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Research paper

# The virtual element method for an obstacle problem of a Kirchhoff-Love plate[☆]

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ABSTRACT

This paper is devoted to the numerical solution of a fourth-order elliptic variational inequality of the first kind by the virtual element method (VEM). The variational inequality models an obstacle problem for the Kirchhoff-Love plate. Both conforming and fully nonconforming VEMs are studied to solve the fourth-order elliptic variational inequality. Optimal order error estimates are derived in the discrete energy norm, under certain solution regularity assumptions. The primal-dual active algorithm is applied to solve the discrete problems. Numerical examples are reported to show the performance of the numerical methods and to illustrate the convergence orders of the numerical solutions.



## 1. Introduction

The pioneering work of virtual element methods (VEMs), which are a generalization of the finite element method, can be found in [1–3]. Subsequently, VEMs have been extended to solving many different kinds of boundary value problems and initial-boundary value problems thanks to their advantages in handling problems with complex geometries and high solution regularity requirements. For example, conforming and nonconforming VEMs are presented for second-order elliptic problems in [4,5], fourth-order problems in [6,7], polyharmonic problems in [8], elasticity problems in [9,10], and Stokes or Navier–Stokes problems in [11–16], and so on.

An important family of nonlinear boundary value and initial-boundary value problems in a wide variety of applications is provided by variational inequalities, which describe physical and engineering problems with more complicated features. A partial list of the applications of variational inequalities includes contact mechanics, flows of non-Newtonian fluid or with nonlinear leak/slip boundary conditions of friction type, mathematical finance, obstacle problems, plasticity, Stefan problems, unilateral problems, and so on. Rigorous mathematical analysis of variational inequalities started in 1960s. Some comprehensive references on the mathematical theory, numerical methods and applications of variational inequalities are [17–23]. Recently, VEMs have also been applied to solve variational inequalities (cf. [24–27]) and closely related hemivariational inequalities (cf. [28–32]). The goal of this paper is to study conforming and fully nonconforming VEMs for solving

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a fourth-order elliptic variational inequality arising in an obstacle problem of a Kirchhoff-Love plate. We note that numerical approximation of fourth-order obstacle plate problems has been previously studied in a couple of publications, e.g., [33–37]. Conforming, nonconforming and discontinuous Galerkin methods were analyzed in [35]; several results proved in [35] will be quoted and used in our analysis of VEMs for the fourth-order elliptic variational inequality, cf. Sections 3 and 4. A quadratic $C^0$ interior penalty method [34] and the Morley finite element method [33] were considered for the obstacle plate problem.

It is known that for numerical analysis of variational inequalities, due to the inequality feature, it is possible to derive optimal order error estimates only for the lowest order elements (cf. [38]). Moreover, since in general, the solution of a variational inequality does not have high regularity, high order elements are rarely used in solving a variational inequality. Therefore, in this paper, we only study lowest order conforming and fully nonconforming virtual element methods for the fourth-order elliptic variational inequality. We first propose an abstract numerical method and develop its error estimate (see (3.9) and Lemma 3.2). Then, in order to bound a key quantity $R_h(u,u_h)$ in Lemma 3.2, we extend an abstract framework of error analysis regarding triangular meshes in [35, Section 3] to polygonal meshes (see (3.17) and (3.23)). The technique is mainly based on the harmonic property of the lowest order conforming virtual element space and the maximum principle for harmonic functions. Incorporated with the conforming and fully nonconforming VEMs for fourth order elliptic problems, we can propose the conforming and fully nonconforming VEMs for the obstacle problem (2.2) in a unified way, and derive optimal order error estimates for the two methods under appropriate solution regularity assumptions. In particular, for deriving the error bound of the latter method, an enriching operator is introduced and its error estimates are developed as well. We mention in passing that the fully nonconforming virtual element method has fewer degrees of freedom than the conforming method at each element of a given mesh, but both methods have the same convergence order in the discrete energy norm in theory which is demonstrated by our numerical experiments too.

The rest of this paper is organized as follows. In Section 2, we introduce the fourth order variational inequality for the plate obstacle problem. The variational inequality is of the first kind, more precisely, it is an inequality posed over a convex constraint set. In Section 3, we present an abstract framework of the VEMs for the obstacle problem. A preliminary result on error estimation is established too. In Section 4, we discuss conforming as well as fully nonconforming VEMs for the obstacle problem, and derive optimal order error bounds under certain solution regularity assumptions. In Section 5, numerical results are presented to show the performance of the VEMs in solving a sample fourth order variational inequality and to provide numerical convergence orders that match the theoretical predictions.

## 2. A fourth-order variational inequality for the plate obstacle problem

Let $\Omega\subset\mathbb{R}^2$ be a bounded convex polygonal domain with the boundary $\Gamma=\partial\Omega$. We will use $(\cdot,\cdot)$ for the $L^2(\Omega)$ inner product, and use $\|\cdot\|_0$ for the standard norm in $L^2(\Omega)$. To define the obstacle problem, we introduce an obstacle function $\psi\in C^2(\Omega)\cap C(\overline{\Omega})$ such that $\psi<0$ on the boundary $\Gamma$ and a corresponding function set

$$K=\{v\in V \mid v\geq\psi \text{ in } \Omega\}, \tag{2.1}$$

where $V=H_0^2(\Omega)$. It is easy to check that $K$ is a non-empty, closed and convex subset of $V$. For any $f\in L^2(\Omega)$, similar to [39], we consider the variational inequality

$$u\in K,\quad a(u,v-u)\geq(f,v-u)\quad\forall v\in K, \tag{2.2}$$

where

$$a(u,v)=\int_\Omega D^2u:D^2v\,dx=\int_\Omega\partial_{ij}u\,\partial_{ij}v\,dx. \tag{2.3}$$

Here and below, $i,j$ take their values in the set $\{1,2\}$, $D^2v$ denotes the Hessian of a given function $v$, the symbol ":" means the Frobenius inner product between two matrices, $\partial_iv=\partial_{x_i}v$ and $\partial_{ij}v=\partial_{x_ix_j}v$. We also use Einstein's convention for summation.

The variational inequality (2.2) can be used to model an obstacle problem for a horizontal, elastic thin plate which is clamped on the boundary and is subject to the action of a vertical force with a re-scaled density $f\in L^2(\Omega)$. In this case, the unknown $u$ stands for the vertical displacement of a Kirchhoff-Love plate and the constraint $u\geq\psi$ in $\Omega$ reflects the fact that the plate lies above the obstacle with the height function $\psi$.

Since $a(\cdot,\cdot):V\times V\to\mathbb{R}$ is bounded and elliptic, and $f\in V^*$, the variational inequality (2.2) admits a unique solution $u\in K$ (cf. [18,40]).

Let $\boldsymbol{n}=(n_1,n_2)$, $\boldsymbol{\tau}=(\tau_1,\tau_2)$ be the unit outward normal and the unit tangent vector on $\Gamma$ such that $(\boldsymbol{n},\boldsymbol{\tau})$ forms a right-hand system. Then, write $\partial_{\boldsymbol{nn}}v=\partial_{ij}v\,n_in_j$ and $\partial_{\boldsymbol{n\tau}}v=\partial_{ij}v\,\tau_in_j$. Note that

$$\partial_iv=n_i\partial_{\boldsymbol{n}}v+\tau_i\partial_{\boldsymbol{\tau}}v.$$

Consequently, by a direct manipulation, for sufficiently smooth functions $u$ and $v$,

$$\begin{aligned}\int_\Omega\partial_{ij}u\,\partial_{ij}v\,dx&=\int_{\partial\Omega}\partial_{ij}u\,\partial_iv\,n_j\,ds-\int_\Omega\partial_iv\,\partial_{ijj}u\,dx.\\&=\int_{\partial\Omega}\partial_{\boldsymbol{nn}}u\,\partial_{\boldsymbol{n}}v\,ds+\int_{\partial\Omega}\partial_{\boldsymbol{n\tau}}u\,\partial_{\boldsymbol{\tau}}v\,ds-\int_\Omega\nabla v\cdot\nabla(\Delta u)\,dx.\end{aligned} \tag{2.4}$$

Choosing $v \in H_0^2(\Omega)$, (2.4) leads to

$$\int_\Omega \partial_{ij}u\, \partial_{ij}v\, dx = -\int_\Omega \nabla v \cdot \nabla(\Delta u)\, dx. \tag{2.5}$$

The above two identities will be used frequently in the forthcoming error estimation.

## 3. Virtual element method for the obstacle problem of a clamped Kirchhoff plate

Let $\{\mathcal{T}_h\}_h$, $\mathcal{T}_h := \{T\}_{T\in\mathcal{T}_h}$, be a family of partitions of $\overline{\Omega}$ into polygons. A generic element in the partition $\mathcal{T}_h$ is denoted by $T$. The boundary of $T$ is denoted by $\partial T$. Define $h_T := \mathrm{diam}(T)$ and $h := \max_{T\in\mathcal{T}_h} h_T$. For a non-negative integer $k$ and an element $T \in \mathcal{T}_h$, denote by $\mathbb{P}_k(T)$ the set of all polynomials on $T$ with a total degree no more than $k$. Let $\mathcal{E}_h$ be the set of all the element edges and denote by $\mathcal{V}_h = \mathcal{V}_h^i \cup \mathcal{V}_h^b$ the set of vertices in $\mathcal{T}_h$, where $\mathcal{V}_h^i$ and $\mathcal{V}_h^b$ are the sets of interior vertices of $\mathcal{T}_h$ and boundary vertices on $\Gamma$, respectively. We denote the traces of $v$ on $e \subset \partial T^+ \cap \partial T^-$ from the interiors of the elements $T^\pm$ by $v^\pm$, respectively. Then, we define the jump of $v$ on the interior edge by $[v] = v^+ - v^-$ and on the boundary edge by $[v] = v|_e$.

Throughout this paper, we will always make the following assumption on the family of partitions $\{\mathcal{T}_h\}_h$ (cf. [41]):

**Assumption (H1).** For each $T \in \mathcal{T}_h$, there exists a "virtual triangulation" $\mathcal{T}_T$ of $T$ such that $\mathcal{T}_T$ is uniformly shape regular and quasi-uniform. The corresponding mesh size of $\mathcal{T}_T$ is bounded from below by a constant multiple of $h_T$. Each edge of $T$ is a side of certain triangle in $\mathcal{T}_T$.

Given a positive integer $m$, a real number $p \ge 1$, and a bounded set $D \subset \mathbb{R}^2$, $W^{m,p}(D)$ denotes the usual Sobolev space with the corresponding norm $\|\cdot\|_{m,p,D}$ and semi-norm $|\cdot|_{m,p,D}$. When $p = 2$, we denote the corresponding norm and semi-norm by $\|\cdot\|_{m,D}$ and $|\cdot|_{m,D}$, respectively. Moreover, if $D = \Omega$, we simply write $\|\cdot\|_m$ and $|\cdot|_m$ for the norm and semi-norm in $W^{m,2}(\Omega)$ (or $H^m(\Omega)$). Introduce the broken Sobolev space

$$H^m(\mathcal{T}_h) = \Pi_{T\in\mathcal{T}_h} H^m(T) = \{v \in L^2(\Omega) \mid v|_T \in H^m(T)\ \forall T \in \mathcal{T}_h\},$$

which is endowed with the broken $H^m$-seminorm

$$|v|_{m,h} = \Big(\sum_{T\in\mathcal{T}_h} |v|_{m,T}^2\Big)^{1/2}. \tag{3.1}$$

Based on the partition $\mathcal{T}_h$, we construct a finite dimensional function space $V_h \subset H^2(\mathcal{T}_h)$, and denote by $V_h^T$ the restriction of $V_h$ on $T$. We assume that $|\cdot|_{2,h}$ is a norm on the space $V_h$. The discrete counterpart of the constraint set $K$ is

$$K_h := \{v \in V_h \mid v(p) \ge \psi(p)\ \forall p \in \mathcal{V}_h\}.$$

On the other hand, the bilinear form from (2.3) can be split as

$$a(u,v) = \sum_{T\in\mathcal{T}_h} a^T(u,v), \quad a^T(u,v) = \int_T D^2u : D^2v\, dx.$$

Evidently,

$$a^T(u,v) \le |u|_{2,T}|v|_{2,T} \quad \forall u, v \in H^2(T), \tag{3.2}$$

$$a^T(v,v) \ge |v|_{2,T}^2 \quad \forall v \in H^2(T). \tag{3.3}$$

The discrete symmetric bilinear form $a_h(\cdot,\cdot)$ over $V_h \times V_h$ will be constructed through the formula

$$a_h(u_h,v_h) = \sum_{T\in\mathcal{T}_h} a_h^T(u_h,v_h) \quad \forall u_h, v_h \in V_h,$$

where $a_h^T(\cdot,\cdot)$ is a bilinear form on $V_h^T \times V_h^T$.

From now on, we will always assume the following conditions hold:

**Assumption (H2).**

- *k-Consistency*: There exists a natural number $k$ such that for all $p \in \mathbb{P}_k(T)$ and for all $v_h \in V_h^T$,

$$a_h^T(p, v_h) = a^T(p, v_h). \tag{3.4}$$

- *Stability*: There exist two positive constants $\alpha_*$ and $\alpha^*$, independent of $h_T$ and $T$, such that

$$\alpha_* a^T(v_h,v_h) \le a_h^T(v_h,v_h) \le \alpha^* a^T(v_h,v_h) \quad \forall v_h \in V_h^T. \tag{3.5}$$

Combining the inequalities (3.5), (3.2) and (3.3), we can derive the following relations:

$$a_h(u_h,v_h) \le \alpha^* |u_h|_{2,h}|v_h|_{2,h} \quad \forall u_h, v_h \in V_h, \tag{3.6}$$

$$a_h(v_h, v_h) \geq \alpha_* |v_h|_{2,h}^2 \quad \forall\, v_h \in V_h. \tag{3.7}$$

Let $f_h \in V_h^*$ be an approximation of $f$ such that

$$(f - f_h, v_h) \lesssim \|f - f_h\|_{V_h^*} |v_h|_{2,h} \quad \forall\, v_h \in V_h. \tag{3.8}$$

Here and below, to simplify the presentation, we use " $\cdots \lesssim \cdots$" to mean "$\cdots \leq C \cdots$", with $C$ a generic constant independent of $h_T$ or $h$, which may take different values at different occurrences. In addition, for any two quantities $A$ and $B$, $A \approx B$ means that both $A \lesssim B$ and $B \lesssim A$ hold.

The discrete approximation of the variational inequality (2.2) is

$$u_h \in K_h, \quad a_h(u_h, v - u_h) \geq (f_h, v - u_h) \quad \forall\, v \in K_h. \tag{3.9}$$

Since assumption (H1) holds, the Scott-Dupont approximation theory yields the following result directly.

**Lemma 3.1.** *For all $T \in \mathcal{T}_h$ and all $v \in H^{k+1}(T)$ with $k$ a natural number, there exist functions $v_\pi \in \mathbb{P}_k(T)$ such that*

$$\sum_{i=0}^{k+1} h_T^i \|v - v_\pi\|_{i,T} \lesssim h_T^{k+1} |v|_{k+1,T}. \tag{3.10}$$

Moreover, it is easy to show the following trace inequality (cf. [41]):

$$\|v\|_{0,\partial T}^2 \lesssim h_T^{-1} \|v\|_{0,T}^2 + h_T |v|_{1,T}^2 \quad \forall\, v \in H^1(T). \tag{3.11}$$

A preliminary error estimate is given next. We define an interpolation operator $I_h : H_0^2(\Omega) \to V_h$ by $(I_h v)|_T := I_T v$, where $I_T$ is the interpolation operator on $T$. When there is no confusion caused, for all $v \in H^3(\Omega)$, we will also use $v_\pi$ to denote a piecewise polynomial with respect to $\mathcal{T}_h$, which is determined by Lemma 3.1 in an elementwise way.

**Lemma 3.2.** *Let $u$ and $u_h$ be the solutions of (2.2) and (3.9), respectively. If $u \in H^{k+1}(\Omega)$ with $k$ a natural number and $I_h u \in K_h$, then*

$$|u - u_h|_{2,h}^2 \lesssim |u - I_h u|_{2,h}^2 + h^{2k-2} |u|_{k+1}^2 + \|f - f_h\|_{V_h^*}^2 + R_h(u, u_h), \tag{3.12}$$

*where*

$$R_h(u, u_h) = \sum_{T \in \mathcal{T}_h} a^T(u, I_h u - u_h) - (f, I_h u - u_h). \tag{3.13}$$

**Proof.** Denote $w_h = I_h u - u_h$. By (3.7),

$$\alpha_* |w_h|_{2,h}^2 \leq a_h(w_h, w_h) = \sum_{T \in \mathcal{T}_h} a_h^T(I_h u, w_h) - a_h(u_h, w_h).$$

Write

$$a_h^T(I_h u, w_h) = a_h^T(I_h u - u_\pi, w_h) + a_h^T(u_\pi, w_h),$$

where $u_\pi \in \mathbb{P}_2(T)$ is given according to Eq. 3.10.

By (3.4),

$$a_h^T(u_\pi, w_h) = a^T(u_\pi, w_h) = a^T(u_\pi - u, w_h) + a^T(u, w_h).$$

So

$$\alpha_* |w_h|_{2,h}^2 \leq \sum_{T \in \mathcal{T}_h} \left[ a_h^T(I_h u - u_\pi, w_h) + a^T(u_\pi - u, w_h) \right] + \sum_{T \in \mathcal{T}_h} a^T(u, w_h) - a_h(u_h, w_h).$$

Then recalling (3.5), we have by the Cauchy-Schwarz inequality that

$$\begin{aligned} |w_h|_{2,h}^2 \lesssim & \left[ \Big( \sum_{T \in \mathcal{T}_h} |I_T u - u_\pi|_{2,T}^2 \Big)^{1/2} + \Big( \sum_{T \in \mathcal{T}_h} |u_\pi - u|_{2,T}^2 \Big)^{1/2} \right] |w_h|_{2,h} \\ & + \sum_{T \in \mathcal{T}_h} a^T(u, w_h) - a_h(u_h, w_h). \end{aligned} \tag{3.14}$$

By (3.9),

$$a_h(u_h, w_h) \geq (f_h, w_h).$$

Then,

$$-a_h(u_h, w_h) \leq -(f_h, w_h) = -(f, w_h) + (f - f_h, w_h)$$

and we use (3.8) to bound the term $(f - f_h, w_h)$. Hence, from (3.14) and the triangle inequality,

$$|w_h|_{2,h}^2 \lesssim \Big[\Big(\sum_{T\in\mathcal{T}_h} |I_T u - u|_{2,T}^2\Big)^{1/2} + \Big(\sum_{T\in\mathcal{T}_h} |u_\pi - u|_{2,T}^2\Big)^{1/2}\Big]|w_h|_{2,h} + \|f - f_h\|_{V_h^*}|w_h|_{2,h} + R_h(u, u_h),$$

where $R_h(u, u_h)$ is defined in (3.13). In view of (3.10) and the modified Cauchy-Schwarz inequality

$$a\,b \le \varepsilon\, b^2 + c_\varepsilon\, a^2, \quad \varepsilon > 0,\ c_\varepsilon = \frac{1}{4\,\varepsilon},\ \forall\, a, b \in \mathbb{R}, \tag{3.15}$$

with $\varepsilon > 0$ sufficiently small, we obtain

$$|w_h|_{2,h}^2 \lesssim \sum_{T\in\mathcal{T}_h} |I_T u - u|_{2,T}^2 + h^{2k-2}|u|_{k+1}^2 + \|f - f_h\|_{V_h^*}^2 + R_h(u, u_h). \tag{3.16}$$

Finally, by the triangle inequality,

$$|u - u_h|_{2,h}^2 = |(u - I_h u) + w_h|_{2,h}^2 \le 2\big(|u - I_h u|_{2,h}^2 + |w_h|_{2,h}^2\big),$$

we obtain the inequality (3.12). □

Thanks to (3.12), our main task next is to bound the term $R_h(u, u_h)$ defined in (3.13). To this end, following the ideas in [33,35], we first introduce an intermediate obstacle problem:

$$\tilde{u}_h \in \tilde{K}_h, \quad a(\tilde{u}_h, v - \tilde{u}_h) \ge (f, v - \tilde{u}_h) \quad \forall\, v \in \tilde{K}_h, \tag{3.17}$$

where $\tilde{K}_h = \{v \in V \mid v(p) \ge \psi(p)\ \forall\, p \in \mathcal{V}_h\}$.

Next, we introduce the lowest order conforming VEM space as a bridge to bound the difference between $u$ and $\tilde{u}_h$. As shown in [2], for all $T \in \mathcal{T}_h$, define

$$V_1(T) = \big\{v \in H^1(T) \mid \Delta v = 0 \text{ in } T,\ v|_{\partial T} \in C(\partial T),\ v|_e \in P_1(e)\ \forall\, e \in \partial T\big\}, \tag{3.18}$$

with the function values at the vertices of $T$ as a set of degrees of freedom. Then the virtual element space $W_h$ is defined by

$$W_h = \{v_h \in C(\overline{\Omega}) \mid v|_T \in V_1(T)\ \forall\, T \in \mathcal{T}_h\} \cap H_0^1(\Omega).$$

Let $I_c^T : H^2(T) \to V_1(T)$ be the nodal interpolation operator and $I_c$ the global counterpart defined by $I_c v|_T = I_c^T(v|_T)\ \forall\, v \in C(\overline{\Omega})$. For the operator $I_c^T$, we can establish the following interpolation error estimates.

**Lemma 3.3.** *For any $T \in \mathcal{T}_h$, there hold*

$$\|v - I_c^T v\|_{0,\infty,T} \lesssim h\|v\|_{2,T} \quad \forall\, v \in H^2(T), \tag{3.19}$$

$$\|v - I_c^T v\|_{0,\infty,T} \lesssim h^2|v|_{2,\infty,T} \quad \forall\, v \in C^2(\overline{T}). \tag{3.20}$$

**Proof.** Let $v_\pi$ be the function in Lemma 3.1 for $k = 1$. Then, using the definition of $I_c^T$ and the maximum principle for harmonic functions (cf. [42]), we know

$$\begin{aligned}\|v - I_c^T v\|_{0,\infty,T} &\le \|v - v_\pi\|_{0,\infty,T} + \|I_c^T(v - v_\pi)\|_{0,\infty,T}\\ &\le \|v - v_\pi\|_{0,\infty,T} + \|I_c^T(v - v_\pi)\|_{0,\infty,\partial T}\\ &\le \|v - v_\pi\|_{0,\infty,T} + \|v - v_\pi\|_{0,\infty,\partial T}\\ &\le 2\,\|v - v_\pi\|_{0,\infty,T}.\end{aligned} \tag{3.21}$$

On the other hand, for all $\tau \in \mathcal{T}_T$, we have by the Sobolev embedding theorem and the scaling argument that

$$\|v - v_\pi\|_{0,\infty,\tau} \lesssim \sum_{i=0}^{2} h_T^{-1+i}|v - v_\pi|_{i,\tau},$$

which implies

$$\|v - v_\pi\|_{0,\infty,T} \lesssim \sum_{i=0}^{2} h_T^{-1+i}|v - v_\pi|_{i,T}.$$

This combined with Lemma 3.1 yields (3.19) readily. The estimate (3.20) can be obtained in a similar manner. □

In addition, by the definition of $I_c$, it is easy to see that $I_c\psi(p) \le I_c\tilde{u}_h(p)\ \forall\, p \in \mathcal{V}_h$, which implies that $I_c\psi \le I_c\tilde{u}_h$ on $\partial T$. Hence, by the maximum principle for harmonic functions, we further have $I_c^T\psi \le I_c^T\tilde{u}_h$ for all $T \in \mathcal{T}_h$, which implies the following important relation:

$$I_c\psi \le I_c\tilde{u}_h \quad \text{on } \overline{\Omega}. \tag{3.22}$$

Let $I$ be the coincidence set of the obstacle problem (2.2) defined by

$$I = \{x \in \Omega \mid u(x) = \psi(x)\}.$$

The boundary condition of $u$ and the assumption on $\psi$ in Section 2 imply that the compact sets $\partial\Omega$ and $I$ are disjoint. For any positive number $\tau$, we define the compact set $I_\tau$ by

$$I_\tau = \{x \in \overline{\Omega} \mid \mathrm{dist}(x, I) \le \tau\},$$

and let $\delta_h = \|(\tilde{u}_h - I_c\tilde{u}_h) + (I_c\psi - \psi)\|_{0,\infty,I_\tau}$. We can choose $\tau > 0$ small enough such that the compact sets $I_\tau$ and $\partial\Omega$ remain disjoint.

Since $u \in C^2(\Omega) \cap H_0^2(\Omega)$, with the help of (3.19) and (3.20), we have $\delta_h \lesssim h^2$ by using the similar arguments for deriving [35, (8.19)]. Further, using the key inequality (3.22) and the similar arguments for proving [35, Lemma 3.4], we have, for sufficiently small $h$,

$$|u - \tilde{u}_h|_2^2 \lesssim \delta_h \lesssim h^2. \tag{3.23}$$

## 4. Error estimates of virtual element solutions

### 4.1. Conforming virtual element method

We study the $C^1$–type conforming virtual element method proposed in [43] for an obstacle problem of a Kirchhoff plate and try to complete the convergence analysis of this method. In this subsection, we replace $V_h$ by the conforming virtual element space $V_h^c$. Moreover, we assume the assumption (H2) in $V_h^c$ such that the properties still hold in $V_h^c$. Denote by $V_h^{T,c}$ for the restriction of $V_h^c$ to the element $T$. We consider the case $k = 2$ only.

**Local construction of $V_h^{T,c}$.**

Let $T$ be a polygonal element. We define the local finite dimensional space $V_h^{T,c}$ on the element $T$ by

$$V_h^{T,c} := \{v \in H^2(T) \mid \Delta^2 v = 0 \text{ in } T,\ v|_e \in \mathbb{P}_3(e),\ \partial_{\boldsymbol{n}} v|_e \in \mathbb{P}_1(e)\ \forall e \subset \partial T\}. \tag{4.1}$$

We choose the following degrees of freedom for a function $v$ in $V_h^{T,c}$ (cf. [43,44]):

**(D1)**. The values of $v(\xi)$ at the vertices of $T$;

**(D2)**. The values of $h_\xi \partial_1 v(\xi)$ and $h_\xi \partial_2 v(\xi)$.

Here and below, $h_\xi$ is a characteristic length attached to each vertex $\xi$ of $T$. For instance, we may take $h_\xi$ to be the average of the diameters of the elements having $\xi$ as a vertex.

**Global construction of $V_h^c$.** For every decomposition $\mathcal{T}_h$, define the global virtual element space $V_h^c$ as

$$V_h^c := \{v \in V \mid v|_T \in V_h^{T,c}\ \forall T \in \mathcal{T}_h\}.$$

Correspondingly,

$$K_h^c = \{v \in V_h^c \mid v(p) \ge \psi(p)\ \forall p \in \mathcal{V}_h\}. \tag{4.2}$$

The discrete approximation of the variational inequality (2.2) becomes

$$u_h \in K_h^c, \quad a_h^c(u_h, v - u_h) \ge (f_h^c, v - u_h) \quad \forall v \in K_h^c. \tag{4.3}$$

We note that the local degrees of freedom in $V_h^{T,c}$ and the construction of $V_h^c$ imply $V_h^c \subset H^2(\Omega)$, indicating that the virtual element space $V_h^c$ is conforming. It is automatic that $|v|_{2,h}$ is a norm on $V_h^c$. In order to construct the local bilinear form $a_h^{T,c}(\cdot,\cdot)$, we define a projection operator $\Pi_2^{T,c} : V_h^{T,c} \to \mathbb{P}_2(T)$ by (cf. [44])

$$\begin{cases} a^T(\Pi_2^{T,c}\psi, q) = a^T(\psi, q) \quad \forall q \in \mathbb{P}_2(T) \quad \forall \psi \in V_h^{T,c}, \\ \int_{\partial T} \nabla \Pi_2^{T,c}\psi\, ds = \int_{\partial T} \nabla\psi\, ds, \\ \int_{\partial T} \Pi_2^{T,c}\psi\, ds = \int_{\partial T} \psi\, ds. \end{cases} \tag{4.4}$$

Following the arguments in [44, Lemma 3.5 and Lemma 3.6], under the mesh assumption (H1), the inverse inequality and norm equivalence can be established in $V_h^{T,c}$. For convenience, we record these results in the following two lemmas.

**Lemma 4.1.** *For all $T \in \mathcal{T}_h$, there hold*

$$|v|_{1,T} \lesssim h_T^{-1}\|v\|_{0,T}, \quad |v|_{2,T} \lesssim h_T^{-2}\|v\|_{0,T} \quad \forall v \in V_h^{T,c}. \tag{4.5}$$

**Lemma 4.2.** *For all $T \in \mathcal{T}_h$, there hold*

$$h_T^{-1}\|v\|_{0,T} \approx \|\boldsymbol{\chi}(v)\|_{l^2} \quad \forall v \in V_h^{T,c}, \tag{4.6}$$

*and*

$$|v|_{2,T} \approx h_T^{-1}\|\boldsymbol{\chi}(v)\|_{l^2} \quad \forall v \in \mathrm{Ker}(\Pi_2^{T,c}), \tag{4.7}$$

*where $\boldsymbol{\chi}(\cdot) = [\boldsymbol{\chi}_v, \boldsymbol{\chi}_\nabla]$ is the d.o.f. vector corresponding to the $H^2$-conforming virtual element space $V_h^{T,c}$. Here, $\boldsymbol{\chi}_v$ collects the degrees of freedom in* (**D1**)*, and $\boldsymbol{\chi}_\nabla$ the degrees of freedom in* (**D2**).

Define

$$a_h^{T,c}(u,v) = a^T(\Pi_2^{T,c}u, \Pi_2^{T,c}v) + S^{T,c}(u-\Pi_2^{T,c}u, v-\Pi_2^{T,c}v) \quad \forall\, u,v \in V_h^{T,c}.$$

We choose the stabilization term as

$$S^{T,c}(v,w) = h_\xi^{-2}\sum_{i=1}^{N^{T,c}} \chi_i(v)\chi_i(w),$$

which ensures properties (3.4) and (3.5) (cf. [44]). Here, $N^{T,c}$ is the number of degrees of freedom in $V_h^{T,c}$, $\chi_i$ is the $i$th local degree of freedom in (**D1**)–(**D2**).

Next, we consider the error estimation for the nodal interpolation operator $I_T$. We first derive a stability estimate.

**Lemma 4.3.** *Let $I_T : H^3(T) \to V_h^{T,c}$ be the standard nodal interpolation operator. For all $T \in \mathcal{T}_h$, there holds*

$$\|I_T v\|_{0,T} \lesssim \sum_{i=0}^{3} h_T^i |v|_{i,T} \quad \forall\, v \in H^3(T). \tag{4.8}$$

**Proof.** According to the norm equivalence in [44], we have

$$\|I_T v\|_{0,T} \eqsim h_T \|\boldsymbol{\chi}(I_T v)\|_{l^2} \eqsim h_T \|\boldsymbol{\chi}(v)\|_{l^2}. \tag{4.9}$$

Owing to the assumption (H1), we have by the Sobolev embedding theorem that, for all vertex $\xi$ of $T$,

$$|\chi_v(\xi)| = |v(\xi)| \le \|v\|_{0,\infty,T} \lesssim \sum_{m=0}^{3} h_T^{m-1}|v|_{m,T}. \tag{4.10}$$

Similarly,

$$|\chi_\nabla(\xi)| \le h_\xi |\nabla v(\xi)| \le h_T |v|_{1,\infty,T} \lesssim \sum_{m=1}^{3} h_T^{m-1}|v|_{m,T}. \tag{4.11}$$

The combination of (4.9) to (4.11) gives (4.8) readily. □

**Lemma 4.4.** *For all $T \in \mathcal{T}_h$, if $v \in H^3(T)$, then we have*

$$\|v - I_T v\|_{i,T} \lesssim h_T^{3-i}|v|_{3,T}, \quad i = 0,1,2. \tag{4.12}$$

**Proof.** By the triangle inequality and (4.8), for all $p \in \mathbb{P}_2(T)$,

$$\|I_T(v-p)\|_{0,T} \lesssim \sum_{m=0}^{3} h_T^m |v-p|_{m,T}, \tag{4.13}$$

which combined with (3.10) implies

$$\|v - I_T v\|_{0,T} \le \|v - v_\pi\|_{0,T} + \|I_T(v-v_\pi)\|_{0,T} \lesssim \sum_{m=0}^{3} h_T^m |v-v_\pi|_{m,T} \lesssim h_T^3|v|_{3,T}. \tag{4.14}$$

On the other hand, using the inverse inequality in [44], we find

$$|I_T(v-v_\pi)|_{i,T} \lesssim h_T^{-i}\|I_T(v-v_\pi)\|_{0,T}, \quad i=1,2,$$

which combined with (3.10) again implies

$$|v - I_T v|_{i,T} \le |v - v_\pi|_{i,T} + |I_T(v-v_\pi)|_{i,T} \lesssim h_T^{3-i}|v|_{3,T},$$

as required. □

For an approximation of the right-hand side $f_h^c$, we define

$$\langle f_h^c, v_h\rangle = \sum_{T\in\mathcal{T}_h}\int_T P_0^T f\, \widehat{v}_h \, dx,$$

where $P_0^T f$ is the $L^2$-projection $f$ onto the space of $P_0(T)$. Here and below, $\widehat{v} = \frac{1}{N}\sum_{i=1}^{N} v(a_i^T)$ with $a_i^T$ the vertices of $T$ and $N$ the number of edges of $T$, respectively. Then, we have the following approximation property (cf. [43])

$$\|f - f_h^c\|_{(V_h^c)^*} \le C\, h\, \|f\|_0. \tag{4.15}$$

For the conforming method, the quantity $R_h(u, u_h)$ of (3.13) is simplified to

$$R_h(u, u_h) = a(u, I_h u - u_h) - (f, I_h u - u_h). \tag{4.16}$$

Let us bound $R_h(u, u_h)$ by making use of the solution $\tilde{u}_h$ of (3.17). We will assume the solution regularity $u \in H^3(\Omega)$. Write

$$a(u, I_h u - u_h) = a(u - \tilde{u}_h, I_h u - u_h) + a(\tilde{u}_h, I_h u - \tilde{u}_h) + a(\tilde{u}_h, \tilde{u}_h - u_h). \tag{4.17}$$

For the first term on the right side of (4.17), in view of (3.23), we have

$$a(u - \tilde{u}_h, I_h u - u_h) \lesssim |u - \tilde{u}_h|_2 |I_h u - u_h|_2 \lesssim h\, |I_h u - u_h|_2. \tag{4.18}$$

Choose $v = u_h$ in (3.17) to get

$$a(\tilde{u}_h, u_h - \tilde{u}_h) \geq (f, u_h - \tilde{u}_h).$$

Thus,

$$a(\tilde{u}_h, \tilde{u}_h - u_h) \leq (f, \tilde{u}_h - I_h u) + (f, I_h u - u_h). \tag{4.19}$$

Moreover,

$$\begin{aligned} a(\tilde{u}_h, I_h u - \tilde{u}_h) &= a(\tilde{u}_h - u, I_h u - \tilde{u}_h) + a(u, I_h u - u) + a(u, u - \tilde{u}_h) \\ &\lesssim |\tilde{u}_h - u|_2 |I_h u - \tilde{u}_h|_2 + a(u, I_h u - u) + a(u, u - \tilde{u}_h). \end{aligned} \tag{4.20}$$

By the triangle inequality and ,Eqn 4.12

$$|\tilde{u}_h - u|_2 |I_h u - \tilde{u}_h|_2 \leq |\tilde{u}_h - u|_2^2 + |\tilde{u}_h - u|_2 |I_h u - u|_2 \lesssim h^2. \tag{4.21}$$

Using integration by part in (2.3) and $u - I_h u \in H_0^2(\Omega)$, by (2.5) and (4.12), we obtain

$$\begin{aligned} a(u, I_h u - u) &= -\int_\Omega \nabla(I_h u - u) \cdot \nabla(\Delta u)\, dx \\ &\lesssim |u - I_h u|_{1,\Omega} |u|_{3,\Omega} \lesssim h^2 |u|_{3,\Omega}^2. \end{aligned} \tag{4.22}$$

Applying (4.17)–(4.22) in (4.16), we find that

$$R_h(u, u_h) \lesssim h\, |I_h u - u_h|_2 + h^2 + a(u, u - \tilde{u}_h) + (f, \tilde{u}_h - I_h u). \tag{4.23}$$

On the other hand, as shown in [35, Section 3], we can find a function $\phi \in C^\infty(\bar{\Omega})$ with the properties

$$\phi \in [0,1] \text{ in } \overline{\Omega}, \quad \phi = 1 \text{ in } I_\tau, \quad \phi = 0 \text{ in } \overline{\Omega} \backslash I_{2\tau}$$

so that

$$\hat{u}_h = \tilde{u}_h + \delta_h \phi \in K.$$

From (2.2) with $v = \hat{u}_h$,

$$a(u, \tilde{u}_h + \delta_h \phi - u) \geq (f, \tilde{u}_h + \delta_h \phi - u),$$

i.e.,

$$a(u, u - \tilde{u}_h) \leq \delta_h a(u, \phi) - (f, \tilde{u}_h - u + \delta_h \phi).$$

So

$$\begin{aligned} a(u, u - \tilde{u}_h) + (f, \tilde{u}_h - I_h u) &\leq (f, u - I_h u) + \delta_h [a(u, \phi) - (f, \phi)] \\ &\lesssim h^3 \|f\|_0 |u|_3 + \delta_h [a(u, \phi) - (f, \phi)]. \end{aligned} \tag{4.24}$$

By (3.23),

$$\delta_h \lesssim h^2.$$

Inserting (4.24) into (4.23), we have

$$R_h(u, u_h) \lesssim h\, |I_h u - u_h|_2 + h^2 + h^3 \|f\|_0 |u|_3. \tag{4.25}$$

Now we are in a position to state and prove an optimal order error estimate for the $C^1$-type VEM.

**Theorem 4.1.** *Let $u_h \in K_h^c$ be the solution of (4.3). If assumptions (H1)–(H2) hold, and $u \in H^3(\Omega)$ is the exact solution of (2.2), then we have the error bound*

$$|u - u_h|_{2,h} \lesssim h.$$

**Proof.** In view of (4.25) and (3.16), we have

$$|I_h u - u_h|_{2,h}^2 \lesssim \sum_{T\in\mathcal{T}_h} |I_T u - u|_{2,T}^2 + h^2|u|_3^2 + \|f - f_h^c\|_{(V_h^c)^*}^2 \\ + h^2 + h|I_h u - u_h|_2 + h^3\|f\|_0|u|_3. \tag{4.26}$$

By applying the modified Cauchy-Schwarz inequality (3.15) with a sufficiently small $\varepsilon$, we obtain from (4.26) that

$$|I_h u - u_h|_{2,h}^2 \lesssim \sum_{T\in\mathcal{T}_h} |I_T u - u|_{2,T}^2 + h^2|u|_3^2 + \|f - f_h^c\|_{(V_h^c)^*}^2 + h^2 + h^3\|f\|_0|u|_3. \tag{4.27}$$

Thanks to (4.12), we have

$$\sum_{T\in\mathcal{T}_h} |u - I_T u|_{2,T}^2 \lesssim h^2|u|_3^2.$$

Together with (4.15), we derive from (4.27) that

$$|I_h u - u_h|_{2,h}^2 \lesssim h^2,$$

i.e.,

$$|I_h u - u_h|_{2,h} \lesssim h.$$

Finally, by the triangle inequality,

$$|u - u_h|_{2,h} \le |u - I_h u|_{2,h} + |I_h u - u_h|_{2,h},$$

and we get the desired result. □

*4.2. Fully nonconforming virtual element method*

We turn to the study of the fully nonconforming virtual element ([6,45,7]) to solve the obstacle problem of the Kirchhoff plate and derive an error bound for the numerical solution. Denote by $V_h^{nc}$ the nonconforming virtual element space. We keep the assumption (H2) on $V_h^{nc}$ and replace $V_h$ by $V_h^{nc}$. The restriction of $V_h^{nc}$ to the element $T$ is written as $V_h^{T,nc}$. Again, we consider the case $k = 2$.

**Local construction of $V_h^{T,nc}$.**

$$V_h^{T,nc} := \{v_h \in H^2(T) \mid \Delta^2 v_h = 0 \text{ in } T,\ \partial_{\boldsymbol{nn}} v_h|_e \in \mathbb{P}_0(e),\ (\partial_{\boldsymbol{n\tau\tau}} v_h + \partial_{\boldsymbol{n}}(\Delta v_h))|_e = 0\ \forall e \in \partial T\}.$$

We choose the degrees of freedom for a function $v_h$ of $V_h^{T,nc}$ as follows:

**(D3)**. The values of $v_h(\xi)$ at the vertices of $T$;

**(D4)**. $\int_e \partial_{\boldsymbol{n}} v_h\, ds$ for any edge $e$ of $\partial T$.

**Global construction of $V_h^{nc}$.**

Built upon the local space $V_h^{T,nc}$, the global nonconforming virtual element space is defined as follows:

$$V_h^{nc} = \{v_h \in L^2(\Omega) \mid v_h|_T \in V_h^{T,nc},\ v_h \text{ is continuous at internal vertices}, \\ v_h(\xi) = 0\ \forall \xi \in \mathcal{V}_h^b,\ \int_e [\partial_{\boldsymbol{n}} v_h]\, ds = 0,\ \forall e \in \mathcal{E}_h\}. \tag{4.28}$$

Correspondingly,

$$K_h^{nc} = \{v \in V_h^{nc} \mid v(p) \ge \psi(p)\ \forall p \in \mathcal{V}_h\}. \tag{4.29}$$

The discrete approximation of the variational inequality (2.2) becomes

$$u_h \in K_h^{nc}, \quad a_h^{nc}(u_h, v - u_h) \ge (f_h^{nc}, v - u_h) \quad \forall v \in K_h^{nc}. \tag{4.30}$$

We observe that by construction, $V_h^{nc} \subset H^{2,nc}(\mathcal{T}_h)$ and $V_h \not\subseteq H^2(\Omega)$. Note that $|\cdot|_{2,h}$ is a norm on the space $V_h^{nc}$ (cf. [46]).

By checking the derivations in [45], we know that all the estimates hold under the assumption (H1). Similar to [45,30], we can derive error estimates of the nodal interpolation operator.

**Lemma 4.5.** *Let $I_T : H^m(T) \to V_h^{T,nc}$, $m = 2, 3$, be the standard nodal interpolation operator. For all $T \in \mathcal{T}_h$, the following error estimate holds*

$$\|v - I_T v\|_{i,T} \lesssim h_T^{m-i}|v|_{m,T}, \quad i = 0, 1, 2. \tag{4.31}$$

In order to construct the local bilinear form $a_h^{T,nc}(\cdot,\cdot)$, we define a projection operator $\Pi_2^{T,nc} : V_h^{T,nc} \to \mathbb{P}_2(T)$ by

$$\begin{cases} a^T(\Pi_2^{T,nc}\psi, q) = a^T(\psi, q) \quad \forall q \in \mathbb{P}_2(T),\ \forall \psi \in V_h^{T,nc}, \\ \widehat{\Pi_2^{T,nc}\psi} = \widehat{\psi}, \\ \sum_{e\subset\partial T} \frac{1}{|e|}\int_e \nabla\Pi_2^{T,nc}\psi\, ds = \sum_{e\subset\partial T} \frac{1}{|e|}\int_e \nabla\psi\, ds. \end{cases} \tag{4.32}$$

It is easy to check that

$$\Pi_2^{T,nc} v = v \quad \forall v \in \mathbb{P}_2(T).$$

We define the local bilinear form by

$$a_h^{T,nc}(u,v) = a^T(\Pi_2^{T,nc}u, \Pi_2^{T,nc}v) + S^{T,nc}(u - \Pi_2^{T,nc}u, v - \Pi_2^{T,nc}v) \quad \forall u, v \in V_h^{T,nc},$$

where the stabilization term is

$$S^{T,nc}(v,w) = h_T^{-2} \sum_{i=1}^{N^{T,nc}} \chi_i(v)\chi_i(w),$$

which ensures the properties (3.4) and (3.5) (cf. [45]). Here, $N^{T,nc}$ is the number of degrees of freedom in $V_h^{T,nc}$, $\chi_i$ is the $i$th local degree of freedom in (**D3**)–(**D4**).

Furthermore, choosing $q = \Pi_2^{T,nc}\psi$ in (4.32), by (3.2) and (3.3), we can verify

$$|\Pi_2^{T,nc}\psi|_{2,T} \lesssim |\psi|_{2,T} \quad \forall \psi \in H^2(T). \tag{4.33}$$

For an approximation of the right-hand side $f_h^{nc}$, we define

$$\langle f_h^{nc}, v_h \rangle = (f, \Pi_h^{nc} v_h), \tag{4.34}$$

where $(\Pi_h^{nc}v)|_T := \Pi_2^{T,nc}(v|_T)$ for $v \in V_h^{nc}$. Then, we have the following approximation property (cf. [45,30])

$$|(f, v_h) - \langle f_h^{nc}, v_h \rangle| = |(f, v_h - \Pi_h^{nc} v_h)| \lesssim h^2 \|f\|_0 |v_h|_{2,h}. \tag{4.35}$$

**Construction of the enriching operator $E_h$.**

For every nonconforming virtual element function $v \in V_h^{nc}$, we need to construct an associated conforming counterpart $E_h v \in V_h^c$. To this end, for any $v \in V_h^{nc}$, we introduce the piecewise $L^2$-projection $P_h$ by

$$P_h v|_T = P^T(v|_T), \quad T \in \mathcal{T}_h,$$

where $P^T$ is the $L^2(T)$-projection from $V_h^{T,nc}$ to $\mathbb{P}_2(T)$ with $V_h^{T,nc}$ the local nonconforming VEM space.

For a vertex $\xi$ of $\mathcal{T}_h$, let $\omega(\xi)$ denote the union of all the elements in $\mathcal{T}_h$ sharing the point $\xi$. Let $N(\xi)$ denote the number of elements in $\omega(\xi)$. For every nonconforming VEM function $v \in V_h^{nc}$, we choose

$$(E_h v)(x) = \sum_{i=1}^{N^c} \mathcal{N}_i(E_h v)\Phi_i(x),$$

with $N^c = \dim(V_h^c)$ and $\{\Phi_i(x)\}$ being a set of shape basis functions of $V_h^c$. Here, the values of degrees of freedom $\mathcal{N} = \mathrm{span}\{\mathcal{N}_a, \mathcal{N}_\nabla\}$ are given by

- The values at interior vertices $\xi \in \mathcal{V}_h^i$:

$$\mathcal{N}_a(E_h v) = E_h v(\xi) := v(\xi).$$

- The gradient values at interior vertices $\xi \in \mathcal{V}_h^i$:

$$\mathcal{N}_\nabla(E_h v) := \frac{1}{N(\xi)} \sum_{T' \in \omega(\xi)} h_\xi \nabla P_h v|_{T'}(\xi).$$

For $v \in V_h^{nc}$, $(P_h v - E_h v)|_T \in V_h^{T,c}$, by the Bramble-Hilbert lemma and (4.6), we derive

$$\begin{aligned} \|v - E_h v\|_{0,T} &\le \|v - P_h v\|_{0,T} + \|P_h v - E_h v\|_{0,T} \\ &\lesssim h_T^2 |v|_{2,T} + h_T \|\boldsymbol{\chi}(P_h v - E_h v)\|_{l^2}. \end{aligned} \tag{4.36}$$

For $\xi \in \mathcal{V}_h^i$, using the similar arguments for deriving (3.19) to get

$$\begin{aligned} |(P_h v - E_h v)|_T(\xi)| &\le \|P_h v - v\|_{0,\infty,T} \\ &\lesssim \sum_{i=0}^{2} h_T^{-1+i} |P_h v - v|_{i,T} \\ &\lesssim h_T |v|_{2,T}. \end{aligned} \tag{4.37}$$

According to [44, Lemma 4.2], there holds

$$|h_\xi \nabla (P_h v - E_h v)|_T(\xi)| \lesssim h_T |v|_{2,T}. \tag{4.38}$$

Then, we have by (4.37) and (4.38) that

$$\|\boldsymbol{\chi}(P_h v - E_h v)\|_{l^2} \lesssim h_T |v|_{2,T}. \tag{4.39}$$

Combining (4.36) and (4.39), one easily finds

$$\sum_{T\in\mathcal{T}_h} h_T^{-4}\|v-E_hv\|_{0,T}^2 \lesssim |v|_{2,h}^2. \tag{4.40}$$

Using an argument similar to that in [47, Lemma 4.2], we obtain

$$\sum_{T\in\mathcal{T}_h} |v-E_hv|_{2,T}^2 \lesssim |v|_{2,h}^2. \tag{4.41}$$

Then, we have the following result.

**Lemma 4.6.** *Let $E_h$ be the above enriching operator from $V_h^{nc}$ to $V_h^c$. For all $T\in\mathcal{T}_h$, there holds*

$$\|v-E_hv\|_{L^2(\Omega)} + h\Big(\sum_{T\in\mathcal{T}_h}|v-E_hv|_{1,T}^2\Big)^{1/2} + h^2|E_hv|_2 \lesssim h^2|v|_{2,h} \quad \forall\, v\in V_h^{nc}. \tag{4.42}$$

**Proof.** In view of the inverse inequality (4.5) and the norm equivalence (4.7), we have

$$\begin{aligned}|v-E_hv|_{1,T} &\le |v-P_hv|_{1,T} + |P_hv-E_hv|_{1,T}\\ &\lesssim |v-P_hv|_{1,T} + h_T^{-1}\|P_hv-E_hv\|_{0,T}\\ &\lesssim h_T|v|_{2,T} + \|\boldsymbol{\chi}(P_hv-E_hv)\|_{l^2},\end{aligned}$$

with $\boldsymbol{\chi}=\{\chi_i\}_{i=1}^{N^{T,c}}$, where $\chi_i$ is the degree of freedom in (**D1**)–(**D2**). Then, (4.39) implies

$$h\Big(\sum_{T\in\mathcal{T}_h}|v-Ev|_{1,T}^2\Big)^{1/2} \lesssim h^2|v|_{2,h}. \tag{4.43}$$

Since

$$|E_hv|_2 \le |E_hv-v|_{2,h} + |v|_{2,h},$$

an application of (4.41) leads to

$$|E_hv|_2 \lesssim |v|_{2,h}. \tag{4.44}$$

From (4.40), (4.43) and (4.44), we obtain the desired result. □

**Lemma 4.7.** *For any $u\in H^3(\Omega)\cap H_0^2(\Omega)$, there holds*

$$\|u-E_hI_hu\|_0 + h|u-E_hI_hu|_1 + h^2|u-E_hI_hu|_2 \lesssim h^3|u|_3. \tag{4.45}$$

**Proof.** For $T\in\mathcal{T}_h$, let $S_T$ be the interior of the union of the closure of all the polygons in $\mathcal{T}_h$ neighboring $T$. By (4.40) and (4.5), we have that for any $v\in V_h^{nc}$,

$$\|v-E_hv\|_{0,T} \lesssim h^2|v|_{H^2(S_T)} \lesssim \|v\|_{L^2(S_T)}.$$

Similar to the proof of (4.44), we derive from the above inequality that

$$\|E_hv\|_{0,T} \lesssim \|v\|_{L^2(S_T)}. \tag{4.46}$$

It is easy to check that $(E_h)|_T\phi=\phi$ for all $\phi\in\mathbb{P}_2(T)$ with $T\in\mathcal{T}_h$. Let $\phi$ be an arbitrary quadratic function on $S_T$. By (4.46),

$$\begin{aligned}\|u-E_hI_hu\|_{0,T} &= \|u-\phi+E_h\phi-E_hI_hu\|_{0,T}\\ &\lesssim \|u-\phi\|_{L^2(S_T)} + \|\phi-I_hu\|_{L^2(S_T)}\\ &\lesssim \|u-\phi\|_{L^2(S_T)} + \|u-I_hu\|_{L^2(S_T)}.\end{aligned} \tag{4.47}$$

Using (4.31) and Lemma 3.1, we obtain

$$\|u-E_hI_hu\|_0 \lesssim h^3|u|_3. \tag{4.48}$$

By a similar technique, we have

$$\begin{aligned}&|u-E_hI_hu|_1 \lesssim h^2|u|_3,\\ &|u-E_hI_hu|_2 \lesssim h\,|u|_3.\end{aligned}$$

Thus, (4.45) holds. □

**Lemma 4.8.** *For all $T\in\mathcal{T}_h$, if $u\in H^3(\Omega)\cap H_0^2(\Omega)$, then we have*

$$\Big|\sum_{T\in\mathcal{T}_h} a^T(u,v-E_hv)\Big| \lesssim h\,|u|_3|v|_{2,h} \quad \forall\, v\in V_h^{nc}. \tag{4.49}$$

**Proof.** We follow an argument given in [33,35]. By integration by parts, it follows that

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} a^T(u, v-E_hv) &= \sum_{T\in\mathcal{T}_h}\int_T \partial_{ij}u\,\partial_{ij}(v-E_hv)\,dx\\ &= -\sum_{T\in\mathcal{T}_h}\int_T \nabla(\Delta u)\cdot\nabla(v-E_hv)\,dx + \sum_{T\in\mathcal{T}_h}\int_{\partial T}\partial_{\boldsymbol{nn}}u\,\partial_{\boldsymbol{n}}(v-E_hv)\,ds\\ &\quad+\sum_{T\in\mathcal{T}_h}\int_{\partial T}\partial_{\boldsymbol{n\tau}}u\,\partial_{\boldsymbol{\tau}}(v-E_hv)\,ds.\end{aligned} \tag{4.50}$$

The first term on the right-hand side of (4.50) can be estimated as follows.

$$\begin{aligned}\Big|\sum_{T\in\mathcal{T}_h}\int_T \nabla(\Delta u)\cdot\nabla(v-E_hv)\,dx\Big| &\le |\Delta u|_{H^1(\Omega)}\Big(\sum_{T\in\mathcal{T}_h}|v-E_hv|^2_{H^1(T)}\Big)^{1/2}\\ &\lesssim h\,|u|_3|v|_{2,h}.\end{aligned} \tag{4.51}$$

Note that

$$\int_e[\partial_{\boldsymbol{n}}(v-E_hv)]\,ds=0,\quad \int_e[\partial_{\boldsymbol{\tau}}(v-E_hv)]\,ds=0,\quad \forall\, e\in\partial T^+\cap\partial T^-.$$

Let $\overline{\partial_{\boldsymbol{nn}}u}$ be the mean value of $\partial_{\boldsymbol{nn}}u$ over the edge $e$. We have

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}\partial_{\boldsymbol{nn}}u\,\partial_{\boldsymbol{n}}(v-E_hv)\,ds=\sum_{e\in\mathcal{E}_h}\int_e(\partial_{\boldsymbol{nn}}u-\overline{\partial_{\boldsymbol{nn}}u})\,[\partial_{\boldsymbol{n}}(v-E_hv)]\,ds.$$

By the Cauchy-Schwarz inequality, (4.42) and the trace theorem, we obtain

$$\begin{aligned}&\sum_{e\in\mathcal{E}_h}\int_e(\partial_{\boldsymbol{nn}}u-\overline{\partial_{\boldsymbol{nn}}u})[\partial_{\boldsymbol{n}}(v-E_hv)]\,ds\\ &\quad\le\Big(\sum_{e\in\mathcal{E}_h}|e|^{-1}\|\partial_{\boldsymbol{nn}}u-\overline{\partial_{\boldsymbol{nn}}u}\|^2_{L^2(e)}\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h}|e|\,\|[\partial_{\boldsymbol{n}}(v-E_hv)]\|^2_{L^2(e)}\Big)^{1/2}\\ &\quad\lesssim h|u|_3|v|_{2,h}.\end{aligned} \tag{4.52}$$

Similar to the derivation of (4.52), we know

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}\partial_{\boldsymbol{n\tau}}u\partial_{\boldsymbol{\tau}}(v-E_hv)\,ds\lesssim h\,|u|_3|v|_{2,h}. \tag{4.53}$$

Inserting (4.51), (4.52) and (4.53) into (4.50), we can derive the inequality (4.49). □

Now, we turn to bound the term $R_h(u,u_h)$ defined in (3.13). Write

$$\sum_{T\in\mathcal{T}_h}a^T(u,I_hu-u_h)=\sum_{T\in\mathcal{T}_h}a^T(u,I_hu-u_h-E_h(I_hu-u_h))+a(u,E_hI_hu-E_hu_h).$$

Applying (4.49), we have

$$\sum_{T\in\mathcal{T}_h}a^T(u,I_hu-u_h)\lesssim h|u|_3|I_hu-u_h|_{2,h}+a(u,E_hI_hu-E_hu_h). \tag{4.54}$$

We will make use of the solution $\tilde{u}_h$ defined by (3.17). Note that

$$a(u,E_hI_hu-E_hu_h)=a(u-\tilde{u}_h,E_hI_hu-E_hu_h)+a(\tilde{u}_h,E_hI_hu-\tilde{u}_h)+a(\tilde{u}_h,\tilde{u}_h-E_hu_h).$$

By (3.23) and (4.42),

$$a(u-\tilde{u}_h,E_hI_hu-E_hu_h)\lesssim|u-\tilde{u}_h|_2|E_h(I_hu-u_h)|_2\lesssim h\,|I_hu-u_h|_{2,h}.$$

From the definition (3.17),

$$a(\tilde{u}_h,\tilde{u}_h-E_hu_h)\le(f,\tilde{u}_h-E_hu_h).$$

Hence

$$a(u,E_hI_hu-E_hu_h)\lesssim h\,|I_hu-u_h|_{2,h}+a(\tilde{u}_h,E_hI_hu-\tilde{u}_h)+(f,\tilde{u}_h-E_hu_h). \tag{4.55}$$

Note that

$$a(\tilde{u}_h,E_hI_hu-\tilde{u}_h)=a(\tilde{u}_h-u,E_hI_hu-\tilde{u}_h)+a(u,E_hI_hu-u)+a(u,u-\tilde{u}_h).$$

Using the Cauchy-Schwarz inequality, (3.23) and (4.45), we obtain

$$\begin{aligned} a(\tilde{u}_h - u, E_h I_h u - \tilde{u}_h) &= a(\tilde{u}_h - u, E_h I_h u - u) + a(\tilde{u}_h - u, u - \tilde{u}_h) \\ &\lesssim |\tilde{u}_h - u|_2 |E_h I_h u - u|_2 \\ &\lesssim h^2 |u|_3. \end{aligned} \tag{4.56}$$

In view of $E_h I_h u - u \in H_0^2(\Omega)$, by (2.5) and (4.45),

$$\begin{aligned} a(u, E_h I_h u - u) &= -\int_\Omega \nabla(\Delta u) \nabla(E_h I_h u - u)\, dx \\ &\lesssim |u|_3 |u - E_h I_h u|_1 \\ &\lesssim h^2 |u|_3^2. \end{aligned} \tag{4.57}$$

Combining (4.55) with (4.56), we get

$$a(\tilde{u}_h, E_h I_h u - \tilde{u}_h) \lesssim h^2 + a(u, u - \tilde{u}_h). \tag{4.58}$$

Inserting (4.58) into (4.55), we derive

$$a(u, E_h I_h u - E_h u_h) \lesssim h\, |I_h u - u_h|_{2,h} + h^2 + a(u, u - \tilde{u}_h) + (f, \tilde{u}_h - E_h u_h).$$

From (4.54), we have

$$\begin{aligned} R_h(u, u_h) \lesssim\ & h|u|_3 |I_h u - u_h|_{2,h} + h\, |I_h u - u_h|_{2,h} + h^2 \\ & + a(u, u - \tilde{u}_h) + (f, \tilde{u}_h - E_h u_h) - (f, I_h u - u_h). \end{aligned} \tag{4.59}$$

As in the conforming case, using the function $\phi$ constructed in [35], we know

$$\hat{u}_h = \tilde{u}_h + \delta_h \phi \in K.$$

Observe that

$$a(u, u - \tilde{u}_h) = a(u, u - \hat{u}_h) + \delta_h a(u, \phi). \tag{4.60}$$

Note that

$$a(u, u - \hat{u}_h) \le (f, u - \hat{u}_h),$$

and

$$(f, u - \hat{u}_h) = (f, u - \tilde{u}_h) - \delta_h (f, \phi).$$

Hence,

$$a(u, u - \tilde{u}_h) \le \delta_h [a(u, \phi) - (f, \phi)] + (f, u - \tilde{u}_h). \tag{4.61}$$

From (4.42) and (4.45), we obtain

$$\begin{aligned} & (f, u - \tilde{u}_h) + (f, \tilde{u}_h - E_h u_h) - (f, I_h u - u_h) \\ &= (f, u - E_h I_h u) + (f, E_h I_h u - \tilde{u}_h) + (f, \tilde{u}_h - E_h u_h) - (f, I_h u - u_h) \\ &= (f, u - E_h I_h u) + \Big(f, E_h (I_h u - u_h) - (I_h u - u_h)\Big) \\ &\lesssim h^3 |u|_3 \|f\|_0 + h^2 \|f\|_0 |I_h u - u_h|_{2,h}. \end{aligned} \tag{4.62}$$

Together with (4.61), (4.62) and (3.23), we have

$$a(u, u - \tilde{u}_h) + (f, \tilde{u}_h - E_h u_h) - (f, I_h u - u_h) \lesssim h^2 + h^3 |u|_3 \|f\|_0 + h^2 \|f\|_0 |I_h u - u_h|_{2,h}. \tag{4.63}$$

Owing to (4.59) and (4.63), we find

$$\begin{aligned} R_h(u, u_h) \lesssim\ & h|u|_3 |I_h u - u_h|_{2,h} + h|I_h u - u_h|_{2,h} + h^2 \\ & + h^3 |u|_3 \|f\|_0 + h^2 \|f\|_0 |I_h u - u_h|_{2,h}. \end{aligned} \tag{4.64}$$

An optimal order error bound for the fully nonconforming VEM is provided in the next result.

**Theorem 4.2.** *Assume (H1)–(H2) and $u \in H^3(\Omega)$ for the true solution of (2.2). Let $u_h \in K_h^{nc}$ be the solution of (4.30). Then we have the error bound*

$$|u - u_h|_{2,h} \lesssim h.$$

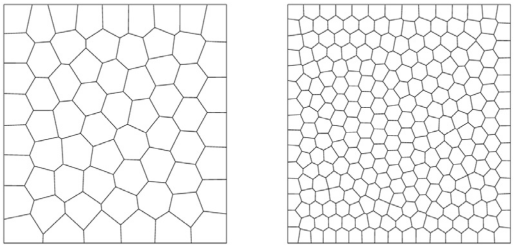

**Fig. 1.** Polygonal meshes with 64 elements (left) and 256 elements (right).

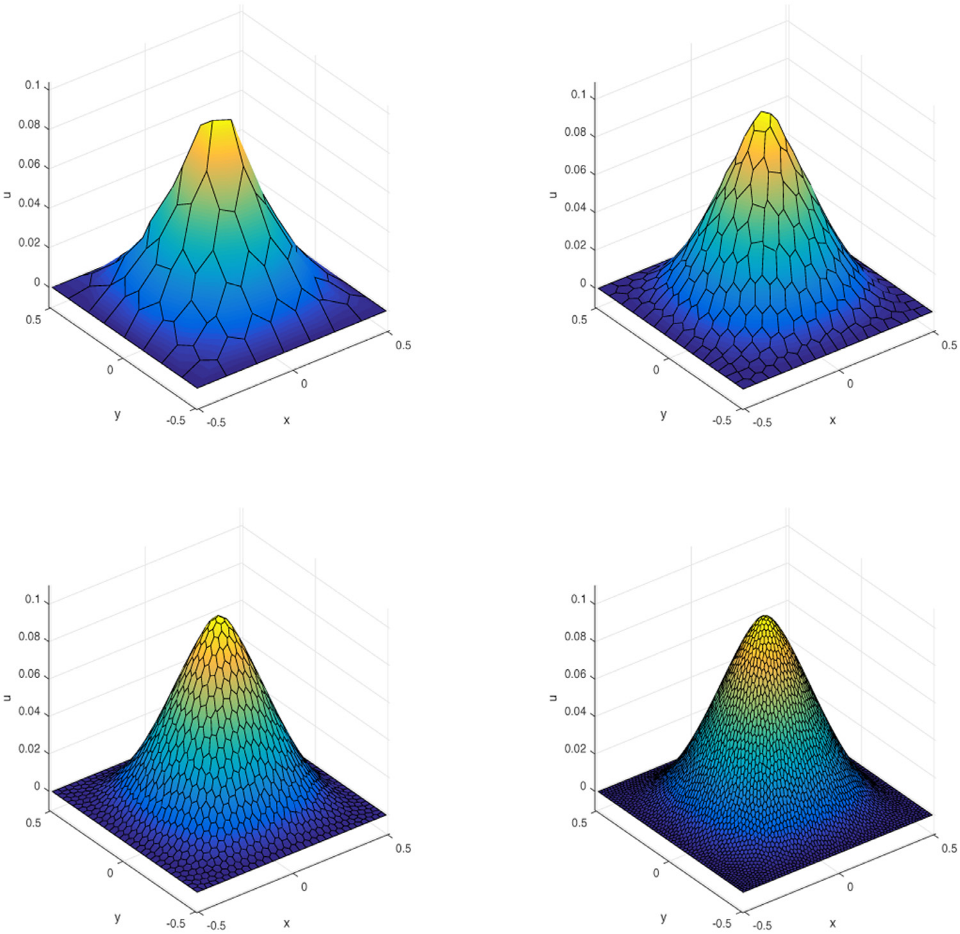


**Fig. 2.** Numerical solutions for $h = 1/8$ (upper left), $1/16$ (upper right), $1/32$ (bottom left), and $1/64$ (bottom right).

**Proof.** In view of (4.64) and (3.16),

$$|I_h u - u_h|_{2,h}^2 \lesssim \sum_{T\in\mathcal{T}_h} |I_T u - u|_{2,T}^2 + h^2|u|_3^2 + \|f - f_h^{nc}\|_{(V_h^{nc})^*}^2 \\ + h|u|_3|I_h u - u_h|_{2,h} + h|I_h u - u_h|_{2,h} + h^2 + h^3|u|_3\|f\|_0 + h^2\|f\|_0|I_h u - u_h|_{2,h}.$$

By an application of the modified Cauchy-Schwarz inequality (3.15) with a sufficiently small $\varepsilon > 0$, we have

$$|I_h u - u_h|_{2,h}^2 \lesssim \sum_{T\in\mathcal{T}_h} |I_T u - u|_{2,T}^2 + h^2|u|_3^2 + \|f - f_h^{nc}\|_{(V_h^{nc})^*}^2 \\ + h^2 + h^3|u|_3\|f\|_0 + h^4\|f\|_0^2. \tag{4.65}$$

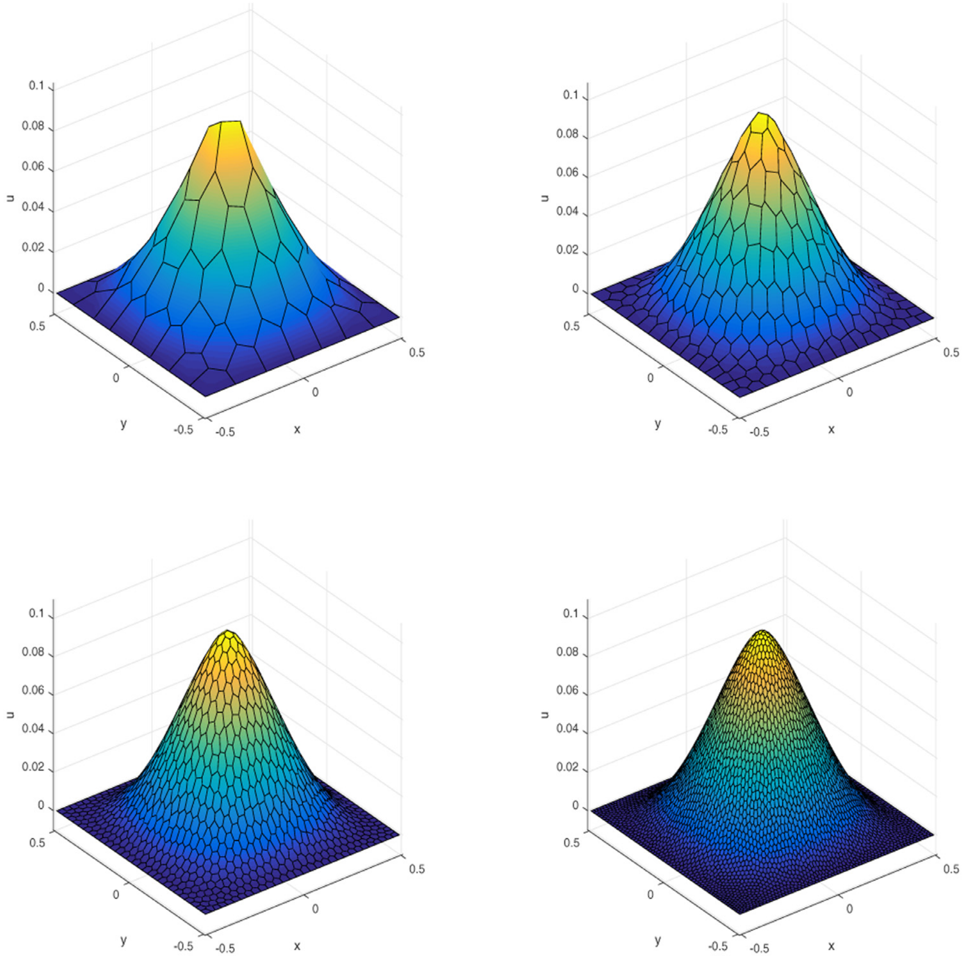


**Fig. 3.** Numerical solutions for $h = 1/8$ (upper left), $1/16$ (upper right), $1/32$ (bottom left) and $1/64$ (bottom right).

By (4.31), it is straightforward to get

$$\sum_{T\in\mathcal{T}_h} |u - I_T u|_{2,T}^2 \lesssim h^2 |u|_3^2.$$

Use (4.35) to bound $\|f - f_h^{nc}\|_{(V_h^{nc})^*}^2$. From (4.65) we conclude that

$$|I_h u - u_h|_{2,h} \lesssim h.$$

Finally, we apply the triangle inequality

$$|u - u_h|_{2,h} \le |u - I_h u|_{2,h} + |I_h u - u_h|_{2,h}$$

to complete the proof. □

## 5. Numerical experiments

For the implementation of the numerical methods, for both the conforming and fully nonconforming VEMs, we first observe that the discrete problem (3.9) is equivalent to the optimization problem

$$u_h = \underset{v\in K_h}{\operatorname{argmin}} \left[ \frac{1}{2} a_h(v, v) - (f_h, v) \right] \tag{5.1}$$

where

$$K_h = \{ v \in V_h \mid v(p) \ge \psi(p) \ \forall\, p \in \mathcal{V}_h \}.$$

Let $N_1$ and $N$ be the number of nodal points and the number of global degree freedoms, respectively. Denote by $I$ the set corresponding to the indices of the nodal points, and denote by $\{\phi_i\}_{i=1}^N$ the shape functions of $V_h$. Expressing a virtual element function in terms of the basis functions, $v = \sum_{i=1}^{N} v_i\phi_i$, and denoting $\boldsymbol{v} = (v_1, v_2, \cdots, v_N)^T$, we can rewrite the minimization problem (5.1) as

$$\min_{\boldsymbol{v}\in\boldsymbol{K}} \frac{1}{2} v^T \boldsymbol{A}\boldsymbol{v} - \boldsymbol{b}^T \boldsymbol{v}, \tag{5.2}$$

where

$$\boldsymbol{A} = (a_h(\phi_i, \phi_j))_{N\times N}, \qquad \boldsymbol{b} = ((f_h, \phi_i))_N,$$

and

$$\boldsymbol{K} = \{\boldsymbol{v} \in \mathbb{R}^N \mid v_i \geq \psi_i \text{ for } i \in I\}.$$

Write $\boldsymbol{v} = (\boldsymbol{v}_1^T, \boldsymbol{v}_2^T)^T$ with $\boldsymbol{v}_1 \in \mathbb{R}^{N_1}$ and partition the matrix $\boldsymbol{A}$ and the vector $\boldsymbol{b}$ accordingly:

$$\boldsymbol{A} := \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{12}^T & \boldsymbol{A}_{22} \end{pmatrix}, \qquad \boldsymbol{b} := \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix}.$$

After eliminating $\boldsymbol{v}_2$, we get the following reduced minimization problem (cf. [28,29]): Find $\boldsymbol{v}_1^* \in \mathbb{R}^{N_1}$ such that

$$\hat{F}(\boldsymbol{v}_1^*) = \min_{\boldsymbol{v}_1\in\boldsymbol{K}_1} \hat{F}(\boldsymbol{v}_1) \tag{5.3}$$

where

$$\hat{F}(\boldsymbol{v}_1) := \frac{1}{2} \boldsymbol{v}_1^T \widetilde{\boldsymbol{A}}_1 \boldsymbol{v}_1 - \widetilde{\boldsymbol{b}}_1^T \boldsymbol{v}_1$$

with

$$\widetilde{\boldsymbol{A}}_1 := \boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{12}^T, \quad \widetilde{\boldsymbol{b}}_1 := \boldsymbol{b}_1 - \boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{b}_2,$$

and

$$\boldsymbol{K}_1 := \{\boldsymbol{v} \in \mathbb{R}^{N_1} \mid v_i \geq \psi_i \text{ for } i \in I\}.$$

After $\boldsymbol{v}_1^*$ is computed, we let

$$\boldsymbol{v}_2^* = \boldsymbol{A}_{22}^{-1}\left(\boldsymbol{b}_2 - \boldsymbol{A}_{12}^T \boldsymbol{v}_1^*\right).$$

By the Karush-Kuhn-Tucker (KKT) condition, the constrained minimization problem (5.3) is equivalent to the following complementarity problem

$$\begin{cases} \widetilde{\boldsymbol{A}}_1 \boldsymbol{v}_1 - \widetilde{\boldsymbol{b}}_1 = \boldsymbol{\lambda} \\ \boldsymbol{v}_1 \geq \boldsymbol{\psi}_1, \quad \boldsymbol{\lambda} \geq \boldsymbol{0}, \quad (\boldsymbol{\lambda}, \boldsymbol{v}_1 - \boldsymbol{\psi}_1) = \boldsymbol{0}, \end{cases} \tag{5.4}$$

where $\boldsymbol{\psi}_1 = (\psi_1, \psi_2, \cdots, \psi_{N_1})^T$ and $\boldsymbol{\lambda} \in \mathbb{R}^{N_1}$. Note that the second line in (5.4) can equivalently be expressed as

$$\mathcal{C}(\boldsymbol{v}_1, \boldsymbol{\lambda}) = \boldsymbol{0},$$

where

$$\mathcal{C}(\boldsymbol{v}_1, \boldsymbol{\lambda}) = \boldsymbol{\lambda} - \max(\boldsymbol{0}, \boldsymbol{\lambda} + c\,(\boldsymbol{\psi}_1 - \boldsymbol{v}_1))$$

for any positive number $c$. We apply the primal-dual active algorithm to solve the discrete system (5.4). Note that the primal-dual active algorithm is equivalent to the semismooth Newton method and has a superlinear convergence order ([47]).

We use the code PolyMesher ([48]) to generate the polygonal meshes and then solve the discrete obstacle problem. Meshes with 64 elements and 256 elements are shown in Fig. 1.

**Example 1.** For the problem data, we use $\Omega = (-0.5, 0.5)^2$, $f(\boldsymbol{x}) = 0$, $\psi(\boldsymbol{x}) = 0.1 - 2\,|\boldsymbol{x}|^2$.

In this example, the area of the domain $\Omega$ is 1, and we define $h = \sqrt{1/n}$, where $n$ is the number of elements of the mesh. Since the exact solution is unknown, we use the numerical solution with a fine mesh as the "reference" solution $u_{ref}$. For this example, the "reference" solution is taken to be the numerical solution with $h = 1/256$. We measure the relative error of the numerical solutions in the discrete energy norm:

$$Err := \left[\frac{a_h(u_{ref} - u_h, u_{ref} - u_h)}{a_h(u_{ref}, u_{ref})}\right]^{1/2}. \tag{5.5}$$

**Table 1**
Numerical errors on square meshes for the $C^1$-VEM with $k = 2$.

| $h$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{16}$ | $\frac{1}{32}$ | $\frac{1}{64}$ |
|---|---|---|---|---|---|
| Err | 5.85e-01 | 6.43e-01 | 4.74e-01 | 2.90e-01 | 1.52e-01 |
| Order | \ | −0.13 | 0.44 | 0.71 | 0.935 |

**Table 2**
Numerical errors on square meshes for the fully nonconforming VEM with $k = 2$.

| $h$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{16}$ | $\frac{1}{32}$ | $\frac{1}{64}$ |
|---|---|---|---|---|---|
| Err | 4.65e-01 | 4.78e-01 | 3.21e-01 | 1.87e-01 | 9.52e-02 |
| Order | \ | −0.0396 | 0.574 | 0.780 | 0.974 |

### *5.1. Results for the conforming $C^1$-type VEM*

The numerical solutions for the conforming $C^1$-type VEM corresponding to meshes with $h = 1/8, 1/16, 1/32$, and $1/64$ are shown in Fig. 2, respectively. A convergence trend is evident for the numerical solutions as the element number increases. In Table 1, we report numerical convergence orders. We observe that as $h$ decreases, the numerical convergence order approaches 1, matching the theoretical prediction in Theorem 4.1.

### *5.2. Results for the fully nonconforming VEM*

The numerical solutions for the fully nonconforming VEM corresponding to meshes with $h = 1/8, 1/16, 1/32$, and $1/64$ are shown in Fig. 3, respectively. A convergence trend is evident for the numerical solutions as the mesh is refined. In Table 2, we report numerical convergence orders. We observe that as $h$ decreases, the numerical convergence order approaches 1, matching the theoretical prediction in Theorem 4.2.

## Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## CRediT authorship contribution statement

**Fang Feng:** Methodology, Formal analysis, Writing – original draft. **Weimin Han:** Formal analysis, Writing – review & editing. **Jianguo Huang:** Methodology, Formal analysis, Writing – review & editing.

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