friction, Commun. Contemp. Math. 15 (2013). http://dx.doi.org/10.1142/S0219199713500168.