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NUMERICAL ANALYSIS OF A HYPERBOLIC HEMIVARIATIONAL INEQUALITY ARISING IN DYNAMIC CONTACT*

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Abstract. In this paper a fully dynamic viscoelastic contact problem is studied. The contact is assumed to be bilateral and frictional, where the friction law is described by a nonmonotone relation between the tangential stress and the tangential velocity. A weak formulation of the problem leads to a second order nonmonotone subdifferential inclusion, also known as a second order hyperbolic hemivariational inequality. We study both semidiscrete and fully discrete approximation schemes and bound the errors of the approximate solutions. Under some regularity assumptions imposed on the true solution, optimal order error estimates are derived for the linear element solution. This theoretical result is illustrated numerically.

Key words. hyperbolic hemivariational inequality, dynamic contact, linearly viscoelastic material, nonmonotone friction law, finite element method, error estimate

AMS subject classifications. 65M60, 65M15, 74M10, 74M15, 74S05, 74S20

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1. Introduction.

This paper provides error analysis for numerical methods to solve a hyperbolic hemivariational inequality arising in a dynamic bilateral contact process for a viscoelastic material. The main mathematical difficulty in the study of the problem is due to the nonmonotonicity of the friction law, and hence, we cannot apply the standard techniques based on convex analysis. We formulate the contact condition corresponding to the friction law by means of an inclusion involving the Clarke subdifferential of a locally Lipschitz potential. Consequently, we deal with a second order evolutionary hemivariational inequality as a starting point to the numerical analysis of the contact problem. For approximation of the hemivariational inequality, we discuss both the spatially semidiscrete and fully discrete schemes. We use the finite element method for the spatial discretization and backward difference to approximate the time derivative. In both cases we derive error estimates that are of optimal order when the linear elements are used, if the true solution has certain regularity. Finally we present results of computer simulations on a two-dimensional contact problem, to show the performance of the numerical methods and to provide

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numerical evidence of the theoretically predicted optimal convergence order of the linear element solutions.

The problem is on the cutting edge of contact mechanics, the theory, numerical analysis, and computer simulations of hemivariational inequalities. The mathematical modeling of contact problems in mechanics has reached a mature level, as is witnessed by the recent large number of publications on its theory and applications in engineering and industry. For details concerning classical contact models and their analysis, we refer to [16, 24, 41], where numerical analysis involving error estimates is also conducted in the case of quasi-static and dynamic problems. For more recent mathematical results devoted to contact mechanics we refer also to [42, 43]. The theory of hemivariational inequalities, which allows us to model nonmonotone and nonsmooth contact problems, is a relatively new approach. Early comprehensive references in the area are [33, 35, 36, 37]. For more recent work, we refer to [30] and the references cited there. In [17], the finite element method is studied for solving some hemivariational inequalities. There are, however, still few publications devoted to the error estimates in the numerical solution of hemivariational inequalities. In [3], numerical approximation for a static hemivariational inequality is studied. In [15], a class of variational-hemivariational inequalities is studied, theoretically and numerically. The numerical analysis presented here is also motivated by techniques used in [7, 8, 9, 41, 5].

The rest of the paper is structured as follows. In section 2 we introduce the notation as well as some preliminary material. In section 3 we present the classical formulation of the frictional contact problem, list assumptions on the data, and present variational formulations of the problem. In section 4 we introduce and analyze a spatially semidiscrete scheme for solving the problem, and in section 5 we study a fully discrete approximation scheme. For both schemes, we derive optimal order error estimates for the linear element solutions under certain solution regularity assumptions. In section 6 we present numerical results in simulations of a two-dimensional contact problem and provide numerical evidence of optimal order convergence for the linear element solutions.

2. Notation and preliminaries. In this section we present the notation and some preliminary material to be used later. For further details we refer the reader to [14, 16, 21, 35].

We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ ($d \le 3$ in applications), and use "$\cdot$" and "$|\cdot|$" for the inner product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively,

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i, \quad |\boldsymbol{v}| = (\boldsymbol{v}\cdot\boldsymbol{v})^{\frac{1}{2}} \quad \forall\, \boldsymbol{u},\boldsymbol{v} \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad |\boldsymbol{\tau}| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{\frac{1}{2}} \quad \forall\, \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathbb{S}^d.$$

Here and below the indices i and j run between 1 and d, and the summation convention over repeated indices is adopted.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ. The unit outward normal vector $\boldsymbol{\nu}$ is defined a.e. on Γ. We introduce the following function spaces:

$$H = L^2(\Omega;\mathbb{R}^d) = \{\boldsymbol{u} = (u_i) \mid u_i \in L^2(\Omega)\}, \quad Q = \{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$
$$H_1 = \{\boldsymbol{u} \in H \mid \boldsymbol{\varepsilon}(\boldsymbol{u}) \in Q\}, \qquad Q_1 = \{\boldsymbol{\sigma} \in Q \mid \mathrm{Div}\,\boldsymbol{\sigma} \in H\}.$$

Here $\boldsymbol{\varepsilon}\colon H_1 \to Q$ and $\mathrm{Div}\colon Q_1 \to H$ are the *deformation* and *divergence* operators, defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \mathrm{Div}\ \boldsymbol{\sigma} = (\sigma_{ij,j}),$$

respectively, where the index following a comma indicates the partial derivative with respect to the corresponding component of the independent variable. The spaces H, Q, H_1, and Q_1 are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u},\boldsymbol{v})_H = \int_\Omega u_i v_i\,dx, \qquad (\boldsymbol{\sigma},\boldsymbol{\tau})_Q = \int_\Omega \sigma_{ij}\tau_{ij}\,dx,$$

$$(\boldsymbol{u},\boldsymbol{v})_{H_1} = (\boldsymbol{u},\boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q, \quad (\boldsymbol{\sigma},\boldsymbol{\tau})_{Q_1} = (\boldsymbol{\sigma},\boldsymbol{\tau})_Q + (\mathrm{Div}\,\boldsymbol{\sigma},\mathrm{Div}\,\boldsymbol{\tau})_H.$$

The associated norms on these spaces are denoted by $\|\cdot\|_H$, $\|\cdot\|_Q$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{Q_1}$, respectively.

Let $H_\Gamma = H^{1/2}(\Gamma;\mathbb{R}^d)$ and let $\bar\gamma : H_1 \to H_\Gamma$ be the trace operator. For every element $\boldsymbol{v} \in H_1$, we use the same symbol $\boldsymbol{v}$ to denote the trace $\bar\gamma\boldsymbol{v}$ of $\boldsymbol{v}$ on Γ, and we denote by v_ν and $\boldsymbol{v}_\tau$ the *normal* and *tangential* components of $\boldsymbol{v}$ on the boundary Γ given by

$$v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}, \quad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}.$$

Let H_Γ^* be the dual of H_Γ and let $\langle\cdot,\cdot\rangle_{H_\Gamma^*\times H_\Gamma}$ denote the duality pairing between H_Γ^* and H_Γ. For every $\boldsymbol{\sigma} \in Q_1$ there exists an element $\boldsymbol{\sigma\nu} \in H_\Gamma^*$ such that

$$(\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (\mathrm{Div}\,\boldsymbol{\sigma},\,\boldsymbol{v})_H = \langle\boldsymbol{\sigma\nu},\bar\gamma\boldsymbol{v}\rangle_{H_\Gamma^*\times H_\Gamma} \quad \forall\,\boldsymbol{v}\in H_1.$$

Moreover, if $\boldsymbol{\sigma}$ is a smooth (say, C^1) function, then

$$\langle\boldsymbol{\sigma\nu},\bar\gamma\boldsymbol{v}\rangle_{H_\Gamma^*\times H_\Gamma} = \int_\Gamma \boldsymbol{\sigma\nu}\cdot\boldsymbol{v}\,d\Gamma \quad \forall\,\boldsymbol{v}\in H_1.$$

We denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the *normal* and *tangential* traces of $\boldsymbol{\sigma}$,

$$\sigma_\nu = (\boldsymbol{\sigma\nu})\cdot\boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu\boldsymbol{\nu}.$$

Next recall the definitions of classical (one-sided) directional derivative and its generalization in the sense of Clarke. Let X be a Banach space and X^* its dual. For a function $\varphi : X \to \mathbb{R}$, the *directional derivative* of φ at $x \in X$ in the direction $v \in X$ is defined by

$$\varphi'(x;v) = \lim_{\lambda\downarrow 0}\frac{\varphi(x+\lambda v)-\varphi(x)}{\lambda}$$

whenever this limit exists. The Clarke generalized directional derivative of a locally Lipschitz function $\varphi : X \to \mathbb{R}$ at the point $x \in X$ in the direction $v \in X$ is defined by

$$\varphi^0(x;v) = \limsup_{y\to x,\lambda\downarrow 0}\frac{\varphi(y+\lambda v)-\varphi(y)}{\lambda}.$$

The Clarke subdifferential of φ at x is a subset of X^* given by

$$\partial\varphi(x) = \{\zeta\in X^* : \varphi^0(x;v) \ge \langle\zeta,v\rangle_{X^*\times X}\ \forall v\in X\}.$$

A locally Lipschitz function $\varphi : X \to \mathbb{R}$ is said to be *regular* (in the sense of Clarke) at $x \in X$ if for all $v \in X$, the directional derivative $\varphi'(x;v)$ exists and $\varphi'(x;v) = \varphi^0(x;v)$. The function φ is regular (in the sense of Clarke) on X if it is regular at every point $x \in X$.

We will need the following discrete Gronwall inequality [16, Chapter 7].

LEMMA 2.1. *Let $T > 0$ be given. For a positive integer N we define $k = T/N$. Assume that $\{g_n\}_{n=1}^N$ and $\{e_n\}_{n=1}^N$ are two sequences of nonnegative numbers satisfying*

$$e_n \le \bar{c} g_n + \bar{c} \sum_{j=1}^{n} k e_j, \quad n = 1, \ldots, N,$$

for a positive constant $\bar{c}$ independent of N or k. Then there exists a positive constant c, independent of N or k, such that

$$\max_{1 \le n \le N} e_n \le c \max_{1 \le n \le N} g_n.$$

3. Mechanical problem and variational formulations. We start with a description of the mechanical problem. A linearly viscoelastic body occupies an open bounded connected set $\Omega \subset \mathbb{R}^d$ with a Lipschitz boundary Γ that is partitioned into three parts $\overline{\Gamma}_1$, $\overline{\Gamma}_2$, and $\overline{\Gamma}_3$ with Γ_1, Γ_2, and Γ_3 being relatively open and mutually disjoint, and $\text{meas}\,(\Gamma_1) > 0$. Let $[0, T]$ be a time interval of interest, $T > 0$.

We assume that the body is clamped on Γ_1 and thus the displacement field vanishes there. A volume force of density $\boldsymbol{f}_0$ acts in Ω and a surface traction of density $\boldsymbol{f}_2$ acts on Γ_2. The body is in frictional contact with an obstacle on Γ_3. We assume the contact is bilateral, i.e., there is no loss of contact during the process. Thus, the normal displacement u_ν vanishes on Γ_3. We model the friction by a nonmonotone friction law. The dynamic process is considered.

The classical formulation of the mechanical problem is the following.

PROBLEM $\mathcal{P}_M$. *Find a displacement $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^d$ such that*

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega \times (0, T), \tag{3.1}$$

$$\rho\, \ddot{\mathbf{u}} = \text{Div}\, \boldsymbol{\sigma} + \boldsymbol{f}_0 \quad \text{in } \Omega \times (0, T), \tag{3.2}$$

$$\boldsymbol{u} = \mathbf{0} \quad \text{on } \Gamma_1 \times (0, T), \tag{3.3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{3.4}$$

$$u_\nu = 0 \quad \text{on } \Gamma_3 \times (0, T), \tag{3.5}$$

$$|\boldsymbol{\sigma}_\tau| \le \mu(0) S \text{ if } \dot{\boldsymbol{u}}_\tau = \mathbf{0},\ -\boldsymbol{\sigma}_\tau = \mu(|\dot{\boldsymbol{u}}_\tau|) S \frac{\dot{\boldsymbol{u}}_\tau}{|\dot{\boldsymbol{u}}_\tau|} \text{ if } \dot{\boldsymbol{u}}_\tau \ne \mathbf{0} \quad \text{on } \Gamma_3 \times (0, T), \tag{3.6}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1 \quad \text{in } \Omega. \tag{3.7}$$

Here, (3.1) is the linearly viscoelastic constitutive law [14, 16], (3.2) is the equation of motion, where ρ is the mass density, (3.3) is the homogeneous displacement boundary condition on Γ_1, (3.4) is the traction boundary condition on Γ_2, (3.5) represents the bilateral contact condition, and (3.7) provides the initial displacement and velocity conditions. In (3.6), $\mu(|\dot{\boldsymbol{u}}_\tau|)S$ represents the magnitude of the limiting friction traction at which slip begins, $S \ge 0$ being given. The friction coefficient μ is allowed to depend on the tangential speed $|\dot{\boldsymbol{u}}_\tau|$. The strict inequality in (3.6) holds in the stick zone and the equality holds in the slip zone. This physical model of slip-dependent friction was introduced in [38] for geophysical context of earthquake modeling and it also was studied in [18, 19, 20, 25, 26, 29, 40].

In the study of the contact problem we need the following assumptions on its data:

$H(\mathcal{A})$: The viscosity operator $\mathcal{A} : \Omega \times [0,T] \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} \text{(a) } \mathcal{A}(\cdot,\cdot,\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \times [0,T] \ \forall\, \boldsymbol{\varepsilon} \in \mathbb{S}^d; \\ \text{(b) } \mathcal{A}(\boldsymbol{x},t,\cdot) \text{ is continuous on } \mathbb{S}^d \text{ for a.e. } (\boldsymbol{x},t) \in \Omega \times [0,T]; \\ \text{(c) } |\mathcal{A}(\boldsymbol{x},t,\boldsymbol{\varepsilon})| \le a_0(\boldsymbol{x},t) + a_1|\boldsymbol{\varepsilon}| \ \forall\, \boldsymbol{\varepsilon} \in \mathbb{S}^d, \\ \qquad \text{a.e. } (\boldsymbol{x},t) \in \Omega \times [0,T], \text{ with } a_0 \in L^2(\Omega \times (0,T)),\ a_0 \ge 0 \text{ and } a_1 > 0; \\ \text{(d) } \mathcal{A}(\boldsymbol{x},t,\boldsymbol{\varepsilon}) : \boldsymbol{\varepsilon} \ge \alpha |\boldsymbol{\varepsilon}|^2 \ \forall\, \boldsymbol{\varepsilon} \in \mathbb{S}^d, \text{ a.e. } (\boldsymbol{x},t) \in \Omega \times [0,T] \text{ with } \alpha > 0; \\ \text{(e) } (\mathcal{A}(\boldsymbol{x},t,\boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x},t,\boldsymbol{\varepsilon}_2)) : (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{A}} |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2 \\ \qquad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } (\boldsymbol{x},t) \in \Omega \times [0,T] \text{ with } m_{\mathcal{A}} > 0; \\ \text{(f) } |\mathcal{A}(\boldsymbol{x},t,\boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x},t,\boldsymbol{\varepsilon}_2)| \le L_{\mathcal{A}} |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| \\ \qquad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } (\boldsymbol{x},t) \in \Omega \times [0,T] \text{ with } L_{\mathcal{A}} > 0. \end{cases}$$

$H(\mathcal{B})$: The elasticity operator $\mathcal{B} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is a bounded, symmetric, nonnegatively definite fourth order tensor, i.e.,

$$\begin{cases} \text{(a) } \mathcal{B}_{ijkl} \in L^\infty(\Omega),\ 1 \le i,j,k,l \le d; \\ \text{(b) } \mathcal{B}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathcal{B}\boldsymbol{\tau} \ \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. in } \Omega; \\ \text{(c) } \mathcal{B}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \ge 0 \ \forall\, \boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. in } \Omega. \end{cases}$$

$H(f)$: The force and the traction densities satisfy

$$\boldsymbol{f}_0 \in L^2(0,T; L^2(\Omega;\mathbb{R}^d)), \quad \boldsymbol{f}_2 \in L^2(0,T; L^2(\Gamma_2;\mathbb{R}^d)).$$

$H(\mu)$: The friction bound $\mu : [0,\infty) \to \mathbb{R}$ satisfies

$$\begin{cases} \text{(a) } \mu \text{ is continuous;} \\ \text{(b) } |\mu(s)| \le c(1+s) \ \forall\, s \ge 0,\ c > 0; \\ \text{(c) } \mu(s_1) - \mu(s_2) \ge -\lambda(s_1 - s_2) \ \forall\, s_1 > s_2 \ge 0 \text{ with } \lambda > 0. \end{cases}$$

Remark 3.1. If $\mathcal{A}(\cdot,\cdot,\boldsymbol{\varepsilon})$ is linear in $\boldsymbol{\varepsilon}$, then $H(\mathcal{A})(d)$ and $H(\mathcal{A})(e)$ are equivalent with $\alpha = m_{\mathcal{A}}$, and $H(\mathcal{A})(f)$ implies $H(\mathcal{A})(c)$ with $a_0 = 0$ and $a_1 = L_{\mathcal{A}}$.

Since μ corresponds to the physical resistance force, it is nonnegative. However, in mathematical analysis of the contact problem, we do not need to impose this condition. The condition (c) is the so-called one-side Lipschitz condition, which allows the function to decrease at a rate not faster than λ.

Using the Clarke subdifferential (cf. [11]), we can express the friction condition (3.6) in another form. Indeed, define a function $j : \mathbb{R}^d \to \mathbb{R}$ by

$$j(\boldsymbol{\xi}) = S \int_0^{|\boldsymbol{\xi}|} \mu(s)\, ds \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^d. \tag{3.8}$$

Then, assuming $H(\mu)$(a) – (b), the condition (3.6) is equivalent to the following subdifferential inclusion:

$$-\boldsymbol{\sigma}_\tau \in \partial j(\dot{\boldsymbol{u}}_\tau) \quad \text{on } \Gamma_3 \times (0,T),$$

where $\partial j(\boldsymbol{\xi})$ denotes the Clarke subdifferential of j at the point $\boldsymbol{\xi} \in \mathbb{R}^d$.

Properties of the function j are summarized in the next lemma.

LEMMA 3.2. *If the assumptions* $H(\mu)$(a)–(b) *hold, then the function* j *defined by* (3.8) *is regular in the sense of Clarke, it is locally Lipschitz, and*

$$|\boldsymbol{\eta}| \le Sc(1 + |\boldsymbol{\xi}|) \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^d, \boldsymbol{\eta} \in \partial j(\boldsymbol{\xi}).$$

If furthermore the assumption $H(\mu)$(c) *holds, then we have*

$$(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \ge -S\lambda|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2 \quad \forall\, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^d,\ \boldsymbol{\eta}_i \in \partial j(\boldsymbol{\xi}_i),\ i = 1, 2. \tag{3.9}$$

Proof. We will show that j is regular in the sense of Clarke. First observe that for $\boldsymbol{\xi} \ne \mathbf{0}$ we have $\partial j(\boldsymbol{\xi}) = \{S\mu(|\boldsymbol{\xi}|)\boldsymbol{\xi}/|\boldsymbol{\xi}|\}$ and so j is regular at $\boldsymbol{\xi}$ [12, Proposition 5.6.15]. Next, consider the case $\boldsymbol{\xi} = \mathbf{0}$. Let $\boldsymbol{v} \in \mathbb{R}^d$. Using $H(\mu)(a)$ we have

$$j'(\mathbf{0}; \boldsymbol{v}) = \lim_{\lambda \downarrow 0} \frac{1}{\lambda} S \int_0^{|\lambda \boldsymbol{v}|} \mu(t)\, dt = S\, \mu(0)\, |\boldsymbol{v}|.$$

By definition,

$$j^0(\mathbf{0}; \boldsymbol{v}) = \limsup_{\boldsymbol{\xi} \to \mathbf{0}, \lambda \downarrow 0} \frac{S}{\lambda} \int_{|\boldsymbol{\xi}|}^{|\boldsymbol{\xi} + \lambda \boldsymbol{v}|} \mu(s)\, ds.$$

Since $\mu \in C([0, \infty))$,

$$\begin{aligned} j^0(\mathbf{0}; \boldsymbol{v}) &= S\, \mu(0) \limsup_{\boldsymbol{\xi} \to \mathbf{0}, \lambda \downarrow 0} \frac{|\boldsymbol{\xi} + \lambda \boldsymbol{v}| - |\boldsymbol{\xi}|}{\lambda} \\ &= S\, \mu(0) \limsup_{\boldsymbol{\xi} \to \mathbf{0}, \lambda \downarrow 0} \frac{2\, \boldsymbol{\xi} \cdot \boldsymbol{v} + \lambda\, |\boldsymbol{v}|^2}{|\boldsymbol{\xi} + \lambda \boldsymbol{v}| + |\boldsymbol{\xi}|} \\ &= S\, \mu(0) \limsup_{\boldsymbol{\xi} \to \mathbf{0}} \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \cdot \boldsymbol{v} \\ &= S\, \mu(0)\, |\boldsymbol{v}|. \end{aligned}$$

So j is regular at $\mathbf{0}$.

The other properties then follow straightforwardly. ∎

To introduce a weak formulation of the mechanical problem $\mathcal{P}_M$, we first define a closed subspace of H_1,

$$V = \{\boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_1,\ v_\nu = 0 \text{ on } \Gamma_3\}.$$

Since $\operatorname{meas}(\Gamma_1) > 0$, Korn's inequality holds [34, p. 79]: for some constant $C_K > 0$, depending only on Ω and Γ_1,

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_Q \ge C_K \|\boldsymbol{v}\|_{H_1} \quad \forall\, \boldsymbol{v} \in V. \tag{3.10}$$

On V, we use the inner product given by

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V \tag{3.11}$$

and let $\|\cdot\|_V$ be the associated norm, i.e.,

$$\|\boldsymbol{v}\|_V = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_Q \quad \forall\, \boldsymbol{v} \in V. \tag{3.12}$$

It follows from (3.10) and (3.12) that $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent norms on V and therefore $(V, \|\cdot\|_V)$ is a real Hilbert space. The duality pairing between V and V^* is denoted by $\langle\cdot,\cdot\rangle$. Identifying H with its dual, we have an evolution triple $V \subset H \subset V^*$ with dense, continuous, and compact embeddings. We denote by $i : V \to H$ the identity mapping and by $i^* : V^* \to H$ its adjoint mapping. By the Sobolev trace theorem and by (3.10) there exists a constant C_0 depending only on the domain Ω, Γ_1, and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \le C_0 \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V. \tag{3.13}$$

By (3.13) there exists a continuous trace operator $\gamma : V \to L^2(\Gamma_3; \mathbb{R}^d)$ and for the function $\boldsymbol{v} \in V$ we still denote by $\boldsymbol{v}$ its trace $\gamma\boldsymbol{v}$. In what follows we need the spaces $\mathcal{V} = L^2(0,T;V)$, $\mathcal{H} = L^2(0,T;H)$, and $\mathcal{W} = \{\boldsymbol{v} \in \mathcal{V} \mid \dot{\boldsymbol{v}} \in \mathcal{V}^*\}$, where the time derivative involved in the definition of $\mathcal{W}$ is understood in the sense of vector valued distributions. Equipped with the norm $\|\boldsymbol{v}\|_{\mathcal{W}} = \left(\|\boldsymbol{v}\|^2_{\mathcal{V}} + \|\dot{\boldsymbol{v}}\|^2_{\mathcal{V}^*}\right)^{1/2}$ the space $\mathcal{W}$ becomes a separable Hilbert space. We also have $\mathcal{W} \subset \mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*$. It is well known that the embeddings $\mathcal{W} \subset C([0,T];H)$ and $\{\boldsymbol{w} \in \mathcal{V} \mid \dot{\boldsymbol{w}} \in \mathcal{W}\} \subset C([0,T];V)$ are continuous. Next we define operators $A : (0,T) \times V \to V^*$ and $B : V \to V^*$ by

$$\langle A(t,\boldsymbol{u}), \boldsymbol{v}\rangle = (\mathcal{A}(t, \boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \quad \text{for } \boldsymbol{u},\ \boldsymbol{v} \in V \text{ and } t \in (0,T), \tag{3.14}$$

$$\langle B\boldsymbol{u}, \boldsymbol{v}\rangle = (\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \quad \text{for } \boldsymbol{u},\ \boldsymbol{v} \in V, \tag{3.15}$$

a functional $J : L^2(\Gamma_3; \mathbb{R}^d) \to \mathbb{R}$ by

$$J(\boldsymbol{v}) = \int_{\Gamma_3} j(\boldsymbol{v})\, d\Gamma \quad \text{for } \boldsymbol{v} \in L^2(\Gamma_3; \mathbb{R}^d), \tag{3.16}$$

and a function $\boldsymbol{f} : (0,T) \to V^*$ by

$$\langle \boldsymbol{f}(t), \boldsymbol{v}\rangle = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\, d\Gamma \quad \text{for } \boldsymbol{v} \in V,\ \text{a.e. } t \in (0,T). \tag{3.17}$$

Assuming $H(\mathcal{A})$, we have the following properties for the operator $A : [0,T] \times V \to V^*$:

$$\begin{cases} \text{(a) } A(\cdot, \boldsymbol{v}) \text{ is measurable on } (0,T)\ \forall \boldsymbol{v} \in V; \\ \text{(b) } A(t,\cdot) \text{ is pseudomonotone on } V \text{ for a.e. } t \in (0,T); \\ \text{(c) } \|A(t,\boldsymbol{v})\|_{V^*} \le a_0(t) + a_1\|\boldsymbol{v}\|_V\ \forall \boldsymbol{v} \in V,\ \text{a.e. } t \in (0,T) \\ \qquad \text{with } a_0 \in L^2(0,T),\ a_0 \ge 0, \text{ and } a_1 > 0; \\ \text{(d) } \langle A(t,\boldsymbol{v}), \boldsymbol{v}\rangle \ge \alpha \|\boldsymbol{v}\|_V^2\ \forall \boldsymbol{v} \in V,\ \text{a.e. } t \in (0,T) \text{ with } \alpha > 0; \\ \text{(e) } \langle A(t,\boldsymbol{v}_1) - A(t,\boldsymbol{v}_2), \boldsymbol{v}_1 - \boldsymbol{v}_2\rangle \ge m_A \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V^2 \\ \qquad \forall\, \boldsymbol{v}_1, \boldsymbol{v}_2 \in V, \text{ a.e. } t \in (0,T) \text{ with } m_A = m_{\mathcal{A}} > 0; \\ \text{(f) } \|A(t,\boldsymbol{v}_1) - A(t,\boldsymbol{v}_2)\|_{V^*} \le L_A \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V \\ \qquad \forall\, \boldsymbol{v}_1, \boldsymbol{v}_2 \in V, \text{ a.e. } t \in (0,T) \text{ with } L_A = L_{\mathcal{A}} > 0. \end{cases}$$

Under the assumption $H(\mathcal{B})$, the operator $B \in L(V, V^*)$ is self-adjoint and monotone. Under the assumption $H(\mu)$, the functional $J : L^2(\Gamma_3; \mathbb{R}^d) \to \mathbb{R}$ is locally Lipschitz, and we have the following inequalities:

$$\|\boldsymbol{\eta}\|_{L^2(\Gamma_3;\mathbb{R}^d)} \le SC(1 + \|\boldsymbol{v}\|_{L^2(\Gamma_3;\mathbb{R}^d)})\ \forall\, \boldsymbol{\eta} \in \partial J(\boldsymbol{v}), \tag{3.18}$$

$$\langle \boldsymbol{\eta}_1 - \boldsymbol{\eta}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2\rangle_{L^2(\Gamma_3;\mathbb{R}^d)} \ge -S\lambda \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|^2_{L^2(\Gamma_3;\mathbb{R}^d)}\ \forall\, \boldsymbol{\eta}_i \in \partial J(\boldsymbol{v}_i),\ i = 1, 2, \tag{3.19}$$

where $C = \sqrt{2}\, c \max\left\{1, \sqrt{\mathrm{meas}_{d-1}(\Gamma_3)}\right\}$. The assumption $H(f)$ implies

$$f \in \mathcal{V}^*.$$

For the initial values, we will assume the following:

H_0: $\boldsymbol{u}_0 \in V$, $\boldsymbol{u}_1 \in H$.

Proceeding in a standard way [16, 30], we obtain the following variational formulation of the frictional Problem $\mathcal{P}_M$.

Problem $\mathcal{P}_V$. *Find a displacement field* $\boldsymbol{u} \in \mathcal{V}$ *with* $\dot{\boldsymbol{u}} \in \mathcal{W}$ *and a friction density* $\boldsymbol{\xi}_\tau \in L^2(0, T; L^2(\Gamma_3; \mathbb{R}^d))$ *such that*

$$\langle \rho\, \ddot{\boldsymbol{u}}(t) + A(t, \dot{\boldsymbol{u}}(t)) + B\boldsymbol{u}(t) - \boldsymbol{f}(t), \boldsymbol{v}\rangle = \int_{\Gamma_3} \boldsymbol{\xi}_\tau(t)\cdot \boldsymbol{v}_\tau \, d\Gamma \quad \forall\, \boldsymbol{v} \in V, \text{ a.e. } t, \tag{3.20}$$

$$-\boldsymbol{\xi}_\tau \in \partial j(\dot{\boldsymbol{u}}_\tau) \quad \text{a.e. on } \Gamma_3 \times (0, T), \tag{3.21}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1.$$

Here and below, "a.e. t" means "a.e. $t \in (0, T)$." The above problem can be expressed equivalently as follows.

Problem $\mathcal{P}_{V,1}$. *Find a displacement field* $\boldsymbol{u} \in \mathcal{V}$ *with* $\dot{\boldsymbol{u}} \in \mathcal{W}$ *such that*

$$\langle \rho\, \ddot{\boldsymbol{u}}(t) + A(t, \dot{\boldsymbol{u}}(t)) + B\boldsymbol{u}(t) - \boldsymbol{f}(t), \boldsymbol{v}\rangle + \int_{\Gamma_3} j^0(\dot{\boldsymbol{u}}_\tau(t); \boldsymbol{v}_\tau)\, d\Gamma \ge 0 \quad \forall\, \boldsymbol{v} \in V, \text{ a.e. } t,$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1.$$

Problem $\mathcal{P}_{V,1}$ is called a boundary hemivariational inequality. Next we define an auxiliary problem.

Problem $\mathcal{P}_{V,2}$. *Find a displacement field* $\boldsymbol{u} \in \mathcal{V}$ *with* $\dot{\boldsymbol{u}} \in \mathcal{W}$ *such that*

$$\rho\, \ddot{\boldsymbol{u}}(t) + A(t, \dot{\boldsymbol{u}}(t)) + B\boldsymbol{u}(t) + \gamma^* \partial J(\gamma \dot{\boldsymbol{u}}_\tau(t)) \ni \boldsymbol{f}(t) \quad \text{a.e. } t \in (0, T),$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1,$$

where γ *is the trace operator on* Γ_3 *and* γ^* *its adjoint, and* $\gamma \dot{\boldsymbol{u}}_\tau$ *means* $(\gamma \dot{\boldsymbol{u}})_\tau$.

A function $\boldsymbol{u} \in \mathcal{V}$ is a solution of Problem $\mathcal{P}_{V,2}$ if and only if $\dot{\boldsymbol{u}} \in \mathcal{W}$ and there exists $\boldsymbol{\eta} \in L^2(0, T; L^2(\Gamma_3; \mathbb{R}^d))$ such that

$$\rho\, \ddot{\boldsymbol{u}}(t) + A(t, \dot{\boldsymbol{u}}(t)) + B\boldsymbol{u}(t) + \boldsymbol{\eta}(t) = \boldsymbol{f}(t) \text{ a.e. } t \in (0, T),$$

$$\boldsymbol{\eta}(t) \in \gamma^* \partial J(t, \gamma \dot{\boldsymbol{u}}_\tau(t)) \text{ a.e. } t \in (0, T),$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1.$$

The hemivariational inequality corresponding to Problem $\mathcal{P}_{V,2}$ reads as follows.

Problem $\mathcal{P}_{V,3}$. *Find a displacement field* $\boldsymbol{u} \in \mathcal{V}$ *with* $\dot{\boldsymbol{u}} \in \mathcal{W}$ *such that*

$$\langle \rho\, \ddot{\boldsymbol{u}}(t) + A(t, \dot{\boldsymbol{u}}(t)) + B\boldsymbol{u}(t) - \boldsymbol{f}(t), \boldsymbol{v}\rangle + J^0(\gamma \dot{\boldsymbol{u}}_\tau(t); \gamma \boldsymbol{v}_\tau) \ge 0 \quad \forall\, \boldsymbol{v} \in V, \text{ a.e. } t,$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1.$$

We complete this section with a result on solution existence and uniqueness for Problem $\mathcal{P}_{V,2}$.

Theorem 3.3. *Assume* $H(\mathcal{A})$, $H(\mathcal{B})$, $H(\mu)$, $H(f)$, H_0, *and*

$$\frac{\alpha}{2} > S\, C\, C_0^2, \quad m_A > S\lambda C_0^2. \tag{3.22}$$

Then Problem $\mathcal{P}_{V,2}$ has a unique solution $\boldsymbol{u}$, and the following bound holds:

$$\|\boldsymbol{u}\|_{C(0,T;V)} + \|\dot{\boldsymbol{u}}\|_{\mathcal{W}} \le \tilde{C}\,(1 + \|\boldsymbol{u}_0\|_V + \|\boldsymbol{u}_1\|_H + \|\boldsymbol{f}\|_{\mathcal{V}^*}) \tag{3.23}$$

with a positive constant $\tilde{C}$.

The proof of this result follows from the arguments used in the proof of Theorem 5.15 (for existence and uniqueness of a solution) and of Lemma 5.8 (for the bound (3.23)) of [30].

Since j is regular (cf. Lemma 3.2), Problems $\mathcal{P}_V$, $\mathcal{P}_{V,1}$, $\mathcal{P}_{V,2}$, and $\mathcal{P}_{V,3}$ are equivalent [28, Remark 4]. In particular, under the assumptions stated in Theorem 3.3, Problem $\mathcal{P}_V$ has a unique solution.

4. Spatially semidiscrete approximation. In this section we introduce and analyze a spatially semidiscrete approximation for Problem $\mathcal{P}_V$.

Let V^h be a finite dimensional subspace of V, where $h > 0$ denotes a spatial discretization parameter. Let $\boldsymbol{u}_0^h, \boldsymbol{u}_1^h \in V^h$ be suitable approximations of $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$, characterized by

$$(\boldsymbol{u}_0^h - \boldsymbol{u}_0, \boldsymbol{v}^h)_V = 0, \quad (\boldsymbol{u}_1^h - \boldsymbol{u}_1, \boldsymbol{v}^h)_H = 0 \quad \forall\, \boldsymbol{v}^h \in V^h. \tag{4.1}$$

It is easy to observe that

$$\|\boldsymbol{u}_0^h\|_V \le \|\boldsymbol{u}_0\| \quad \text{and} \quad \|\boldsymbol{u}_1^h\|_H \le \|\boldsymbol{u}_1\|_H. \tag{4.2}$$

Then we have the following semidiscrete approximation of Problem $\mathcal{P}_V$.

PROBLEM $\mathcal{P}_V^h$. *Find a displacement field $\boldsymbol{u}^h \in L^2\ (0,T;V^h)$ with $\dot{\boldsymbol{u}}^h, \ddot{\boldsymbol{u}}^h \in L^2(0,T;V^h)$, and a friction density $\boldsymbol{\xi}_\tau^h \in L^2(0,T;L^2(\Gamma_3;\mathbb{R}^d))$ such that*

$$\begin{aligned} &\langle \rho\, \ddot{\boldsymbol{u}}^h(t) + A(t, \dot{\boldsymbol{u}}^h(t)) + B\boldsymbol{u}^h(t) - \boldsymbol{f}(t), \boldsymbol{v}^h\rangle \\ &\qquad = \int_{\Gamma_3} \boldsymbol{\xi}_\tau^h(t)\cdot \boldsymbol{v}_\tau^h\, d\Gamma\ \forall\, \boldsymbol{v}^h \in V^h,\ \text{a.e. } t, \end{aligned} \tag{4.3}$$

$$-\boldsymbol{\xi}_\tau^h(t) \in \partial j(\dot{\boldsymbol{u}}_\tau^h(t)) \text{ a.e. on } \Gamma_3 \times (0,T), \tag{4.4}$$

$$\boldsymbol{u}^h(0) = \boldsymbol{u}_0^h, \quad \dot{\boldsymbol{u}}^h(0) = \boldsymbol{u}_1^h. \tag{4.5}$$

Under the assumptions of Theorem 3.3, we also have the existence and uniqueness of a solution to Problem $\mathcal{P}_V^h$. Moreover, similar to (3.23), and thanks to (4.2), we have the bound

$$\|\boldsymbol{u}^h\|_{C(0,T;V)} + \|\dot{\boldsymbol{u}}^h\|_{\mathcal{W}} \le \tilde{C}\,(1 + \|\boldsymbol{u}_0\|_V + \|\boldsymbol{u}_1\|_H + \|\boldsymbol{f}\|_{\mathcal{V}^*}) \tag{4.6}$$

with a positive constant $\tilde{C}$.

We provide a result on the error estimates between the solutions of Problems $\mathcal{P}_V$ and $\mathcal{P}_V^h$.

THEOREM 4.1. *Assume that $H(\mathcal{A})$, $H(\mathcal{B})$, $H(\mu)$, $H(f)$, H_0, and* (3.22) *hold. Let $\boldsymbol{u}$ and $\boldsymbol{u}^h$ be solutions of Problems $\mathcal{P}_V$ and $\mathcal{P}_V^h$, respectively. Then there exists a positive constant c depending only on the data of the problem, such that for any $\boldsymbol{v}^h \in L^2(0,T;V^h) \cap \mathcal{W}$,*

$$\begin{aligned} &\|\boldsymbol{u}-\boldsymbol{u}^h\|^2_{C(0,T;V)} + \|\dot{\boldsymbol{u}}-\dot{\boldsymbol{u}}^h\|^2_{C(0,T;H)} + \|\dot{\boldsymbol{u}}-\dot{\boldsymbol{u}}^h\|^2_{\mathcal{V}} \\ &\quad \le c\big(\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V^2 + \|\boldsymbol{u}_1-\boldsymbol{u}_1^h\|_H\|\boldsymbol{u}_1-\boldsymbol{v}^h(0)\|_H \\ &\qquad + \|\dot{\boldsymbol{u}}-\boldsymbol{v}^h\|^2_{\mathcal{V}} + \|\ddot{\boldsymbol{u}}-\dot{\boldsymbol{v}}^h\|^2_{\mathcal{V}^*} + \|\dot{\boldsymbol{u}}_\tau-\boldsymbol{v}^h_\tau\|_{L^2(0,T;L^2(\Gamma_3;\mathbb{R}^d))}\big). \end{aligned} \tag{4.7}$$

Proof. Let us define the functions $\boldsymbol{w}(t) = \dot{\boldsymbol{u}}(t)$ and $\boldsymbol{w}^h(t) = \dot{\boldsymbol{u}}^h(t)$ for all $t \in [0,T]$. Then,

$$\begin{aligned} \boldsymbol{u}(t) &= (I\boldsymbol{w})(t) = \boldsymbol{u}_0 + \int_0^t \boldsymbol{w}(s)\,ds, \\ \boldsymbol{u}^h(t) &= (I^h\boldsymbol{w}^h)(t) = \boldsymbol{u}_0^h + \int_0^t \boldsymbol{w}^h(s)\,ds, \end{aligned} \tag{4.8}$$

and we can express (3.20)–(3.21) and (4.3)–(4.4) as follows:

$$\begin{aligned} &\langle \rho\,\dot{\boldsymbol{w}}(t) + A(t,\boldsymbol{w}(t)) + B(I\boldsymbol{w})(t) - \boldsymbol{f}(t), \boldsymbol{v}\rangle \\ &\qquad = \int_{\Gamma_3} \boldsymbol{\xi}_\tau(t)\cdot \boldsymbol{v}_\tau\,d\Gamma \quad \forall\, \boldsymbol{v}\in V, \text{ a.e. } t, \end{aligned} \tag{4.9}$$

$$-\boldsymbol{\xi}_\tau \in \partial j(\boldsymbol{w}_\tau) \text{ a.e. on } \Gamma_3\times(0,T), \tag{4.10}$$

$$\boldsymbol{w}(0) = \boldsymbol{u}_1, \tag{4.11}$$

$$\begin{aligned} &\langle \rho\,\dot{\boldsymbol{w}}^h(t) + A(t,\boldsymbol{w}^h(t)) + B(I^h\boldsymbol{w}^h)(t) - \boldsymbol{f}(t), \boldsymbol{v}^h\rangle \\ &\qquad = \int_{\Gamma_3} \boldsymbol{\xi}^h_\tau(t)\cdot \boldsymbol{v}^h_\tau\,d\Gamma \quad \forall\, \boldsymbol{v}^h\in V^h, \text{ a.e. } t, \end{aligned} \tag{4.12}$$

$$-\boldsymbol{\xi}^h_\tau \in \partial j(\boldsymbol{w}^h_\tau) \text{ a.e. on } \Gamma_3\times(0,T), \tag{4.13}$$

$$\boldsymbol{w}^h(0) = \boldsymbol{u}^h_1. \tag{4.14}$$

For any $\boldsymbol{v}^h \in V^h$, we have from (4.9) and (4.12) that for a.e. $t\in(0,T)$,

$$\begin{aligned} &\rho\langle \dot{\boldsymbol{w}}(t) - \dot{\boldsymbol{w}}^h(t), \boldsymbol{v}^h\rangle + \langle A(t,\boldsymbol{w}(t)) - A(t,\boldsymbol{w}^h(t)), \boldsymbol{v}^h\rangle \\ &\qquad + \langle B(I\boldsymbol{w})(t) - B(I^h\boldsymbol{w}^h)(t), \boldsymbol{v}^h\rangle + \int_{\Gamma_3} (\boldsymbol{\xi}^h_\tau(t) - \boldsymbol{\xi}_\tau(t))\cdot \boldsymbol{v}^h_\tau\,d\Gamma = 0. \end{aligned} \tag{4.15}$$

Note that

$$\langle \dot{\boldsymbol{w}}(t) - \dot{\boldsymbol{w}}^h(t), \boldsymbol{w}(t) - \boldsymbol{w}^h(t)\rangle = \frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}(t) - \boldsymbol{w}^h(t)\|^2_H. \tag{4.16}$$

From the strong monotonicity of A, we have

$$m_A\|\boldsymbol{w}(t) - \boldsymbol{w}^h(t)\|^2_V \le \langle A\boldsymbol{w}(t) - A\boldsymbol{w}^h(t), \boldsymbol{w}(t) - \boldsymbol{w}^h(t)\rangle. \tag{4.17}$$

By the symmetry of B, we have

$$\begin{aligned}(4.18)\qquad &\langle B(I\boldsymbol{w})(t) - B(I^h\boldsymbol{w}^h)(t), \boldsymbol{w}(t) - \boldsymbol{w}^h(t)\rangle\\ &= \langle B\boldsymbol{u}(t) - B\boldsymbol{u}^h(t), \dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}^h(t)\rangle\\ &= \frac{1}{2}\frac{d}{dt}\langle B\boldsymbol{u}(t) - B\boldsymbol{u}^h(t), \boldsymbol{u}(t) - \boldsymbol{u}^h(t)\rangle.\end{aligned}$$

From (4.10), (4.13), (3.9), and (3.13), we obtain

$$(4.19)\qquad \int_{\Gamma_3} (\boldsymbol{\xi}_\tau^h(t) - \boldsymbol{\xi}_\tau(t)) \cdot (\boldsymbol{w}_\tau(t) - \boldsymbol{w}_\tau^h(t))\, d\Gamma \le S\lambda C_0^2 \|\boldsymbol{w}(t) - \boldsymbol{w}^h(t)\|_V^2.$$

Denote $c_0 = m_A - S\lambda C_0^2$. From (4.16)–(4.19) we obtain

$$\begin{aligned}(4.20)\qquad &\frac{1}{2}\rho\frac{d}{dt}\|\boldsymbol{w}(t) - \boldsymbol{w}^h(t)\|_H^2 + c_0\|\boldsymbol{w}(t) - \boldsymbol{w}^h(t)\|_V^2 + \frac{1}{2}\frac{d}{dt}\langle B\boldsymbol{u}(t) - B\boldsymbol{u}^h(t), \boldsymbol{u}(t) - \boldsymbol{u}^h(t)\rangle\\ &\le \rho\langle \dot{\boldsymbol{w}}(t) - \dot{\boldsymbol{w}}^h(t), \boldsymbol{w}(t) - \boldsymbol{w}^h(t)\rangle + \langle A(t, \boldsymbol{w}(t)) - A(t, \boldsymbol{w}^h(t)), \boldsymbol{w}(t) - \boldsymbol{w}^h(t)\rangle\\ &\quad + \langle B(I\boldsymbol{w})(t) - B(I^h\boldsymbol{w}^h)(t), \boldsymbol{w}(t) - \boldsymbol{w}^h(t)\rangle\\ &\quad + \int_{\Gamma_3} (\boldsymbol{\xi}_\tau^h(t) - \boldsymbol{\xi}_\tau(t)) \cdot (\boldsymbol{w}_\tau(t) - \boldsymbol{w}_\tau^h(t))\, d\Gamma\\ &= \rho\langle \dot{\boldsymbol{w}}(t) - \dot{\boldsymbol{w}}^h(t), \boldsymbol{w}(t) - \boldsymbol{v}^h(t)\rangle + \langle A(t, \boldsymbol{w}(t)) - A(t, \boldsymbol{w}^h(t)), \boldsymbol{w}(t) - \boldsymbol{v}^h(t)\rangle\\ &\quad + \langle B(I\boldsymbol{w})(t) - B(I^h\boldsymbol{w}^h)(t), \boldsymbol{w}(t) - \boldsymbol{v}^h(t)\rangle\\ &\quad + \int_{\Gamma_3} (\boldsymbol{\xi}_\tau^h(t) - \boldsymbol{\xi}_\tau(t)) \cdot (\boldsymbol{w}_\tau(t) - \boldsymbol{v}_\tau^h(t))\, d\Gamma,\end{aligned}$$

where $\boldsymbol{v}^h(t) \in V^h$ for a.e. $t \in (0, T)$ is arbitrary and the last equality follows from (4.15). For $t \in (0, T)$, assuming $\boldsymbol{v}^h \in \mathcal{W}$ we perform integration by parts [13, Proposition 8.4.14]:

$$\begin{aligned}\int_0^t \langle \dot{\boldsymbol{w}}(s) - \dot{\boldsymbol{w}}^h(s), \boldsymbol{w}(s) - \boldsymbol{v}^h(s)\rangle\, ds = &\big(\boldsymbol{w}(t) - \boldsymbol{w}^h(t), \boldsymbol{w}(t) - \boldsymbol{v}^h(t)\big)_H\\ &- \big(\boldsymbol{w}(0) - \boldsymbol{w}^h(0), \boldsymbol{w}(0) - \boldsymbol{v}^h(0)\big)_H\\ &- \int_0^t \Big\langle \boldsymbol{w}(s) - \boldsymbol{w}^h(s), \dot{\boldsymbol{w}}(s) - \dot{\boldsymbol{v}}^h(s)\Big\rangle\, ds.\end{aligned}$$

Thus,

$$\begin{aligned}(4.21)\qquad &\int_0^t \langle \dot{\boldsymbol{w}}(s) - \dot{\boldsymbol{w}}^h(s), \boldsymbol{w}(s) - \boldsymbol{v}^h(s)\rangle\, ds\\ &\le \|\boldsymbol{w}(t) - \boldsymbol{w}^h(t)\|_H \|\boldsymbol{w}(t) - \boldsymbol{v}^h(t)\|_H + \|\boldsymbol{u}_1 - \boldsymbol{u}_1^h\|_H \|\boldsymbol{u}_1 - \boldsymbol{v}^h(0)\|_H\\ &\quad + \int_0^t \Big\langle \boldsymbol{w}(s) - \boldsymbol{w}^h(s), \dot{\boldsymbol{w}}(s) - \dot{\boldsymbol{v}}^h(s)\Big\rangle\, ds\\ &\le \frac{1}{4}\|\boldsymbol{w}(t) - \boldsymbol{w}^h(t)\|_H^2 + \|\boldsymbol{w}(t) - \boldsymbol{v}^h(t)\|_H^2 + \|\boldsymbol{u}_1 - \boldsymbol{u}_1^h\|_H \|\boldsymbol{u}_1 - \boldsymbol{v}^h(0)\|_H\\ &\quad + \varepsilon\|\boldsymbol{w} - \boldsymbol{w}^h\|_{\mathcal{V}}^2 + \frac{1}{4\varepsilon}\|\dot{\boldsymbol{w}} - \dot{\boldsymbol{v}}^h\|_{\mathcal{V}^*}^2.\end{aligned}$$

Using the Lipschitz continuity of A,

$$\begin{aligned}\int_0^t \langle A(s,\boldsymbol{w}(s)) - A(s,\boldsymbol{w}^h(s)), \boldsymbol{w}(s)-\boldsymbol{v}^h(s)\rangle\, ds \\ \le \int_0^t L_A \|\boldsymbol{w}(s)-\boldsymbol{w}^h(s)\|_V \|\boldsymbol{w}(s)-\boldsymbol{v}^h(s)\|_V\, ds \\ \le \varepsilon \|\boldsymbol{w}-\boldsymbol{w}^h\|_{\mathcal{V}}^2 + \frac{L_A}{4\varepsilon}\|\boldsymbol{w}-\boldsymbol{v}^h\|_{\mathcal{V}}^2.\end{aligned} \tag{4.22}$$

Using the properties of B,

$$\begin{aligned}&\int_0^t \langle B(I\boldsymbol{w})(s) - B(I^h\boldsymbol{w}^h)(s), \boldsymbol{w}(s)-\boldsymbol{v}^h(s)\rangle \\ &\quad\le \int_0^t \|B\|_{\mathcal{L}(V,V^*)} \|\boldsymbol{u}(s)-\boldsymbol{u}^h(s)\|_V \|\boldsymbol{w}(s)-\boldsymbol{v}^h(s)\|_V \\ &\quad\le \varepsilon\|\boldsymbol{u}-\boldsymbol{u}^h\|_{\mathcal{V}}^2 + \frac{\|B\|_{\mathcal{L}(V,V^*)}}{4\varepsilon}\|\boldsymbol{w}-\boldsymbol{v}^h\|_{\mathcal{V}}^2 \\ &\quad\le \varepsilon 2T\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V^2 + \varepsilon 2T\|\boldsymbol{w}-\boldsymbol{w}^h\|_{\mathcal{V}}^2 + \frac{\|B\|_{\mathcal{L}(V,V^*)}}{4\varepsilon}\|\boldsymbol{w}-\boldsymbol{v}^h\|_{\mathcal{V}}^2.\end{aligned} \tag{4.23}$$

It remains to bound the last term of (4.20). From (3.18) and (3.23), we have

$$\begin{aligned}&\int_0^t \int_{\Gamma_3} (\boldsymbol{\xi}_\tau^h(s) - \boldsymbol{\xi}_\tau(s))\cdot(\boldsymbol{w}_\tau(s)-\boldsymbol{v}_\tau^h(s))\, d\Gamma\, ds \\ &\quad\le \int_0^t \Big(\|\boldsymbol{\xi}_\tau^h(s)\|_{L^2(\Gamma_3;\mathbb{R}^d)} + \|\boldsymbol{\xi}_\tau(s)\|_{L^2(\Gamma_3;\mathbb{R}^d)}\Big) \|\boldsymbol{w}_\tau(s)-\boldsymbol{v}_\tau^h(s)\|_{L^2(\Gamma_3;\mathbb{R}^d)}\, ds \\ &\quad\le \int_0^t SC\left(2 + C_0(\|\boldsymbol{w}(t)\|_V + \|\boldsymbol{w}^h(t)\|_V)\right) \|\boldsymbol{w}_\tau(s)-\boldsymbol{v}_\tau^h(s)\|_{L^2(\Gamma_3;\mathbb{R}^d)}\, ds \\ &\quad\le 2SC\left(\sqrt{T} + C_0(\|\boldsymbol{w}\|_{\mathcal{V}} + \|\boldsymbol{w}^h\|_{\mathcal{V}})\right)\|\boldsymbol{w}_\tau-\boldsymbol{v}_\tau^h\|_{L^2(0,T;L^2(\Gamma_3;\mathbb{R}^d))} \\ &\quad\le 2SC\left(\sqrt{T} + 2\tilde{C}(1+\|\boldsymbol{u}_0\|_V + \|\boldsymbol{u}_1\|_H + \|\boldsymbol{f}\|_{\mathcal{V}^*})\right)\|\boldsymbol{w}_\tau-\boldsymbol{v}_\tau^h\|_{L^2(0,T;L^2(\Gamma_3;\mathbb{R}^d))}.\end{aligned} \tag{4.24}$$

Denote

$$\begin{aligned}r = {}& \|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V^2 + \|\boldsymbol{u}_1-\boldsymbol{u}_1^h\|_H\|\boldsymbol{u}_1-\boldsymbol{v}^h(0)\|_H + \|\boldsymbol{w}-\boldsymbol{v}^h\|_{\mathcal{V}}^2 \\ &+ \|\dot{\boldsymbol{w}}-\dot{\boldsymbol{v}}^h\|_{\mathcal{V}^*}^2 + \|\boldsymbol{w}_\tau-\boldsymbol{v}_\tau^h\|_{L^2(0,T;L^2(\Gamma_3;\mathbb{R}^d))}.\end{aligned}$$

We integrate (4.20) and apply (4.21)–(4.24) to get

$$\frac{1}{2}\rho\|\boldsymbol{w}(t)-\boldsymbol{w}^h(t)\|_H^2 + 2\left(c_0-(2+2T)\varepsilon\right)\int_0^t \|\boldsymbol{w}(s)-\boldsymbol{w}^h(s)\|_V^2\, ds \le c_1 r, \tag{4.25}$$

where the constant c_1 depends only on the data of the problem. Since $t\in(0,T)$ is arbitrary, with ε small enough, we obtain from (4.25) that

$$\|\boldsymbol{w}-\boldsymbol{w}^h\|_{C(0,T;H)}^2 + \|\boldsymbol{w}-\boldsymbol{w}^h\|_{\mathcal{V}}^2 \le c_2 r \tag{4.26}$$

with $c_2 > 0$. For any $t \in (0,T)$,

$$\|\boldsymbol{u}(t) - \boldsymbol{u}^h(t)\|_V \le \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V + \int_0^T \|\boldsymbol{w}(t) - \boldsymbol{w}^h(t)\|_V \, dt$$
$$\le \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V + \sqrt{T}\|\boldsymbol{w} - \boldsymbol{w}^h\|_{\mathcal{V}}.$$

Thus,

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|^2_{C(0,T;V)} \le 2\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|^2 + 2\sqrt{T}\|\boldsymbol{w} - \boldsymbol{w}^h\|^2_{\mathcal{V}} \le c_3 r \tag{4.27}$$

with a positive constant c_3. From (4.26)–(4.27) we obtain the result (4.7). ☐

Remark 4.2. It follows from (3.23) and (4.6) that $\boldsymbol{u}, \boldsymbol{u}^h \in C(0,T;V)$. Since the embedding $\mathcal{W} \subset C(0,T;H)$ is continuous, again from (3.23) and (4.6), we have $\dot{\boldsymbol{u}}, \dot{\boldsymbol{u}}^h \in C(0,T;H)$. Thus it is reasonable to estimate the norms $\|\boldsymbol{u}-\boldsymbol{u}^h\|_{C(0,T;V)}$ and $\|\dot{\boldsymbol{u}} - \dot{\boldsymbol{u}}^h\|_{C(0,T;H)}$ in (4.7).

Theorem 4.1 is valid for any finite dimensional subspace V^h of V. In applications, V^h is usually taken to be a finite element space. As a particular example, assume Ω is a polygonal/polyhedral domain and let $\{\mathcal{T}^h\}$ be a regular family of finite element triangulations of $\overline{\Omega}$ into triangles ($d = 2$) or tetrahedrons ($d = 3$). For an element $T \in \mathcal{T}^h$, denote by $P_1(T)$ the space of polynomials of a total degree less than or equal to one in T. Then we can use the linear element space of continuous piecewise affine functions:

$$V^h = \{\boldsymbol{v}^h \in [C(\overline{\Omega})]^d : \boldsymbol{v}^h|_T \in [P_1(T)]^d \ \forall T \in \mathcal{T}^h,\ \boldsymbol{v}^h = \boldsymbol{0} \text{ on } \Gamma_1,\ v_\nu^h = 0 \text{ on } \Gamma_3\}. \tag{4.28}$$

In the numerical simulations presented in section 6, this linear element space with $d = 2$ is used.

Corollary 4.3. *Keep the assumptions stated in Theorem* 4.1. *Assume Ω is a polygonal/polyhedral domain, and let $\{V^h\}$ be the family of linear element spaces defined by* (4.28)*, corresponding to a regular family of finite element triangulations of $\overline{\Omega}$ into triangles or tetrahedrons. Let $\boldsymbol{u}$ and $\boldsymbol{u}^h$ be solutions of Problems $\mathcal{P}_V$ and $\mathcal{P}_V^h$, respectively. Assume $\boldsymbol{u}_0 \in H^2(\Omega;\mathbb{R}^d)$, $\boldsymbol{u}_1 \in H^1(\Omega;\mathbb{R}^d)$, and take $\boldsymbol{u}_0^h, \boldsymbol{u}_1^h \in V^h$ to be projections of $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$, defined by* (4.1). *Under the regularity condition*

$$\dot{\boldsymbol{u}} \in L^2(0,T;H^2(\Omega;\mathbb{R}^d)),\ \ddot{\boldsymbol{u}} \in L^2(0,T;H^2(\Omega;\mathbb{R}^d)),\ \dot{\boldsymbol{u}}_\tau \in L^2(0,T;H^2(\Gamma_3;\mathbb{R}^d)),$$

we have the optimal order error estimate

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_{C(0,T;V)} + \|\dot{\boldsymbol{u}} - \dot{\boldsymbol{u}}^h\|_{C(0,T;H)} + \|\dot{\boldsymbol{u}} - \dot{\boldsymbol{u}}^h\|_{\mathcal{V}} \le c\, h \tag{4.29}$$

for a constant c independent of h.

Proof. Note that under the stated regularity assumptions, for a.e. $t \in [0,T]$, $\dot{\boldsymbol{u}}(t)$, $\ddot{\boldsymbol{u}}(t)$ are continuous on $\overline{\Omega}$, and $\dot{\boldsymbol{u}}_\tau(t)$ is continuous on Γ_3. Let $\boldsymbol{v}^h(t) = \Pi^h\dot{\boldsymbol{u}}(t) \in V^h$ be the finite element interpolant of $\dot{\boldsymbol{u}}(t)$, a.e. $t \in [0,T]$. Note that $\boldsymbol{v}_\tau^h(t) = (\Pi^h\dot{\boldsymbol{u}}(t))_\tau$ is the continuous piecewise linear interpolant of $\dot{\boldsymbol{u}}_\tau(t)$ on Γ_3. Moreover, $\dot{\boldsymbol{v}}^h(t)$ is the continuous piecewise linear interpolant of $\ddot{\boldsymbol{u}}(t)$. Then by the standard finite element interpolation error estimates [2, 6, 10], we have the following approximation properties:

$$\|\dot{\boldsymbol{u}}(t) - \boldsymbol{v}^h(t)\|_V \le ch\|\dot{\boldsymbol{u}}(t)\|_{H^2(\Omega;\mathbb{R}^d)},$$
$$\|\ddot{\boldsymbol{u}}(t) - \dot{\boldsymbol{v}}^h(t)\|_{V^*} \le ch\|\ddot{\boldsymbol{u}}(t)\|_{H^2(\Omega;\mathbb{R}^d)},$$
$$\|\dot{\boldsymbol{u}}_\tau(t) - \boldsymbol{v}_\tau^h(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)} \le ch^2\|\dot{\boldsymbol{u}}_\tau\|_{H^2(\Gamma_3;\mathbb{R}^d)}$$

and

$$\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V \le ch\,\|\boldsymbol{u}_0\|_{H^2(\Omega;\mathbb{R}^d)},$$
$$\|\boldsymbol{u}_1 - \boldsymbol{u}_1^h\|_H \le ch\,\|\boldsymbol{u}_1\|_{H^1(\Omega;\mathbb{R}^d)}.$$

It follows that

$$\|\dot{\boldsymbol{u}} - \boldsymbol{v}^h\|_{\mathcal{V}} \le ch\|\dot{\boldsymbol{u}}\|_{L^2(0,T;H^2(\Omega;\mathbb{R}^d))},$$
$$\|\ddot{\boldsymbol{u}} - \dot{\boldsymbol{v}}^h\|_{\mathcal{V}^*} \le ch\|\ddot{\boldsymbol{u}}\|_{L^2(0,T;H^2(\Omega;\mathbb{R}^d))},$$
$$\|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}_\tau^h\|_{L^2(0,T;L^2(\Gamma_3;\mathbb{R}^d))} \le ch^2\|\dot{\boldsymbol{u}}_\tau\|_{L^2(0,T;H^2(\Gamma_3;\mathbb{R}^d))}.$$

Then the error bound (4.29) follows from (4.7). ∎

Note that for other choices of the finite element space V^h, Theorem 4.1 can be applied similarly to derive error estimates of the finite element solutions, under certain corresponding regularity assumptions on the true solution $\boldsymbol{u}$.

5. Fully discrete error estimates. In this section we introduce a fully discrete approximation of Problem $\mathcal{P}_V$ and bound the error of the fully discrete solutions. For simplicity in exposition, we assume

$$A(\cdot,\boldsymbol{v}) \in C(0,T;V^*) \quad \forall\,\boldsymbol{v}\in V, \qquad \boldsymbol{f}\in C(0,T;V^*). \tag{5.1}$$

In addition to the finite dimensional subspace $V^h \subset V$ for spatial discretization, we need temporal discretization. We define a uniform partition of $[0,T]$ denoted by $0 = t_0 < t_1 < \cdots < t_N = T$. Let $k = T/N$ be a time step size and for a continuous function g we denote $g_n = g(t_n)$. For a sequence $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/k$ for $n = 1,\ldots,N$ the backward divided difference. With the backward Euler scheme for the time derivative, the fully discrete approximation of the Problem $\mathcal{P}_V$ is the following.

Problem $\mathcal{P}_V^{kh}$. *Find a velocity field* $\{\boldsymbol{w}_n^{hk}\}_{n=0}^N \subset V^h$ *and a friction density* $\{\boldsymbol{\xi}_n^{hk}\}_{n=0}^N \subset L^2(\Gamma_3;\mathbb{R}^d)$ *such that*

$$\langle \rho\,\delta\boldsymbol{w}_n^{hk} + A(t_n,\boldsymbol{w}_n^{hk}) + B\boldsymbol{u}_n^{hk} - \boldsymbol{f}_n, \boldsymbol{v}^h\rangle = \int_{\Gamma_3} \boldsymbol{\xi}_{n\,\tau}^{hk}\cdot\boldsymbol{v}_\tau^h\,d\Gamma \quad \forall\,\boldsymbol{v}^h\in V^h, \tag{5.2}$$

$$-\boldsymbol{\xi}_{n\,\tau}^{hk} \in \partial j(\boldsymbol{w}_{n\,\tau}^{hk}) \text{ a.e. on } \Gamma_3,\ n = 1,\ldots,N, \tag{5.3}$$

and

$$\boldsymbol{w}_0^{hk} = \boldsymbol{u}_1^h, \tag{5.4}$$

where the discrete displacement field $\{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset V^h$ *is given by*

$$\boldsymbol{u}_n^{hk} = \boldsymbol{u}_0^h + \sum_{j=1}^n k\boldsymbol{w}_j^{hk}. \tag{5.5}$$

Under the assumptions of Theorem 3.3, there exists a unique solution of Problem $\mathcal{P}_V^{kh}$. The following boundedness property on the numerical solution will be needed in error estimation.

Theorem 5.1. *Assume* $H(\mathcal{A})$, $H(\mathcal{B})$, $H(\mu)$, $H(f)$, H_0, *and* (3.22). *Then for some constant* $C > 0$,

$$k\sum_{n=1}^N \|\boldsymbol{w}_n^{hk}\|^2 \le C. \tag{5.6}$$

Proof. Taking $\boldsymbol{v}^h = \boldsymbol{w}_n^{hk}$ in (5.2) and using (5.5) we have

$$(5.7)\qquad \rho(\boldsymbol{w}_n^{hk} - \boldsymbol{w}_{n-1}^{hk}, \boldsymbol{w}_n^{hk})_H + k\langle A(t_n, \boldsymbol{w}_n^{hk}), \boldsymbol{w}_n^{hk}\rangle + \langle B\boldsymbol{u}_n^{hk}, \boldsymbol{u}_n^{hk} - \boldsymbol{u}_{n-1}^{hk}\rangle = k\langle \boldsymbol{f}_n, \boldsymbol{w}_n^{hk}\rangle + k\int_{\Gamma_3} \boldsymbol{\xi}_{n\,\tau}^{hk} \cdot \boldsymbol{w}_{n\tau}^{hk}\, d\Gamma.$$

Note that

$$(\boldsymbol{w}_n^{hk} - \boldsymbol{w}_{n-1}^{hk}, \boldsymbol{w}_n^{hk})_H = \frac{1}{2}\|\boldsymbol{w}_n^{hk}\|_H^2 - \frac{1}{2}\|\boldsymbol{w}_{n-1}^{hk}\|_H^2 + \frac{1}{2}\|\boldsymbol{w}_n^{hk} - \boldsymbol{w}_{n-1}^{hk}\|_H^2.$$

From the property (d) of the operator A, we get

$$(5.8)\qquad \langle A(t_n, \boldsymbol{w}_n^{hk}), \boldsymbol{w}_n^{hk}\rangle \geq \alpha\|\boldsymbol{w}_n^{hk}\|_V^2.$$

From the properties of B, we obtain

$$(5.9)\qquad \begin{aligned} \langle B\boldsymbol{u}_n^{hk}, \boldsymbol{u}_n^{hk} - \boldsymbol{u}_{n-1}^{hk}\rangle &= \frac{1}{2}\langle B\boldsymbol{u}_n^{hk}, \boldsymbol{u}_n^{hk}\rangle - \frac{1}{2}\langle B\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{u}_{n-1}^{hk}\rangle \\ &\quad + \frac{1}{2}\langle B(\boldsymbol{u}_n^{hk} - \boldsymbol{u}_{n-1}^{hk}), \boldsymbol{u}_n^{hk} - \boldsymbol{u}_{n-1}^{hk}\rangle \\ &\geq \frac{1}{2}\langle B\boldsymbol{u}_n^{hk}, \boldsymbol{u}_n^{hk}\rangle - \frac{1}{2}\langle B\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{u}_{n-1}^{hk}\rangle. \end{aligned}$$

Moreover,

$$(5.10)\qquad \langle \boldsymbol{f}_n, \boldsymbol{w}_n^{hk}\rangle \leq \|\boldsymbol{f}_n\|_{V^*}\|\boldsymbol{w}_n^{hk}\|_V \leq \frac{\alpha}{4}\|\boldsymbol{w}_n^{hk}\|_V^2 + \frac{1}{\alpha}\|\boldsymbol{f}_n\|_{V^*}^2.$$

From (5.3) and (3.18), we get

$$(5.11)\qquad \int_{\Gamma_3} \boldsymbol{\xi}_{n\,\tau}^{hk} \cdot \boldsymbol{w}_{n\tau}^{hk}\, d\Gamma \leq \frac{1}{\alpha}SCC_0^2 + \left(SCC_0^2 + \frac{\alpha}{4}\right)\|\boldsymbol{w}_n^{hk}\|_V^2.$$

Using (5.7)–(5.11) we obtain, with $c_0 = \alpha/2 - SCC_0^2$,

$$\begin{aligned} &\rho\frac{1}{2}\|\boldsymbol{w}_n^{hk}\|_H^2 + \rho\frac{1}{2}\|\boldsymbol{w}_n^{hk} - \boldsymbol{w}_{n-1}^{hk}\|_H^2 + c_0 k\,\|\boldsymbol{w}_n^{hk}\|_V^2 + \frac{1}{2}\langle B\boldsymbol{u}_n^{hk}, \boldsymbol{u}_n^{hk}\rangle \\ &\quad \leq k\frac{1}{\alpha}\|\boldsymbol{f}_n\|_{V^*}^2 + k\frac{1}{\alpha}SCC_0^2 + \rho\frac{1}{2}\|\boldsymbol{w}_{n-1}^{hk}\|_H^2 + \frac{1}{2}\langle B\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{u}_{n-1}^{hk}\rangle. \end{aligned}$$

Summing up the last inequality for $n = 1, \ldots, N$ we obtain

$$\begin{aligned} &\rho\frac{1}{2}\|\boldsymbol{w}_N^{hk}\|_H^2 + \rho\frac{1}{2}\sum_{n=1}^{N}\|\boldsymbol{w}_n^{hk} - \boldsymbol{w}_{n-1}^{hk}\|_H^2 + c_0 k\sum_{n=1}^{N}\|\boldsymbol{w}_n^{hk}\|_V^2 \\ &\quad \leq \frac{1}{\alpha}\|\boldsymbol{f}\|_{\mathcal{V}^*}^2 + T\frac{1}{\alpha}SCC_0^2 + \rho\frac{1}{2}\|\boldsymbol{u}_1^h\|_H^2 + \frac{1}{2}\langle B\boldsymbol{u}_0^h, \boldsymbol{u}_0^h\rangle. \end{aligned}$$

From the last inequality and (3.22) we obtain (5.6). ∎

Now we state a result on error estimation.

THEOREM 5.2. *Assume $H(\mathcal{A})$, $H(\mathcal{B})$, $H(\mu)$, $H(f)$, H_0, and* (3.22)*, and for the solution $\boldsymbol{u}$ of Problem $\mathcal{P}_V$,*

$$(5.12)\qquad \boldsymbol{u} \in C^2(0,T;H) \cap C^1(0,T;V), \quad \dot{\boldsymbol{u}}_\tau \in C(0,T;L^2(\Gamma_3;\mathbb{R}^d)).$$

Let $\{\boldsymbol{w}_n^{hk}\}_{n=0}^N$ be the solution of Problem $\mathcal{P}_V^{kh}$ and let $\{\boldsymbol{u}_n^{hk}\}_{n=0}^N$ be given by (5.5). Then the following bound holds for all $\{\boldsymbol{v}_j^h\}_{j=1}^N \subset V^h$:

$$
\begin{aligned}
(5.13)\quad & \max_{1\le n\le N}\left\{\|\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}\|_H^2+\sum_{j=1}^n k\|\boldsymbol{w}_j-\boldsymbol{w}_j^{hk}\|_V^2\right\}\\
&\le c\Bigg[k\sum_{j=1}^N\left(\|\dot{\boldsymbol{w}}_j-\delta\boldsymbol{w}_j\|_H^2+\|\boldsymbol{w}_j-\boldsymbol{v}_j^h\|_V^2\right)+\max_{1\le n\le N}\|\boldsymbol{w}_{\tau n}-\boldsymbol{v}_{\tau n}^h\|_{L^2(\Gamma_3;\mathbb{R}^d)}\\
&\quad+\frac{1}{k}\sum_{j=1}^{N-1}\|(\boldsymbol{w}_j-\boldsymbol{v}_j^h)-(\boldsymbol{w}_{j+1}-\boldsymbol{v}_{j+1}^h)\|_H^2+\max_{1\le n\le N}\|\boldsymbol{w}_n-\boldsymbol{v}_n^h\|_H^2\\
&\quad+\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V^2+k^2\|\boldsymbol{u}\|_{H^2(0,T;V)}^2+\|\boldsymbol{w}_0-\boldsymbol{u}_1^h\|_H^2\Bigg].
\end{aligned}
$$

Proof. Taking the same $\boldsymbol{v}^h\in V^h$ in (3.20) and (5.2) we obtain for $n=1,\dots,N$,

$$
\begin{aligned}
(5.14)\quad & \left(\rho(\dot{\boldsymbol{w}}_n-\delta\boldsymbol{w}_n^{hk}),\boldsymbol{v}^h\right)_H+\langle A_n(\boldsymbol{w}_n)-A_n(\boldsymbol{w}_n^{hk}),\boldsymbol{v}^h\rangle\\
&+\langle B(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\boldsymbol{v}^h\rangle+\int_{\Gamma_3}(\boldsymbol{\xi}_{n\,\tau}^{hk}-\boldsymbol{\xi}_{n\tau})\cdot\boldsymbol{v}_\tau^h\,d\Gamma=0.
\end{aligned}
$$

From (5.14) we get

$$
\begin{aligned}
&\left(\rho(\dot{\boldsymbol{w}}_n-\delta\boldsymbol{w}_n^{hk}),\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}\right)_H+\langle A_n(\boldsymbol{w}_n)-A_n(\boldsymbol{w}_n^{hk}),\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}\rangle\\
&\quad+\langle B(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}\rangle+\int_{\Gamma_3}(\boldsymbol{\xi}_{n\,\tau}^{hk}-\boldsymbol{\xi}_{n\tau})\cdot(\boldsymbol{w}_{n\tau}-\boldsymbol{w}_{n\,\tau}^{hk})\,d\Gamma\\
&\quad=\left(\rho(\dot{\boldsymbol{w}}_n-\delta\boldsymbol{w}_n^{hk}),\boldsymbol{w}_n-\boldsymbol{v}^h\right)_H+\langle A_n(\boldsymbol{w}_n)-A_n(\boldsymbol{w}_n^{hk}),\boldsymbol{w}_n-\boldsymbol{v}^h\rangle\\
&\qquad+\langle B(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\boldsymbol{w}_n-\boldsymbol{v}^h\rangle+\int_{\Gamma_3}(\boldsymbol{\xi}_{n\,\tau}^{hk}-\boldsymbol{\xi}_{n\tau})\cdot(\boldsymbol{w}_{n\tau}-\boldsymbol{v}_\tau^h)\,d\Gamma.
\end{aligned}
$$

After some reformulation we obtain

(5.15)

$$
\begin{aligned}
&\left(\rho(\delta\boldsymbol{w}_n-\delta\boldsymbol{w}_n^{hk}),\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}\right)_H+\langle A_n(\boldsymbol{w}_n)-A_n(\boldsymbol{w}_n^{hk}),\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}\rangle\\
&\quad+\int_{\Gamma_3}(\boldsymbol{\xi}_{n\,\tau}^{hk}-\boldsymbol{\xi}_{n\tau})\cdot(\boldsymbol{w}_{n\tau}-\boldsymbol{w}_{n\,\tau}^{hk})\,d\Gamma=\left(\rho(\delta\boldsymbol{w}_n-\delta\boldsymbol{w}_n^{hk}),\boldsymbol{w}_n-\boldsymbol{v}^h\right)\\
&\quad+\left(\rho(\dot{\boldsymbol{w}}_n-\delta\boldsymbol{w}_n),(\boldsymbol{w}_n-\boldsymbol{v}^h)+(\boldsymbol{w}_n^{hk}-\boldsymbol{w}_n)\right)_H+\langle A_n(\boldsymbol{w}_n)-A_n(\boldsymbol{w}_n^{hk}),\boldsymbol{w}_n-\boldsymbol{v}^h\rangle\\
&\quad+\langle B(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),(\boldsymbol{w}_n-\boldsymbol{v}^h)+(\boldsymbol{w}_n^{hk}-\boldsymbol{w}_n)\rangle+\int_{\Gamma_3}(\boldsymbol{\xi}_{n\,\tau}^{hk}-\boldsymbol{\xi}_{n\tau})\cdot(\boldsymbol{w}_{n\tau}-\boldsymbol{v}_\tau^h)\,d\Gamma.
\end{aligned}
$$

Using the formula $2(a-b,a)_H=\|a-b\|_H^2+\|a\|_H^2-\|b\|_H^2$ for $a=\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}$ and $b=\boldsymbol{w}_{n-1}-\boldsymbol{w}_{n-1}^{hk}$ we obtain

$$
(5.16)\quad \frac{1}{2k}\left(\|\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}\|_H^2-\|\boldsymbol{w}_{n-1}-\boldsymbol{w}_{n-1}^{hk}\|_H^2\right)\le\left(\rho(\delta\boldsymbol{w}_n-\delta\boldsymbol{w}_n^{hk}),\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}\right)_H.
$$

By the Lipschitz continuity of A,

$$
(5.17)\quad \langle A_n(\boldsymbol{w}_n)-A_n(\boldsymbol{w}_n^{hk}),\boldsymbol{w}_n-\boldsymbol{v}^h\rangle\le L_A\|\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}\|_V\|\boldsymbol{w}_n-\boldsymbol{v}^h\|_V.
$$

From (3.18) and (3.13) we also have

$$\int_{\Gamma_3} (\boldsymbol{\xi}_{n\,\tau}^{hk} - \boldsymbol{\xi}_{n\tau}) \cdot (\boldsymbol{w}_{n\tau} - \boldsymbol{v}_\tau^h)\, d\Gamma \le C(1 + \|\boldsymbol{w}_n\|_V + \|\boldsymbol{w}_n^{hk}\|_V) \|\boldsymbol{w}_{n\tau} - \boldsymbol{v}_\tau^h\|_{L^2(\Gamma_3;\mathbb{R}^d)}. \tag{5.18}$$

In further estimations we use (5.16), strong monotonicity of A and (3.19) for the left-hand side of (5.15) and (5.17), (5.18), the properties of B, the inequalities $ab \le 2a^2 + 2b^2$ or $ab \le \epsilon a^2 + b^2/4\epsilon$, for $a, b, \epsilon > 0$, and (3.23) for its right-hand side. Thus we get, with $c_0 = m_A - S\lambda C_0^2 > 0$ and $\epsilon < c_0$,

$$\begin{aligned} &\frac{1}{2k}\left(\|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_H^2 - \|\boldsymbol{w}_{n-1} - \boldsymbol{w}_{n-1}^{hk}\|_H^2\right) + c_0\|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_V^2 \\ &\quad \le C\left(\|\dot{\boldsymbol{w}}_n - \delta\boldsymbol{w}_n\|_H^2 + \|\boldsymbol{w}_n - \boldsymbol{v}^h\|_V^2 + \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2\right) \\ &\qquad + (1 + \|\boldsymbol{w}_n\|_V + \|\boldsymbol{w}_n^{hk}\|_V)\|\boldsymbol{w}_{n\,\tau} - \boldsymbol{v}_\tau^h\|_{L^2(\Gamma_3;\mathbb{R}^d)} \\ &\qquad + \epsilon\|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_V^2 + \left(\rho(\delta\boldsymbol{w}_n - \delta\boldsymbol{w}_n^{hk}), \boldsymbol{w}_n - \boldsymbol{v}^h\right)_H. \end{aligned} \tag{5.19}$$

We replace n by j in the relation (5.19) and sum over j from 1 to n to obtain

(5.20)

$$\begin{aligned} &\|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_H^2 + 2k\,(c_0 - \epsilon)\sum_{j=1}^n \|\boldsymbol{w}_j - \boldsymbol{w}_j^{hk}\|_V^2 \\ &\quad \le \|\boldsymbol{w}_0 - \boldsymbol{w}_0^{hk}\|_V^2 + Ck\sum_{j=1}^n \left(\|\dot{\boldsymbol{w}}_j - \delta\boldsymbol{w}_j\|_H^2 + \|\boldsymbol{w}_j - \boldsymbol{v}_j^h\|_V^2 + \|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\|_V^2\right) \\ &\qquad + Ck\sum_{j=1}^n (1 + \|\boldsymbol{w}_j\|_V + \|\boldsymbol{w}_j^{hk}\|_V)\|\boldsymbol{w}_{j\tau} - \boldsymbol{v}_{j\tau}^h\|_{L^2(\Gamma_3;\mathbb{R}^d)} \\ &\qquad + 2k\sum_{j=1}^n \left(\rho(\delta\boldsymbol{w}_j - \delta\boldsymbol{w}_j^{hk}), \boldsymbol{w}_j - \boldsymbol{v}_j^h\right)_H \end{aligned}$$

for all $\{\boldsymbol{v}_j^h\}_{j=1}^n \subset V^h$. We also have

$$\begin{aligned} &\sum_{j=1}^n k\left(\rho(\delta\boldsymbol{w}_j - \delta\boldsymbol{w}_j^{hk}), \boldsymbol{w}_j - \boldsymbol{v}_j^h\right)_H \\ &\quad = \sum_{j=1}^n \left(\rho(\boldsymbol{w}_j - \boldsymbol{w}_j^{hk} - (\boldsymbol{w}_{j-1} - \boldsymbol{w}_{j-1}^{hk})), \boldsymbol{w}_j - \boldsymbol{v}_j^h\right)_H \\ &\quad \le \epsilon\|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_H^2 + C\|\boldsymbol{w}_n - \boldsymbol{v}_n^h\|_H^2 + c\|\boldsymbol{w}_0 - \boldsymbol{w}_0^h\|_H^2 + c\|\boldsymbol{w}_1 - \boldsymbol{v}_1^h\|_H^2 \\ &\qquad + \sum_{j=1}^{n-1} \rho\|\boldsymbol{w}_j - \boldsymbol{w}_j^{hk}\|_H\,\|\boldsymbol{w}_j - \boldsymbol{v}_j^h - (\boldsymbol{w}_{j+1} - \boldsymbol{v}_{j+1}^h)\|_H \\ &\quad \le \epsilon\|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_H^2 + C\|\boldsymbol{w}_n - \boldsymbol{v}_n^h\|_H^2 + c\|\boldsymbol{w}_0 - \boldsymbol{w}_0^h\|_H^2 + c\|\boldsymbol{w}_1 - \boldsymbol{v}_1^h\|_H^2 \\ &\qquad + \sum_{j=1}^{n-1} 4\rho k\|\boldsymbol{w}_j - \boldsymbol{w}_j^{hk}\|_H^2 + \frac{1}{k}\sum_{j=1}^{n-1} \|\boldsymbol{w}_j - \boldsymbol{v}_j^h - (\boldsymbol{w}_{j+1} - \boldsymbol{v}_{j+1}^h)\|_H^2. \end{aligned} \tag{5.21}$$

Taking (4.8) at time $t = t_j$ and subtracting it from (5.5) we find that

$$\|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\|_V \le \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V + \sum_{l=1}^j k\|\boldsymbol{w}_l - \boldsymbol{w}_l^{hk}\|_V + I_j, \tag{5.22}$$

where I_j is the integration error given by

$$I_j = \left\| \int_0^{t_j} \boldsymbol{w}(s)\, ds - \sum_{l=1}^{j} k \boldsymbol{w}_l \right\|_V .$$

We know that [16]

$$I_j \le k \|\boldsymbol{u}\|_{H^2(0,T;V)}.$$

From (5.22) we get

$$\|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\|_V^2 \le C \left(\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + j \sum_{l=1}^{j} k^2 \|\boldsymbol{w}_l - \boldsymbol{w}_l^{hk}\|_V^2 + k^2 \|\boldsymbol{u}\|_{H^2(0,T;V)}^2 \right);$$

using inequality $j \le n \le N$ and the fact that $Nk = T$ we estimate

$$\sum_{j=1}^{n} k \|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\|_V^2 \le CT \left(\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + k^2 \|\boldsymbol{u}\|_{H^2(0,T;V)}^2 \right) + T \sum_{j=1}^{n} k \sum_{l=1}^{j} \|\boldsymbol{w}_l - \boldsymbol{w}_l^{hk}\|_V^2. \tag{5.23}$$

Denote $e_n := \|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_H^2 + \sum_{j=1}^n k \|\boldsymbol{w}_j - \boldsymbol{w}_j^{hk}\|_V^2$ and

$$\begin{aligned} g_n := {} & \|\boldsymbol{w}_0 - \boldsymbol{w}_0^{hk}\|_V^2 + k \sum_{j=1}^{n} \left(\|\dot{\boldsymbol{w}}_j - \delta \boldsymbol{w}_j\|_H^2 + \|\boldsymbol{w}_j - \boldsymbol{v}_j^h\|_V^2 \right) \\ & + Ck \sum_{j=1}^{n} \left(1 + \|\boldsymbol{w}_j\|_V + \|\boldsymbol{w}_j^{hk}\|_V \right) \|\boldsymbol{w}_{j\tau} - \boldsymbol{v}_{j\tau}^h\|_{L^2(\Gamma_3;\mathbb{R}^d)} \\ & + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + k^2 \|\boldsymbol{u}\|_{H^2(0,T;V)}^2 + \|\boldsymbol{w}_n - \boldsymbol{v}_n^h\|_H^2 + \|\boldsymbol{w}_0 - \boldsymbol{w}_0^h\|_H^2 \\ & + \|\boldsymbol{w}_1 - \boldsymbol{v}_1^h\|_H^2 + \frac{1}{k} \sum_{j=1}^{n-1} \|\boldsymbol{w}_j - \boldsymbol{v}_j^h - (\boldsymbol{w}_{j+1} - \boldsymbol{v}_{j+1}^h)\|_H^2. \end{aligned}$$

Then, from (5.20), (5.21), and (5.23),

$$e_n \le C g_n + \sum_{j=1}^{n} k e_j \quad \text{for } n = 1, \dots, N \tag{5.24}$$

with $C > 0$. Note that from (5.6),

$$\begin{aligned} k \sum_{j=1}^{n} (1 + \|\boldsymbol{w}_j\|_V + \|\boldsymbol{w}_j^{hk}\|_V) &= nk + k \sum_{j=1}^{n} \|\boldsymbol{w}_j\|_V + k \sum_{j=1}^{n} \|\boldsymbol{w}_j^{hk}\|_V \\ &\le T + T \|\boldsymbol{w}\|_{C(0.T;V)} + \sqrt{T} \sqrt{k \sum_{j=1}^{N} \|\boldsymbol{w}_j^{hk}\|_V^2} \\ &\le C. \end{aligned} \tag{5.25}$$

From (5.24), Lemma 2.1, and (5.25) we obtain (5.13) which completes the proof of the theorem. ∎

Similar to Theorem 4.1, Theorem 5.2 can be used to produce convergence order error estimates for the fully discrete approximations with particular choices of the finite dimensional subspace V^h. As a sample result, we consider using the linear element spaces $\{V^h\}$ of (4.28).

COROLLARY 5.3. *Keep the assumptions stated in Theorem* 5.2. *Assume* Ω *is a polygonal/polyhedral domain, and let* $\{V^h\}$ *be the family of linear element spaces defined by* (4.28)*, corresponding to a regular family of finite element triangulations of* $\overline{\Omega}$ *into triangles or tetrahedrons. Let* $\boldsymbol{u}$ *and* $\{\boldsymbol{w}_n^{hk}\}_{n=0}^N$ *be solutions of Problems* $\mathcal{P}_V$ *and* $\mathcal{P}_V^{kh}$*, respectively. Assume* $\boldsymbol{u}_0 \in H^2(\Omega;\mathbb{R}^d)$*,* $\boldsymbol{u}_1 \in H^1(\Omega;\mathbb{R}^d)$*, and let* $\boldsymbol{u}_0^h, \boldsymbol{u}_1^h \in V^h$ *be defined by* (4.1)*. Let* $\{\boldsymbol{u}_n^{hk}\}_{n=0}^N$ *be defined by* (5.5)*. Under the regularity conditions*

$$\boldsymbol{u} \in C^1(0,T;H^2(\Omega;\mathbb{R}^d)) \cap H^3(0,T;H), \quad \dot{\boldsymbol{u}}_\tau \in C(0,T;H^2(\Gamma_3;\mathbb{R}^d)),$$

we have the optimal order error estimate

$$\max_{1\le n\le N}\{\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V + \|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_H\} \le c(h+k). \tag{5.26}$$

Proof. Let $\boldsymbol{v}_j^h \in V^h$ be the finite element interpolant of $\boldsymbol{u}_j$, $t \in [0,T]$, $1 \le j \le N$. Note that [16]

$$k\sum_{j=1}^N \|\dot{\boldsymbol{w}}_j - \delta \boldsymbol{w}_j\|_H^2 \le ck^2\|\boldsymbol{u}\|_{H^2(0,T;H)}^2,$$

$$\frac{1}{k}\sum_{j=1}^{N-1} \|(\boldsymbol{w}_j - \boldsymbol{v}_j^h) - (\boldsymbol{w}_{j+1} - \boldsymbol{v}_{j+1}^h)\|_H^2 \le ch^2\|\boldsymbol{u}\|_{H^2(0,T;V)}^2.$$

Then similar to the proof of Corollary 4.3, we obtain (5.26) from (5.13). ∎

6. Numerical simulations. The aim of this section is to present some numerical results to illustrate the behavior of the solution of the frictional contact problem Problem $\mathcal{P}_V$. We pay particular attention to the numerical convergence order.

The numerical solution of Problem $\mathcal{P}_V$ is based on the backward Euler divided difference for the time discretization and the finite element approximation using the linear element space (4.28) for the spatial discretization. To solve the discrete problems, we use a "convexification" iterative procedure [3, 4], which leads to a sequence of convex programming problems. For each "convexification" iteration, the coefficient of friction $\mu(|\dot{\boldsymbol{u}}_\tau|)$ is fixed to a given value depending on the tangential velocity solution $\dot{\boldsymbol{u}}_\tau$ found in the previous iteration. Then, the resulting nonsmooth convex iterative problems are solved. The frictional bilateral condition is treated by using an augmented Lagrangian approach. For details about this numerical method, we refer the reader to [1, 3, 4, 45]. For practical implementation of the method, we use additional fictitious nodes for the Lagrange multiplier in the initial mesh. Construction of these nodes depends on the contact elements used for the geometrical discretization of the interface Γ_3. In our numerical example, the discretization is based on "node-to-rigid" contact element, which is composed of one node of Γ_3 and one Lagrange multiplier node. To keep this paper to a reasonable length, we skip the details of the numerical algorithms and implementation; details on the discretization step and computational contact mechanics, including algorithms similar to that used here, can be found in [22, 23, 27, 45]. Different numerical methods in the study of such frictional problems, including the proximal bundle methods, also can be found in [17, 31, 32, 44].

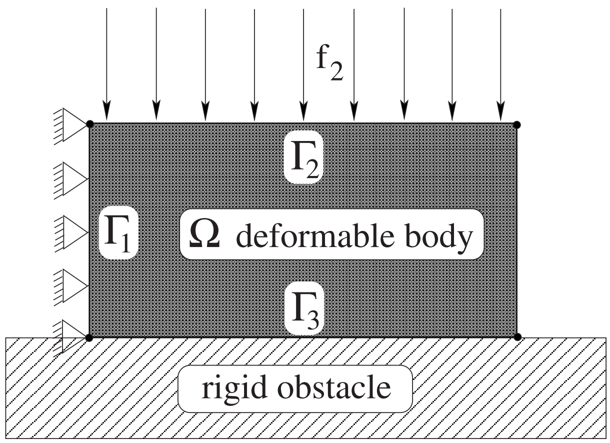

FIG. 1. *Reference configuration of the two-dimensional example.*

Numerical example. We consider the physical setting shown in Figure 1. There, $\Omega = (0, L_1) \times (0, L_2) \subset \mathbb{R}^2$ with $L_1, L_2 > 0$ and

$$\Gamma_1 = \{0\} \times [0, L_2],\ \Gamma_2 = ([0, L_1] \times \{L_2\}) \cup (\{L_1\} \times [0, L_2]),\ \Gamma_3 = [0, L_1] \times \{0\}.$$

The domain Ω represents the cross section of a three-dimensional linearly viscoelastic body subjected to the action of tractions in such a way that a plane stress hypothesis is valid. On $\Gamma_1 = \{0\} \times [0, L_2]$ the body is clamped, i.e., the displacement field vanishes there. Vertical compressions act on the part $[0, L_1] \times \{L_2\}$ of the boundary and the part $\{L_1\} \times [0, L_2]$ is traction free. No body forces are assumed to act on the elastic body during the process. The body is in frictional bilateral contact with an obstacle on the part $\Gamma_3 = [0, L_1] \times \{0\}$ of the boundary. The friction follows a nonmonotone law in which the friction coefficient depends on the tangential velocity $|\dot{\boldsymbol{u}}_\tau|$. For the coefficient of friction we choose a function $\mu : \mathbb{R}^d \to \mathbb{R}$ of the form

$$\mu(|\dot{\boldsymbol{u}}_\tau|) = (a - b)\, e^{-\alpha\, |\dot{\boldsymbol{u}}_\tau|} + b \tag{6.1}$$

with $a, b, \alpha > 0$, $a \geq b$. Note that the friction law (3.6) with (6.1) describes the slip weakening phenomenon which appears in the study of geophysical problems; see [39] for details. The coefficient of friction decreases with the slip rate from the value a to the limit value b. For this reason, the corresponding friction law is nonmonotone.

The compressible material response is governed by a linearly viscoelastic constitutive law in which the viscosity tensor $\mathcal{A}$ and the elasticity tensor $\mathcal{B}$ are given by

$$(\mathcal{A}\boldsymbol{\tau})_{\alpha\beta} = \mu_1(\tau_{11} + \tau_{22})\delta_{\alpha\beta} + \mu_2\tau_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq 2,\ \forall\, \boldsymbol{\tau} \in \mathbb{S}^2,$$
$$(\mathcal{B}\boldsymbol{\tau})_{\alpha\beta} = \frac{E\kappa}{(1+\kappa)(1-2\kappa)}(\tau_{11} + \tau_{22})\delta_{\alpha\beta} + \frac{E}{1+\kappa}\tau_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq 2,\ \forall\, \boldsymbol{\tau} \in \mathbb{S}^2,$$

where μ_1 and μ_2 are viscosity constants, E and κ are Young's modulus and Poisson's ratio of the material, and $\delta_{\alpha\beta}$ denotes the Kronecker symbol.

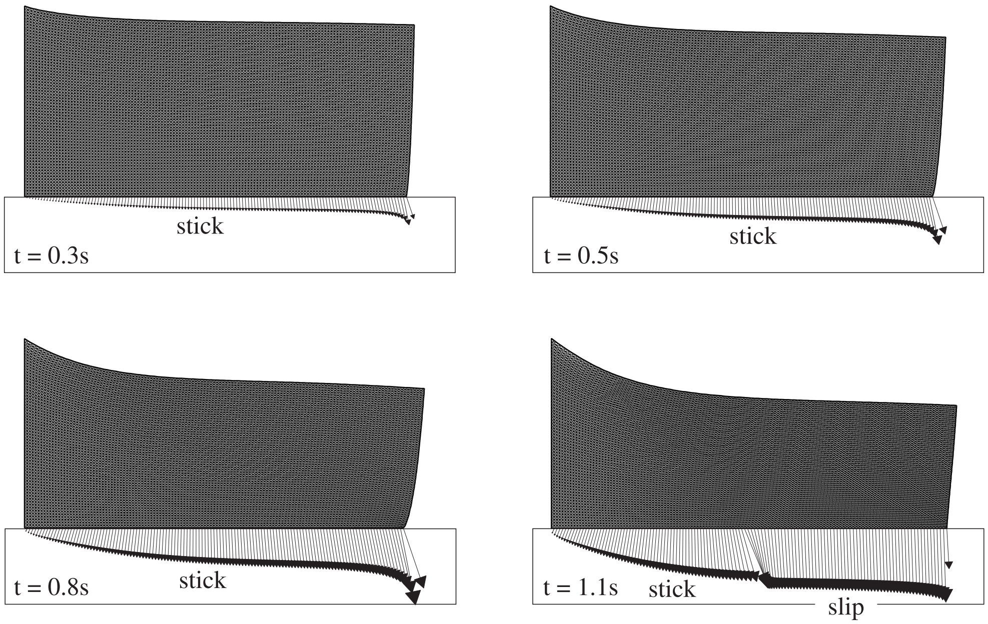

FIG. 2. *Evolution of deformed meshes and frictional contact forces during the dynamic compression process.*

For computation we use the following data:

$$L_1 = 1\,m, \quad L_2 = 0.5\,m, \quad \rho = 1000\,kg/m^3, \quad T = 1.1\,s,$$

$$\mu_1 = 50\,N/m^2, \quad \mu_2 = 100\,N/m^2, \quad E = 2000\,N/m^2, \quad \kappa = 0.3,$$

$$\boldsymbol{f}_0 = (0, -10^{-5})\,N/m^2, \qquad \boldsymbol{f}_2 = \begin{cases} (0,0)\,N/m & \text{on } \{L\} \times [0, L], \\ (0, -600\,t)\,N/m & \text{on } [0, L_1] \times \{L_2\}, \end{cases}$$

$$a = 1, \quad b = 0.1, \quad \alpha = 200.$$

Our results are presented in Figures 2, 3, and 4 and are explained below.

Mechanical behavior of the solution. In Figure 2 we plot the deformed configuration as well as the interface forces on Γ_3 during the dynamic compression process at times $t = 0.3\,s$, $t = 0.5\,s$, $t = 0.8\,s$, and $t = 1.1\,s$. At the beginning of the process, the contact nodes are in status of stick, and then at the end of the process, on the right side of Γ_3, a large proportion of contact nodes switches to status of slip when the compression of the domain is stronger. There, the friction bound has decreased with respect to the evolution of $\mu(|\dot{\boldsymbol{u}}_\tau|)$ and is reached.

In Figure 3 we plot the deformed meshes and the interface forces on Γ_3 for two different values of the coefficients a and b, respectively. Note that in the case $a = 1$ and $b = 0.1$ considered in Figure 2 the coefficient of friction is a nonmonotone function with respect to the slip rate, while in the cases $a = b = 0.1$ and $a = b = 1$ it is a constant. In the case $a = b = 0.1$ we note that all the contact nodes are in slip contact since, there, the friction bound is low and, therefore, is reached. In contrast, in the case $a = b = 1$ the friction bound is higher and, as a consequence, all the contact nodes are in stick status.

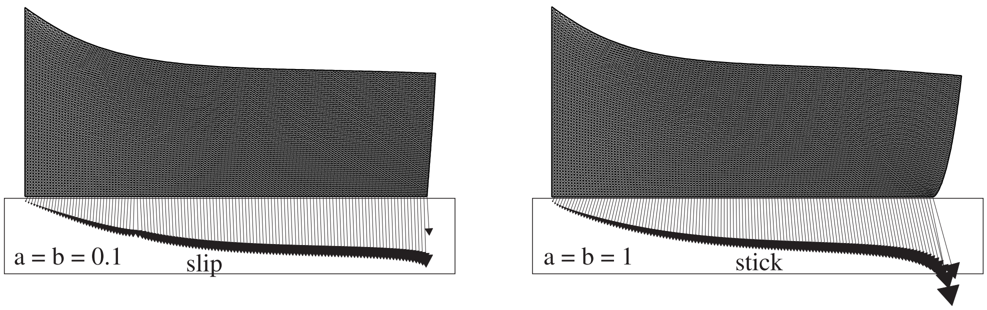

FIG. 3. *Deformed meshes and interface forces on Γ_3 corresponding to different values of the coefficients a and b.*

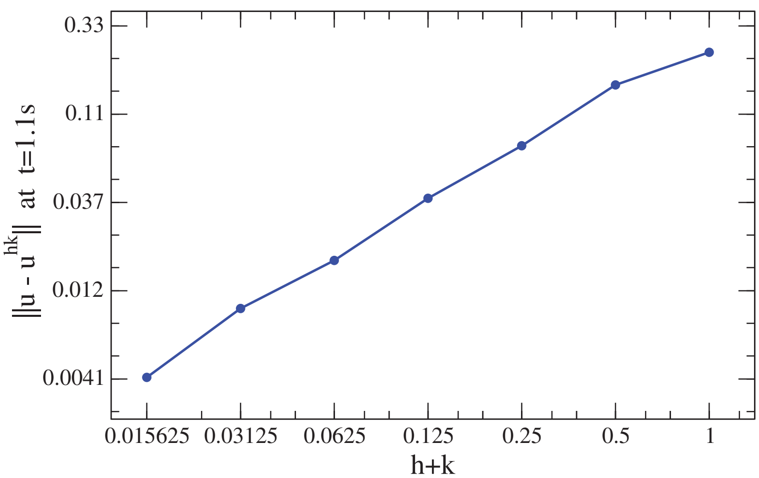

FIG. 4. *Numerical errors.*

Numerical convergence order. In order to check the convergence of the discrete scheme and to illustrate the optimal error estimate obtained in section 5, we computed a sequence of numerical solutions by using uniform discretizations of the problem domain according to the spatial discretization parameter h and time step k. For instance, the deformed configuration and the interface forces plotted in Figure 2 correspond to the choices $h = 1/128$ and $k = 1/128$.

The numerical error $\|\boldsymbol{u}-\boldsymbol{u}^{hk}\|_V$ is computed for several discretization parameters of h and k. Here, the boundary Γ of Ω is divided into $1/h$ equal parts. We start with $h = 1/2$ and $k = 1/2$, which are successively halved. The numerical solution corresponding to $h = 1/256$ and $k = 1/256$ was taken as the "exact" solution, used to compute the errors of the numerical solutions; this fine discretization corresponds to a problem with $133,896$ degrees of freedom at each time level. The numerical results are presented in Figure 4, where the dependence of the error estimate $\|\boldsymbol{u} - \boldsymbol{u}^{hk}\|_V$ with respect to h and k is plotted. A first order convergence is clearly observed, providing numerical evidence of the theoretical optimal order error estimate obtained in section 5.

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