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Finite element method for a stationary Stokes hemivariational inequality with slip boundary condition

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This paper is devoted to the study of a hemivariational inequality problem for the stationary Stokes equations with a nonlinear slip boundary condition. The hemivariational inequality is formulated with the use of the generalized directional derivative and generalized gradient in the sense of Clarke. We provide an existence and uniqueness result for the hemivariational inequality. Then we apply the finite element method to solve the hemivariational inequality. The incompressibility constraint is treated through a mixed formulation. Error estimates are derived for numerical solutions. Numerical simulation results are reported to illustrate the theoretically predicted convergence orders.

Keywords: Stokes equations; hemivariational inequality; existence; uniqueness; finite element; error estimate.

1. Introduction

Variational and hemivariational inequalities have emerged as an important tool in studying a wide range of nonlinear problems in science and engineering. Since the 1960s there has been extensive research on the modelling, theoretical analysis and numerical simulations of variational inequalities; see for instance Duvaut & Lions (1976), Kinderlehrer & Stampacchia (1980) and Baiocchi & Capelo

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(1984) on mathematical theories; Glowinski *et al.* (1981), Glowinski (1984), Hlaváček *et al.* (1988) and Haslinger *et al.* (1996) on numerical solutions and Kikuchi & Oden (1988), Han & Sofonea (2002) and Wriggers (2006) on applications in contact mechanics. While variational inequalities are concerned with nonsmooth convex energy functionals (potentials), hemivariational inequalities are mathematical problems concerning nonsmooth and nonconvex energy functionals (superpotentials). The notion of hemivariational inequalities was first introduced by Panagiotopoulos in the early 1980s (Panagiotopoulos, 1983) and is closely related to the development of the concept of the generalized gradient of a locally Lipschitz functional developed by Clarke (1975, 1983). Since then hemivariational inequalities have attracted much interest from the research community. Some comprehensive references on hemivariational inequalities include Panagiotopoulos (1993), Naniewicz & Panagiotopoulos (1995), Haslinger *et al.* (1999), Carl *et al.* (2007), Migórski *et al.* (2013) and Sofonea and Migórski (2018). In recent years optimal-order error estimates have been derived for numerical solutions of hemivariational inequalities arising in solid mechanics (cf. Han *et al.* 2014; Barboteu *et al.* 2015; Han *et al.* 2017; Han 2018; Han *et al.* 2018; Han *et al.* 2019; Han & Sofonea 2019).

Fujita (1993, 1994) investigated the boundary value problem for steady motions of viscous incompressible fluid, where he introduced some slip or leak boundary conditions of friction type. Subsequently, many theoretical results on the properties of the solution, for example, existence, uniqueness, regularity and continuous dependence on data, for Stokes problems are presented in Fujita *et al.* (1995), Fujita & Kawarada (1998), Saito & Fujita (2001), Saito (2004), Le Roux (2005), Saidi (2007) and Fang & Han (2016). The finite element approximation of the problems can be found in Li & Li (2008, 2010) and Kashiwabara (2013a,b). In these references the weak formulations of the problems are variational inequalities. In this paper we study a hemivariational inequality problem for the stationary Stokes equations with a nonlinear slip boundary condition.

Let $\Omega \subset \mathbb{R}^d$ ($d \le 3$ in applications) be an open bounded connected set with a Lipschitz boundary $\partial\Omega$. The boundary consists of two parts: $\partial\Omega = \overline{\Gamma \cup S}$ with $\mathrm{meas}(\Gamma) > 0$, $\mathrm{meas}(S) > 0$ and $\overline{\Gamma} \cap \overline{S} = \emptyset$. Denote by $\boldsymbol{n} = (n_1, \cdots, n_d)^{\mathrm{T}}$ the unit outward normal on the boundary $\partial\Omega$. For a vector-valued function $\boldsymbol{u}$ on the boundary let $u_n = \boldsymbol{u} \cdot \boldsymbol{n}$ and $\boldsymbol{u}_\tau = \boldsymbol{u} - u_n\boldsymbol{n}$ be the normal component and the tangential component, respectively. With the flow velocity field $\boldsymbol{u}$ and the pressure p we define the strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathrm{T}})$ and the stress tensor $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\nu\boldsymbol{\varepsilon}(\boldsymbol{u})$, where $\boldsymbol{I}$ is the identity matrix. Let $\sigma_n = \boldsymbol{n} \cdot \boldsymbol{\sigma}\boldsymbol{n}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{n} - \sigma_n\boldsymbol{n}$ be the normal component and the tangential component of $\boldsymbol{\sigma}$.

We consider the Stokes problem

$$-\nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{1.2}$$

with the following boundary conditions:

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma, \tag{1.3}$$

$$u_n = 0, \quad -\boldsymbol{\sigma}_\tau \in \partial j(\boldsymbol{u}_\tau) \quad \text{on } S. \tag{1.4}$$

Here, $j(\boldsymbol{u}_\tau)$ is a shorthand notation for $j(\boldsymbol{x}, \boldsymbol{u}_\tau)$; $j : S \times \mathbb{R}^d \to \mathbb{R}$ is assumed locally Lipschitz and ∂j is the subdifferential of $j(\boldsymbol{x}, \cdot)$ in the sense of Clarke (cf. Section 2), ν is a positive quantity representing the viscosity coefficient and $\boldsymbol{f}$ is the density of external forces. In the literature (1.4) is known as a slip boundary condition. The first part $u_n = 0$ means that the normal velocity is zero on the boundary S, so the fluid cannot pass through S outside the domain. The second part represents a friction condition,

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relating the friction $\boldsymbol{\sigma}_\tau$ and the tangential velocity $\boldsymbol{u}_\tau$. This relation is of nonmonotone type when the potential j is not a convex function.

The organization of this paper is as follows. In Section 2 we present some definitions and auxiliary material. In Section 3 we introduce several different variational formulations of the Stokes problem, establish their equivalence and study the well-posedness of the weak formulations. In Section 4 we apply the finite element method to solve the hemivariational inequality and derive error bounds. In Section 5 we present numerical examples to illustrate the theoretically predicted convergence orders.

2. Preliminaries

For a normed space X we denote by $\|\cdot\|_X$ its norm, by X^* its topological dual and by $\langle\cdot,\cdot\rangle_{X^*\times X}$ the duality pairing between X^* and X. The symbol X_w is used for the space X endowed with the weak topology. Weak convergence will be indicated by the symbol $\rightharpoonup$. We denote the Euclidean norm in $\mathbb{R}^n$ by $|\cdot|$. The symbol 2^{X^*} represents the set of all subsets of X^*. For simplicity in exposition, in the following, we always assume X is a Banach space, unless stated otherwise.

We first recall the definitions of generalized directional derivative and generalized gradient in the sense of Clarke for a locally Lipschitz function.

Definition 2.1 (Clarke, 1983) Let $f : X \to \mathbb{R}$ be a locally Lipschitz function. The generalized directional derivative of f at $x \in X$ in the direction $v \in X$, denoted by $f^0(x;v)$, is defined by

$$f^0(x;v) = \limsup_{y\to x,\lambda\to 0+} \frac{f(y+\lambda v)-f(y)}{\lambda}.$$

The generalized gradient or subdifferential of f at x, denoted by $\partial f(x)$, is a subset of the dual space X^* given by

$$\partial f(x) = \{\zeta \in X^* \mid f^0(x;v) \geq \langle\zeta, v\rangle_{X^*\times X}\ \forall\, v\in X\}. \tag{2.1}$$

A locally Lipschitz function f is said to be regular (in the sense of Clarke) at $x\in X$ if for all $v \in X$, the one-sided directional derivative $f'(x;v)$ exists and $f^0(x;v) = f'(x;v)$.

We then recall the definition of pseudomonotonicity, first for a single-valued operator.

Definition 2.2 (Zeidler, 1990) An operator $F : X \to X^*$ is said to be pseudomonotone, if

(i) F is bounded (i.e., it maps bounded subsets of X into bounded subsets of X^*);

(ii) $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty}\langle Fu_n, u_n-u\rangle_{X^*\times X} \leq 0$ imply

$$\langle Fu, u-v\rangle_{X^*\times X} \leq \liminf_{n\to\infty}\langle Fu_n, u_n-v\rangle_{X^*\times X} \quad \forall\, v\in X.$$

It can be proved (see Migórski & Ochal, 2005, for example) that an operator $F : X \to X^*$ is pseudomonotone iff it is bounded and $u_n \rightharpoonup u$ in X together with $\limsup_{n\to\infty}\langle Fu_n, u_n-u\rangle_{X^*\times X} \leq 0$ implies $Fu_n \rightharpoonup Fu$ in X^* and $\lim_{n\to\infty}\langle Fu_n, u_n-u\rangle_{X^*\times X} = 0$.

We will apply the following surjectivity result, adapted from that found in Migórski *et al.* (2017) and Han *et al.* (2017).

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THEOREM 2.3 Let X be a reflexive Banach space, X_j a Banach space, $\gamma_j \in \mathcal{L}(X, X_j)$ and denote by $\|\gamma_j\|$ the operator norm of γ_j. Assume $A\colon X \to X^*$ is pseudomonotone and strongly monotone: for a constant $m_A > 0$,

$$\langle Av_1 - Av_2, v_1 - v_2\rangle \geq m_A \|v_1 - v_2\|_X^2 \quad \forall\, v_1, v_2 \in X. \tag{2.2}$$

Further assume $J\colon X_j \to \mathbb{R}$ is locally Lipschitz, and there are constants $c_0, c_1, \alpha_j \geq 0$ such that

$$\|\partial J(z)\|_{X_j^*} \leq c_0 + c_1 \|z\|_{X_j} \quad \forall\, z \in X_j, \tag{2.3}$$

$$J^0(z_1; z_2 - z_1) + J^0(z_2; z_1 - z_2) \leq \alpha_j \|z_1 - z_2\|_{X_j}^2 \quad \forall\, z_1, z_2 \in X_j. \tag{2.4}$$

Then under the assumption

$$\alpha_j \|\gamma_j\|^2 < m_A, \tag{2.5}$$

for any $f \in X^*$ there is a unique solution $u \in X$ to the problem

$$\langle Au, v\rangle + J^0(\gamma_j u; \gamma_j v) \geq \langle f, v\rangle \quad \forall\, v \in X. \tag{2.6}$$

3. Variational formulations

We denote by $\mathbb{S}^d$ the space of second-order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The canonical inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \quad \|\boldsymbol{v}\|_{\mathbb{R}^d} = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2} \quad \text{for all } \boldsymbol{u} = (u_i),\ \boldsymbol{v} = (v_i) \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\sigma}\|_{\mathbb{S}^d} = (\boldsymbol{\sigma} : \boldsymbol{\sigma})^{1/2} \quad \text{for all } \boldsymbol{\sigma} = (\sigma_{ij}),\ \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d.$$

Here and below the indices i and j run between 1 and d, and the summation convention over repeated indices is used.

The space $(H^m(\Omega))^d$ $(m \geq 1)$ is denoted by $\boldsymbol{H}^m(\Omega)$. For an analysis of the problem defined by (1.1)–(1.4) we introduce the following function spaces:

$$\boldsymbol{V} := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{v}|_\Gamma = \boldsymbol{0},\ v_n|_S = 0\},\ \mathbf{V}_0 := H_0^1(\Omega)^d,\ \boldsymbol{V}_\sigma := \{\boldsymbol{v} \in \boldsymbol{V}\colon\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\},$$
$$\boldsymbol{H} := L^2(\Omega)^d,\ \mathcal{H} := \{\boldsymbol{\sigma} = (\sigma_{ij})_{d\times d}\colon \sigma_{ij} = \sigma_{ji} \in L^2(\Omega),\ 1 \leq i,j \leq d\},$$
$$M := L_0^2(\Omega) = \left\{q \in L^2(\Omega) : \textstyle\int_\Omega q \,\mathrm{d}x = 0\right\},$$
$$\boldsymbol{H}_\sigma^1(\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)\colon\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\},\ \boldsymbol{H}_{0,\sigma}^1(\Omega) := \boldsymbol{V}_0 \cap \boldsymbol{H}_\sigma^1(\Omega).$$

Let

$$\mathcal{H}_1 := \{\boldsymbol{\sigma} \in \mathcal{H} : \operatorname{Div} \boldsymbol{\sigma} \in \boldsymbol{H}\}.$$

Define $\boldsymbol{\varepsilon} : \boldsymbol{H}^1(\Omega) \to \mathcal{H}$ and $\mathrm{Div} : \mathcal{H}_1 \to \boldsymbol{H}$ by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})),\ \varepsilon_{ij}(\boldsymbol{u}) = \tfrac{1}{2}(u_{i,j} + u_{i,j}), \quad \mathrm{Div}\boldsymbol{\sigma} = (\sigma_{ij,j})_d.$$

Recall the following formulas (Migórski *et al.*, 2013, Chapter 2):

$$\int_\Omega (u \operatorname{div} \boldsymbol{v} + \nabla u \cdot \boldsymbol{v})\, \mathrm{d}x = \int_{\partial\Omega} u\, v_n \mathrm{d}s \quad \forall\, u \in H^1(\Omega),\ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega), \tag{3.1}$$

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v})\mathrm{d}x + \int_\Omega \mathrm{Div}\boldsymbol{\sigma} \cdot \boldsymbol{v}\, \mathrm{d}x = \int_{\partial\Omega} \boldsymbol{\sigma}\boldsymbol{n} \cdot \boldsymbol{v}\, \mathrm{d}s \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}^1(\Omega),\ \boldsymbol{\sigma} \in H^1(\Omega; \mathbb{S}^d). \tag{3.2}$$

It is well known that the spaces $\boldsymbol{H}$ and $\mathcal{H}$ are Hilbert spaces equipped with the inner products

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{\boldsymbol{H}} = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}\, \mathrm{d}x, \quad \langle \boldsymbol{\sigma}, \boldsymbol{\tau}\rangle_{\mathcal{H}} = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\, \mathrm{d}x.$$

Let $\|\cdot\|_1$ be the norm in Hilbert space $\boldsymbol{H}^1(\Omega)$. Since $\mathrm{meas}(\Gamma) > 0$ the following Korn inequality (cf. Kikuchi & Oden, 1988, Lemma 6.2) holds:

$$\|\boldsymbol{v}\|_1 \le C_1 \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{3.3}$$

where C_1 depends only on Ω and Γ. This implies that the norm $\|\cdot\|_{\boldsymbol{V}} = \|\boldsymbol{\varepsilon}(\cdot)\|_{\mathcal{H}}$ is equivalent on $\boldsymbol{V}$ with the norm $\|\cdot\|_1$. Therefore, $(\boldsymbol{V}, \|\cdot\|_{\boldsymbol{V}})$ is a Hilbert space.

The duality pairing between $\boldsymbol{V}$ and $\boldsymbol{V}^*$ is denoted by $\langle\cdot,\cdot\rangle$. Identifying $\boldsymbol{H}$ with its dual we have an evolution triple $\boldsymbol{V} \subset \boldsymbol{H} \subset \boldsymbol{V}^*$ with dense, continuous and compact embeddings. We denote by $i : \boldsymbol{V} \to \boldsymbol{H}$ the identity mapping and by $i^* : \boldsymbol{V}^* \to \boldsymbol{H}$ its adjoint mapping. By the Sobolev trace theorem and (3.3) there exists a constant $C_2 > 0$ depending only on the domain Ω, Γ and S such that

$$\|\boldsymbol{v}\|_{L^2(S)^d} \le C_2 \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}. \tag{3.4}$$

By (3.4) there exists a continuous trace operator $\gamma : \boldsymbol{V} \to \boldsymbol{L}^2(S) := L^2(S)^d$ and for $\boldsymbol{v} \in \boldsymbol{V}$ we still denote by $\boldsymbol{v}$ its trace $\gamma\boldsymbol{v}$.

Introduce the following bilinear forms:

$$a(\boldsymbol{u}, \boldsymbol{v}) = 2\nu\langle \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{\mathcal{H}} \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbf{V}, \tag{3.5}$$

$$b(\boldsymbol{v}, p) = \int_\Omega p \operatorname{div}\boldsymbol{v}\, \mathrm{d}x \quad \forall\, \boldsymbol{v} \in \boldsymbol{V},\ p \in M \tag{3.6}$$

and a linear form

$$\langle \boldsymbol{f}, \boldsymbol{v}\rangle = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}x. \tag{3.7}$$

As a consequence of Korn's inequality (3.3), $a(\cdot,\cdot)$ is coercive on $\boldsymbol{V}$, that is,

$$a(\boldsymbol{v}, \boldsymbol{v}) = 2\nu\|\boldsymbol{v}\|_{\boldsymbol{V}}^2 \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}. \tag{3.8}$$

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Concerning the superpotential j we assume the following properties:
$H(j)$. $j : S \times \mathbb{R}^d \to \mathbb{R}$ is such that

(i) $j(\cdot, \boldsymbol{\xi})$ is measurable on S for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and there exists $\mathbf{e} \in \boldsymbol{L}^2(S)$ such that $j(\cdot, e(\cdot)) \in L^1(S)$;

(ii) $j(\mathbf{x}, \cdot)$ is locally Lipschitz on $\mathbb{R}^d$ for a.e. $\boldsymbol{x} \in S$;

(iii) $\|\boldsymbol{\eta}\|_{\mathbb{R}^d} \le c_0 + c_1 \|\boldsymbol{\xi}\|_{\mathbb{R}^d}$ for all $\boldsymbol{\xi} \in \mathbb{R}^d, \boldsymbol{\eta} \in \partial j(\mathbf{x}, \boldsymbol{\xi})$ a.e. $\boldsymbol{x} \in S$ with $c_0, c_1 \ge 0$;

(iv) $(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \ge -m_\tau \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|^2_{\mathbb{R}^d}$ for all $\boldsymbol{\eta}_i, \boldsymbol{\xi}_i \in \mathbb{R}^d$, $\boldsymbol{\eta}_i \in \partial j(\mathbf{x}, \boldsymbol{\xi}_i)$, $i = 1, 2$, a.e. $\mathbf{x} \in S$ with $m_\tau \ge 0$.

It can be verified that the assumption $H(j)$(iv) is equivalent to

$$j^0(\boldsymbol{\xi}_1; \boldsymbol{\xi}_2 - \boldsymbol{\xi}_1) + j^0(\boldsymbol{\xi}_2; \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \le m_\tau \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|^2_{\mathbb{R}^d} \quad \forall\, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^d. \tag{3.9}$$

Now we consider the functional $J : \boldsymbol{L}^2(S) \to \mathbb{R}$ defined by

$$J(\boldsymbol{u}) = \int_S j(\boldsymbol{x}, \boldsymbol{u})\, \mathrm{d}s, \quad \boldsymbol{u} \in \boldsymbol{L}^2(S). \tag{3.10}$$

Using arguments similar to those in the proof of Migórski *et al.* (2013, Theorem 4.20) we have the following result.

Lemma 3.1 Assume that $j : S \times \mathbb{R}^d \to \mathbb{R}$ has the properties $H(j)$. Then the functional J defined by (3.10) satisfies
$H(J)$.

(i) $J(\cdot)$ is locally Lipschitz on $\boldsymbol{L}^2(S)$;

(ii) $\|z\|_{\boldsymbol{L}^2(S)} \le \bar{c}_0 + \bar{c}_1 \|\boldsymbol{u}\|_{\boldsymbol{L}^2(S)}$ for all $\mathbf{z} \in \partial J(\boldsymbol{u})$, $\boldsymbol{u} \in L^2(S)^d$ with $\bar{c}_0 = \sqrt{3\,\mathrm{meas}(S)}\, c_0$ and $\bar{c}_1 = \sqrt{3}\, c_1$;

(iii) $\langle \mathbf{z}_1 - \mathbf{z}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2 \rangle_{L^2(S)^d} \ge -m_\tau \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{L^2(S;\mathbb{R}^d)}$ for all $z_i \in \partial J(\boldsymbol{u}_i)$, $\boldsymbol{u}_i \in \boldsymbol{L}^2(S)$, $i = 1, 2$.

We comment that in applying Theorem 2.3 later to the hemivariational inequality considered in this paper, $H(J)$(ii) corresponds to (2.3), whereas $H(J)$(iii) corresponds to (2.4) via (3.9).

Now we derive weak formulations of the boundary value problems (1.1)–(1.4). Note that the incompressibility constraint (1.2) implies

$$\Delta \boldsymbol{u} = 2\mathrm{Div}\boldsymbol{\varepsilon}(\boldsymbol{u}).$$

From the above equation and (1.1) we have

$$-2\,\nu\, \mathrm{Div}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \nabla p = \boldsymbol{f} \quad \text{in } \Omega. \tag{3.11}$$

Multiply (3.11) by an arbitrary $\boldsymbol{V} \in \boldsymbol{V}$ and integrate over Ω to get

$$-2\,\nu \int_\Omega \mathrm{Div}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{v}\, \mathrm{d}x + \int_\Omega \nabla p \cdot \boldsymbol{v}\, \mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}x. \tag{3.12}$$

Note that

$$\boldsymbol{\sigma n} \cdot \boldsymbol{v} = \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau + \sigma_n v_n.$$

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Performing integration by parts on the left side of (3.12), applying the Green-type formulas (3.1) and (3.2) and taking into account the boundary conditions (1.3) and (1.4) we obtain

$$2\nu \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x - \int_\Omega p\,\mathrm{div}\boldsymbol{v}\,\mathrm{d}x - \int_S \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau\,\mathrm{d}s = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x. \tag{3.13}$$

In view of the definition (2.1) of the Clarke subdifferential from (1.4) we have

$$-\int_S \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau\,\mathrm{d}s \le \int_S j^0(\boldsymbol{u}_\tau;\boldsymbol{v}_\tau)\,\mathrm{d}s. \tag{3.14}$$

Consequently, from (3.13), (3.14) and (1.2) we can derive the following weak formulations:

PROBLEM 3.2 Find $(\boldsymbol{u},p) \in \boldsymbol{V}\times M$ such that

$$a(\boldsymbol{u},\boldsymbol{v}) - b(\boldsymbol{v},p) + \int_S j^0(\boldsymbol{u}_\tau;\boldsymbol{v}_\tau)\mathrm{d}s \ge \langle \boldsymbol{f},\boldsymbol{v}\rangle \quad \forall\,\boldsymbol{v}\in\boldsymbol{V}, \tag{3.15}$$

$$b(\boldsymbol{u},q) = 0 \quad \forall\,q\in M. \tag{3.16}$$

PROBLEM 3.3 Find $\boldsymbol{u}\in\boldsymbol{V}_\sigma$ such that

$$a(\boldsymbol{u},\boldsymbol{v}) + \int_S j^0(\boldsymbol{u}_\tau;\boldsymbol{v}_\tau)\,\mathrm{d}s \ge \langle \boldsymbol{f},\boldsymbol{v}\rangle \quad \forall\,\boldsymbol{v}\in\boldsymbol{V}_\sigma. \tag{3.17}$$

Let us recall the well-known inf-sup condition (Temam, 1979):

$$\beta_1\|p\|_{L^2(\Omega)} \le \sup_{v\in V_0}\frac{b(\boldsymbol{v},p)}{\|\boldsymbol{v}\|_V} \quad \forall\,p\in M,$$

where $\beta_1 > 0$ is a constant.

Next we show that Problems 3.2 and 3.3 are equivalent.

THEOREM 3.4 Problems 3.2 and 3.3 are equivalent.

Proof. It is easy to see that if $(\boldsymbol{u},p)\in\boldsymbol{V}\times M$ is a solution of Problem 3.2, then $\boldsymbol{u}\in\boldsymbol{V}_\sigma$ is a solution of Problem 3.3.

Conversely, suppose that $\boldsymbol{u}\in\boldsymbol{V}_\sigma$ is a solution of Problem 3.3. Then

$$a(\boldsymbol{u},\boldsymbol{v}) = \langle \boldsymbol{f},\boldsymbol{v}\rangle \quad \forall\,\boldsymbol{v}\in\boldsymbol{H}^1_{0,\sigma}(\Omega).$$

Thus, by a classical result (Boffi *et al.*, 2013, Chapter 4), we know that there exists a function $p\in M$ such that

$$a(\boldsymbol{u},\boldsymbol{v}) - b(\boldsymbol{v},p) = \langle \boldsymbol{f},\boldsymbol{v}\rangle \quad \forall\,\boldsymbol{v}\in\boldsymbol{V}_0. \tag{3.18}$$

Let $\boldsymbol{V}\in\boldsymbol{V}$ be arbitrary and fixed. Since $b(\cdot,\cdot)$ satisfies the inf-sup condition there is a function $\boldsymbol{v}_1\in\boldsymbol{V}_0$ such that

$$b(\boldsymbol{v}_1,q) = b(\boldsymbol{v},q) \quad \forall\,q\in M.$$

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Denoting $\boldsymbol{v}_2 = \boldsymbol{v} - \boldsymbol{v}_1$ we easily get $\boldsymbol{v}_2 \in \boldsymbol{V}_\sigma$. Thus, it follows from (3.17) that

$$a(\boldsymbol{u}, \boldsymbol{v}_2) + \int_S j^0(\boldsymbol{u}_\tau; \boldsymbol{v}_{2,\tau})\mathrm{d}s \geq \langle \boldsymbol{f}, \boldsymbol{v}_2 \rangle. \tag{3.19}$$

In view of (3.18) we have

$$a(\boldsymbol{u}, \boldsymbol{v}_1) - b(\boldsymbol{v}_1, p) = \langle \boldsymbol{f}, \boldsymbol{v}_1 \rangle. \tag{3.20}$$

Therefore, from (3.19) and (3.20) we obtain

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p) + \int_S j^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\,\mathrm{d}s &= a(\boldsymbol{u}, \boldsymbol{v}_1) - b(\boldsymbol{v}_1, p) + a(\boldsymbol{u}, \boldsymbol{v}_2) + \int_S j^0(u_n; v_{2,n})\,\mathrm{d}s \\ &\geq \langle \boldsymbol{f}, \boldsymbol{v}_1 \rangle + \langle \boldsymbol{f}, \boldsymbol{v}_2 \rangle \\ &= \langle \boldsymbol{f}, \boldsymbol{v} \rangle. \end{aligned}$$

Hence, (3.15) holds. □

We are now in a position to state and prove the following existence and uniqueness result of Problem 3.3.

THEOREM 3.5 Let Ω be a bounded and connected open subset of $\mathbb{R}^d$. Suppose that $\boldsymbol{f} \in \boldsymbol{V}^*$, $H(j)$ and

$$2\,\nu > m_\tau \|\gamma\|^2. \tag{3.21}$$

Then Problem 3.3 has a unique solution $\boldsymbol{u}$ and the following bound holds:

$$\|\boldsymbol{u}\|_V \leq c\,(1 + \|\boldsymbol{f}\|_{-1}) \tag{3.22}$$

with a constant $c > 0$. Moreover, the solution $\boldsymbol{u}$ depends Lipschitz continuously on $\boldsymbol{f}$: there exists a constant $\widetilde{c} > 0$ such that for solutions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ of the problem corresponding to $\boldsymbol{f} = \boldsymbol{f}_1$ and $\boldsymbol{f}_2$,

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \leq \widetilde{c}\,\|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_{-1}. \tag{3.23}$$

Proof. For the bilinear form $a(\cdot,\cdot)$ we associate a linear continuous operator $A \in \mathcal{L}(\boldsymbol{V}, \boldsymbol{V}^*)$ defined by

$$\langle A\boldsymbol{u}, \boldsymbol{v} \rangle = a(\boldsymbol{u}, \boldsymbol{v}) \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbf{V}. \tag{3.24}$$

Then Problem 3.3 may be equivalently written in the following form: find $\boldsymbol{u} \in \boldsymbol{V}_\sigma$ such that

$$\langle A\boldsymbol{u}, \boldsymbol{v} \rangle + \int_S j^0(\boldsymbol{u}; \boldsymbol{v})\mathrm{d}s \geq \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall\, \boldsymbol{v} \in \mathbf{V}_\sigma. \tag{3.25}$$

Since

$$a(\boldsymbol{v}, \boldsymbol{v}) = 2\nu \|\boldsymbol{v}\|_V^2 \quad \forall\, \boldsymbol{v} \in \mathbf{V}, \tag{3.26}$$

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$a(\cdot,\cdot)$ is coercive. Since A is bounded, continuous and monotone, from Zeidler (1990, Proposition 27.6), we deduce that the operator A is pseudomonotone. Since $A \in \mathcal{L}(\boldsymbol{V}, \boldsymbol{V}^*)$, from (3.24) and (3.26), we know that A is coercive and strongly monotone with constant $2\nu > 0$. By applying Theorem 2.3 with $X = \boldsymbol{V}$, $X_j = L^2(S)^d$, $\gamma_j = \gamma$, $\alpha_j = m_\tau$, $m_A = 2\,\nu$, and recalling Lemma 3.1, we know that the problem

$$\boldsymbol{u} \in \boldsymbol{V}_\sigma, \quad \langle A\boldsymbol{u}, \boldsymbol{v}\rangle + J^0(\boldsymbol{u};\boldsymbol{v}) \geq \langle \boldsymbol{f}, \boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_\sigma \tag{3.27}$$

has a unique solution and (3.22) holds. Since

$$J^0(\boldsymbol{u};\boldsymbol{v}) \leq \int_S j^0(\boldsymbol{u};\boldsymbol{v})\,\mathrm{d}s$$

the solution of problem (3.27) is also a solution of problem (3.25). Through a standard argument it can be shown that a solution of problem (3.25) is unique. The bounds (3.22) and (3.23) can be derived by standard arguments; cf. Migórski *et al.* (2013, proof of Theorem 4.20). □

REMARK 3.6 By virtue of Theorem 3.4, Problem 3.2 also admits a unique solution.

In the case where the functional j is convex, Problem 3.2 reduces to a variational inequality problem studied by Fujita and other researchers. Note that in this case $m_\tau = 0$ for $H(j)$(iv) and (3.21) is trivially satisfied.

4. Finite element approximation

For simplicity in discussion we assume Ω is a polygonal/polyhedral domain in this section. Let $\{\mathcal{T}^h\}$ be a regular family of triangular partitions of $\overline{\Omega}$ into triangles. The diameter of an element $T \in \mathcal{T}^h$ is denoted by h_K, and the mesh size h is defined by $h = \max_{T\in\mathcal{T}^h} h_K$. Corresponding to the partition $\mathcal{T}^h$ we introduce finite element spaces $\boldsymbol{V}_h \subset \boldsymbol{V}$ and $M_h \subset M$ such that the discrete inf-sup condition holds: for a constant $c_0 > 0$ independent of h,

$$c_0\|q_h\|_{L^2(\Omega)} \leq \sup_{\boldsymbol{v}_h\in\boldsymbol{V}_{h0}} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_V} \quad \forall\, q_h \in M_h, \tag{4.1}$$

where

$$\boldsymbol{V}_{h0} = \boldsymbol{V}_h \cap \boldsymbol{V}_0.$$

As examples, we may use P1b/P1 finite elements (Arnold *et al.*, 1984),

$$\boldsymbol{V}_h = \{\boldsymbol{v}_h \in \boldsymbol{V} \cap C^0(\overline{\Omega})^d : \boldsymbol{v}_h|_T \in [P_1(T)]^d \oplus B(T)\ \forall\, T \in \mathcal{T}^h\}, \tag{4.2}$$

$$M_h = \{q_h \in M \cap C^0(\overline{\Omega}) : q_h|_T \in P_1(T)\ \forall\, T \in \mathcal{T}^h\}, \tag{4.3}$$

or P2/P1 finite elements (Girault & Raviart, 1986, Chapter II, Corollary 4.1),

$$\boldsymbol{V}_h = \{\boldsymbol{v}_h \in \mathbf{V} \cap C^0(\overline{\Omega})^d : \boldsymbol{v}_h|_T \in [P_2(T)]^d\ \forall\, T \in \mathcal{T}^h\}, \tag{4.4}$$

$$M_h = \{q_h \in M \cap C^0(\overline{\Omega}) : q_h|_T \in P_1(T)\ \forall\, T \in \mathcal{T}^h\}, \tag{4.5}$$

where $P_k(T)$ represents the space of polynomials of total degree less than or equal to k in T and $B(T)$ is the space of bubble functions on T.

Then we introduce the following discrete approximation of Problem 3.2.

PROBLEM 4.1 Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times M_h$ such that

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h) - \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle + \int_S j^0(\boldsymbol{u}_{h,\tau}; \boldsymbol{v}_{h,\tau})\, \mathrm{d}s \geq 0 \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{4.6}$$

$$b(\boldsymbol{u}_h, q_h) = 0 \quad \forall\, q_h \in M_h. \tag{4.7}$$

Denote

$$\boldsymbol{V}_{h,\sigma} = \left\{ \boldsymbol{v}_h \in \boldsymbol{V}_h \colon b(\boldsymbol{v}_h, q_h) = 0 \,\forall\, q_h \in M_h \right\}.$$

Eliminating the unknown p_h we obtain the following variant of Problem 4.1.

PROBLEM 4.2 Find $\boldsymbol{u}_h \in \boldsymbol{V}_{h,\sigma}$ such that

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) - \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle + \int_S j^0(\boldsymbol{u}_{h,\tau}; \boldsymbol{v}_{h,\tau})\, \mathrm{d}s \geq 0 \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_{h,\sigma}. \tag{4.8}$$

Similarly to Theorem 3.4 we have the following result.

THEOREM 4.3 Problems 4.1 and 4.2 are equivalent.

We have the existence and uniqueness of a solution to Problem 4.1.

THEOREM 4.4 Under the assumptions of Theorem 3.5, Problem 4.1 has a unique solution $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times M_h$, which satisfies the bound

$$\|\boldsymbol{u}_h\|_V \leq c\,(1 + \|\boldsymbol{f}\|_{-1}) \tag{4.9}$$

with a constant $c > 0$.

Next we present a Céa-type inequality for error estimation.

THEOREM 4.5 Suppose that the assumptions of Theorem 3.5 hold. Let $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}_h, p_h)$ be solutions of Problems 3.2 and 4.1, respectively. Then there exists a positive constant c depending only on the data of the problem such that for all $\boldsymbol{V}_h \in \boldsymbol{V}_h$ and $q_h \in M_h$,

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_V + \|p - p_h\|_{L^2(\Omega)} \leq c \left(\|\boldsymbol{u} - \boldsymbol{v}_h\|_V + \|p - q_h\|_{L^2(\Omega)} + \|\boldsymbol{u}_\tau - \boldsymbol{v}_{h\tau}\|^{1/2}_{\boldsymbol{L}^2(S)} \right). \tag{4.10}$$

Proof. From (3.15) we have

$$a(\boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_0. \tag{4.11}$$

From (4.6) we have

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h) = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_{h0}. \tag{4.12}$$

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Subtracting (4.12) from (4.11) with $\boldsymbol{v} = \boldsymbol{v}_h$ we obtain

$$a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p - p_h) = 0 \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_{h0}. \tag{4.13}$$

Therefore,

$$b(\boldsymbol{v}_h, p_h - q_h) = b(\boldsymbol{v}_h, p_h - p) + b(\boldsymbol{v}_h, p - q_h) = a(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p - q_h) \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{v}_{h0}. \tag{4.14}$$

Thus, from (4.14) and the discrete inf-sup condition (4.1) we have

$$\begin{aligned} c_0 \|p_h - q_h\|_{L^2(\Omega)} &\le \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_{h0}} \frac{b(\boldsymbol{v}_h, p_h - q_h)}{\|\boldsymbol{v}_h\|_V} \\ &= \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_{h0}} \frac{a(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p - q_h)}{\|\boldsymbol{v}_h\|_V} \\ &\le 2\nu \|\boldsymbol{u} - \boldsymbol{u}_h\|_V + \|p - q_h\|_{L^2(\Omega)}. \end{aligned}$$

Consequently,

$$\|p - p_h\|_{L^2(\Omega)} \le \|p - q_h\|_{L^2(\Omega)} + \|q_h - p_h\|_{L^2(\Omega)} \le c \left(\|p - q_h\|_{L^2(\Omega)} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_V \right). \tag{4.15}$$

For any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ we write

$$\begin{aligned} a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{u}_h) &= a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{v}_h) + a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) \\ &= a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{v}_h) + a(\boldsymbol{u}, \boldsymbol{v}_h - \boldsymbol{u}) + a(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_h) - a(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h). \end{aligned} \tag{4.16}$$

From (3.15),

$$a(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_h) \le -b(\boldsymbol{u}_h - \boldsymbol{u}, p) - \langle \boldsymbol{f}, \boldsymbol{u}_h - \boldsymbol{u} \rangle + \int_S j^0(\boldsymbol{u}_\tau; \boldsymbol{u}_{h\tau} - \boldsymbol{u}_\tau)\, \mathrm{d}s.$$

From (4.6),

$$-a(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) \le -b(\boldsymbol{v}_h - \boldsymbol{u}_h, p_h) - \langle \boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}_h \rangle + \int_S j^0(\boldsymbol{u}_{h\tau}; \boldsymbol{v}_{h\tau} - \boldsymbol{u}_{h\tau})\, \mathrm{d}s.$$

Use these inequalities and recall (3.16) and (4.7) in (4.16),

$$\begin{aligned} a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{u}_h) \le{}& a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{v}_h) + a(\boldsymbol{u}, \boldsymbol{v}_h - \boldsymbol{u}) - b(\boldsymbol{u}_h, p) - b(\boldsymbol{v}_h, p_h) - \langle \boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u} \rangle \\ &+ \int_S \left[j^0(\boldsymbol{u}_\tau; \boldsymbol{u}_{h\tau} - \boldsymbol{u}_\tau) + j^0(\boldsymbol{u}_{h\tau}; \boldsymbol{v}_{h\tau} - \boldsymbol{u}_{h\tau}) \right] \mathrm{d}s. \end{aligned}$$

Since

$$j^0(\boldsymbol{u}_{h\tau}; \boldsymbol{v}_{h\tau} - \boldsymbol{u}_{h\tau}) \le j^0(\boldsymbol{u}_{h\tau}; \boldsymbol{v}_{h\tau} - \boldsymbol{u}_\tau) + j^0(\boldsymbol{u}_{h\tau}; \boldsymbol{u}_\tau - \boldsymbol{u}_{h\tau})$$

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and by (3.9),

$$j^0(\boldsymbol{u}_\tau;\boldsymbol{u}_{h\tau}-\boldsymbol{u}_\tau)+j^0(\boldsymbol{u}_{h\tau};\boldsymbol{u}_\tau-\boldsymbol{u}_{h\tau})\le m_\tau\|\boldsymbol{u}_\tau-\boldsymbol{u}_{h\tau}\|^2,$$

we then have

$$\begin{aligned}a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}-\boldsymbol{u}_h)\le\ & a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}-\boldsymbol{v}_h)+a(\boldsymbol{u},\boldsymbol{v}_h-\boldsymbol{u})-b(\boldsymbol{u}_h,p)-b(\boldsymbol{v}_h,p_h)-\langle\boldsymbol{f},\boldsymbol{v}_h-\boldsymbol{u}\rangle\\ &+\int_S j^0(\boldsymbol{u}_{h\tau};\boldsymbol{v}_{h\tau}-\boldsymbol{u}_\tau)\,\mathrm{d}s+m_\tau\|\gamma\|^2\|\boldsymbol{u}-\boldsymbol{u}_h\|_V^2.\end{aligned}\tag{4.17}$$

Perform an integration by parts on (4.11),

$$-\int_\Omega \operatorname{div}\boldsymbol{\sigma}(\boldsymbol{u})\cdot\boldsymbol{v}\,\mathrm{d}x=\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x\quad\forall\,\boldsymbol{v}\in\boldsymbol{V}_0.$$

Thus,

$$-\operatorname{div}\boldsymbol{\sigma}(\boldsymbol{u})=\boldsymbol{f}\quad\text{a.e. in }\Omega.$$

Now multiply the above equation by an arbitrary $\boldsymbol{V}\in\boldsymbol{V}$ and integrate over Ω to obtain

$$-\int_S \boldsymbol{\sigma}_\tau\cdot\boldsymbol{v}_\tau\,\mathrm{d}s+\int_\Omega\boldsymbol{\sigma}(\boldsymbol{u}):\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x=\int_\Omega\boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x,$$

i.e.,

$$a(\boldsymbol{u},\boldsymbol{v})-b(\boldsymbol{v},p)=\langle\boldsymbol{f},\boldsymbol{v}\rangle+\int_S\boldsymbol{\sigma}_\tau\cdot\boldsymbol{v}_\tau\,\mathrm{d}s\quad\forall\,\boldsymbol{v}\in\boldsymbol{V}.\tag{4.18}$$

Take $\boldsymbol{v}=\boldsymbol{v}_h-\boldsymbol{u}$ in (4.18):

$$a(\boldsymbol{u},\boldsymbol{v}_h-\boldsymbol{u})-b(\boldsymbol{v}_h-\boldsymbol{u},p)=\langle\boldsymbol{f},\boldsymbol{v}_h-\boldsymbol{u}\rangle+\int_S\boldsymbol{\sigma}_\tau\cdot(\boldsymbol{v}_{h\tau}-\boldsymbol{u}_\tau)\,\mathrm{d}s.$$

Use this relation in (4.17),

$$\begin{aligned}a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}-\boldsymbol{u}_h)\le\ & a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}-\boldsymbol{v}_h)+b(\boldsymbol{v}_h-\boldsymbol{u},p-p_h)+b(\boldsymbol{u}-\boldsymbol{u}_h,p-q_h)\\ &+\int_S\Big[\boldsymbol{\sigma}_\tau\cdot(\boldsymbol{v}_{h\tau}-\boldsymbol{u}_\tau)+j^0(\boldsymbol{u}_{h\tau};\boldsymbol{v}_{h\tau}-\boldsymbol{u}_\tau)\Big]\,\mathrm{d}s+m_\tau\|\gamma\|^2\|\boldsymbol{u}-\boldsymbol{u}_h\|_V^2.\end{aligned}\tag{4.19}$$

Since

$$\begin{aligned}|b(\boldsymbol{v}_h-\boldsymbol{u},p-p_h)|&\le\|p-p_h\|_{L^2(\Omega)}\|\boldsymbol{u}-\boldsymbol{v}_h\|_V,\\ |b(\boldsymbol{u}-\boldsymbol{u}_h,p-q_h)|&\le\|p-q_h\|_{L^2(\Omega)}\|\boldsymbol{u}-\boldsymbol{u}_h\|_V\end{aligned}$$

and

$$\begin{aligned}a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}-\boldsymbol{u}_h)&=2\nu\|\boldsymbol{u}-\boldsymbol{u}_h\|_V^2,\\ a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}-\boldsymbol{v}_h)&\le2\nu\|\boldsymbol{u}-\boldsymbol{u}_h\|_V\|\boldsymbol{u}-\boldsymbol{v}_h\|_V,\end{aligned}$$

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we derive from (4.19) that

$$(2\nu - m_\tau\|\gamma\|^2)\|\boldsymbol{u}-\boldsymbol{u}_h\|_V^2 \le \left(2\nu\|\boldsymbol{u}-\boldsymbol{v}_h\|_V + \|p-q_h\|_{L^2(\Omega)}\right)\|\boldsymbol{u}-\boldsymbol{u}_h\|_V$$

$$+ \|p-p_h\|_{L^2(\Omega)}\|\boldsymbol{u}-\boldsymbol{v}_h\|_V$$

$$+ \left(2\sqrt{2S}c_0 + 2\sqrt{2}cc_1\|\gamma\|(1+\|\boldsymbol{f}\|_{-1})\right)\|\boldsymbol{u}_\tau - \boldsymbol{v}_{h\tau}\|_{\boldsymbol{L}^2(S)}. \tag{4.20}$$

Applying Young's inequality, from (3.21), (4.15) and (4.20), we get the bound (4.10). □

Applying Theorem 4.5, when the P1b/P1 elements (4.2)and (4.3) are used, we have following error bound.

THEOREM 4.6 Suppose that the assumptions of Theorem 4.5 hold. Let $(\boldsymbol{u}, p)$ be the solution of Problem 3.2 and $(\boldsymbol{u}_h, p_h)$ be that of Problem 4.1 with the P1b/P1 elements (4.2) and (4.3). Assume the solution regularities $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$, $\boldsymbol{u}_\tau|_S \in \boldsymbol{H}^2(S)$ and $p \in H^1(\Omega)$. Then we have

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_V + \|p-p_h\|_{L^2(\Omega)} \le c\,h. \tag{4.21}$$

Proof. We take $\boldsymbol{v}_h = \Pi_h\boldsymbol{u} \in \boldsymbol{V}_h$ to be the finite element interpolant of $\boldsymbol{u}$ and $q_h = P_h p \in M_h$ to be the L^2-projection of p in (4.10). By the standard finite element approximation theory (cf. Brenner & Scott, 2008; Ciarlet, 1978),

$$\|\boldsymbol{u}-\Pi_h\boldsymbol{u}\|_V \le c\,h\,\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)},$$

$$\|p-P_hp\|_{L^2(\Omega)} \le c\,h\,\|p\|_{\boldsymbol{H}^1(\Omega)},$$

$$\|\boldsymbol{u}_\tau - (\Pi_h\boldsymbol{u})_\tau\|_{\boldsymbol{L}^2(S)} \le c\,h^2\|\boldsymbol{u}_\tau\|_{\boldsymbol{H}^2(S)}.$$

Then we get (4.21) from (4.10). □

REMARK 4.7 The solution regularity assumption $\boldsymbol{u}_\tau|_S \in \boldsymbol{H}^2(S)$ can be replaced by its weaker piecewise counterpart. More precisely, express S as the union of closed flat components with disjoint interiors

$$S = \cup_{i=1}^{i_0} S_i,$$

where each S_i is either a line segment ($d = 2$) or a polygon ($d = 3$). Assume the finite element partition is compatible in the sense that if the intersection of one side/face of an element with one set S_i has a positive $(d-1)$-dimensional measure, then the side/face lies entirely in S_i. Then we can replace the assumption $\boldsymbol{u}_\tau|_S \in \boldsymbol{H}^2(S)$ by $\boldsymbol{u}_\tau|_{S_i} \in \boldsymbol{H}^2(S_i)$, $1 \le i \le i_0$.

In the statement of Theorem 4.6, if we drop the solution regularity assumption for $\boldsymbol{u}_\tau$ on S or S_i, $1 \le i \le i_0$, then (4.10) will lead to a nonoptimal-order error bound $c\,h^{3/4}$.

Similarly to Theorem 4.6 we have the next result.

THEOREM 4.8 Suppose that the assumptions of Theorem 4.5 hold. Let $(\boldsymbol{u}, p)$ be the solution of Problem 3.2 and $(\boldsymbol{u}_h, p_h)$ be that of Problem 4.1 with the P2/P1 elements (4.4) and (4.5). Assume the solution

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regularities $\boldsymbol{u} \in \boldsymbol{H}^3(\Omega)$, $\boldsymbol{u}_\tau|_S \in \boldsymbol{H}^3(S)$ and $p \in \mathbf{H}^2(\Omega)$. Then we have

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_V + \|p - p_h\|_{L^2(\Omega)} \le c\, h^{3/2}. \tag{4.22}$$

5. Numerical experiment

Let Ω be a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$, which is decomposed into two parts: Γ for the Dirichlet boundary and S for the slip boundary. Let

$$J(\boldsymbol{v}) = \int_S j(\boldsymbol{v})\,\mathrm{d}s, \quad j(\boldsymbol{v}) = \int_0^{|v_\tau|} \mu(t)\,\mathrm{d}t. \tag{5.1}$$

Then the boundary condition on S can be also expressed as

$$u_n = 0, \quad |\sigma_\tau| \le \mu(0) \text{ if } u_\tau = 0, \quad \sigma_\tau = -\mu(|u_\tau|)\,\mathrm{sgn}(u_\tau) \text{ if } u_\tau \ne 0. \tag{5.2}$$

Here, σ_τ is the tangential component of the vector $\boldsymbol{\sigma n}$, and

$$\mu(t) = (a - b)\, e^{-\alpha\, t} + b, \tag{5.3}$$

where $a > b$ and α are non-negative constants. It can be verified that $H(j)$ is satisfied with $m_\tau = \alpha\,(a-b)$ for $H(j)$(iv).

5.1 *Numerical implementation*

We use the P1b/P1 elements for the discretization. The unknowns in the numerical problem are the node values of the approximate velocity and pressure. Let n_u be the number of finite element nodes in $\overline{\Omega}\backslash\Gamma$ and we denote by $\mathbf{u} \in \mathbb{R}^{2n_u}$ the solution vector of the node values of the velocity. In the numerical implementation, the functional $J(\boldsymbol{v})$ is approximated by $J^h(\mathbf{v})$ using the trapezoidal rule for the integration over the slip boundary. Note that the numerical integration for the functional J introduces an additional source of error in the numerical solution. In the context of a related variational inequality problem for the Stokes problem with a slip boundary condition, the error due to the quadrature was explicitly analysed in Kashiwabara (2013a).

After discretization the hemivariational inequality problem can be expressed as the following constrained minimization problem: find $\mathbf{u} \in \mathbb{V}$ such that

$$\boldsymbol{u} = \mathrm{argmin}_{\mathbf{v}\in\mathbb{V}} \left(\tfrac{1}{2}\mathbf{v}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{v} - \mathbf{v}^{\mathrm{T}}\boldsymbol{l} + J^h(\boldsymbol{v})\right), \tag{5.4}$$

where

$$\mathbb{V} = \{\mathbf{v} \in \mathbb{R}^{2n_u} \mid \mathbf{N}\mathbf{v} = \mathbf{0} \text{ and } \mathbf{B}\mathbf{v} = \mathbf{0}\}.$$

In the definition of the space $\mathbb{V}$ the impermeability constraint on S is represented by $\mathbf{N}\mathbf{v} = 0$, whereas the discrete incompressibility constraint is represented by $\mathbf{B}\mathbf{v} = 0$. The presence of the nonsmooth and nonconvex function $J^h(\cdot)$ complicates the minimization problem.

The nonconvexity is dealt with by approximating the minimization (5.4) by a sequence of convex problems as was originally presented in Mistakidis & Panagiotopoulos (1998). This results in the

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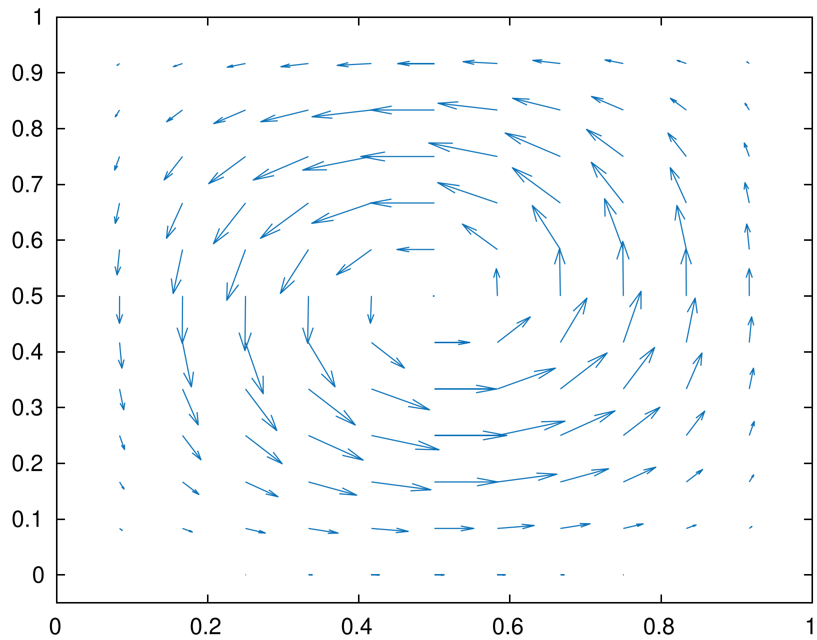

FIG. 1. Plot of velocity over Ω.

following iteration method: $\mathbf{u}^{(1)} = \mathbf{0}$, then for $k = 1, 2, \ldots,$

$$\boldsymbol{u}^{(k+1)} := \operatorname{argmin}_{\boldsymbol{v}\in\mathbb{V}} \left\{ \tfrac{1}{2}\mathbf{v}^{\mathrm{T}}\mathbf{A}\mathbf{v} - \mathbf{v}^{\mathrm{T}}\mathbf{l} + \textstyle\sum_{i=1}^{n_c} \eta_i^{(k)} |\mathbf{v}_\tau(\boldsymbol{x}_{r(i)})| \right\}, \tag{5.5}$$

$$\eta_i^{(k+1)} := \mu\left(|\boldsymbol{u}_\tau^{(k+1)}(\boldsymbol{x}_{r(i)})| \right) w_i, \quad i = 1, 2, \ldots, n_c, \tag{5.6}$$

where w_i is the weight corresponding to the trapezoidal rule applied over the boundary S, n_c is the number of nodal values on the slip boundary and r is an index map from the set $\{1, 2, \ldots, n_c\}$ to the indices corresponding to nodes on the slip boundary. The symbol $\boldsymbol{v}_\tau(\mathbf{x}_{r(i)})$ is used to denote the component of the vector $\boldsymbol{v}$ that represents the tangential component of the corresponding finite element function $\boldsymbol{v}^h$ at the node $\boldsymbol{x}_{r(i)}$. The iteration is stopped after the successive iterates for $\mathbf{u}$ and $\boldsymbol{\eta}$ are less than some tolerance ϵ. In our examples we choose $\epsilon = 10^{-6}$.

Each convex problem corresponds to an ordinary variational inequality and can be solved using the corresponding methods (e.g., Kashiwabara, 2013b; Kucera *et al.*, 2018). We solve the minimization by reformulating the problem in terms of its dual, as is done in Kucera *et al.* (2018). Let $\mathbf{E} = [\mathbf{N}^{\mathrm{T}}, \mathbf{T}^{\mathrm{T}}, \mathbf{B}^{\mathrm{T}}]^{\mathrm{T}}$ with corresponding dual variable $\boldsymbol{\lambda} = [\boldsymbol{\lambda}_{\mathrm{N}}^{\mathrm{T}}, \boldsymbol{\lambda}_{\mathrm{T}}^{\mathrm{T}}, \mathbf{p}^{\mathrm{T}}]^{\mathrm{T}}$. The dual variable $\boldsymbol{\lambda}_N$ corresponds to the impermeability constraint, $\boldsymbol{\lambda}_T$ corresponds to the slip boundary condition and $\mathbf{p}$ is a vector of nodal values of the discrete pressure function. By considering the saddle point problem associated with (5.5) and making the velocity substitution (5.8) below, the following bound-constrained minimization

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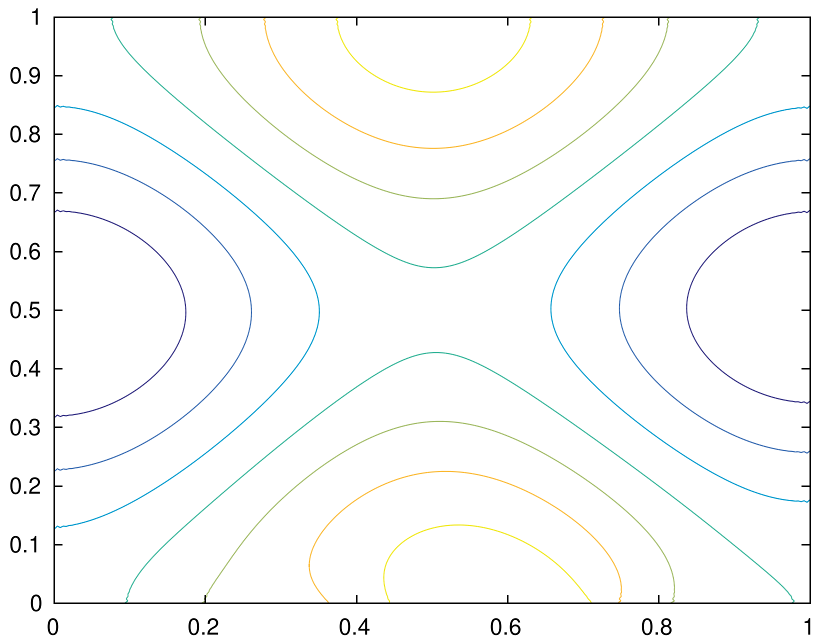

FIG. 2. Plot of pressure over Ω.

problem is obtained:

$$\boldsymbol{\lambda}^{(k+1)} = \operatorname{argmin}_{\boldsymbol{\lambda}\in\Lambda(k)} \left(\tfrac{1}{2}\boldsymbol{\lambda}^{\mathrm{T}}\mathbf{E}\mathbf{A}^{-1}\mathbf{E}^{\mathrm{T}}\boldsymbol{\lambda} - \boldsymbol{\lambda}^{\mathrm{T}}\mathbf{E}\mathbf{A}^{-1}\mathbf{l}\right),$$

where $\Lambda(k) := \left\{\boldsymbol{\lambda}_N, \boldsymbol{\lambda}_T \in \mathbb{R}^{n_c}, \mathbf{p} \in \mathbb{R}^{n_p} \mid |\boldsymbol{\lambda}_T| \le \boldsymbol{\eta}^{(k)}\right\}$, n_p is the number of nodal values in the discrete pressure solution and an inequality between two vectors is understood componentwise. The iteration (5.5–5.6) takes the following form: $\boldsymbol{\lambda}^{(1)} = \mathbf{0}$, then for $k = 1, 2, \ldots,$

$$\boldsymbol{\lambda}^{(k+1)} := \operatorname{argmin}_{\boldsymbol{\lambda}\in\Lambda(k)} \left(\tfrac{1}{2}\boldsymbol{\lambda}^{\mathrm{T}}\mathbf{E}\mathbf{A}^{-1}\mathbf{E}^{\mathrm{T}}\boldsymbol{\lambda} - \boldsymbol{\lambda}^{\mathrm{T}}\mathbf{E}\mathbf{A}^{-1}\mathbf{l}\right), \tag{5.7}$$

$$\boldsymbol{u}^{(k+1)} := \mathbf{A}^{-1}(\mathbf{l} - \mathbf{E}^{\mathrm{T}}\boldsymbol{\lambda}^{(k+1)}), \tag{5.8}$$

$$\eta_i^{(k+1)} := \mu\left(|\mathbf{u}_\tau^{(k+1)}(\boldsymbol{x}_{r(i)})|\right) w_i, \quad i = 1, 2, \ldots, n_c. \tag{5.9}$$

The minimization (5.7) can be solved in a number of ways; see Kucera *et al.* (2018) for two examples. In our experiments we employ the MATLAB interior point solver for bound constrained problems `fmincon`.

5.2 *Numerical results.* We take Ω to be a square domain $\Omega = (0,1) \times (0,1)$ and $\nu = 1$. The homogeneous Dirichlet boundary condition Γ corresponds to the top, left and right of the domain, and

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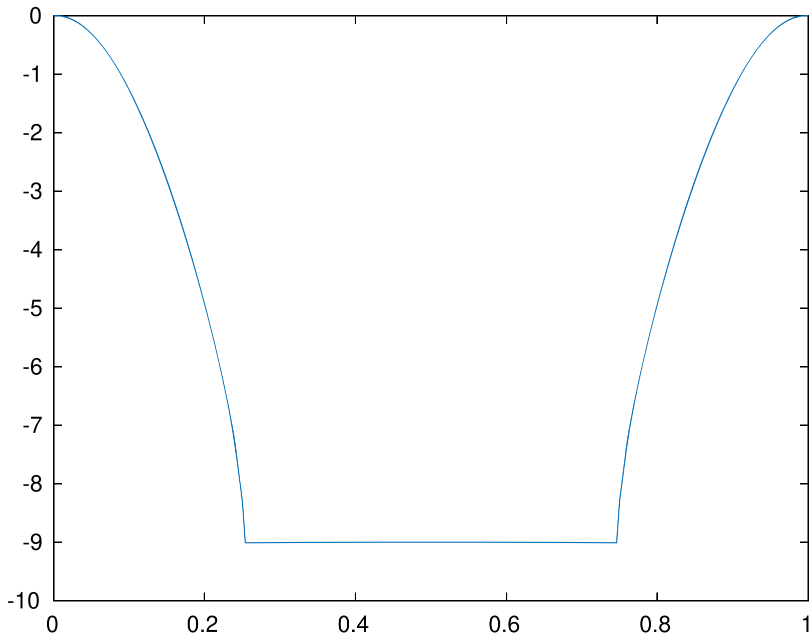

FIG. 3. Plot of σ_τ along the slip boundary.

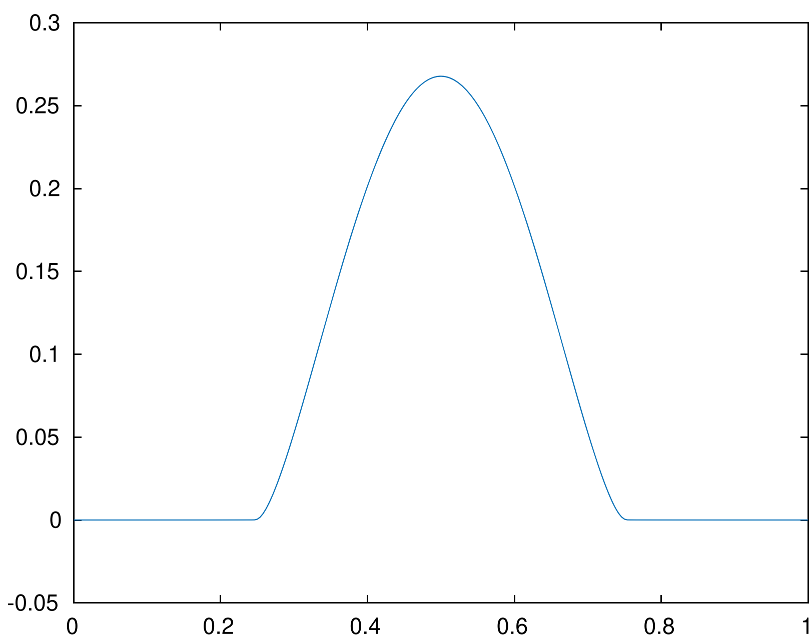

FIG. 4. Plot of u_τ along the slip boundary.

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TABLE 1 *Numerical convergence orders*

h	$\lVert \boldsymbol{u}^h - \boldsymbol{u}^* \rVert_{L^2}$	Order(h)	$\lVert \boldsymbol{u}^h - \boldsymbol{u}^* \rVert_{H^1}$	Order(h)	$\lVert p^h - p^* \rVert_{L^2}$	Order(h)
$\frac{\sqrt{2}}{4}$	7.6871e-1	—	7.0990	—	2.4621	—
$\frac{\sqrt{2}}{8}$	2.6804e-1	1.5200	4.1873	7.6159e-1	1.2244	1.0077
$\frac{\sqrt{2}}{16}$	7.2864e-2	1.8792	2.2792	8.7752e-1	4.1932e-1	1.5460
$\frac{\sqrt{2}}{32}$	1.8418e-2	1.9841	9.2515e-1	1.3007	1.3978e-1	1.5849
$\frac{\sqrt{2}}{64}$	4.4197e-3	2.0591	4.6392e-1	9.9581e-1	4.8017e-2	1.5416

the slip boundary corresponds to the bottom of the domain, $S = (0,1) \times \{0\}$. The source function is defined by $\boldsymbol{f} = -\Delta \hat{\boldsymbol{u}} + \nabla \hat{p}$ where

$$\hat{\boldsymbol{u}}(x,y) = \begin{pmatrix} -\cos(2\pi x)\sin(2\pi y) + \sin(2\pi y) \\ \sin(2\pi x)\cos(2\pi y) - \sin(2\pi x) \end{pmatrix},$$

$$\hat{p}(x,y) = 2\pi(\cos(2\pi y) - \cos(2\pi x)).$$

We use uniform triangular meshes with the interval $[0,1]$ being split into m equal subintervals with a sequence of positive integers m, then the mesh size is $h = \sqrt{2}/m$. The reference solution $(\boldsymbol{u}^*, p^*)$ is taken to be $(\boldsymbol{u}^h, p^h)$ with $h = \sqrt{2}/256$, and it is used to compute the numerical solution errors. The convergence order is defined as

$$\text{Order}(h) = \log_2\left(\frac{\lVert \boldsymbol{u}^{2h} - \boldsymbol{u}^* \rVert}{\lVert \boldsymbol{u}^h - \boldsymbol{u}^* \rVert}\right),$$

in either the $L^2(\Omega)$- or the $H^1(\Omega)$-norm.

In the simulation we take $a = 9.01$, $b = 9.0$, $\alpha = 10$. The numerical results are shown in Table 1 and Figs 1–4.

We observe that the convergence order of the velocity in the H^1-norm is around 1; this is in agreement with the theoretical error bound (4.21), even though there are no mathematical results available in the literature on the solution regularity for Problem 3.2 that is required in Theorem 4.6 or in Remark 4.7. However, the numerical convergence order for the pressure appears to be higher than the predicted order of 1.

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