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WELL-POSEDNESS AND OPTIMAL CONTROL OF A HEMIVARIATIONAL INEQUALITY FOR NONSTATIONARY STOKES FLUID FLOW

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Abstract. A time-dependent Stokes fluid flow problem is studied with nonlinear boundary conditions described by the Clarke subdifferential. We present equivalent weak formulations of the problem, one of them in the form of a hemivariational inequality. The existence of a solution is shown through a limiting procedure based on temporally semi-discrete approximations. Uniqueness of the solution and its continuous dependence on data are also established. Finally, we present a result on the existence of a solution to an optimal control problem for the hemivariational inequality.

1. **Introduction.** Let Ω be a bounded simply connected domain in $\mathbb{R}^d$ ($d = 2$ or 3) with a C^2 boundary Γ. Let $T_0 > 0$ and define $Q = \Omega \times (0, T_0)$. In this paper, we consider hemivariational inequalities for the nonstationary Stokes system

$$\boldsymbol{u}_t - \nu \Delta \boldsymbol{u} + \nabla h = \boldsymbol{f} \quad \text{in } Q, \tag{1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } Q, \tag{2}$$

where $\boldsymbol{u}$ is the flow velocity field, $\nu > 0$ the kinematic viscosity, $h = p + |\boldsymbol{u}|^2/2$ the dynamic pressure (p the pressure), $\boldsymbol{f}$ the density of external forces. The system (1)–(2) is to be supplemented by initial and boundary conditions. For simplicity in writing, we use $\boldsymbol{u}(t)$ to stand for the function $\Omega \ni \boldsymbol{x} \mapsto \boldsymbol{u}(\boldsymbol{x}, t)$. Let $\boldsymbol{u}_0$ denote the initial velocity. Then the initial condition is

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega. \tag{3}$$

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For boundary conditions, we consider the normal direction and tangential direction separately. Let $\boldsymbol{n} = (n_1, \cdots, n_d)^T$ be the unit outward normal on the boundary Γ. For a vector $\boldsymbol{u}$, $u_N = \boldsymbol{u} \cdot \boldsymbol{n}$ denotes the normal component, whereas $\boldsymbol{u}_T = \boldsymbol{u} - u_N \boldsymbol{n}$ is the tangential component. Denote $\Sigma = \Gamma \times (0, T_0)$. Then the boundary conditions considered in this paper are

$$\boldsymbol{u}_T = \boldsymbol{0} \quad \text{on } \Sigma, \tag{4}$$

$$h(t) \in \partial j(t, u_N(t)) \quad \text{on } \Sigma. \tag{5}$$

Here $j(t, u_N(t))$ is a short-hand notation for $j(\boldsymbol{x}, t, u_N(\boldsymbol{x}, t))$ and $j : \Omega \times (0, T_0) \times \mathbb{R} \to \mathbb{R}$ is called a superpotential. We assume the function j is locally Lipschitz in its third argument and write ∂j for the subdifferential of $j(\boldsymbol{x}, t, \cdot)$ in the sense of Clarke. The condition (4) models a non-slip boundary condition. The boundary condition (5) arises in the motion of a fluid through a tube or channel: the fluid pumped into Ω can leave the tube at the boundary orifices while a device can change the sizes of the latter. In this problem we regulate the normal velocity of the fluid on the boundary to reduce the total pressure on Γ.

Hemivariational inequalities were first studied by P. D. Panagiotopoulos in early eighties as weak formulations for several classes of mechanical problems with nonsmooth and nonconvex energy superpotentials. Since that time many papers and monographs on hemivariational inequalities have appeared, see for example [15, 16, 25, 27, 29, 30].

Recently, inequality problems for the time-dependent Stokes equations have been studied in [12, 13, 19, 32]. In all these papers, since the function $j(\boldsymbol{x}, t, \cdot)$ is convex, the considered problems were formulated as variational inequalities involving maximal monotone operators. In this paper, due to the lack of convexity of the superpotential j, our problem is formulated as a hemivariational inequality. To show the solution existence, we use a sequence of temporally semi-discrete approximation problems, known as the Rothe method in some references. The main idea is to replace time derivative with the backward difference scheme and to solve the associated elliptic problem at every time step to find the solution at the consecutive points of the time mesh. As long as one can solve the underlying elliptic problems, this method does not require any smoothing or other additional regularizing conditions. The Rothe method has been used in studying a variety of nonlinear problems, see for example [31, 28, 18, 5].

The mathematical theory of optimal control has in the past few decades rapidly developed into an important and seperate field of applied mathematics. In a wide range of applications, such as robotics, aviation and space technology, heat conduction, electromagnetic waves and fluid flows, there are many interesting problems in which a given cost functional has to be minimized subject to differential equations and other constraints. There is a large literature on optimal control problems. For optimal control problems for systems described by ordinary differential equations see [7], for partial differential equations see [20, 35], for variational inequalities see [4, 34] and for hemivariational inequalities see [9, 17, 21, 22, 23, 24]. In this paper, we consider an optimal control problem associated with the hemivariational inequality. The existence of an optimal solution to the control problem is shown.

The organization of this paper is as follows. In Section 2 we introduce some definitions and auxiliary material. The problem setup and some assumptions on the data are presented in Section 3. In Section 4 we show the solution existence. Solution uniqueness and continuous dependence results are established in Section

5. Section 6 is devoted to the optimal control problem for which we establish the existence of an optimal solution.

2. **Preliminaries.** For a normed space X, we denote by $\|\cdot\|_X$ its norm, by X^* its topological dual, and by $\langle\cdot,\cdot\rangle_{X^*\times X}$ the duality pairing between X^* and X. The symbol X_w is used for the space X endowed with the weak topology. Weak convergence will be indicated by the symbol $\rightharpoonup$. We denote the Euclidean norm in $\mathbb{R}^n$ by $|\cdot|$. The symbol 2^{X^*} represents the set of all subsets of X^*. We always assume X is a Banach space, unless stated otherwise. We first recall some definitions.

Let $f : X \to \mathbb{R}$ be a locally Lipschitz function. Following [8], we define the generalized directional derivative of f at $x \in X$ in the direction $v \in X$ by

$$f^0(x; v) = \limsup_{y\to x,\lambda\to 0} \frac{f(y+\lambda v) - f(y)}{\lambda}.$$

We then define the generalized gradient or subdifferential of f at x by

$$\partial f(x) = \{\zeta \in X^* \mid f^0(x; v) \geq \langle\zeta, v\rangle_{X^*\times X}\ \forall\, v \in X\}.$$

We say f is regular (in the sense of Clarke) at $x \in X$ if for all $v \in X$, the one-sided directional derivative $f'(x; v)$ exists and $f^0(x; v) = f'(x; v)$.

The concept of pseudomonotonicity plays an important role in this paper. We say a single-valued operator $F : X \to X^*$ is pseudomonotone, if

(i) F is bounded (i.e., it maps bounded subsets of X into bounded subsets of X^*);

(ii) $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty}\langle Fu_n, u_n - u\rangle_{X^*\times X} \leq 0$ imply

$$\langle Fu, u - v\rangle_{X^*\times X} \leq \liminf_{n\to\infty}\langle Fu_n, u_n - v\rangle_{X^*\times X} \quad \forall\, v \in X.$$

It can be proved (see [25], for example) that an operator $F : X \to X^*$ is pseudomonotone iff it is bounded and $u_n \rightharpoonup u$ in X together with $\limsup_{n\to\infty}\langle Fu_n, u_n - u\rangle_{X^*\times X} \leq 0$ imply $Fu_n \rightharpoonup Fu$ in X^* and $\lim_{n\to\infty}\langle Fu_n, u_n - u\rangle_{X^*\times X} = 0$.

Now let X be a reflexive Banach space. We say a multi-valued operator $F : X \to 2^{X^*}$ is pseudomonotone if

(a) F has values which are nonempty, bounded, closed and convex;

(b) F is upper semicontinuous from each finite dimensional subspace of X into X^*_w;

(c) for any sequences $\{u_n\} \subset X$ and $\{u_n^*\} \subset X^*$ such that $u_n \rightharpoonup u$ in X, $u_n^* \in Fu_n$ and $\limsup_{n\to\infty}\langle u_n^*, u_n - u\rangle_{X^*\times X} \leq 0$, we have that for every $v \in X$, there exists $u^*(v) \in Fu$ such that

$$\langle u^*(v), u - v\rangle_{X^*\times X} \leq \liminf_{n\to\infty}\langle u_n^*(v), u - v\rangle_{X^*\times X}.$$

The following proposition is usually used to check the pseudomonotonicity of a operator.

Proposition 1. ([11]) *Let X be a real reflexive Banach space, and assume that $F : X \to 2^{X^*}$ satisfies the following conditions:*

(i) *for each $v \in X$, $F(v)$ is a nonempty, closed and convex subset of X^*;*

(ii) *F is bounded;*

(iii) *if $v_n \rightharpoonup v$ in X, $v_n^* \rightharpoonup v^*$ in X^* with $v_n^* \in F(v_n)$, and $\limsup_{n\to\infty}\langle v_n^*, v_n - v\rangle_{X^*\times X} \leq 0$, then $v^* \in F(v)$ and $\langle v_n^*, v_n\rangle \to \langle v^*, v\rangle_{X^*\times X}$.*

Then the operator F is pseudomonotone.

We need the notion of coercivity. We say an operator $F : X \to 2^{X^*}$ is coercive if either $D(F)$ is bounded or $D(F)$ is unbounded and

$$\lim_{\|u\|_X \to \infty, u \in D(F)} \frac{\inf\{\langle u^*, u\rangle_{X^* \times X} \mid u^* \in Fu\}}{\|u\|_X} = +\infty.$$

The following surjectivity result for pseudomonotone and coercive operators will be applied later in the paper.

Theorem 2.1. ([11]) *Let X be a reflexive Banach space and $F : X \to 2^{X^*}$ be pseudomonotone and coercive. Then F is surjective, i.e., $R(F) = X^*$.*

For a Banach space X and a finite time interval $I = (0, T_0)$, we will use the spaces $L^p(I; X)$, $1 \le p \le \infty$. Denote by $BV(I; X)$ the space of functions of bounded total variation on I defined as follows. Let π denote a finite partition of $\overline{I}$: $0 = a_0 < a_1 < \cdots < a_n = T_0$, and let $\mathcal{F}$ be the collection of all such partitions. Then we define the total variation as

$$\|x\|_{BV(I;X)} = \sup_{\pi \in \mathcal{F}} \sum_{i=1}^{n} \|x(a_i) - x(a_{i-1})\|_X.$$

For $1 \le q < \infty$, we similarly define

$$\|x\|^q_{BV^q(I;X)} = \sup_{\pi \in \mathcal{F}} \sum_{i=1}^{n} \|x(a_i) - x(a_{i-1})\|^q_X.$$

Now for Banach spaces X, Z such that $X \subset Z$ we introduce a vector space

$$M^{p,q}(I; X, Z) = L^p(I; X) \cap BV^q(I; Z).$$

It is a Banach space for $1 \le p, q < \infty$ with the norm given by $\|\cdot\|_{L^p(I;X)} + \|\cdot\|_{BV^q(I;Z)}$.

The following result is crucial in proving the convergence of the Rothe method (cf. Theorem 4.5).

Theorem 2.2. ([18]) *Let $1 \le p, q < \infty$. Let $X_1 \subset X_2 \subset X_3$ be real Banach spaces such that X_1 is reflexive, the embedding $X_1 \subset X_2$ is compact and the embedding $X_2 \subset X_3$ is continuous. Then a bounded subset of $M^{p,q}(I; X_1, X_3)$ is relatively compact in $L^p(I; X_2)$.*

The following Aubin-Cellina convergence theorem will be used.

Theorem 2.3. ([2]) *Let $F : X \to 2^Y$ be an upper semicontinuous multifunction from a Hausdorff locally convex space X to the closed convex subsets of a Banach space Y endowed with the weak topology. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of functions such that*

(a) *$x_n : (0, T_0) \to X$ and $y_n : (0, T_0) \to Y$ are measurable functions, for all $n \in \mathbb{N}$;*

(b) *for almost all $t \in (0, T_0)$ and for every neighborhood $\mathcal{N}(0)$ of 0 in $X \times Y$ there exists $n_0 \in \mathbb{N}$ such that $(x_n(t), y_n(t)) \in \mathrm{Gr}(F) + \mathcal{N}(0)$ for all $n \ge n_0$;*

(c) *$x_n(t) \to x(t)$ for a.e. $t \in (0, T_0)$, where $x : (0, T_0) \to X$;*

(d) *$y_n \in L^1(0, T_0; Y)$ and $y_n \rightharpoonup y$ in $L^1(0, T_0; Y)$, where $y \in L^1(0, T_0; Y)$.*

Then $(x(t), y(t)) \in \mathrm{Gr}(F)$, i.e. $y(t) \in F(x(t))$ for a.e. $t \in (0, T_0)$.

3. **Weak formulations.** We introduce the weak formulations of the problem (1)–(5) in this section. Let

$$\mathcal{M} = \{\boldsymbol{v} \in C^\infty(\overline{\Omega}; \mathbb{R}^d) \mid \mathrm{div}\boldsymbol{v} = 0 \text{ in } \Omega,\ \boldsymbol{v}_T = \boldsymbol{0} \text{ on } \Gamma\}.$$

We denote by V and H the closure of $\mathcal{M}$ in the norms of $H^1(\Omega; \mathbb{R}^d)$ and $L^2(\Omega; \mathbb{R}^d)$, respectively, and identity H with its dual H^*. We define the space Z to be the closure of $\mathcal{M}$ in the norm of $H^\delta(\Omega; \mathbb{R}^d)$ with some $\delta \in (\frac{1}{2}, 1)$. Note the relations

$$V \subset Z \subset H = H^* \subset Z^* \subset V^*$$

with all embeddings being dense and compact. We denote by $\langle \cdot, \cdot \rangle$ the duality of V and V^*, by $(\cdot, \cdot)$ the scalar product in H. The norms in V and H we denote by $\| \cdot \|_V$ and $\| \cdot \|_H$. Denoting by $i : V \to Z$ the embedding injection and by $\gamma : Z \to L^2(\Gamma; \mathbb{R}^d)$ and $\gamma_0 : H^1(\Omega; \mathbb{R}^d) \to H^{1/2}(\Gamma; \mathbb{R}^d) \subset L^2(\Gamma; \mathbb{R}^d)$ the trace operators, for all $\boldsymbol{v} \in V$ we have $\gamma_0 \boldsymbol{v} = \gamma(i\boldsymbol{v})$. For simplicity we omit the notation of the embedding i and write $\gamma_0 \boldsymbol{v} = \gamma \boldsymbol{v}$. Denoting by $\iota : V \to H$ the embedding injection. For $T_0 > 0$, we define the spaces $\mathcal{V} = L^2(0, T_0; V)$, $\mathcal{Z} = L^2(0, T_0; Z)$, $\mathcal{H} = L^2(0, T_0; H)$, $\mathcal{U} = L^2(0, T_0; L^2(\Gamma; \mathbb{R}^d))$, $\mathcal{V}^* = L^2(0, T_0; V^*)$, $\mathcal{Z}^* = L^2(0, T_0; Z^*)$ and $\mathcal{W} = \{\boldsymbol{v} \in \mathcal{V} \mid \boldsymbol{v}' \in \mathcal{V}^*\}$, where $\boldsymbol{v}' = \boldsymbol{v}_t$ is the time derivative of $\boldsymbol{v}$, understood in the sense of distributions. The space $\mathcal{W}$ is embedded continuously in $C(0, T_0; H)$, the space of all continuous functions $v : [0, T_0] \to H$ with the norm

$$\|v\|_{C(0,T_0;H)} = \max_{t \in [0,T_0]} \|v(t)\|_H.$$

Concerning the data, we assume

$$\boldsymbol{f} \in \mathcal{V}^*, \quad \boldsymbol{u}_0 \in H, \tag{6}$$

and

$H(j)$: $j : \Gamma \times (0, T_0) \times \mathbb{R} \to \mathbb{R}$ is such that

(*i*) $j(\cdot, \cdot, \xi)$ is measurable on Σ for all $\xi \in \mathbb{R}$ and there exists $e \in L^2(\Gamma)$ such that $j(\cdot, \cdot, e(\cdot)) \in L^1(\Sigma)$;

(*ii*) $j(\boldsymbol{x}, t, \cdot)$ is locally Lipschitz on $\mathbb{R}$ for a.e. $(\boldsymbol{x}, t) \in \Sigma$;

(*iii*) $|\eta| \le c_0(1 + |\xi|)$ for all $\xi \in \mathbb{R}$, $\eta \in \partial j(\boldsymbol{x}, t, \xi)$, a.e. $(\boldsymbol{x}, t) \in \Sigma$ with $c_0 > 0$;

(*iv*) $(\eta_1 - \eta_2)(\xi_1 - \xi_2) \ge -m_N |\xi_1 - \xi_2|^2$ for all $\eta_i \in \partial j(\boldsymbol{x}, t, \xi_i)$, $\xi_i \in \mathbb{R}$, $i = 1, 2$, a.e. $(\boldsymbol{x}, t) \in \Sigma$ with $m_N > 0$.

We proceed to derive weak formulations of the problem (1)–(5). Recall the identity (see [14])

$$-\Delta \boldsymbol{u} = \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - \nabla \mathrm{div}\boldsymbol{u},$$

where the symbol **curl** denotes the curl operator (see [14] for its definition). From (1)–(2) we derive that

$$\boldsymbol{u}_t + \nu\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} + \nabla h = \boldsymbol{f} \quad \text{in } Q. \tag{7}$$

Let $\boldsymbol{v}, \boldsymbol{w} \in V$. We define $A : V \to V^*$ by

$$\langle A\boldsymbol{v}, \boldsymbol{w} \rangle = \nu \int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{w}\, dx.$$

It is known from [33] that in the case of simply connected domain Ω, the bilinear form

$$((\boldsymbol{v}, \boldsymbol{w}))_V = \int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{w}\, dx$$

generates a norm in V, $\|\boldsymbol{v}\|_V = ((\boldsymbol{v},\boldsymbol{v}))_V^{1/2}$, which is equivalent to the $H^1(\Omega;\mathbb{R}^d)$-norm.

Multiplying the equation of motion (7) by $\boldsymbol{v} \in V$ and applying the Green formula, we obtain

$$\langle \boldsymbol{u}_t + A\boldsymbol{u}(t), \boldsymbol{v}\rangle + \int_\Gamma h v_N d\Gamma = \langle \boldsymbol{f}(t), \boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in V, \text{ a.e. } t \in (0, T_0).$$

From the relation (5), by using the definition of the Clarke subdifferential, we have

$$\int_\Gamma h v_N d\Gamma \le \int_\Gamma j^0(t, u_N(t); v_N) d\Gamma,$$

where $j^0(t,\xi;\eta) \equiv j^0(\boldsymbol{x},t,\xi;\eta)$ denotes the directional derivative of $j(\boldsymbol{x},t,\cdot)$ at the point $\xi \in \mathbb{R}$ in the direction $\eta \in \mathbb{R}$. The last two relations yield the following weak formulation.

Problem 3.1. *Find $\boldsymbol{u} \in \mathcal{W}$ such that*

$$\begin{cases} \langle \boldsymbol{u}_t + A\boldsymbol{u}(t), \boldsymbol{v}\rangle + \int_\Gamma j^0(t, u_N(t); v_N) d\Gamma \ge \langle \boldsymbol{f}(t), \boldsymbol{v}\rangle \ \forall\, \boldsymbol{v} \in V, \text{ a.e. } t \in (0, T_0), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0. \end{cases} \tag{8}$$

Corresponding to the superpotential j, we define a functional $J : (0, T_0) \times L^2(\Gamma;\mathbb{R}^d) \to \mathbb{R}$ by

$$J(t, \boldsymbol{u}) = \int_\Gamma j(\boldsymbol{x}, t, u_N(\boldsymbol{x}))\, d\Gamma, \quad \boldsymbol{u} \in L^2(\Gamma;\mathbb{R}^d), \text{ a.e. } t \in (0, T_0). \tag{9}$$

The following result holds.

Lemma 3.2. *Assume that $j : \Gamma \times (0, T_0) \times \mathbb{R} \to \mathbb{R}$ has the properties $H(j)$. Then the functional J defined by* (9) *satisfies*

$H(J)$: *(i) $J(\cdot, \boldsymbol{u})$ is measurable on $(0, T_0)$ for all $\boldsymbol{u} \in L^2(\Gamma;\mathbb{R}^d)$;*

(ii) $J(t,\cdot)$ is locally Lipschitz on $L^2(\Gamma;\mathbb{R}^d)$ for a.e. $t \in (0, T_0)$;

(iii) $\|\boldsymbol{\eta}\|_{L^2(\Gamma;\mathbb{R}^d)} \le \bar{c}\,(1 + \|\boldsymbol{u}\|_{L^2(\Gamma;\mathbb{R}^d)})$ for all $\boldsymbol{\eta} \in \partial J(t, \boldsymbol{u}), \boldsymbol{u} \in L^2(\Gamma;\mathbb{R}^d)$, a.e. $t \in (0, T_0)$ with $\bar{c} = \sqrt{2}\, c_0 \max\{\sqrt{\text{meas}(\Gamma)}, 1\}$;

(iv) $J^0(t, \boldsymbol{u}; \boldsymbol{v}) \le \int_\Gamma j^0(t, u_N(\boldsymbol{x}); v_N(\boldsymbol{x})) d\Gamma$ for all $\boldsymbol{u}, \boldsymbol{v} \in L^2(\Gamma;\mathbb{R}^d)$, a.e. $t \in (0, T_0)$;

(v) $\langle \boldsymbol{z}_1(t) - \boldsymbol{z}_2(t), \boldsymbol{u}_1 - \boldsymbol{u}_2\rangle_{L^2(\Gamma;\mathbb{R}^d)} \ge -m_N \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{L^2(\Gamma;\mathbb{R}^d)}$ for all $\boldsymbol{z}_i(t) \in \partial J(t, \boldsymbol{u}_i)$, $\boldsymbol{u}_i \in L^2(\Gamma;\mathbb{R}^d)$, $\boldsymbol{z}_i \in L^2(0, T_0; L^2(\Gamma;\mathbb{R}^d))$, $i = 1, 2$, a.e. $t \in (0, T_0)$.

Proof. We define $\widetilde{j} : \Gamma \times (0, T) \times \mathbb{R}^d \to \mathbb{R}$ by $\widetilde{j}(\boldsymbol{x}, t, \boldsymbol{\eta}) = j(\boldsymbol{x}, t, \eta_N)$ for $(\boldsymbol{x}, t, \boldsymbol{\eta}) \in \Gamma \times (0, T_0) \times \mathbb{R}^d$. Then, $\widetilde{j}(\boldsymbol{x}, t, \boldsymbol{\eta}) = j(\boldsymbol{x}, t, L\boldsymbol{\eta})$, where $L \in \mathcal{L}(\mathbb{R}^d, \mathbb{R})$, $L\boldsymbol{\eta} = \eta_N = \boldsymbol{\eta}\cdot\boldsymbol{n}$ and that $L^* \in \mathcal{L}(\mathbb{R}, \mathbb{R}^d)$ is given by $L^* r = r\boldsymbol{n}$ for $r \in \mathbb{R}$ ([25, Lemma 13]). Then, in view of [27, Proposition 3.37, Theorem 3.47], we obtain the conclusions (i)–(iv). Using arguments similar to those in the proof of Theorem 4.20 in [27], we get (v). □

We consider the following inclusion:

Problem 3.3. *Find $\boldsymbol{u} \in \mathcal{W}$ such that*

$$\begin{cases} \boldsymbol{u}'(t) + A\boldsymbol{u}(t) + \gamma^*\partial J(\gamma\boldsymbol{u}(t)) \ni \boldsymbol{f}(t), \text{ a.e. } t \in (0, T_0), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \end{cases} \tag{10}$$

where $\partial J(\gamma\boldsymbol{u}(t)) \equiv \partial J(t, \gamma\boldsymbol{u}(t))$ and $\gamma^* : L^2(\Gamma;\mathbb{R}^d) \to Z^*$ is the adjoint operator to γ.

We will refer to the following equivalent formulation of Problem 3.3.

Problem 3.4. *Find* $(\boldsymbol{u},\boldsymbol{\eta}) \in \mathcal{W}\times\mathcal{U}$ *such that*

$$\begin{cases} \boldsymbol{u}'(t) + A\boldsymbol{u}(t) + \gamma^*\boldsymbol{\eta}(t) = \boldsymbol{f}(t) \text{ for a.e. } t\in(0,T_0) \\ \boldsymbol{\eta}(t) \in \partial J(\gamma\boldsymbol{u}(t)) \text{ for a.e. } t\in(0,T_0) \\ \boldsymbol{u}(0) = \boldsymbol{u}_0. \end{cases} \tag{11}$$

Remark 1. If the functional J is of the form (9) and $H(j)$ holds, it is clear that every solution to Problem 3.3 (or Problem 3.4) is also a solution to Problem 3.1. If either j or $-j$ is regular, then the converse is also true. Indeed, from [27, Theorem 3.47(vii)] we have, for all $\boldsymbol{v}\in V$ and a.e. $t\in(0,T_0)$,

$$\langle \boldsymbol{f}(t) - \boldsymbol{u}'(t) - A\boldsymbol{u}(t), \boldsymbol{v}\rangle \le \int_\Gamma j^0(t, u_N(t); v_N)\, d\Gamma = J^0(t,\gamma\boldsymbol{u}(t);\gamma\boldsymbol{v}).$$

By Proposition 3.37(ii) in [27], we obtain

$$\boldsymbol{f}(t) - \boldsymbol{u}'(t) - A\boldsymbol{u}(t) \in \partial(J\circ\gamma)(\boldsymbol{u}(t)) = \gamma^*\partial J(\gamma\boldsymbol{u}(t)), \quad \text{a.e. } t\in(0,T_0),$$

which implies (10).

Note that from our problem setting, we have the following properties:
$H(A)$: $A: V\to V^*$ is a linear, continuous, symmetric operator such that

$$\langle A\boldsymbol{v},\boldsymbol{v}\rangle = \nu\|\boldsymbol{v}\|_V^2 \quad \forall\, \boldsymbol{v}\in V;$$

$H(\gamma)$: the Nemytskii operator $\overline{\gamma}: M^{2,2}(0,T_0;V,V^*)\to\mathcal{U}$ defined by $(\overline{\gamma}\boldsymbol{v})(t) = \gamma\boldsymbol{v}(t)$ is compact.

4. **Solution existence.** In this section we show the existence of a solution to Problem 3.4. This is achieved through the consideration of a temporally semi-discrete approximation of Problem 3.4 based on the backward Euler difference for the time derivative; such an approximation is also known as the Rothe method. For a fixed $N\in\mathbb{N}$, define the time step-size $\tau = T_0/N$. Introduce the piecewise constant interpolant of $\boldsymbol{f}$ by

$$\boldsymbol{f}_\tau^k = \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} \boldsymbol{f}(t)\, dt, \quad k = 1,\dots,N.$$

We approximate the initial condition by elements of V. Namely, let $\{\boldsymbol{u}_\tau^0\}\subset V$ be such that $\boldsymbol{u}_\tau^0 \to \boldsymbol{u}_0$ in H as $\tau\to 0$, and $\|\boldsymbol{u}_\tau^0\|_V \le C/\sqrt{\tau}$ for some constant $C>0$. Since V is dense in H, such a sequence $\{\boldsymbol{u}_\tau^0\}$ exists (cf. [31, Theorem 8.9]).

The semi-discrete approximation of Problem 3.4 is the following.

Problem 4.1. *Find* $\{\boldsymbol{u}_\tau^k\}_{k=0}^N \subset V$, *and* $\{\boldsymbol{\eta}_\tau^k\}_{k=0}^N \subset L^2(\Gamma;\mathbb{R}^d)$ *such that for* $k = 1,\dots,N$,

$$\begin{cases} \frac{1}{\tau}(\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1},\boldsymbol{v}) + \langle A\boldsymbol{u}_\tau^k,\boldsymbol{v}\rangle + \langle\boldsymbol{\eta}_\tau^k,\gamma\boldsymbol{v}\rangle_{L^2(\Gamma;\mathbb{R}^d)} = \langle\boldsymbol{f}_\tau^k,\boldsymbol{v}\rangle \quad \forall\,\boldsymbol{v}\in V, \\ \boldsymbol{\eta}_\tau^k \in \partial J(\gamma\boldsymbol{u}_\tau^k). \end{cases} \tag{12}$$

Denote by λ the trace constant of $V\to L^2(\Gamma;\mathbb{R}^d)$:

$$\|\boldsymbol{v}\|_{L^2(\Gamma;\mathbb{R}^d)} \le \lambda\,\|\boldsymbol{v}\|_V \quad \forall\,\boldsymbol{v}\in V.$$

First we show an existence result for Problem 4.1.

Theorem 4.2. *Assume* (6), $H(j)$, *and* $\nu > \overline{c}\,\lambda^2$. *Then there exists a solution to Problem 3.4.*

Proof. It is sufficient to prove that for a given $\boldsymbol{u}_\tau^{k-1} \in V$, there exist $\boldsymbol{u}_\tau^k \in V$ and $\boldsymbol{\eta}_\tau^k \in L^2(\Gamma;\mathbb{R}^d)$ satisfying (12). Note that (12) is equivalent to

$$L\boldsymbol{u}_\tau^k \ni \boldsymbol{f}_\tau^k + \frac{\iota^*\iota}{\tau}\boldsymbol{u}_\tau^{k-1},$$

where the multivalued operator $L : V \to 2^{V^*}$ is defined by

$$L\boldsymbol{v} = \frac{\iota^*\iota}{\tau}\boldsymbol{v} + A\boldsymbol{v} + \gamma^*\partial J(\gamma\boldsymbol{v}), \quad \boldsymbol{v} \in V.$$

Note that it is enough to prove the surjectivity of L. In view of Theorem 2.1, we will show that L is pseudomonotone and coercive.

First, we prove the coercivity of L. Let $\boldsymbol{v} \in V$ and $\boldsymbol{v}^* \in L\boldsymbol{v}$. Then

$$\boldsymbol{v}^* = \frac{\iota^*\iota}{\tau}\boldsymbol{v} + A\boldsymbol{v} + \gamma^*\boldsymbol{\eta},$$

where $\boldsymbol{\eta} \in \partial J(\gamma\boldsymbol{v})$. Using $H(A)$, we have

$$\begin{aligned} \langle \boldsymbol{v}^*, \boldsymbol{v}\rangle &= (\frac{\iota^*\iota}{\tau}\boldsymbol{v}, \boldsymbol{v}) + \langle A\boldsymbol{v}, \boldsymbol{v}\rangle + \langle \boldsymbol{\eta}, \gamma\boldsymbol{v}\rangle_{L^2(\Gamma;\mathbb{R}^d)} \\ &\geq \frac{1}{\tau}\|\boldsymbol{v}\|_H^2 + \nu\|\boldsymbol{v}\|_V^2 + \langle \boldsymbol{\eta}, \gamma\boldsymbol{v}\rangle_{L^2(\Gamma;\mathbb{R}^d)}. \end{aligned} \tag{13}$$

From $H(J)(iii)$, we have

$$\begin{aligned} \langle \boldsymbol{\eta}, \gamma\boldsymbol{v}\rangle_{L^2(\Gamma;\mathbb{R}^d)} &\geq -\|\boldsymbol{\eta}\|_{L^2(\Gamma;\mathbb{R}^d)}\|\boldsymbol{v}\|_{L^2(\Gamma;\mathbb{R}^d)} \\ &\geq -\bar{c}\,\|\boldsymbol{v}\|_{L^2(\Gamma;\mathbb{R}^d)}(1 + \|\boldsymbol{v}\|_{L^2(\Gamma;\mathbb{R}^d)}) \\ &\geq -\bar{c}\,\lambda^2\|\boldsymbol{v}\|_V^2 - \bar{c}\,\lambda\,\|\boldsymbol{v}\|_V. \end{aligned} \tag{14}$$

It follows from (13) and (14) that

$$\langle \boldsymbol{v}^*, \boldsymbol{v}\rangle \geq \frac{1}{\tau}\|\boldsymbol{v}\|_H^2 + (\nu - \bar{c}\lambda^2)\|\boldsymbol{v}\|_V^2 - \bar{c}\lambda\,\|\boldsymbol{v}\|_V.$$

Therefore, the operator L is coercive.

Next we prove that L is pseudomonotone. Since the operator $\frac{\iota^*\iota}{\tau}$ is bounded, continuous and monotone, from Theorem 3.69(i) in [27] we deduce that the operator $\frac{\iota^*\iota}{\tau}$ is pseudomonotone. Since the trace operator $\gamma : V \to L^2(\Gamma;\mathbb{R}^d)$ is compact, from Lemma 2 in [18] we obtain that $\gamma^*\partial J(\gamma\cdot)$ is pseudomonotone. Since the sum of two pseudomonotone operators remains pseudomonotone (cf. [11, Proposition 1.3.68]), L is pseudomonotone. □

Let us establish a boundedness result for the semi-discrete solutions.

Lemma 4.3. *Under the assumptions of Theorem 4.2, there is a constant $M_1 > 0$, independent of τ, such that*

$$\max_{k=1,\ldots,N}\|\boldsymbol{u}_\tau^k\|_H + \sum_{k=1}^N \|\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}\|_H^2 + \tau\sum_{k=1}^N \|\boldsymbol{u}_\tau^k\|_V^2 \leq M_1. \tag{15}$$

Proof. Take $\boldsymbol{v} = \boldsymbol{u}_\tau^k$ in (4.1),

$$\frac{1}{\tau}(\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}, \boldsymbol{u}_\tau^k) + \langle A\boldsymbol{u}_\tau^k, \boldsymbol{u}_\tau^k\rangle + \langle \boldsymbol{\eta}_\tau^k, \gamma\boldsymbol{u}_\tau^k\rangle_{L^2(\Gamma;\mathbb{R}^d)} = \langle \boldsymbol{f}_\tau^k, \boldsymbol{u}_\tau^k\rangle. \tag{16}$$

We have

$$(\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}, \boldsymbol{u}_\tau^k) = \frac{1}{2}\|\boldsymbol{u}_\tau^k\|_H^2 - \frac{1}{2}\|\boldsymbol{u}_\tau^{k-1}\|_H^2 + \frac{1}{2}\|\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}\|_H^2.$$

From $H(A)$,

$$\langle A\boldsymbol{u}_\tau^k, \boldsymbol{u}_\tau^k\rangle = \nu\|\boldsymbol{u}_\tau^k\|_V^2.$$

For any $\varepsilon > 0$, we have

$$\langle \boldsymbol{f}_\tau^k, \boldsymbol{u}_\tau^k\rangle \le \|\boldsymbol{f}_\tau^k\|_{V^*}\|\boldsymbol{u}_\tau^k\|_V \le \frac{\varepsilon}{2}\|\boldsymbol{u}_\tau^k\|_V^2 + \frac{1}{2\varepsilon}\|\boldsymbol{f}_\tau^k\|_{V^*}^2.$$

Recall (see (14)) that

$$\langle \boldsymbol{\eta}_\tau^k, \gamma\boldsymbol{u}_\tau^k\rangle_{L^2(\Gamma;\mathbb{R}^d)} \ge -\bar{c}\lambda^2\|\boldsymbol{u}_\tau^k\|_V^2 - \bar{c}\lambda\,\|\boldsymbol{u}_\tau^k\|_V.$$

Therefore, for any $\varepsilon > 0$,

$$\begin{aligned}\langle \boldsymbol{\eta}_\tau^k, \gamma\boldsymbol{u}_\tau^k\rangle_{L^2(\Gamma;\mathbb{R}^d)} &\ge -\bar{c}\lambda^2\|\boldsymbol{u}_\tau^k\|_V^2 - \frac{\varepsilon}{2}\|\boldsymbol{u}_\tau^k\|_V^2 - \frac{\bar{c}^2\lambda^2}{2\varepsilon}\\ &= (-\bar{c}\lambda^2 - \frac{\varepsilon}{2})\|\boldsymbol{u}_\tau^k\|_V^2 - \frac{\bar{c}^2\lambda^2}{2\varepsilon}.\end{aligned}$$

Thus, from (16), we have

$$\|\boldsymbol{u}_\tau^k\|_H^2 - \|\boldsymbol{u}_\tau^{k-1}\|_H^2 + \|\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}\|_H^2 + c_1\tau\|\boldsymbol{u}_\tau^k\|_V^2 \le \frac{\tau}{\varepsilon}\,\|\boldsymbol{f}_\tau^k\|_{V^*}^2 + c_2\tau, \tag{17}$$

where $c_1 = 2\left(\nu - \varepsilon - \bar{c}\lambda^2\right)$, $c_2 = \bar{c}^2\lambda^2/\varepsilon$, and $\varepsilon > 0$ is chosen so that $c_1 > 0$, e.g., $\varepsilon = \left(\nu - \bar{c}\lambda^2\right)/2$.

For $1 \le n \le N$, we sum the inequality (17) for $k = 1, \ldots, n$ to obtain

$$\begin{aligned}\|\boldsymbol{u}_\tau^n\|_H^2 + \sum_{k=1}^n \|\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}\|_H^2 + c_1\tau\sum_{k=1}^n\|\boldsymbol{u}_\tau^k\|_V^2 &\le \|\boldsymbol{u}_\tau^0\|_H^2 + \frac{1}{\varepsilon}\tau\sum_{k=1}^n\|\boldsymbol{f}_\tau^k\|_{V^*}^2 + c_2T\\ &\le \|\boldsymbol{u}_\tau^0\|_H^2 + \frac{1}{\varepsilon}\|\boldsymbol{f}\|_{\mathcal{V}^*}^2 + c_2T.\end{aligned} \tag{18}$$

From (18) we obtain the bound (15). This completes the proof. □

We now construct piecewise linear and piecewise constant interpolants $\boldsymbol{u}_\tau \in C([0,T_0];V)$ and $\overline{\boldsymbol{u}}_\tau \in L^\infty(0,T_0;V)$ by the formulae

$$\boldsymbol{u}_\tau(t) = \boldsymbol{u}_\tau^k + (\frac{t}{\tau} - k)(\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}) \text{ for } t \in ((k-1)\tau, k\tau],\ k = 1,\ldots,N,$$

$$\overline{\boldsymbol{u}}_\tau(t) = \begin{cases}\boldsymbol{u}_\tau^k,\ t \in ((k-1)\tau, k\tau],\ k = 1,\ldots,N,\\ \boldsymbol{u}_\tau^0,\ t = 0.\end{cases}$$

The piecewise constant function $\overline{\boldsymbol{\eta}}_\tau : (0,T_0] \to L^2(\Gamma;\mathbb{R}^d)$ is given by

$$\overline{\boldsymbol{\eta}}_\tau(t) = \boldsymbol{\eta}_\tau^k \quad \text{for } t \in ((k-1)\tau, k\tau],\ k = 1,\ldots,N.$$

Moreover, we define $\boldsymbol{f}_\tau : (0,T_0] \to V^*$ as follows

$$\boldsymbol{f}_\tau(t) = \boldsymbol{f}_\tau^k \quad \text{for } t \in ((k-1)\tau, k\tau],\ k = 1,\ldots,N.$$

By [6, Lemma 3.3], we know that $\boldsymbol{f}_\tau \to \boldsymbol{f}$ in $\mathcal{V}^*$ as $\tau \to 0$. We observe that the distributional derivative of $\boldsymbol{u}_\tau$ is given by $\boldsymbol{u}_\tau'(t) = (\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1})/\tau$ for $t \in ((k-1)\,\tau, k\,\tau)$, $k = 1,\ldots,N$. Thus, (4.1) can be rewritten as

$$\begin{cases}(\boldsymbol{u}_\tau'(t), \boldsymbol{v}) + \langle A\overline{\boldsymbol{u}}_\tau(t), \boldsymbol{v}\rangle + \langle\overline{\boldsymbol{\eta}}_\tau(t), \gamma\boldsymbol{v}\rangle_{L^2(\Gamma;\mathbb{R}^d)} = \langle\overline{\boldsymbol{f}}_\tau(t), \boldsymbol{v}\rangle,\\ \forall\,\boldsymbol{v} \in V,\ \text{a.e. } t \in (0,T_0),\\ \overline{\boldsymbol{\eta}}_\tau(t) \in \partial J(\gamma\overline{\boldsymbol{u}}_\tau(t)),\ \text{a.e. } t \in (0,T_0).\end{cases} \tag{19}$$

We define the Nemytskii operator $\mathcal{A} : \mathcal{V} \to \mathcal{V}^*$ by $(\mathcal{A}\boldsymbol{v})(t) = A(\boldsymbol{v}(t))$ for $\boldsymbol{v} \in \mathcal{V}$ and $\overline{\gamma} : \mathcal{V} \to \mathcal{U}$ by $(\overline{\gamma}\boldsymbol{v})(t) = \gamma\boldsymbol{v}(t)$ for $\boldsymbol{v} \in \mathcal{V}$. Observe that the problem (19) is equivalent to

$$\begin{cases} (\boldsymbol{u}_\tau', \boldsymbol{v})_{\mathcal{H}} + \langle \mathcal{A}\overline{\boldsymbol{u}}_\tau, \boldsymbol{v}\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle \overline{\boldsymbol{\eta}}_\tau, \overline{\gamma}\boldsymbol{v}\rangle_{\mathcal{U}} = \langle \overline{\boldsymbol{f}}_\tau, \boldsymbol{v}\rangle_{\mathcal{V}^*\times\mathcal{V}} \ \forall\, \boldsymbol{v} \in \mathcal{V}, \\ \overline{\boldsymbol{\eta}}_\tau(t) \in \partial J((\overline{\gamma}\,\overline{\boldsymbol{u}}_\tau)(t)), \ \text{a.e. } t \in (0, T_0). \end{cases} \tag{20}$$

Lemma 4.4. *Under the assumptions of Theorem 4.2, there is a constant $M_2 > 0$, independent of τ, such that*

$$\begin{aligned} &\|\overline{\boldsymbol{u}}_\tau\|_{\mathcal{V}} + \|\overline{\boldsymbol{u}}_\tau\|_{L^\infty(0,T_0;H)} + \|\boldsymbol{u}_\tau\|_{C(0,T_0;H)} + \|\boldsymbol{u}_\tau\|_{\mathcal{V}} \\ &\quad + \|\boldsymbol{u}_\tau'\|_{\mathcal{V}^*} + \|\overline{\boldsymbol{\eta}}_\tau\|_{\mathcal{U}} + \|\overline{\boldsymbol{u}}_\tau\|_{M^{2,2}(0,T_0;V,V^*)} \le M_2. \end{aligned} \tag{21}$$

Proof. Bounds on $\|\overline{\boldsymbol{u}}_\tau\|_{L^\infty(0,T_0;H)}$ and $\|\boldsymbol{u}_\tau\|_{C(0,T_0;H)}$ follow directly from (15). Since

$$\|\overline{\boldsymbol{u}}_\tau\|_{\mathcal{V}}^2 = \tau \sum_{k=1}^N \|\boldsymbol{u}_\tau^k\|_V^2,$$

we obtain the bound on $\|\overline{\boldsymbol{u}}_\tau\|_{\mathcal{V}}$ from (15). A simple calculation shows that

$$\|\boldsymbol{u}_\tau\|_{\mathcal{V}}^2 \le \tau \sum_{k=0}^N \|\boldsymbol{u}_\tau^k\|_V^2.$$

Thus, from (15) and the fact

$$\|\boldsymbol{u}_\tau^0\|_V \le C/\sqrt{\tau},$$

we get the bound on $\|\boldsymbol{u}_\tau\|_{\mathcal{V}}$.

Next, using $H(J)(iii)$ we have

$$\begin{aligned} \|\overline{\boldsymbol{\eta}}_\tau\|_{\mathcal{U}}^2 &= \int_0^{T_0} \|\overline{\boldsymbol{\eta}}_\tau(t)\|_{L^2(\Gamma;\mathbb{R}^d)}^2 dt \\ &\le \int_0^{T_0} \left(2\overline{c}^2 + 2\overline{c}^2\, \|\overline{\boldsymbol{u}}_\tau(t)\|_{L^2(\Gamma;\mathbb{R}^d)}^2\right) dt \\ &\le \int_0^{T_0} \left(2\overline{c}^2 + 2\overline{c}^2\lambda^2 \|\overline{\boldsymbol{u}}_\tau(t)\|_V^2\right) dt \\ &= 2T_0\overline{c}^2 + 2\overline{c}^2\lambda^2\|\overline{\boldsymbol{u}}_\tau\|_{\mathcal{V}}^2, \end{aligned}$$

and hence from the bound on $\|\overline{\boldsymbol{u}}_\tau\|_{\mathcal{V}}$ we get the bound on $\|\overline{\boldsymbol{\eta}}_\tau\|_{\mathcal{U}}$.

Using $H(A)$, from (20) we have

$$\begin{aligned} \|\boldsymbol{u}_\tau'\|_{\mathcal{V}^*} &= \sup_{\|\boldsymbol{v}\|_{\mathcal{V}}\le 1} |\langle \boldsymbol{u}_\tau', \boldsymbol{v}\rangle_{\mathcal{V}^*\times\mathcal{V}}| = \sup_{\|\boldsymbol{v}\|_{\mathcal{V}}\le 1} |(\boldsymbol{u}_\tau', \boldsymbol{v})_{\mathcal{H}}| \\ &= \sup_{\|\boldsymbol{v}\|_{\mathcal{V}}\le 1} |\langle \overline{\boldsymbol{f}}_\tau, \boldsymbol{v}\rangle_{\mathcal{V}^*\times\mathcal{V}} - \langle \mathcal{A}\overline{\boldsymbol{u}}_\tau, \boldsymbol{v}\rangle_{\mathcal{V}^*\times\mathcal{V}} - \int_0^{T_0} \langle \overline{\boldsymbol{\eta}}_\tau(t), \gamma\boldsymbol{v}(t)\rangle_{L^2(\Gamma;\mathbb{R}^d)} dt| \\ &\le \|\overline{\boldsymbol{f}}_\tau\|_{\mathcal{V}^*} + (\int_0^{T_0} \|A\overline{\boldsymbol{u}}_\tau(t)\|_{V^*}^2 dt)^{\frac{1}{2}} + \lambda\, \|\overline{\boldsymbol{\eta}}_\tau\|_{\mathcal{U}} \\ &\le \|\overline{\boldsymbol{f}}_\tau\|_{\mathcal{V}^*} + \|A\|_{\mathcal{L}(V,V^*)}\|\overline{\boldsymbol{u}}_\tau\|_{\mathcal{V}} + \lambda\, \|\overline{\boldsymbol{\eta}}_\tau\|_{\mathcal{U}}. \end{aligned}$$

Thus, using the bounds on $\|\overline{\boldsymbol{u}}_\tau\|_{\mathcal{V}}$ and $\|\overline{\boldsymbol{\eta}}_\tau\|_{\mathcal{U}}$ we get the bound on $\|\boldsymbol{u}_\tau'\|_{\mathcal{V}^*}$.

Suppose the $BV^2(0, T_0; V^*)$ seminorm of piecewise constant function $\overline{\boldsymbol{u}}_\tau$ is obtained by some division $0 = a_0 < a_1 < \ldots < a_n = T_0$, and each a_i is in different interval $((m_i-1)\tau, m_i\tau]$, such that $\overline{\boldsymbol{u}}_\tau(a_i) = \boldsymbol{u}_\tau^{m_i}$ with $m_0 = 0, m_n = N$ and $m_{i+1} > m_i$ for $i = 1, \dots, N-1$. Thus, from the bound on $\|\boldsymbol{u}'_\tau\|_{\mathcal{V}^*}$ we have

$$
\begin{aligned}
\|\overline{\boldsymbol{u}}_\tau\|^2_{BV^2(0,T_0;V^*)} &= \sum_{i=1}^n \|\boldsymbol{u}_\tau^{m_i} - \boldsymbol{u}_\tau^{m_i-1}\|^2_{V^*} \\
&\le \sum_{i=1}^n ((m_i - m_{i-1}) \sum_{k=m_{i-1}+1}^{m_i} \|\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}\|^2_{V^*}) \\
&\le N \sum_{i=1}^n \sum_{k=m_{i-1}+1}^{m_i} \|\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}\|^2_{V^*} \\
&= N \sum_{i=1}^N \|\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}\|^2_{V^*} = T_0\tau \sum_{i=1}^N \|\frac{\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}}{\tau}\|^2_{V^*} \\
&= T_0 \int_0^{T_0} \|\boldsymbol{u}'_\tau(t)\|^2_{V^*} dt = T_0 \|\boldsymbol{u}'_\tau\|^2_{\mathcal{V}^*}.
\end{aligned}
$$

Thus, from the bounds on $\|\overline{\boldsymbol{u}}_\tau\|_{\mathcal{V}}$ and $\|\boldsymbol{u}'_\tau\|_{\mathcal{V}^*}$, we deduce that $\overline{\boldsymbol{u}}_\tau$ is bounded in $M^{2,2}(0, T_0; V, V^*)$. Hence, the bound on $\|\overline{\boldsymbol{u}}_\tau\|_{M^{2,2}(0,T_0;V,V^*)}$ is proved. This completes the proof. □

Theorem 4.5. *Keep the assumptions made in Theorem 4.2. Then there exists a pair* $(\boldsymbol{u}, \boldsymbol{\eta}) \in \mathcal{W} \times \mathcal{U}$ *such that for a subsequence,* $\boldsymbol{u}_\tau \rightharpoonup \boldsymbol{u}$ *in* $\mathcal{W}$, $\boldsymbol{u}_\tau \rightharpoonup^* \boldsymbol{u}$ *in* $L^\infty(0, T_0; H)$, $\overline{\boldsymbol{u}}_\tau \rightharpoonup \boldsymbol{u}$ *in* $\mathcal{V}$, $\overline{\boldsymbol{u}}_\tau \rightharpoonup^* \boldsymbol{u}$ *in* $L^\infty(0, T_0; H)$ *and* $\overline{\boldsymbol{\eta}}_\tau \rightharpoonup \boldsymbol{\eta}$ *in* $\mathcal{U}$. *Moreover,* $(\boldsymbol{u}, \boldsymbol{\eta})$ *is a solution of Problem 3.4.*

Proof. From (21), we know that there exist $\overline{\boldsymbol{u}} \in \mathcal{V} \cap L^\infty(0, T_0; H)$, $\boldsymbol{u} \in \mathcal{V} \cap L^\infty(0, T_0; H)$, $\boldsymbol{u}_1 \in \mathcal{V}^*$ and $\boldsymbol{\eta} \in \mathcal{U}$ such that, passing to a subsequence if necessary, the following convergence holds

$$\overline{\boldsymbol{u}}_\tau \to \overline{\boldsymbol{u}} \textit{ weakly in } \mathcal{V} \textit{ and weakly}^* \textit{ in } L^\infty(0, T_0; H), \tag{22}$$

$$\boldsymbol{u}_\tau \to \boldsymbol{u} \textit{ weakly in } \mathcal{V} \textit{ and weakly}^* \textit{ in } L^\infty(0, T_0; H), \tag{23}$$

$$\boldsymbol{u}'_\tau \rightharpoonup \boldsymbol{u}_1 \textit{ in } \mathcal{V}^*, \tag{24}$$

$$\overline{\boldsymbol{\eta}}_\tau \rightharpoonup \boldsymbol{\eta} \textit{ in } \mathcal{U}. \tag{25}$$

First we show that $\overline{\boldsymbol{u}} = \boldsymbol{u}$. Note that

$$\|\overline{\boldsymbol{u}}_\tau - \boldsymbol{u}_\tau\|^2_{\mathcal{V}^*} = \sum_{k=1}^N \int_{(k-1)\tau}^{k\tau} (k\tau - t)^2 \|\frac{\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}}{\tau}\|^2_{V^*} dt = \frac{\tau^2}{3} \|\boldsymbol{u}'_\tau\|^2_{\mathcal{V}^*}.$$

Thus, $\overline{\boldsymbol{u}}_\tau - \boldsymbol{u}_\tau \to \boldsymbol{0}$ in $\mathcal{V}^*$ as $\tau \to 0$. On the other hand, from (22) and (23) we have $\overline{\boldsymbol{u}}_\tau - \boldsymbol{u}_\tau \rightharpoonup \overline{\boldsymbol{u}} - \boldsymbol{u}$ in $\mathcal{V}$. Since the embedding $\mathcal{V} \subset \mathcal{V}^*$ is continuous, we also have $\overline{\boldsymbol{u}}_\tau - \boldsymbol{u}_\tau \rightharpoonup \overline{\boldsymbol{u}} - \boldsymbol{u}$ in $\mathcal{V}^*$. Therefore, $\overline{\boldsymbol{u}} - \boldsymbol{u} = \boldsymbol{0}$, i.e. $\overline{\boldsymbol{u}} = \boldsymbol{u}$. Since $\boldsymbol{u}_\tau \rightharpoonup \boldsymbol{u}$ in $\mathcal{V}$ and $\boldsymbol{u}'_\tau \rightharpoonup \boldsymbol{u}_1$ in $\mathcal{V}^*$, we conclude (cf. [16, Proposition 1.2]) that $\boldsymbol{u}_1 = \boldsymbol{u}'$. Thus, for all $\boldsymbol{v} \in \mathcal{V}$, we obtain

$$(\boldsymbol{u}'_\tau, \boldsymbol{v})_{\mathcal{H}} = \langle \boldsymbol{u}'_\tau, \boldsymbol{v} \rangle_{\mathcal{V}^* \times \mathcal{V}} \to \langle \boldsymbol{u}', \boldsymbol{v} \rangle_{\mathcal{V}^* \times \mathcal{V}} = (\boldsymbol{u}', \boldsymbol{v})_{\mathcal{H}}. \tag{26}$$

From $H(A)$, it is clear that $\mathcal{A}$ is linear and continuous operator from $\mathcal{V}$ to $\mathcal{V}^*$ and thus also weakly continuous. Since $\overline{\boldsymbol{u}}_\tau \rightharpoonup \boldsymbol{u}$ in $\mathcal{V}$, we get

$$\langle \mathcal{A}\overline{\boldsymbol{u}}_\tau, \boldsymbol{v} \rangle_{\mathcal{V}^* \times \mathcal{V}} \to \langle \mathcal{A}\boldsymbol{u}, \boldsymbol{v} \rangle_{\mathcal{V}^* \times \mathcal{V}}. \tag{27}$$

From (25) we get

$$\langle \overline{\boldsymbol{\eta}}_\tau, \overline{\gamma}\boldsymbol{v}\rangle_{\mathcal{U}} \to \langle \boldsymbol{\eta}, \overline{\gamma}\boldsymbol{v}\rangle_{\mathcal{U}}. \tag{28}$$

Since $\boldsymbol{f}_\tau \to \boldsymbol{f}$ in $\mathcal{V}^*$, we have

$$\langle \boldsymbol{f}_\tau, \boldsymbol{v}\rangle_{\mathcal{V}^*\times\mathcal{V}} \to \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{\mathcal{V}^*\times\mathcal{V}}. \tag{29}$$

Using (26)–(29), we can pass to the limit in (20) and obtain

$$(\boldsymbol{u}', \boldsymbol{v})_{\mathcal{H}} + \langle \mathcal{A}\boldsymbol{u}, \boldsymbol{v}\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle \boldsymbol{\eta}, \overline{\gamma}\boldsymbol{v}\rangle_{\mathcal{U}} = \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{\mathcal{V}^*\times\mathcal{V}} \quad \forall\, \boldsymbol{v} \in \mathcal{V}. \tag{30}$$

Since $\overline{\boldsymbol{u}}_\tau \rightharpoonup \boldsymbol{u}$ in $\mathcal{V}$, from $H(\gamma)$ we have $\overline{\gamma}\,\overline{\boldsymbol{u}}_\tau \to \overline{\gamma}\boldsymbol{u}$ in $\mathcal{U}$. Thus, for a subsequence, $\overline{\gamma}\,\overline{\boldsymbol{u}}_\tau(t) \to \overline{\gamma}\boldsymbol{u}(t)$ in $L^2(\Gamma;\mathbb{R}^d)$ for a.e. $t \in (0,T_0)$. Since $\partial J : L^2(\Gamma;\mathbb{R}^d) \to 2^{L^2(\Gamma;\mathbb{R}^d)}$ has nonempty, closed and convex values, and is upper semicontinuous from $L^2(\Gamma;\mathbb{R}^d)$ furnished with strong topology into $L^2(\Gamma;\mathbb{R}^d)$ furnished with weak topology (cf. [10, Proposition 5.6.10]), from (25) and Theorem 2.3 we have

$$\boldsymbol{\eta}(t) \in \partial J(\gamma \boldsymbol{u}(t)), \text{ a.e. } t \in (0,T_0). \tag{31}$$

Finally, we pass to the limit with the initial conditions on the function $\boldsymbol{u}_\tau$. Since $\boldsymbol{u}_\tau \rightharpoonup \boldsymbol{u}$ in $\mathcal{V}$ and $\boldsymbol{u}_\tau' \rightharpoonup \boldsymbol{u}'$ in $\mathcal{V}^*$ and the embedding $\mathcal{W} \subset C(0,T_0;H)$ is continuous, we have $\boldsymbol{u}_\tau(t) \rightharpoonup \boldsymbol{u}(t)$ in H for all $t \in [0,T_0]$ (cf. [26, Lemma 4(b)]). Therefore, $\boldsymbol{u}_\tau^0 = \boldsymbol{u}_\tau(0) \rightharpoonup \boldsymbol{u}(0)$ in H. Since $\boldsymbol{u}_\tau^0 \to \boldsymbol{u}_0$ in H, we have $\boldsymbol{u}(0) = \boldsymbol{u}_0$. This completes the proof. □

5. **Uniqueness and continuous dependence on data.** In this section we study the uniqueness of a solution to Problem 3.4 and continuous dependence of the solution on $\boldsymbol{f}$ and $\boldsymbol{u}_0$.

Theorem 5.1. *Keep the assumptions of Theorem 4.2. Then, there exists a constant $C > 0$ such that for any solution $\boldsymbol{u} \in \mathcal{V}$ to Problem 3.4.*

$$\|\boldsymbol{u}\|_{\mathcal{V}} \le C. \tag{32}$$

If $\nu - m_N\lambda^2 > 0$, where $m_N > 0$ is the constant from $H(J)(v)$, then the solution to Problem 3.4 is unique.

Proof. First we prove the priori estimate (32). Since $\boldsymbol{u} \in \mathcal{V}$ solves Problem 3.4, we have

$$(\boldsymbol{u}'(t), \boldsymbol{u}(t)) + \langle A\boldsymbol{u}(t), \boldsymbol{u}(t)\rangle + \langle \boldsymbol{\eta}(t), \gamma\boldsymbol{u}(t)\rangle_{L^2(\Gamma;\mathbb{R}^2)} = \langle \boldsymbol{f}(t), \boldsymbol{u}(t)\rangle, \text{ a.e. } t \in (0,T_0),$$

where $\boldsymbol{\eta}(t) \in \partial J(\gamma\boldsymbol{u}(t))$ for a.e. $t \in (0,T_0)$.

Recall that

$$\langle \boldsymbol{\eta}(t), \gamma\boldsymbol{u}(t)\rangle_{L^2(\Gamma;\mathbb{R}^d)} \ge -\bar{c}\lambda^2\|\boldsymbol{u}(t)\|_V^2 - \bar{c}\lambda\,\|\boldsymbol{u}(t)\|_V, \text{ a.e. } t \in (0,T_0).$$

Thus, from $H(A)$ we have

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}(t)\|_H^2 + (\nu - \bar{c}\lambda^2)\|\boldsymbol{u}(t)\|_V^2 - \bar{c}\lambda\,\|\boldsymbol{u}(t)\|_V \le \langle \boldsymbol{f}(t), \boldsymbol{u}(t)\rangle, \text{ a.e. } t \in (0,T_0).$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}(t)\|_H^2 + (\nu - \bar{c}\lambda^2)\|\boldsymbol{u}(t)\|_V^2 \le \bar{c}\lambda\,\|\boldsymbol{u}(t)\|_V + \langle \boldsymbol{f}(t), \boldsymbol{u}(t)\rangle, \text{ a.e. } t \in (0,T_0). \tag{33}$$

Integrating (33) from 0 to T_0, we obtain

$$\begin{aligned}\frac{1}{2}\|\boldsymbol{u}(T_0)\|_H^2-\frac{1}{2}\|\boldsymbol{u}_0\|_H^2+(\nu-\bar{c}\lambda^2)\|\boldsymbol{u}\|_{\mathcal{V}}^2 &\le \bar{c}\lambda\int_0^{T_0}\|\boldsymbol{u}(t)\|_V\,dt+\int_0^{T_0}\langle \boldsymbol{f}(t),\boldsymbol{u}(t)\rangle dt\\ &\le \bar{c}\lambda\sqrt{T_0}\,\|\boldsymbol{u}\|_{\mathcal{V}}+\|\boldsymbol{f}\|_{\mathcal{V}^*}\|\boldsymbol{u}\|_{\mathcal{V}}.\end{aligned}$$

Hence,

$$\frac{1}{2}\|\boldsymbol{u}(T_0)\|_H^2+(\nu-\bar{c}\lambda^2)\|\boldsymbol{u}\|_{\mathcal{V}}^2\le(\bar{c}\lambda\sqrt{T}+\|\boldsymbol{f}\|_{\mathcal{V}^*})\|\boldsymbol{u}\|_{\mathcal{V}}+\frac{1}{2}\|\boldsymbol{u}_0\|_H^2. \tag{34}$$

Thus, (32) holds.

Next let $(\boldsymbol{u}_1,\boldsymbol{\eta}_1)$, $(\boldsymbol{u}_2,\boldsymbol{\eta}_2)$ be two solutions of Problem 3.4. Then, for a.e. $t\in(0,T_0)$, we have

$$(\boldsymbol{u}_1'(t)-\boldsymbol{u}_2'(t),\boldsymbol{v})+\langle A(\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)),\boldsymbol{v}\rangle+\langle\boldsymbol{\eta}_1(t)-\boldsymbol{\eta}_2(t),\gamma\boldsymbol{v}\rangle_{L^2(\Gamma;\mathbb{R}^2)}=0\,\forall\,\boldsymbol{v}\in V. \tag{35}$$

Taking $\boldsymbol{v}=\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)$ in (35), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_H^2+\langle A(\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)),\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\rangle\\ &\quad+\langle\boldsymbol{\eta}_1(t)-\boldsymbol{\eta}_2(t),\gamma(\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t))\rangle_{L^2(\Gamma;\mathbb{R}^2)}=0,\ \text{a.e. } t\in(0,T_0).\end{aligned} \tag{36}$$

By $H(J)(v)$, we have

$$\begin{aligned}\langle\boldsymbol{\eta}_1(t)-\boldsymbol{\eta}_2(t),\gamma(\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t))\rangle_{L^2(\Gamma;\mathbb{R}^2)}&\ge -m_N\|\gamma(\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t))\|_{L^2(\Gamma;\mathbb{R}^d)}^2\\ &\ge -m_N\lambda^2\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2.\end{aligned}$$

From the above inequality and (36) we obtain for a.e. $t\in(0,T_0)$

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_H^2+(\nu-m_N\lambda^2)\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2\le 0.$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_H^2\le 0. \tag{37}$$

Integrating (37) from 0 to t, we get

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_H^2\le\|\boldsymbol{u}_1(0)-\boldsymbol{u}_2(0)\|_H^2=0.$$

So $\boldsymbol{u}_1=\boldsymbol{u}_2$, and this completes the proof. □

Next we establish the continuous dependence of solution of Problem 3.4 on $\boldsymbol{f}$ and $\boldsymbol{u}_0$.

Theorem 5.2. *Assume (6), $H(j)$, and $\nu-m\lambda^2>0$ with $m=\max\{\bar{c},m_N\}$. Then the mapping $(\boldsymbol{f},\boldsymbol{u}_0)\mapsto\boldsymbol{u}:\mathcal{V}\times H\to C(0,T_0;H)$ is Lipschitz continuous, where $\boldsymbol{u}$ denotes the unique solution to Problem 3.4.*

Proof. Consider two solutions $\boldsymbol{u}_1,\boldsymbol{u}_2\in\mathcal{W}$ of Problem 3.4 corresponding to two right-hand sides $\boldsymbol{f}_1,\boldsymbol{f}_2\in\mathcal{V}$ and two initial conditions $\boldsymbol{u}_{1,0},\boldsymbol{u}_{2,0}\in H$. Similar to (36), we have

$$\begin{aligned}&(\boldsymbol{u}_1'(t)-\boldsymbol{u}_2'(t),\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t))+\langle A(\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)),\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\rangle\\ &\quad+\langle\boldsymbol{\eta}_1(t)-\boldsymbol{\eta}_2(t),\gamma(\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t))\rangle_{L^2(\Gamma;\mathbb{R}^d)}=\langle\boldsymbol{f}_1(t)-\boldsymbol{f}_2(t),\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\rangle,\end{aligned}$$

where $\boldsymbol{\eta}_1(t) \in \partial J(\gamma \boldsymbol{u}_1(t))$ and $\boldsymbol{\eta}_2(t) \in \partial J(\gamma \boldsymbol{u}_2(t))$ for a.e. $t \in (0, T_0)$. Similar to the proof of the uniqueness in Theorem 5.1, from (38) we get

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_H^2 \leq \|\boldsymbol{u}_{1,0} - \boldsymbol{u}_{2,0}\|_H^2 + \int_0^t \|\boldsymbol{f}_1(s) - \boldsymbol{f}_2(s)\|_{V^*}^2 ds.$$

Thus,

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{C(0,T;H)}^2 \leq \|\boldsymbol{u}_{1,0} - \boldsymbol{u}_{2,0}\|_H^2 + \|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_{\mathcal{V}^*}^2,$$

and the proof is completed. □

6. An optimal control problem.

The optimal control problem studied here arises in some important models such as artificial heart. It is known (cf. [1]) that hemolysis is caused largely by excessive shear stresses and vortices. Blood clot may be caused by recirculation and stagnation. Hence, an artificial heart must be designed so as to minimize shear stresses, vortices, and stagnation. Thus, a meaningful cost functional may be given by

$$P(\boldsymbol{u}, \boldsymbol{f}) = \frac{1}{2} \int_0^{T_0} [Q_1(\nabla \boldsymbol{u}) + Q_2(\mathbf{curl}\,(\boldsymbol{u})) + Q_3(\boldsymbol{u} - \boldsymbol{u}^s) + Q_4(\boldsymbol{f})]\, dt, \tag{38}$$

where Q_i, $1 \leq i \leq 4$, are quadratic functionals of their arguments, $\boldsymbol{u}$ is the velocity field, $\boldsymbol{f}$ is the control, and $\boldsymbol{u}^s$ is an ideal velocity distribution. In (38), Q_1, Q_2 and Q_3 are assumed to be positive semidefinite whereas Q_4 is assumed to be strictly coercive. The problem is to determine a control function in such a way that the cost functional is minimized subject to certain constraints on $\boldsymbol{u}$ and $\boldsymbol{f}$. We will consider the optimal control problem with a general functional

$$P(\boldsymbol{u}, \boldsymbol{v}) = \int_0^{T_0} R(t, \boldsymbol{u}(t), \boldsymbol{v}(t))\, dt,$$

where $R : [0, T_0] \times L^2(\Omega; \mathbb{R}^d) \times L^2(\Omega; \mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}$.

In this section we shall study an optimal control problem for a system described by a hemivariational inequality in Problem 3.3.

Denote $L^2(\Omega; \mathbb{R}^d)$ by $\boldsymbol{H}$. We suppose that $\widehat{\mathcal{U}} = L^2(0, T_0; \boldsymbol{H})$ represents the control space. Let $\widehat{\mathcal{U}}_0 \subset \widehat{\mathcal{U}}$ be the set of admissible controls and $P : \mathcal{W} \times \widehat{\mathcal{U}} \to \mathbb{R}$ be the objective functional.

The control problem is the following:

$$\begin{cases} P(\boldsymbol{u}, \boldsymbol{f}) = \int_0^{T_0} R(t, \boldsymbol{u}(t), \boldsymbol{f}(t))\, dt \to \inf = m \quad \text{such that} \\ \boldsymbol{u}'(t) + A\boldsymbol{u}(t) + \gamma^* \partial J(\gamma \boldsymbol{u}(t)) \ni \boldsymbol{f}(t) \text{ a.e. } t \in (0, T_0), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \ \boldsymbol{f} \in \widehat{\mathcal{U}}_0. \end{cases} \tag{39}$$

In what follows we need the following hypotheses:

$H(\widehat{\mathcal{U}})$: $\widehat{\mathcal{U}}_0$ is a bounded and weakly closed subset of $\widehat{\mathcal{U}}$;

$H(R)$: $R : [0, T_0] \times \boldsymbol{H} \times \boldsymbol{H} \to \mathbb{R} \cup \{+\infty\}$ is a measurable function which satisfies the following three conditions:

(*i*) $R(t, \cdot, \cdot)$ is sequentially lower semicontinuous on $\boldsymbol{H} \times \boldsymbol{H}$, a.e. $t \in (0, T_0)$;

(*ii*) $R(t, \boldsymbol{u}, \cdot)$ is convex on $\boldsymbol{H}$, for all $\boldsymbol{u} \in \boldsymbol{H}$ and a.e. t;

(*iii*) there exist $M > 0$ and $\phi \in L^1(0, T_0)$ such that for all $\boldsymbol{u}, \boldsymbol{f} \in \boldsymbol{H}$ and a.e. t, we have $R(t, \boldsymbol{u}, \boldsymbol{f}) \geq \phi(t) - M(\|\boldsymbol{u}\|_{\boldsymbol{H}} + \|\boldsymbol{f}\|_{\boldsymbol{H}})$.

The following example illustrates the existence of the functional R satisfying the assumption $H(R)$ (cf. [20]).

Example 6.1. Take $R(t,\boldsymbol{u},\boldsymbol{f}) = \|B\boldsymbol{u} - \widetilde{\boldsymbol{u}}\|_{\boldsymbol{H}}^2 + (Q\boldsymbol{f},\boldsymbol{f})_{\boldsymbol{H}}$, where $B \in \mathcal{L}(\boldsymbol{H},\boldsymbol{H})$ is the "observation" operator, $Q \in \mathcal{L}(\boldsymbol{H},\boldsymbol{H})$, $(Q\boldsymbol{f},\boldsymbol{f})_{\boldsymbol{H}} \geq \mu\|\boldsymbol{f}\|_{\boldsymbol{H}}^2$ with $\mu > 0$, and $\widetilde{\boldsymbol{u}} \in \boldsymbol{H}$ is the desired terminal output. Then the hypothesis $H(R)$ is satisfied.

Theorem 6.2. *Assume (6), $H(j)$, $\nu > \bar{c}\lambda^2$, $H(\widehat{\mathcal{U}})$, and $H(R)$. Then, there exists an optimal "state-control" pair $(\boldsymbol{u},\boldsymbol{f}) \in \mathcal{W} \times \widehat{\mathcal{U}}_0$ for (39).*

Proof. Let $\{(\boldsymbol{u}_k,\boldsymbol{f}_k)\} \subset \mathcal{W}\times\widehat{\mathcal{U}}_0$ be a minimizing sequence, i.e. $\lim_{k\to\infty} P(\boldsymbol{u}_k,\boldsymbol{f}_k) = m$. Then,

$$\boldsymbol{u}_k'(t) + A\boldsymbol{u}_k(t) + \gamma^*\boldsymbol{\eta}_k(t) = \boldsymbol{f}_k(t) \text{ a.e. } t \in (0,T_0) \tag{40}$$

with $\boldsymbol{\eta}_k(t) \in \partial J(\gamma\boldsymbol{u}_k(t))$ a.e. $t \in (0,T_0)$ and $\boldsymbol{u}_k(0) = \boldsymbol{u}_0$.

From $H(\widehat{\mathcal{U}})$ it follows that the sequence $\{\boldsymbol{f}_k\}$ belongs to a bounded subset of $\widehat{\mathcal{U}}$. Therefore, by the reflexivity of $\widehat{\mathcal{U}}$, we may assume, by passing to a subsequence if necessary, that for some $\boldsymbol{f} \in \widehat{\mathcal{U}}$, $\boldsymbol{f}_k \rightharpoonup \boldsymbol{f}$ in $\widehat{\mathcal{U}}$. The weak closedness of $\widehat{\mathcal{U}}_0$ implies that the limit $\boldsymbol{f} \in \widehat{\mathcal{U}}_0$. Similarly to the proof of Theorem 5.1 (cf. (34)), we conclude that

$$\|\boldsymbol{u}_k\|_{\mathcal{V}} \leq c_1(1 + \|\boldsymbol{f}_k\|_{\mathcal{V}^*}) \tag{41}$$

for some constant $c_1 > 0$. From (40) we have

$$\|\boldsymbol{u}_k'\|_{\mathcal{V}^*} \leq \|\boldsymbol{f}_k\|_{\mathcal{V}^*} + \|A\|_{\mathcal{L}(V,V^*)}\|\boldsymbol{u}_k\|_{\mathcal{V}} + \|\gamma^*\boldsymbol{\eta}_k\|_{\mathcal{V}^*}. \tag{42}$$

Since $\boldsymbol{\eta}_k(t) \in \partial J(\gamma\boldsymbol{u}_k(t))$ for a.e. $t \in (0,T_0)$, from $H(J)(iii)$ we have

$$\|\gamma^*\boldsymbol{\eta}_k\|_{\mathcal{V}^*} \leq c_2\|\gamma^*\boldsymbol{\eta}_k\|_{\mathcal{Z}^*} \leq c_2\|\gamma^*\|\|\boldsymbol{\eta}_k\|_{\mathcal{U}} \leq c_2\|\gamma^*\| \left(\sqrt{2T_0}\,\bar{c} + \sqrt{2}\,\bar{c}\,\lambda\,\|\gamma\|\|\boldsymbol{u}_k\|_{\mathcal{V}}\right), \tag{43}$$

where $c_2 > 0$ is an embedding constant of $V \subset Z$. Using (41) and (43), from (42) we deduce that for some constant $c_4 > 0$,

$$\|\boldsymbol{u}_k'\|_{\mathcal{V}^*} \leq c_4(1 + \|\boldsymbol{f}_k\|_{\mathcal{V}^*}). \tag{44}$$

From (41) and (44) we conclude that $\{\boldsymbol{u}_k\}$ is bounded in $\mathcal{W}$. Thus, by passing to a subsequence if necessary, we have that $\boldsymbol{u}_k \rightharpoonup \boldsymbol{u}$ in $\mathcal{W}$ for some $\boldsymbol{u} \in \mathcal{W}$. Since the embedding of V into $\boldsymbol{H}$ is compact, so is the embedding of $\mathcal{W}$ into $\widehat{\mathcal{U}}$. Therefore, $\boldsymbol{u}_k \to \boldsymbol{u}$ in $\widehat{\mathcal{U}}$. Using [3, Theorem 2.1], we obtain

$$P(\boldsymbol{u},\boldsymbol{f}) = \int_0^{T_0} R(t,\boldsymbol{u}(t),\boldsymbol{f}(t))\,dt \leq \liminf_{k\to\infty} \int_0^{T_0} R(t,\boldsymbol{u}_k(t),\boldsymbol{f}_k(t))\,dt = m. \tag{45}$$

Since the embedding of $\mathcal{W}$ into $C(0,T_0;H)$ is continuous, $\boldsymbol{u}_k(t) \rightharpoonup \boldsymbol{u}(t)$ in H for all $t \in [0,T_0]$. Hence, we have that $\boldsymbol{u}_k(0) \rightharpoonup \boldsymbol{u}(0) = \boldsymbol{u}_0$ in H. Similarly as in the proof of Lemma 4.4, using $H(J)(iii)$ we get

$$\begin{aligned}\|\boldsymbol{\eta}_k\|_{\mathcal{U}}^2 &\leq 2T_0\bar{c}^2 + 2\bar{c}^2\lambda^2\|\gamma\|^2\|\boldsymbol{u}_k\|_{\mathcal{V}}^2 \\ &\leq 2T_0\bar{c}^2 + 2\bar{c}^2\lambda^2\|\gamma\|^2\|\boldsymbol{u}_k\|_{\mathcal{W}}^2 \\ &\leq M_3,\end{aligned} \tag{46}$$

where $M_3 > 0$. Thus, we may assume that $\boldsymbol{\eta}_k \rightharpoonup \boldsymbol{\eta}$ in $\mathcal{U}$. Hence, $\gamma^*\boldsymbol{\eta}_k \rightharpoonup \gamma^*\boldsymbol{\eta}$ in $\mathcal{V}$. Since $\mathcal{A}$ is linear and continuous, it is weakly continuous. Hence, $\mathcal{A}\boldsymbol{u}_k \rightharpoonup \mathcal{A}\boldsymbol{u}$ in $\mathcal{V}^*$.

Let $\Phi(\boldsymbol{x},t)=\phi(t)\,\boldsymbol{v}(\boldsymbol{x})$ with $\phi\in C_0^\infty(0,T_0)$ and $\boldsymbol{v}\in V$. Multiplying (40) with Φ and integrating on $(0,T_0)$, we have

$$\langle \boldsymbol{u}_k'+\mathcal{A}\boldsymbol{u}_k+\gamma^*\boldsymbol{\eta}_k-\boldsymbol{f}_k,\Phi\rangle_{\mathcal{V}^*\times\mathcal{V}}=0.$$

Letting $k\to\infty$, we obtain

$$\langle \boldsymbol{u}'+\mathcal{A}\boldsymbol{u}+\gamma^*\boldsymbol{\eta}-\boldsymbol{f},\Phi\rangle_{\mathcal{V}^*\times\mathcal{V}}=0,$$

i.e.,

$$\int_0^{T_0}\langle \boldsymbol{u}'(t)+A\boldsymbol{u}(t)+\gamma^*\boldsymbol{\eta}(t)-\boldsymbol{f}(t),\boldsymbol{v}\rangle_{V^*\times V}\phi(t)\,dt=0.$$

Since $\phi\in C_0^\infty(0,T_0)$ is arbitrary, we deduce that

$$\langle \boldsymbol{u}'(t)+A\boldsymbol{u}(t)+\gamma^*\boldsymbol{\eta}(t)-\boldsymbol{f}(t),\boldsymbol{v}\rangle_{V^*\times V}=0 \text{ a.e. } t\in(0,T_0).$$

Since $\boldsymbol{v}\in V$ is arbitrary, we get

$$\boldsymbol{u}'(t)+A\boldsymbol{u}(t)+\gamma^*\boldsymbol{\eta}(t)=\boldsymbol{f}(t) \text{ a.e. } t\in(0,T_0).$$

Continuing as in the proof of Theorem 4.5, we have $\boldsymbol{\eta}(t)\in\partial J(\gamma\boldsymbol{u}(t))$ a.e. $t\in(0,T_0)$. This together with $\boldsymbol{u}(0)=\boldsymbol{u}_0$ shows that the pair $(\boldsymbol{u},\boldsymbol{f})$ is admissible. Recalling (45), we have $P(\boldsymbol{u},\boldsymbol{f})=m$. This completes the proof. ☐

For any given $\boldsymbol{f}\in\widehat{\mathcal{U}}_0$, we denote by $S(\boldsymbol{f})$ the set of solutions in $\mathcal{W}$ of the problem

$$\begin{cases}\boldsymbol{u}'(t)+A\boldsymbol{u}(t)+\gamma^*\partial J(\gamma\boldsymbol{u}(t))\ni\boldsymbol{f}(t) \text{ a.e. } t\in(0,T_0),\\ \boldsymbol{u}(0)=\boldsymbol{u}_0.\end{cases}\tag{47}$$

Next we present a result on the closedness of the graph of the map $\widehat{\mathcal{U}}\supset\widehat{\mathcal{U}}_0\ni\boldsymbol{f}\mapsto S(\boldsymbol{f})\subset\mathcal{W}$ in suitable topologies.

Theorem 6.3. *Assume $H(j)$, $\nu>\bar{c}\lambda^2$, $H(\widehat{\mathcal{U}})$, and $\boldsymbol{u}_0\in H$. Then the multivalued mapping*

$$\widehat{\mathcal{U}}\supset\widehat{\mathcal{U}}_0\ni\boldsymbol{f}\mapsto S(\boldsymbol{f})\subset\mathcal{W}$$

has a closed graph in $\widehat{\mathcal{U}}_w\times\mathcal{W}_w$.

Proof. Let $\boldsymbol{f}_k\in\widehat{\mathcal{U}}_0$, $\boldsymbol{u}_k\in S(\boldsymbol{f}_k)$,

$$\boldsymbol{f}_k\rightharpoonup\boldsymbol{f} \text{ in } \widehat{\mathcal{U}} \quad\text{and}\quad \boldsymbol{u}_k\rightharpoonup\boldsymbol{u} \text{ in } \mathcal{W}.\tag{48}$$

Then

$$\begin{aligned}&\boldsymbol{u}_k'(t)+A\boldsymbol{u}_k(t)+\gamma^*\boldsymbol{\eta}_k(t)=\boldsymbol{f}_k(t) \text{ a.e. } t\in(0,T_0),\\ &\boldsymbol{u}_k(0)=\boldsymbol{u}_0,\end{aligned}$$

where

$$\boldsymbol{\eta}_k(t)\in\partial J(\gamma\boldsymbol{u}_k(t)) \text{ a.e. } t\in(0,T_0).\tag{49}$$

We shall show that $\boldsymbol{u}\in S(\boldsymbol{f})$. The weak closedness of $\widehat{\mathcal{U}}_0$ implies that the limit $\boldsymbol{f}\in\widehat{\mathcal{U}}_0$. Since the embedding of $\mathcal{W}$ into $C(0,T_0;H)$ is continuous, $\boldsymbol{u}_k(t)\rightharpoonup\boldsymbol{u}(t)$ in H for all $t\in[0,T_0]$. Hence, we have

$$\boldsymbol{u}_k(0)\rightharpoonup\boldsymbol{u}(0)=\boldsymbol{u}_0 \text{ in } H.\tag{50}$$

Since $\{\boldsymbol{u}_k\}$ is bounded in $\mathcal{W}$, from (46) we may assume that $\boldsymbol{\eta}_k \rightharpoonup \boldsymbol{\eta}$ in $\mathcal{U}$. Thus,

$$\gamma^* \boldsymbol{\eta}_k \rightharpoonup \gamma^* \boldsymbol{\eta} \text{ in } \mathcal{V}. \tag{51}$$

Since $\mathcal{A}$ is linear and continuous, it is weakly continuous. Therefore,

$$\mathcal{A}\boldsymbol{u}_k \rightharpoonup \mathcal{A}\boldsymbol{u} \text{ in } \mathcal{V}^*. \tag{52}$$

Similarly as in the proof of Theorem 4.5, from (49) we have

$$\boldsymbol{\eta}(t) \in \partial J(\gamma \boldsymbol{u}(t)) \text{ a.e. } t \in (0, T_0). \tag{53}$$

Similar to the proof of Theorem 6.2 and using (48), (51), (52), from (49) we obtain

$$\boldsymbol{u}'(t) + (\mathcal{A}\boldsymbol{u})(t) + \gamma^* \boldsymbol{\eta}(t) = \boldsymbol{f}(t) \text{ a.e. } t \in (0, T_0).$$

This together with (50) and (53) gives $\boldsymbol{u} \in S(\boldsymbol{f})$, and the proof is completed. □

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