Journal of Scientific Computing (2022) 90:89
https://doi.org/10.1007/s10915-022-01759-1

A Nonconforming Virtual Element Method for a Fourth-order Hemivariational Inequality in Kirchhoff Plate Problem

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Received: 5 May 2021 / Revised: 5 December 2021 / Accepted: 6 January 2022

Abstract
This paper is devoted to a fourth-order hemivariational inequality for a Kirchhoff plate problem. A solution existence and uniqueness result is proved for the hemivariational inequality through the analysis of a corresponding minimization problem. A nonconforming virtual element method is developed to solve the hemivariational inequality. An optimal order error estimate in a broken H^2-norm is derived for the virtual element solutions under appropriate solution regularity assumptions. The discrete problem can be formulated as an optimization problem for a difference of two convex (DC) functions and a convergent algorithm is used to solve it. Computer simulation results on a numerical example are reported, providing numerical convergence orders that match the theoretical prediction.

Keywords Kirchhoff plate problem · Hemivariational inequality · Well-posedness · Nonconforming virtual element method · Error analysis · Double bundle algorithm

1 Introduction

Since the early 1980s, many challenging nonsmooth problems in a wide range of applications have been formulated and studied as hemivariational inequalities (HVIs). Modeling, well-posedness analysis and numerical solutions of HVIs can be found in numerous references, cf. comprehensive references [16, 31, 34–37]. Optimal order error analysis for finite element solutions of HVIs started with [28], followed by numerous publications. We refer the reader to the survey paper [29] on recent progress of numerical analysis of HVIs. Compared to vari-

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Published online: 02 February 2022

ational inequalities, a distinguished feature of HVIs is their capability to treat non-monotone, non-smooth, and set-valued relations between physical quantities in application problems.

The pioneering work of virtual element methods (VEMs) is found in [1, 6, 8]. Because of the advantages in handling problems with complex geometries or requiring high-regularity approximations, the VEM has been applied successfully to a wide variety of scientific and engineering problems. For example, conforming and nonconforming VEMs are presented for second-order elliptic problems ([1, 5, 6, 15]), elasticity problems ([7, 42]), fourth-order elliptic problems ([2, 14, 41, 43]) and polyharmonic problems ([3, 18]). Moreover, the method has been applied to solve elliptic variational inequalities and elliptic hemivariational inequalities (cf. [23, 24, 39, 40]). We refer to [9, 10, 12, 17, 18] for a comprehensive understanding of mathematical theories of VEMs.

In this paper, we apply the nonconforming virtual element method to solve a Kirchhoff plate problem, which is formulated as a fourth-order HVI. We provide a result on the existence of a unique solution to the HVI. In most of the existing literature, solution existence for HVIs is shown through an application of an abstract surjectivity result for coercive, pseudomonotone operators. In this paper, we adopt an approach presented in [27] and prove the solution existence through the study of a corresponding minimization problem, thus avoiding the notion of pseudomonotone operators and employment of abstract surjectivity results for such operators. An optimal order error estimate is derived for the virtual element solutions under appropriate solution regularity assumptions. The discrete problem is converted into a DC (difference of convex functions) programming. Then, the double bundle method ([33]) is applied to find the solution of the discretized hemivariational inequality. Numerical results are reported to illustrate computational performance of the VEM studied in this paper.

The rest of the paper is organized as follows. In Sect. 2, we recall notions and basic properties of the generalized directional derivative and subdifferential in the sense of Clarke. In Sect. 3, we introduce a fourth-order HVI for a frictional contact problem of a Kirchhoff plate problem, and prove the unique solvability of the HVI through the analysis of a corresponding minimization problem. In Sect. 4, we apply a nonconforming virtual element method to solve the HVI and provide its error analysis. In Sect. 5, we present a solution algorithm for the discrete problem and report simulation results on a numerical example.

2 Preliminaries

All linear spaces in this paper are real. For a normed space X, we denote by $\|\cdot\|_X$ its norm, by X^* its topological dual, and by $\langle\cdot,\cdot\rangle_{X^*\times X}$ the duality pairing between X^* and X. In the description of the hemivariational inequality, we need the notions of the generalized (Clarke) directional derivative and the generalized gradient of a locally Lipschitz continuous functional (cf. [20]).

Definition 2.1 Let $\psi : X \to \mathbb{R}$ be a locally Lipschitz functional on a Banach space X. The generalized (Clarke) directional derivative of ψ at $x \in X$ in the direction $v \in X$ is defined by

$$\psi^0(x; v) = \limsup_{y\to x,\, \lambda\downarrow 0} \frac{\psi(y+\lambda v)-\psi(y)}{\lambda}.$$

The generalized gradient (subdifferential) of ψ at x is defined by

$$\partial\psi(x) = \{\zeta \in X^* : \psi^0(x; v) \ge \langle\zeta, v\rangle \ \forall\, v \in X\}.$$

We recall some properties of the generalized directional derivative and the generalized subdifferential (cf. [34, Propositions 3.23 & 3.32]).

Proposition 2.1 *Suppose $\psi : X \to \mathbb{R}$ is a locally Lipschitz continuous functional on a Banach space X.*

(1) *For every $x \in X$, the function $\psi^0(x;\cdot) : X \to \mathbb{R}$ is positively homogeneous and subadditive:*

$$\begin{aligned}\psi^0(x;\lambda v) &= \lambda\,\psi^0(x;v) \quad \forall\,\lambda \geq 0,\ v \in X,\\ \psi^0(x;v_1+v_2) &\leq \psi^0(x;v_1)+\psi^0(x;v_2) \quad \forall\,v_1,v_2 \in X.\end{aligned} \tag{2.1}$$

(2) *For any $x, v \in X$,*

$$\psi^0(x;v) = \max\{\langle\zeta,v\rangle : \zeta \in \partial\psi(x)\}. \tag{2.2}$$

(3) *If ψ is convex, then the subdifferential in the sense of Clarke coincides with the subdifferential in the sense of convex analysis.*

Another property we will need is the following result (cf. [34, Proposition 3.35]).

Proposition 2.2 *If $\psi_1, \psi_2 : X \to \mathbb{R}$ are locally Lipschitz continuous on a Banach space X, then*

$$\partial(\psi_1+\psi_2)(x) \subset \partial\psi_1(x)+\partial\psi_2(x) \quad \forall\,x \in X,$$

or equivalently,

$$(\psi_1+\psi_2)^0(x;v) \leq \psi_1^0(x;v)+\psi_2^0(x;v) \quad \forall\,x,v \in X.$$

It is convenient to record an elementary result to be used later:

$$a,b,x \geq 0 \text{ and } x^2 \leq a\,x+b \quad \Longrightarrow \quad x^2 \leq a^2+2\,b. \tag{2.3}$$

Remark 2.1 To simplify the presentation, similar to other papers in numerical methods for hemivariational inequalities, we use the convention that for a function f, ∂f denotes its generalized subdifferential, while for a bounded domain D, ∂D means its boundary.

3 Kirchhoff Plate Problems

In this section, after reviewing a classical Kirchhoff plate bending problem and a fourth-order variational inequality in frictional contact problem for the plate, we introduce a hemivariational inequality (HVI) for a frictional contact problem of the plate where the frictional contact condition is allowed to be non-monotone. We proceed to prove the existence and uniqueness of a solution to the HVI. Contrary to the common approach adopted in most of the existing literature where an abstract surjectivity result on pseudomonotone operators is needed, here we show the solution existence directly through the analysis of a corresponding minimization problem.

3.1 A Classical Kirchhoff Plate Bending Problem

Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain with a boundary Γ. For an elastic thin plate clamped on the boundary and acted by a vertical load with density $f \in L^2(\Omega)$, its deflection u is the solution of the minimization problem (cf. [19, §6.8], [25, Chapter 3, §4])

$$u \in H_0^2(\Omega), \quad E_1(u) = \inf \left\{ E_1(v) : v \in H_0^2(\Omega) \right\} \tag{3.1}$$

where the energy functional

$$E_1(v) = \frac{1}{2} \int_\Omega \mathcal{M}_{\alpha\beta}(v)\, \mathcal{K}_{\alpha\beta}(v)\, dx - \int_\Omega f\, v\, dx$$

in which,

$$\mathcal{M}_{\alpha\beta}(v) = (1-\nu)\mathcal{K}_{\alpha\beta}(v) + \nu\mathcal{K}_{\mu\mu}(v)\delta_{\alpha\beta}, \quad \mathcal{K}_{\alpha\beta}(v) = -\partial_{\alpha\beta} v, \quad 1 \le \alpha, \mu, \beta \le 2,$$

and $\nu \in (0, 1/2)$ is the Poisson ratio of the plate. The quantities $\mathcal{M}_{\alpha\beta}(v)$ are known as the moments. We adopt Einstein's convention for summation over a repeated index, e.g.,

$$\mathcal{K}_{\mu\mu}(v) = \mathcal{K}_{11}(v) + \mathcal{K}_{22}(v).$$

It is a classical result that the problem (3.1) has a unique solution $u \in H_0^2(\Omega)$, and the minimization problem (3.1) is equivalent to the weak formulation

$$u \in H_0^2(\Omega), \qquad a(u, v) = (f, v) \quad \forall\, v \in H_0^2(\Omega), \tag{3.2}$$

where (f, v) stands for the ordinary $L^2(\Omega)$-inner product of f and v, and the bilinear form

$$a(u, v) = \int_\Omega \mathcal{M}_{\alpha\beta}(u)\, \mathcal{K}_{\alpha\beta}(v)\, dx. \tag{3.3}$$

If the solution u of the problem (3.2) is sufficiently smooth, then u satisfies the following pointwise relations:

$$\begin{aligned} -\mathcal{M}_{\alpha\beta,\alpha\beta}(u) &= f && \text{in } \Omega, \\ u = \partial_{\boldsymbol{n}} u &= 0 && \text{on } \Gamma, \end{aligned} \tag{3.4}$$

where $\boldsymbol{n} = (n_1, n_2)$ is the unit outward normal vector defined a.e. on Γ. We use $\boldsymbol{\tau} = (\tau_1, \tau_2)$ for the unit tangential vector a.e. on Γ such that $(\boldsymbol{n}, \boldsymbol{\tau})$ forms a right-hand system.

3.2 A Frictional Contact Problem

In [30], a variational inequality for frictional contact of the Kirchhoff plate is studied. To describe the problem, we decompose the boundary $\Gamma = \partial\Omega$ of the plate as $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ such that Γ_1, Γ_2, and Γ_3 are relatively closed with mutually non-overlapping relative interiors, and meas $(\Gamma_1) > 0$. We assume the plate is clamped on Γ_1, free on Γ_2, and subject to a friction effect following a simplified friction law on Γ_3. Then the deflection u is a solution of the following minimization problem

$$u \in V, \quad E_2(u) = \inf \{E_2(v) : v \in V\} \tag{3.5}$$

where the function space is

$$V = \{v \in H^2(\Omega) : v = \partial_{\boldsymbol{n}} v = 0 \text{ on } \Gamma_1\}, \tag{3.6}$$

and the energy functional is

$$E_2(v) = \frac{1}{2}\int_\Omega \mathcal{M}_{\alpha\beta}(v)\,\mathcal{K}_{\alpha\beta}(v)\,dx + \int_{\Gamma_3} g\,|v|\,ds - \int_\Omega f\,v\,dx,$$

in which $g > 0$ is the frictional bound (see an interpretation on this claim below). The contribution of the frictional effect to the energy functional is reflected by the integral term on Γ_3. By [19, Theorem 6.8-4], $\|v\|_V := |v|_{H^2(\Omega)}$ defines a norm on V which is equivalent to the norm $\|v\|_{H^2(\Omega)}$ over V. The problem (3.5) has a unique solution $u \in V$, which is also the unique solution of the variational inequality

$$u \in V, \quad a(u, v-u) + \int_{\Gamma_3} g\,(|v| - |u|)\,ds \geq (f, v-u) \quad \forall\, v \in V, \tag{3.7}$$

where the bilinear form $a(\cdot,\cdot)$ is defined in (3.3).

For convenience, given a deflection function v, introduce the quantities

$$\begin{aligned} &\mathcal{M}_{\boldsymbol{nn}}(v) := \mathcal{M}_{\alpha\beta}(v) n_\alpha n_\beta, \qquad \mathcal{M}_{\boldsymbol{\tau n}}(v) := \mathcal{M}_{\alpha\beta}(v)\tau_\alpha n_\beta, \\ &\quad \mathcal{Q}_\alpha(v) := \partial_\beta \mathcal{M}_{\alpha\beta}(v), \quad \mathcal{Q}_{\boldsymbol{n}}(v) := \mathcal{Q}_\alpha(v) n_\alpha, \quad \mathcal{N}(v) := \mathcal{Q}_{\boldsymbol{n}}(v) + \partial_{\boldsymbol{\tau}} \mathcal{M}_{\boldsymbol{\tau n}}(v), \end{aligned}$$

where $\mathcal{M}_{\boldsymbol{nn}}(v)$, $-\mathcal{M}_{\boldsymbol{\tau n}}(v)$, $\mathcal{Q}_{\boldsymbol{n}}(v)$, and $\mathcal{N}(v)$ respectively denote the bending moment, the twist moment, the transverse shearing force, and the effective shear force in elastic mechanics (cf. [25, pp. 180–184]).

If the solution $u \in V$ of the problem (3.7) is sufficiently smooth, then it can be shown that u satisfies the equations (3.4) in Ω and the following boundary conditions:

$$\begin{aligned} &u = \partial_{\boldsymbol{n}} u = 0 && \text{on } \Gamma_1, \\ &\mathcal{M}_{\boldsymbol{nn}}(u) = \mathcal{N}(u) = 0 && \text{on } \Gamma_2, \\ &\mathcal{M}_{\boldsymbol{nn}}(u) = 0,\ \mathcal{N}(u) \in \partial\,(g\,|u|) && \text{on } \Gamma_3, \end{aligned} \tag{3.8}$$

where ∂ stands for the subdifferential in convex analysis ([21]). Note that the condition $\mathcal{N}(u) \in \partial\,(g\,|u|)$ is equivalent to the following relations:

$$|\mathcal{N}(u)| \leq g, \quad |\mathcal{N}(u)| < g \;\Rightarrow\; u = 0, \quad |\mathcal{N}(u)| = g \;\Rightarrow\; u = \lambda\,\mathcal{N}(u) \text{ for some } \lambda \geq 0.$$

Thus, $g > 0$ can be interpreted as the frictional bound.

3.3 An Elliptic Hemivariational Inequality for Kirchhoff Plate

We proceed to consider the frictional contact plate problem for which the boundary condition (3.8) takes a more general form

$$\mathcal{M}_{\boldsymbol{nn}}(u) = 0, \quad \mathcal{N}(u) \in \partial j(u) \quad \text{on } \Gamma_3, \tag{3.9}$$

where $j : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function, and $\partial j(u)$ is the generalized subdifferential of j at u in the sense of Clarke. As is common in the literature, we suppress $\boldsymbol{x}$ in $j(\boldsymbol{x}, u)$ and simply write $j(u)$. Unlike the condition $\mathcal{N}(u) \in \partial\,(g\,|u|)$ from (3.8) that gives a monotonic relation between $\mathcal{N}(u)$ and u, in general, the condition $\mathcal{N}(u) \in \partial j(u)$ from (3.9) allows a non-monotonic relation between $\mathcal{N}(u)$ and u. With the function space V defined by (3.6), the bilinear form $a(\cdot,\cdot)$ defined by (3.3), $f \in L^2(\Omega)$ given, we have the following minimization problem.

Problem ($\tilde{\mathrm{P}}$) Find an element $u \in V$ such that

$$E_3(u) = \inf \{E_3(v) : v \in V\}$$

where the energy functional is

$$E_3(v) = \frac{1}{2} \int_\Omega \mathcal{M}_{\alpha\beta}(v)\, \mathcal{K}_{\alpha\beta}(v)\, dx + \int_{\Gamma_3} j(v)\, ds - \int_\Omega f\, v\, dx. \tag{3.10}$$

The corresponding weak formulation is stated next.

Problem (P) Find an element $u \in V$ such that

$$a(u, v) + \int_{\Gamma_3} j^0(u; v)\, ds \geq (f, v) \quad \forall\, v \in V. \tag{3.11}$$

The relation between Problem (P) and Problem ($\tilde{\mathrm{P}}$) is explored in Theorem 3.1 below. On the function j, we impose the following condition:

Assumption (H_0). $j(\cdot, z)$ is measurable on Γ_3 for any $z \in \mathbb{R}$ and there exists $z_0 \in L^2(\Gamma_3)$ such that $j(\cdot, z_0(\cdot)) \in L^1(\Gamma_3)$. $j(\boldsymbol{x}, \cdot)$ is locally Lipschitz on $\mathbb{R}$ for a.e. $\boldsymbol{x} \in \Gamma_3$, and there are constants $c_0, c_1, \alpha_j \geq 0$ such that

$$|\partial j(z)| \leq c_0 + c_1 |z| \quad \forall\, z \in \mathbb{R}, \tag{3.12}$$

$$j^0(z_1; z_2 - z_1) + j^0(z_2; z_1 - z_2) \leq \alpha_j |z_1 - z_2|^2 \quad \forall\, z_1, z_2 \in \mathbb{R}. \tag{3.13}$$

We comment that (3.12) is a short-hand notation for the property

$$|\xi| \leq c_0 + c_1 |z| \quad \forall\, z \in \mathbb{R},\ \forall\, \xi \in \partial j(z).$$

It is known (cf. [34, Theorem 3.47]) that under the assumption (H_0), the integral

$$J(v) = \int_{\Gamma_3} j(v)\, ds$$

is well-defined and locally Lipschitz continuous on $L^2(\Gamma_3)$. Moreover,

$$\partial J(v) \subset \int_{\Gamma_3} \partial j(v)\, ds$$

in the sense that for $v^* \in \partial J(v)$, there exists $\zeta \in L^2(\Gamma_3)$ such that $\zeta(\boldsymbol{x}) \in \partial j(v(\boldsymbol{x}))$ for a.e. $\boldsymbol{x} \in \Gamma_3$, and

$$\langle v^*, v \rangle_{L^2(\Gamma_3) \times L^2(\Gamma_3)} = \int_{\Gamma_3} \zeta(\boldsymbol{x})\, v(\boldsymbol{x})\, ds.$$

Define a linear operator $A : V \to V^*$ by

$$\langle Au, v \rangle = a(u, v), \quad u, v \in V.$$

Then $A \in \mathcal{L}(V, V^*)$. The bilinear form (3.3) is coercive on V since

$$a(v, v) \geq (1 - \nu)\, \|v\|_V^2 \quad \forall\, v \in V.$$

This implies that A is strongly monotone with a monotonicity constant $(1-\nu)$. Define $\overline{f} \in V^*$ by

$$\langle \overline{f}, v \rangle = (f, v) \quad \forall\, v \in V.$$

By the Sobolev trace theorem, there exists a constant $c_\gamma > 0$ such that

$$\|v\|_{L^2(\Gamma_3)} \le c_\gamma \|v\|_V \quad \forall\, v \in V. \tag{3.14}$$

The best constant c_γ in the inequality (3.14) is $c_\gamma = \lambda_{\min}^{-1/2}$ where $\lambda_{\min} > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\int_\Omega \nabla^2 u : \nabla^2 v \, dx = \lambda \int_{\Gamma_3} u\, v\, ds \quad \forall\, v \in V.$$

We assume

$$\alpha_j c_\gamma^2 < 1 - \nu. \tag{3.15}$$

Such a condition is known as a smallness assumption in the literature (cf. [34]).

We present a property of the energy functional E_3, which is obviously Lipschitz continuous on V.

Lemma 3.1 *Under the assumptions* (H_0) *and* (3.15)*, the energy functional* E_3 *defined in* (3.10) *is strongly convex on* V.

Proof Let $u, v \in V$ and let $\xi \in \partial E_3(u)$, $\eta \in \partial E_3(v)$. By Proposition 2.2,

$$\partial E_3(v) \subset Av + \partial J(v) - \overline{f}, \quad \partial J(v) \subset \int_{\Gamma_3} \partial j(v)\, ds.$$

Thus, we can write, for any $w \in V$,

$$\begin{aligned}
\langle \xi, w\rangle &= \langle Au, w\rangle + \int_{\Gamma_3} \xi_1(\boldsymbol{x})\, w(\boldsymbol{x})\, ds - \langle \overline{f}, w\rangle,\\
\langle \eta, w\rangle &= \langle Av, w\rangle + \int_{\Gamma_3} \eta_1(\boldsymbol{x})\, w(\boldsymbol{x})\, ds - \langle \overline{f}, w\rangle
\end{aligned}$$

for some $\xi_1, \eta_1 \in L^2(\Gamma_3)$ with $\xi_1(\boldsymbol{x}) \in \partial j(u(\boldsymbol{x}))$ and $\eta_1(\boldsymbol{x}) \in \partial j(v(\boldsymbol{x}))$ for a.e. $\boldsymbol{x} \in \Gamma_3$. Hence,

$$\begin{aligned}
\langle \xi - \eta, u - v\rangle &= \langle A(u - v), u - v\rangle + \int_{\Gamma_3} (\xi_1(\boldsymbol{x}) - \eta_1(\boldsymbol{x}))\,(u(\boldsymbol{x}) - v(\boldsymbol{x}))\, ds\\
&\ge (1-\nu)\,\|u - v\|_V^2 - \alpha_j \int_{\Gamma_3} |u(\boldsymbol{x}) - v(\boldsymbol{x})|^2\, ds\\
&\ge \left(1 - \nu - \alpha_j c_\gamma^2\right) \|u - v\|_V^2.
\end{aligned}$$

Applying [22, Theorem 3.4], we know that E_3 is strongly convex on V. □

We are now in a position to present a solution existence and uniqueness result for Problems (P) and Problem ($\tilde{\mathrm{P}}$).

Theorem 3.1 *Assume* (H_0) *and* (3.15)*. Then Problem* (P) *has a unique solution. Moreover,* $u \in V$ *is the solution of Problem* (P) *if and only if it is the solution of Problem* ($\tilde{\mathrm{P}}$)*.*

Proof Thanks to Lemma 3.1, the functional E_3 is strictly convex and coercive on V (cf. [27, Proposition 2.5]). Thus by a standard result on convex minimization (cf. [4, §3.3.2]), Problem ($\tilde{\mathrm{P}}$) has a unique solution.

Now we turn to prove the equivalence between Problem (P) and Problem ($\tilde{\text{P}}$). Denote by $u \in V$ the unique solution of Problem ($\tilde{\text{P}}$). Then,

$$0 \in \partial E_3(u).$$

Since

$$\partial E_3(u) \subset Au + \partial J(u) - \overline{f}, \quad \partial J(u) \subset \int_{\Gamma_3} \partial j(u)\, ds,$$

there is a function $\xi \in L^2(\Gamma_3)$ such that $\xi(\boldsymbol{x}) \in \partial j(u(\boldsymbol{x}))$ for a.e. $\boldsymbol{x} \in \Gamma_3$ and

$$\langle Au, v\rangle + \int_{\Gamma_3} \xi(\boldsymbol{x})\, v(\boldsymbol{x})\, ds = \langle \overline{f}, v\rangle \quad \forall\, v \in V. \tag{3.16}$$

Since $\xi(\boldsymbol{x}) \in \partial j(u(\boldsymbol{x}))$ for a.e. $\boldsymbol{x} \in \Gamma_3$,

$$\xi(\boldsymbol{x})\, v(\boldsymbol{x}) \le j^0(u(\boldsymbol{x}); v(\boldsymbol{x})) \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3.$$

Thus, from (3.16), we obtain

$$a(u, v) + \int_{\Gamma_3} j^0(u(\boldsymbol{x}); v(\boldsymbol{x}))\, ds \ge (f, v) \quad \forall\, v \in V.$$

In other words, $u \in V$ is a solution of Problem (P). Let us prove that a solution of Problem (P) is unique. For this purpose, denote by $\tilde{u}$ another solution of Problem (P). Then,

$$a(\tilde{u}, v) + \int_{\Gamma_3} j^0(\tilde{u}; v)\, ds \ge (f, v) \quad \forall\, v \in V. \tag{3.17}$$

Take $v = \tilde{u} - u$ in (3.11), take $v = u - \tilde{u}$ in (3.17), and add the two resulting inequalities,

$$a(u - \tilde{u}, u - \tilde{u}) \le \int_{\Gamma_3} \left[j^0(u; \tilde{u} - u) + j^0(\tilde{u}; u - \tilde{u}) \right] ds.$$

Then,

$$(1 - \nu)\, \|u - \tilde{u}\|_V^2 \le \alpha_j \|u - \tilde{u}\|_{L^2(\Gamma_3)}^2 \le \alpha_j c_\gamma^2 \|u - \tilde{u}\|_V^2.$$

Recalling the smallness condition (3.15), we find from the above inequality that $\tilde{u} = u$.

In conclusion, Problem ($\tilde{\text{P}}$) has a unique solution, Problem (P) also has a unique solution, and the two solutions are equal. □

In the next section, we will develop and analyze a nonconforming virtual element method to solve Problem (P). For this purpose, we present some preliminary results.

We will assume the solution regularity

$$u \in H^3(\Omega). \tag{3.18}$$

We choose arbitrary $v \in C_0^\infty(\Omega)$ in (3.11) to obtain

$$a(u, v) = (f, v) \quad \forall\, v \in C_0^\infty(\Omega).$$

Recall that the bilinear form $a(\cdot, \cdot)$ is defined by (3.3). Thus, the above identity implies

$$-\mathcal{M}_{\alpha\beta,\alpha\beta}(u) = f \quad \text{in the sense of distributions.}$$

Since $f \in L^2(\Omega)$, we have $-\mathcal{M}_{\alpha\beta,\alpha\beta}(u) \in L^2(\Omega)$ and

$$-\mathcal{M}_{\alpha\beta,\alpha\beta}(u) = f \quad \text{a.e. in } \Omega. \tag{3.19}$$

Multiply the equality (3.19) by an arbitrary function $v \in H^1(\Omega)$, integrate over Ω, and perform an integration by parts to get

$$\int_\Omega \mathcal{Q}_\alpha(u)\partial_\alpha v\,dx - (f, v) = \langle \mathcal{Q}_{\boldsymbol{n}}(u), v\rangle_{1/2,\Gamma} \quad \forall\, v \in H^1(\Omega), \tag{3.20}$$

where $\langle\cdot,\cdot\rangle_{1/2,\Gamma}$ denotes the duality pair between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ and

$$\mathcal{Q}_{\boldsymbol{n}}(u) := \mathcal{Q}_\alpha(u)n_\alpha \in H^{-1/2}(\partial\Omega).$$

Now for $v \in V$, $v = \partial_{\boldsymbol{n}} v = 0$ on Γ_1 and we have

$$a(u, v) = \int_\Omega \mathcal{Q}_\alpha(u)\partial_\alpha v\,dx - \int_{\Gamma_2\cup\Gamma_3} [\mathcal{M}_{\boldsymbol{nn}}(u)\partial_{\boldsymbol{n}} v + \mathcal{M}_{\boldsymbol{\tau n}}(u)\partial_{\boldsymbol{\tau}} v]\,ds. \tag{3.21}$$

According to (3.20) and (3.21),

$$a(u, v) = (f, v) + \langle \mathcal{Q}_{\boldsymbol{n}}(u), v\rangle_{1/2,\Gamma} - \int_{\Gamma_2\cup\Gamma_3} [\mathcal{M}_{\boldsymbol{nn}}(u)\partial_{\boldsymbol{n}} v + \mathcal{M}_{\boldsymbol{\tau n}}(u)\partial_{\boldsymbol{\tau}} v]\,ds.$$

Then (3.11) is reduced to

$$\begin{aligned} &\langle \mathcal{Q}_{\boldsymbol{n}}(u), v\rangle_{1/2,\Gamma} - \int_{\Gamma_2\cup\Gamma_3} [\mathcal{M}_{\boldsymbol{nn}}(u)\partial_{\boldsymbol{n}} v + \mathcal{M}_{\boldsymbol{\tau n}}(u)\partial_{\boldsymbol{\tau}} v]\,ds \\ &\quad + \int_{\Gamma_3} j^0(u; v)\,ds \ge 0 \quad \forall\, v \in V. \end{aligned} \tag{3.22}$$

By taking $v \in V$ such that $v = 0$ on Γ and $\partial_{\boldsymbol{n}} v$ arbitrary on $\Gamma_2 \cup \Gamma_3$, it can be shown that

$$\mathcal{M}_{\boldsymbol{nn}}(u) = 0 \quad \text{a.e. on } \Gamma_2 \cup \Gamma_3. \tag{3.23}$$

Thus, from (3.22),

$$\langle \mathcal{Q}_{\boldsymbol{n}}(u), v\rangle_{1/2,\Gamma} - \int_{\Gamma_2\cup\Gamma_3} \mathcal{M}_{\boldsymbol{\tau n}}(u)\,\partial_{\boldsymbol{\tau}} v\,ds + \int_{\Gamma_3} j^0(u; v)\,ds \ge 0 \quad \forall\, v \in V.$$

Replacing v by $-v$ in the above inequality, we have

$$\langle \mathcal{Q}_{\boldsymbol{n}}(u), v\rangle_{1/2,\Gamma} - \int_{\Gamma_2\cup\Gamma_3} \mathcal{M}_{\boldsymbol{\tau n}}(u)\,\partial_{\boldsymbol{\tau}} v\,ds \le \int_{\Gamma_3} j^0(u; -v)\,ds \quad \forall\, v \in V. \tag{3.24}$$

The closure of V in $H^1(\Omega)$ is

$$H^1_{\Gamma_1}(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ a.e. on } \Gamma_1\}.$$

Denote

$$\tilde{H}^1_{\Gamma_1}(\Omega) = \{v \in H^1_{\Gamma_1}(\Omega) : \partial_{\boldsymbol{\tau}} v \in L^2(\Gamma)\}.$$

Then from (3.24), we conclude that

$$\langle \mathcal{Q}_{\boldsymbol{n}}(u), v\rangle_{1/2,\Gamma} - \int_{\Gamma_2\cup\Gamma_3} \mathcal{M}_{\boldsymbol{\tau n}}(u)\partial_{\boldsymbol{\tau}} v\,ds \le \int_{\Gamma_3} j^0(u; -v)\,ds \quad \forall\, v \in \tilde{H}^1_{\Gamma_1}(\Omega). \tag{3.25}$$

4 Nonconforming Virtual Element Discretization and Approximation

Given a positive integer m and a bounded set $S \subset \mathbb{R}^2$, $H^m(S)$ denotes the usual Sobolev space with the corresponding norm $\|\cdot\|_{m,S}$ and semi-norm $|\cdot|_{m,S}$. For simplicity, we assume Ω is a polygonal domain. Let $\{\mathcal{T}_h\}_h$, $\mathcal{T}_h := \{K\}_{K\in\mathcal{T}_h}$, be a family of partitions of $\overline{\Omega}$ into polygons, with a generic element denoted by K; $h := \max_{K\in\mathcal{T}_h} h_K$ and $h_K := \mathrm{diam}(K)$. Let $\mathcal{E}_h$ be the set of all the element edges, $\mathcal{E}_h^b$ the set of all the element edges that lie on the boundary Γ, $\mathcal{E}_h^i$ the set of all interior edges, and $\mathcal{E}_{h,\Gamma_2,\Gamma_3}^b$ the set of all the edges that lie on Γ_2 and Γ_3. Similarly, we denote by $\mathcal{V}_h = \mathcal{V}_h^i \cup \mathcal{V}_h^b$ the set of vertices in $\mathcal{T}_h$, where $\mathcal{V}_h^i$ and $\mathcal{V}_h^b$ are the sets of interior and boundary vertices on Γ, respectively, and by $\mathcal{V}_{h,\Gamma_1}^b \subset \mathcal{V}_h^b$ the set of boundary vertices on Γ_1. For an element K and an edge e on the boundary ∂K of K, $|K|$ and $|e| = h_e$ denote the area of K and the length of e, respectively. We denote the traces of a piecewise smooth function v on $e \subset \partial K^+ \cap \partial K^-$ from the interior of $K^\pm$ by $v^\pm$. Then, we define the jump of v on the interior edge by $[v] = v^+ - v^-$ and on the boundary edge by $[v] = v|_e$. We make the following assumption on the family of decompositions (cf. [17]):

Assumption (H_1). For each $K \in \mathcal{T}_h$, there exists a "virtual triangulation" $\mathcal{T}_K$ of K such that $\mathcal{T}_K$ is uniformly shape regular and quasi-uniform. The corresponding mesh size of $\mathcal{T}_K$ is bounded from below by a constant multiple of h_K. Each edge of K is a side of certain triangle in $\mathcal{T}_K$.

Throughout this paper, for any two quantities a and b, the notation "$a \lesssim b$" stands for "$a \leq Cb$", where C or c (with or without subscript) denotes a positive constant independent of h_K or h, which may take on different values at different occurrences.

For a nonnegative integer ℓ and an open set D, $\mathbb{P}_\ell(D)$ denotes the set of all polynomials on D with the total degree no more than ℓ. Moreover, we use $\mathcal{T}_h^*$ to denote a triangulation of Ω, which comprises all triangles in $\mathcal{T}_K$ for all $K \in \mathcal{T}_h$. Evidently, $\{\mathcal{T}_h^*\}$ is regular and quasi-uniform with respect to the mesh size h.

For any integer $m > 0$, let

$$H^m(\mathcal{T}_h) = \Pi_{K\in\mathcal{T}_h} H^m(K) = \{v \in L^2(\Omega) : v|_K \in H^m(K) \text{ for any } K \in \mathcal{T}_h\}$$

be a broken Sobolev space and endow it with the broken H^m-seminorm

$$|v|_{m,h} := \Big(\sum_{K\in\mathcal{T}_h} |v|_{m,K}^2\Big)^{1/2}.$$

Then we introduce the nonconforming space $H^{2,nc}(\mathcal{T}_h) \subset H^2(\mathcal{T}_h)$ by

$$H^{2,nc}(\mathcal{T}_h) = \Big\{v \in H^2(\mathcal{T}_h) :\ v \text{ continuous at internal vertices},\ v(P_i) = 0\ \forall\, P_i \in \mathcal{V}_{h,\Gamma_1}^b,$$
$$\int_e [\partial_{\boldsymbol{n}} v]\, ds = 0\ \forall\, e \in \mathcal{E}_h \backslash \mathcal{E}_{h,\Gamma_2,\Gamma_3}^b\Big\}.$$

Lemma 4.1 *The quantity $|\cdot|_{2,h}$ is a norm on the spaces V and $H^{2,nc}(\mathcal{T}_h)$.*

Proof It is well-known that $|\cdot|_{2,h}$ is a norm on the space V, cf. [19, Theorem 6.8-4]. So we only need to prove that $|\cdot|_{2,h}$ is a norm on $H^{2,nc}(\mathcal{T}_h)$.

According to [13, Corollary 4.2 & Example 2.4],

$$\|v_h\|_{0,\Omega}^2 + |v_h|_{1,h}^2 \lesssim |v_h|_{2,h}^2 + \sum_{e\in\mathcal{E}_h^i} \left(h_e^{-3}\|\mathrm{P}_1^e[v_h]\|_e^2 + h_e^{-1}\left\|\mathrm{P}_0^e[\partial_{\boldsymbol{n}} v_h]\right\|_e^2\right) \quad \forall\, v_h \in H^{2,nc}(\mathcal{T}_h), \tag{4.1}$$

where P_m^e is the L^2-projection from $L^2(e)$ to $\mathbb{P}_m(e)$. By the definition of $H^{2,nc}(\mathcal{T}_h)$, we have

$$\int_e [\partial_{\boldsymbol{n}} v_h]\, ds = 0 \quad \forall\, e \in \mathcal{E}_h^i,$$

which implies $\mathrm{P}_0^e[\partial_{\boldsymbol{n}} v_h] = 0$. Note that $[v_h]_e$ vanishes at the endpoints of e. If v_h^I is the usual linear interpolant of v_h on e, then $[v_h^I]_e = 0$. So

$$\|\mathrm{P}_1^e[v_h]\|_e^2 \le \|[v_h]\|_e^2 = \|[v_h - v_h^I]\|_e^2 \lesssim h_e^3 \sum_{\tau\in\partial^{-1}e} |v_h|_{2,\tau}^2,$$

where $\partial^{-1}e$ denotes the set of all triangles in $\mathcal{T}_h^*$ with e as one side. Thus, we obtain from (4.1) that

$$\|v_h\|_{0,\Omega}^2 + |v_h|_{1,h}^2 \lesssim |v_h|_{2,h}^2 \quad \forall\, v_h \in H^{2,nc}(\mathcal{T}_h), \tag{4.2}$$

which implies that $|\cdot|_{2,h}$ is a norm on the space $H^{2,nc}(\mathcal{T}_h)$. □

Now we introduce the local and global nonconforming virtual element spaces. The local virtual element space is defined as follows (cf. [2, 43]):

$$V_h^K = \{v_h \in H^2(K) : \Delta^2 v_h = 0 \text{ in } K,\ \mathcal{M}_{\boldsymbol{nn}}(v_h)|_e \in \mathbb{P}_0(e),\ \mathcal{N}(v_h)|_e = 0\ \forall\, e \subset \partial K\}.$$

We choose the following degrees of freedom in V_h^K:

(D1) $v_h(P_i)$ for any vertex P_i of K,
(D2) $\int_e \partial_{\boldsymbol{n}} v_h\, ds$ for any edge e of ∂K,

which are unisolvent for V_h^K. Building upon the local space V_h^K, the global nonconforming virtual element space is then defined as follows:

$$V_h = \Big\{v_h \in L^2(\Omega) :\ v_h|_K \in V_h^K,\ v_h \text{ continuous at internal vertices}, \\ v_h(P_i) = 0\ \forall\, P_i \in \mathcal{V}_{h,\Gamma_1}^b,\ \int_e [\partial_{\boldsymbol{n}} v_h]\, ds = 0\ \forall\, e \in \mathcal{E}_h \backslash \mathcal{E}_{h,\Gamma_2,\Gamma_3}^b\Big\}. \tag{4.3}$$

We observe that by construction, $V_h \subset H^{2,nc}(\mathcal{T}_h)$ and $V_h \not\subseteq H^2(\Omega)$.

Let $f_h \in L^2(\Omega)$ represent a function to be constructed to satisfy the condition

$$(f_h, v) \le c\, \|f\|_{0,\Omega}\|v\|_{0,\Omega} \quad \forall\, v \in V_h. \tag{4.4}$$

Since the bilinear form $a(\cdot,\cdot)$ can be split as

$$a(u, v) = \sum_{K\in\mathcal{T}_h} a^K(u, v), \quad a^K(u, v) = \int_K \mathcal{M}_{\alpha\beta}(u)\,\mathcal{K}_{\alpha\beta}(v)\, dx,$$

we construct the discrete symmetric bilinear form $a_h(\cdot,\cdot)$ over $V_h \times V_h$ in the form

$$a_h(u_h, v_h) = \sum_{K\in\mathcal{T}_h} a_h^K(u_h, v_h) \quad \forall\, u_h, v_h \in V_h.$$

To define the local bilinear form $a_h^K(\cdot,\cdot)$ on $V_h^K \times V_h^K$, introduce a projection operator $\Pi^K : V_h^K \to \mathbb{P}_2(K)$ by the relations (cf. [18])

$$\begin{cases} a^K(\Pi^K\psi, q) = a^K(\psi, q) \quad \forall q \in \mathbb{P}_2(K), \\ \widehat{\Pi^K\psi} = \widehat{\psi}, \\ \displaystyle\sum_{e\subset\partial K} \frac{1}{|e|}\int_e \nabla\Pi^K\psi\, ds = \sum_{e\subset\partial K} \frac{1}{|e|}\int_e \nabla\psi\, ds, \end{cases} \tag{4.5}$$

where

$$\widehat{\psi} = \frac{1}{n}\sum_{i=1}^{n} \psi(P_i^K),$$

and $\{P_i^K\}$ are the vertices of K. Note that (4.5) implies

$$\Pi^K q = q \quad \forall q \in \mathbb{P}_2(K), \tag{4.6}$$

and Π^K is computable from the degrees of freedom (**D1**)–(**D2**). Then the local bilinear form is defined by the formula

$$a_h^K(u_h, v_h) := a^K\left(\Pi^K u_h, \Pi^K v_h\right) + S^K\left(u_h - \Pi^K u_h, v_h - \Pi^K v_h\right) \quad \forall u_h, v_h \in V_h^K, \tag{4.7}$$

where the stabilization term is

$$S^K(w, v) := h_K^{-2} \sum_{i=1}^{N^K} \chi_i(w)\chi_i(v) \quad \forall w, v \in V_h^K,$$

N^K being the number of degrees of freedom, $\{\chi_1, \cdots, \chi_{N^K}\}$ being the local degrees of freedom from (**D1**)–(**D2**).

With the above preparation, our numerical method for Problem (P) is the following.

Problem (P_h) Find an element $u_h \in V_h$ such that

$$a_h(u_h, v_h) + \int_{\Gamma_3} j^0(u_h; v_h)\, ds \ge (f_h, v_h) \quad \forall v_h \in V_h. \tag{4.8}$$

The rest of the section is devoted to an analysis of Problem (P_h).

For a polygon $K \in \mathcal{T}_h$ satisfying assumption (H_1), the following trace inequality holds naturally (cf. [17]):

$$\|v\|_{0,\partial K}^2 \lesssim h_K^{-1}\|v\|_{0,K}^2 + h_K|v|_{1,K}^2 \quad \forall v \in H^1(K). \tag{4.9}$$

By examining the derivations in [18], we find that all the estimates there hold under assumption (H_1) in our context. This implies that $S^K(\cdot,\cdot)$ is a symmetric bilinear form satisfying

$$a^K(v_h, v_h) \lesssim S^K(v_h, v_h) \lesssim a^K(v_h, v_h) \quad \forall v_h \in \ker(\Pi^K). \tag{4.10}$$

From the definition of Π^K and (4.7), and the relations (4.6) and (4.10), we deduce a consistency property:

$$a_h^K(p, v_h) = a^K(p, v_h) \quad \forall p \in \mathbb{P}_2(K),\ v_h \in V_h^K, \tag{4.11}$$

and a stability property: for two positive constants α_* and α^*, independent of h_K and K,

$$\alpha_* a^K(v_h, v_h) \leq a_h^K(v_h, v_h) \leq \alpha^* a^K(v_h, v_h) \quad \forall\, v_h \in V_h^K. \tag{4.12}$$

From the definition of $a^K(\cdot,\cdot)$,

$$a^K(v_h, v_h) = (1-\nu)\int_K \mathcal{K}_{\alpha\beta}(v_h)\mathcal{K}_{\alpha\beta}(v_h)\,dx + \nu\int_K (\Delta v_h)^2\,dx \geq (1-\nu)|v_h|_{2,K}^2. \tag{4.13}$$

From (4.12), we have

$$a_h(v_h, v_h) \geq \sum_{K\in\mathcal{T}_h} \alpha_* a^K(v_h, v_h) \geq \alpha_*(1-\nu)|v_h|_{2,h}^2. \tag{4.14}$$

For later use, we denote

$$\tilde{m}_A = \alpha_*(1-\nu). \tag{4.15}$$

Similarly, for all $u_h, v_h \in V_h^K$, we have

$$a^K(u_h, v_h) = \int_K \mathcal{M}_{\alpha\beta}(u_h)\mathcal{K}_{\alpha\beta}(v_h)\,dx \lesssim |u_h|_{2,K}|v_h|_{2,K}. \tag{4.16}$$

Choosing $q = \Pi^K\psi$ in the first relation in (4.5), and making use of (4.13) and (4.16), we have

$$|\Pi^K\psi|_{2,K} \lesssim |\psi|_{2,K} \quad \forall\,\psi \in H^2(K).$$

Define $\Pi_h : V_h \to \mathbb{P}_2(\mathcal{T}_h)$ by $(\Pi_h v)|_K := \Pi^K(v|_K)$ for each $K \in \mathcal{T}_h$, where

$$\mathbb{P}_2(\mathcal{T}_h) := \{v \in L^2(\Omega) : v|_K \in \mathbb{P}_2(K) \text{ for each } K \in \mathcal{T}_h\}.$$

Under assumption (H_1), we can directly obtain the following result in view of the Scott-Dupont approximation theory (cf. [11]).

Lemma 4.2 *For every $v \in H^3(K)$, there exists a function $v_\pi \in \mathbb{P}_2(K)$ such that*

$$\|v - v_\pi\|_{i,K} \lesssim h_K^{3-i}|v|_{3,K}, \quad i = 0, 1, 2. \tag{4.17}$$

From [18, Lemma A.5], we can also obtain the local Poincaré inequality with the hidden constant independent of the geometric nature of K.

Lemma 4.3 *For any $K \in \mathcal{T}_h$, there holds*

$$\|v\|_{0,K} + h_K|v|_{1,K} \lesssim h_K^2|v|_{2,K} \quad \forall\, v \in \ker(\Pi^K).$$

The above result can be used in derivation of error estimates of the nodal interpolation operator in lower order norms (cf. [18]).

Lemma 4.4 *Let $I_K : H^3(K) \to V_h^K$ be the standard nodal interpolation operator. Then the following error estimates hold:*

$$\|v - I_K v\|_{i,K} \lesssim h_K^{3-i}|v|_{3,K}, \quad i = 0, 1, 2. \tag{4.18}$$

From now on, we write v_I for the global interpolant of v, i.e., for all $K \in \mathcal{T}_h$, $v_I(x)$ is equal to $I_K v(x)$ for $x \in K$.

Lemma 4.5 *There exists some constant* $\tilde{c}_\gamma > 0$ *such that*

$$\|v\|_{L^2(\Gamma_3)} \le \tilde{c}_\gamma |v|_{2,h} \quad \forall\, v \in H^{2,nc}(\mathcal{T}_h) + V. \tag{4.19}$$

Proof Let I_T be the usual interpolation operator from $C(\bar{\Omega})$ into the continuous piecewise linear finite element space related to the triangulation $\mathcal{T}_h^*$. Since the function values at vertices are available for all $v \in H^{2,nc}(\mathcal{T}_h)$, this operator is also well defined in $H^{2,nc}(\mathcal{T}_h)$. For $v \in H^{2,nc}(\mathcal{T}_h) + V$, since $v = 0$ on Γ_1, we have $I_T v = 0$ on Γ_1. By the trace theorem,

$$\|I_T v\|_{L^2(\Gamma_3)} \lesssim \|I_T v\|_{1,\Omega}. \tag{4.20}$$

By (4.20), (4.9) and Lemma 4.1, we obtain

$$\begin{aligned}
\|v\|_{L^2(\Gamma_3)} &\le \|v - I_T v\|_{L^2(\Gamma_3)} + \|I_T v\|_{L^2(\Gamma_3)} \lesssim \Big(\sum_{e\subset\Gamma_3} \|v - I_T v\|_{0,e}^2 \Big)^{1/2} + \|I_T v\|_{1,\Omega} \\
&\lesssim \Big(\sum_{e\subset\Gamma_3} \sum_{\tau_e} \big(h_\tau^{-1} \|v - I_T v\|_{0,\tau}^2 + h_\tau \|I_T v - v\|_{1,\tau}^2 \big) \Big)^{1/2} \\
&\quad + \Big(\sum_{K\in\mathcal{T}_h} \|I_T v - v\|_{1,K}^2 \Big)^{1/2} + \Big(\sum_{K\in\mathcal{T}_h} \|v\|_{1,K}^2 \Big)^{1/2} \\
&\lesssim \Big(\sum_{K\in\mathcal{T}_h} h_K^3 |v|_{2,K}^2 \Big)^{1/2} + \Big(\sum_{K\in\mathcal{T}_h} h_K^2 |v|_{2,K}^2 \Big)^{1/2} + \Big(\sum_{K\in\mathcal{T}_h} |v|_{2,K}^2 \Big)^{1/2} \\
&\lesssim |v|_{2,h},
\end{aligned}$$

where τ_e refers to the only triangle in $\mathcal{T}_h^*$ with e as one side. Consequently, there exists a positive constant $\tilde{c}_\gamma > 0$ independent of h such that (4.19) holds. □

Similar to Problem ($\tilde{\mathrm{P}}$), we introduce a minimization problem related to Problem (P_h).

Problem ($\tilde{\mathrm{P}}_h$) Find $u_h \in V_h$ such that

$$E_3^h(u_h) = \inf \left\{ E_3^h(v_h) : v_h \in V_h \right\},$$

where

$$E_3^h(v_h) = \frac{1}{2} a_h(v_h, v_h) + \int_{\Gamma_3} j(v_h)\, ds - (f_h, v_h).$$

We have the following existence and uniqueness result.

Theorem 4.1 *Assume* (H_0) *and* (H_1), (4.4), *and*

$$\alpha_j \tilde{c}_\gamma^2 < \tilde{m}_A. \tag{4.21}$$

Then Problem (P_h) is equivalent to Problem ($\tilde{\mathrm{P}}_h$), and both problems have the same unique solution $u_h \in V_h$.

Proof Due to assumption (H_1) and (4.19), we have

$$\|v_h\|_{L^2(\Gamma_3)} \le \tilde{c}_\gamma |v_h|_{2,h} \quad \forall\, v_h \in V_h. \tag{4.22}$$

Combining (4.14) with (4.15), we obtain

$$a_h(v_h, v_h) \ge \tilde{m}_A |v_h|_{2,h}^2 \quad \forall\, v_h \in V_h. \tag{4.23}$$

By (H_0), (4.4), (4.21), (4.22) and (4.23), similar to Theorem 3.1, we conclude that Problem ($\tilde{\mathrm{P}}_h$) has a unique solution, which is also the unique solution of Problem (P_h). □

We provide a uniform boundedness result on the numerical solutions, which will be needed in error estimation later.

Lemma 4.6 *Under the assumptions* (H_0) *and* (H_1)*,* (4.4)*, and* (4.21)*, we have*

$$|u_h|_{2,h} \lesssim \|f\|_{0,\Omega} + 1.$$

Proof We take $v_h = -u_h$ in (4.8),

$$a_h(u_h, -u_h) + \int_{\Gamma_3} j^0(u_h; -u_h)\, ds \geq \langle f_h, -u_h \rangle,$$

which is rewritten as

$$a_h(u_h, u_h) \leq \int_{\Gamma_3} j^0(u_h; -u_h)\, ds + \langle f_h, u_h \rangle. \tag{4.24}$$

By (4.14) and (4.15),

$$a_h(u_h, u_h) \geq \tilde{m}_A |u_h|_{2,h}^2, \tag{4.25}$$

and from (3.12) and (3.13),

$$j^0(u_h; -u_h) \leq \alpha_j |u_h|^2 - j^0(0; u_h) \leq \alpha_j |u_h|^2 + c_0 |u_h|.$$

According to (4.19), there is a constant $c > 0$,

$$\int_{\Gamma_3} j^0(u_h; -u_h)\, ds \leq \alpha_j \tilde{c}_\gamma^2 |u_h|_{2,h}^2 + c\, |u_h|_{2,h}. \tag{4.26}$$

Use (4.25), (4.26) and (4.4) in (4.24) to obtain

$$\tilde{m}_A |u_h|_{2,h}^2 \leq c\, \|f\|_{0,\Omega} \|u_h\|_{0,\Omega} + \alpha_j \tilde{c}_\gamma^2 |u_h|_{2,h}^2 + c\, |u_h|_{2,h}.$$

Due to (4.21) and (4.2), we obtain

$$|u_h|_{2,h} \lesssim \|f\|_{0,\Omega} + 1,$$

i.e., $u_h \in V_h$ is uniformly bounded independent of h. □

Using the intrinsic arguments for error analysis of plate elements in [26, 32] as well as the technique developed in [24], we can obtain a Céa-type estimate for our VEM method.

Theorem 4.2 *Under the assumptions* (H_0)*,* (H_1)*,* (3.18)*,* (4.4)*, and* (4.21)*, we have the following inequality:*

$$|u - u_h|_{2,h} \lesssim h + \|u - u_I\|_{L^2(\Gamma_3)}^{1/2} + \|f - f_h\|_{V_h^*}, \tag{4.27}$$

where

$$\|f - f_h\|_{V_h^*} = \sup_{v_h \in V_h} \frac{\langle f - f_h, v_h \rangle}{\|v_h\|_{V_h}}.$$

Proof By Lemma 4.2, for each element K, we can find $u_\pi \in \mathbb{P}_2(K)$ such that

$$\|u - u_\pi\|_{i,K} \lesssim h_K^{3-i} |u|_{3,K}, \quad i = 0, 1, 2. \tag{4.28}$$

Let $w_h = u_I - u_h$. By (4.11), we obtain

$$\tilde{m}_A |w_h|_{2,h}^2 \le \sum_{K\in\mathcal{T}_h} \alpha_* a^K(w_h, w_h) \le \sum_{K\in\mathcal{T}_h} a_h^K(w_h, w_h) = a_h(u_I, w_h) - a_h(u_h, w_h)$$
$$\le \sum_{K\in\mathcal{T}_h} \big(a_h^K(I_K u - u_\pi, w_h) + a^K(u_\pi - u, w_h)\big) + \sum_{K\in\mathcal{T}_h} a^K(u, w_h) - a_h(u_h, w_h). \tag{4.29}$$

By (4.8),

$$-a_h(u_h, w_h) \le -(f, w_h) + (f - f_h, w_h) + \int_{\Gamma_3} j^0(u_h; u_I - u_h)\, ds.$$

Use this inequality in (4.29) to get

$$\tilde{m}_A |w_h|_{2,h}^2 \le \mathrm{I}_1 + \mathrm{I}_2 + (f - f_h, w_h) + \int_{\Gamma_3} j^0(u_h; u_I - u_h)\, ds, \tag{4.30}$$

where

$$\mathrm{I}_1 := \sum_{K\in\mathcal{T}_h} \big(a_h^K(I_K u - u_\pi, w_h) + a^K(u_\pi - u, w_h)\big),$$
$$\mathrm{I}_2 := \sum_{K\in\mathcal{T}_h} a^K(u, w_h) - (f, w_h). \tag{4.31}$$

It follows from (4.28) and (4.18) that

$$\mathrm{I}_1 \le c|w_h|_{2,h}\Big[\Big(\sum_{K\in\mathcal{T}_h} |I_K u - u_\pi|_{2,K}^2\Big)^{1/2} + \Big(\sum_{K\in\mathcal{T}_h} |u - u_\pi|_{2,K}^2\Big)^{1/2}\Big] \le ch|w_h|_{2,h}|u|_{3,\Omega}. \tag{4.32}$$

Using integration by part in (4.31), we obtain

$$\mathrm{I}_2 = \sum_{K\in\mathcal{T}_h} \Big[\int_K \mathcal{Q}_\alpha(u)\partial_\alpha w_h\, dx - \int_{\partial K} \big(\mathcal{M}_{\boldsymbol{nn}}(u)\partial_{\boldsymbol{n}} w_h + \mathcal{M}_{\boldsymbol{\tau n}}(u)\partial_{\boldsymbol{\tau}} w_h\big)\, ds\Big] - (f, w_h). \tag{4.33}$$

Let w_h^I be the nodal interpolant of w_h in the lowest-order H^1-conforming virtual element space presented in [6, 23]. Write

$$\sum_{K\in\mathcal{T}_h} \int_K \mathcal{Q}_\alpha(u)\partial_\alpha w_h\, dx = \sum_{K\in\mathcal{T}_h} \int_K \mathcal{Q}_\alpha(u)\partial_\alpha (w_h - w_h^I)\, dx + \int_\Omega \mathcal{Q}_\alpha(u)\partial_\alpha w_h^I\, dx.$$

Then, by (3.20),

$$\mathrm{I}_2 = \sum_{K\in\mathcal{T}_h} \int_K \mathcal{Q}_\alpha(u)\partial_\alpha (w_h - w_h^I)\, dx + (f, w_h^I - w_h) + \langle \mathcal{Q}_{\boldsymbol{n}}(u), w_h^I\rangle_{1/2,\Gamma}$$
$$- \sum_{K\in\mathcal{T}_h} \int_{\partial K} \big(\mathcal{M}_{\boldsymbol{nn}}(u)\partial_{\boldsymbol{n}} w_h + \mathcal{M}_{\boldsymbol{\tau n}}(u)\partial_{\boldsymbol{\tau}} w_h\big)\, ds.$$

Thus,

$$\mathrm{I}_2 = \mathrm{II} + \mathrm{III} + \langle \mathcal{Q}_{\boldsymbol{n}}(u), w_h^I\rangle_{1/2,\Gamma}, \tag{4.34}$$

where

$$\mathrm{II} := \sum_{K\in\mathcal{T}_h} \int_K \mathcal{Q}_\alpha(u)\partial_\alpha(w_h - w_h^I)\,dx + (f, w_h^I - w_h),$$

$$\mathrm{III} := -\sum_{K\in\mathcal{T}_h} \int_{\partial K} \big(\mathcal{M}_{\boldsymbol{nn}}(u)\partial_{\boldsymbol{n}} w_h + \mathcal{M}_{\boldsymbol{\tau n}}(u)\partial_{\boldsymbol{\tau}} w_h\big)\,ds.$$

It follows from error estimates for nodal interpolation operators (cf. [17]) that

$$|\mathrm{II}| \lesssim \sum_{K\in\mathcal{T}_h} |u|_{3,K}|w_h - w_h^I|_{1,K} + h^2\|f\|_{0,\Omega}|w_h|_{2,h} \lesssim (h|u|_{3,\Omega} + h^2\|f\|_{0,\Omega})|w_h|_{2,h}. \tag{4.35}$$

On the other hand, for any $v \in L^2(e)$ with $e \in \mathcal{E}$, let $R_0^e v = v - \mathrm{P}_0^e v$. Then we have by (4.9) and the Scott-Dupont approximation theory (cf. [11]) that

$$\|R_0^e v\|_{0,e} = \min_{w\in\mathbb{P}_0(K)} \|v - w\|_{0,e} \lesssim h_K^{1/2}|v|_{1,K} \quad \forall\, v \in H^1(K). \tag{4.36}$$

For any $v_h \in V_h$, by the definition (4.3),

$$\mathrm{P}_0^e[\partial_{\boldsymbol{n}} v_h] = 0 \quad \forall\, e \in \mathcal{E}_h\backslash\mathcal{E}_{h,\Gamma_2,\Gamma_3}^b. \tag{4.37}$$

Recalling (3.23), we have

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} \mathcal{M}_{\boldsymbol{nn}}(u)\,\partial_{\boldsymbol{n}} w_h\,ds = \sum_{e\in\mathcal{E}_h\backslash\mathcal{E}_{h,\Gamma_2,\Gamma_3}^b} \int_e \mathcal{M}_{\boldsymbol{nn}}(u)\big[\partial_{\boldsymbol{n}} w_h\big]\,ds.$$

By making use of the property (4.37),

$$\begin{aligned}\int_e \mathcal{M}_{\boldsymbol{nn}}(u)\big[\partial_{\boldsymbol{n}} w_h\big]\,ds &= \int_e \mathcal{M}_{\boldsymbol{nn}}(u)\left(\big[\partial_{\boldsymbol{n}} w_h\big] - \mathrm{P}_0^e\big[\partial_{\boldsymbol{n}} w_h\big]\right)ds\\ &= \int_e R_0^e\mathcal{M}_{\boldsymbol{nn}}(u)\,R_0^e[\partial_{\boldsymbol{n}} w_h]\,ds.\end{aligned}$$

So it follows from (4.36) that

$$\begin{aligned}\Big|\sum_{K\in\mathcal{T}_h} \int_{\partial K} \mathcal{M}_{\boldsymbol{nn}}(u)\,\partial_{\boldsymbol{n}} w_h\,ds\Big| &\le \sum_{e\in\mathcal{E}_h\backslash\mathcal{E}_{h,\Gamma_2,\Gamma_3}^b} \|R_0^e\mathcal{M}_{\boldsymbol{nn}}(u)\|_e\|R_0^e[\partial_{\boldsymbol{n}} w_h]\|_e\\ &\lesssim \sum_{K\in\mathcal{T}_h} h_K|u|_{3,K}|w_h|_{2,K} \lesssim h|u|_{3,\Omega}|w_h|_{2,h}.\end{aligned} \tag{4.38}$$

Write

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} \int_{\partial K} \mathcal{M}_{\boldsymbol{\tau n}}(u)\partial_{\boldsymbol{\tau}} w_h\,ds &= \sum_{K\in\mathcal{T}_h} \int_{\partial K} \mathcal{M}_{\boldsymbol{\tau n}}(u)\partial_{\boldsymbol{\tau}} w_h^I\,ds\\ &\quad + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \mathcal{M}_{\boldsymbol{\tau n}}(u)\partial_{\boldsymbol{\tau}}(w_h - w_h^I)\,ds.\end{aligned}$$

Then, using the fact $w_h^I \in C(\overline{\Omega})$ and $w_h^I = 0$ on Γ_1, we have

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} \mathcal{M}_{\boldsymbol{\tau n}}(u)\partial_{\boldsymbol{\tau}} w_h^I\,ds = \sum_{e\subset\Gamma_2\cup\Gamma_3} \int_e \mathcal{M}_{\boldsymbol{\tau n}}(u)\partial_{\boldsymbol{\tau}} w_h^I\,ds. \tag{4.39}$$

Similar to the derivation of (4.38), we find that

$$\begin{aligned}\Big|\sum_{K\in\mathcal{T}_h}\int_{\partial K}\mathcal{M}_{\boldsymbol{\tau n}}(u)\partial_{\boldsymbol{\tau}}(w_h-w_h^I)\,ds\Big| &\le \sum_{K\in\mathcal{T}_h}\int_{\partial K}\Big|\mathcal{M}_{\boldsymbol{\tau n}}(u)\big(\partial_{\boldsymbol{\tau}}(w_h-w_h^I)\big)\Big|\,ds\\ &\lesssim \sum_{K\in\mathcal{T}_h}|u|_{3,K}|w_h-w_h^I|_{1,K}\lesssim h|u|_{3,\Omega}|w_h|_{2,h}.\end{aligned} \tag{4.40}$$

From (4.34)–(4.40),

$$\mathrm{I}_2\le c\left(h|u|_{3,\Omega}+h^2\|f\|_{0,\Omega}\right)|w_h|_{2,h}+\langle\mathcal{Q}_{\boldsymbol{n}}(u),w_h^I\rangle_{1/2,\Gamma}-\sum_{e\subset\Gamma_2\cup\Gamma_3}\int_e\mathcal{M}_{\boldsymbol{\tau n}}(u)\partial_{\boldsymbol{\tau}}w_h^I\,ds.$$

Using (3.25) to get

$$\langle\mathcal{Q}_{\boldsymbol{n}}(u),w_h^I\rangle_{1/2,\Gamma}-\sum_{e\subset\Gamma_2\cup\Gamma_3}\int_e\mathcal{M}_{\boldsymbol{\tau n}}(u)\partial_{\boldsymbol{\tau}}w_h^I\,ds\le\int_{\Gamma_3}j^0(u;-w_h^I)\,ds.$$

Therefore,

$$\mathrm{I}_2\le c\left(h|u|_{3,\Omega}+h^2\|f\|_{0,\Omega}\right)|w_h|_{2,h}+\int_{\Gamma_3}j^0(u;-w_h^I)\,ds. \tag{4.41}$$

Consequently, we combine (4.30), (4.32) with (4.41) to obtain

$$\begin{aligned}\tilde{m}_A|w_h|_{2,h}^2\le c&\left[h\,|u|_{3,\Omega}|w_h|_{2,h}+h^2\|f\|_{0,\Omega}|w_h|_{2,h}+(f-f_h,w_h)\right]\\ &+\int_{\Gamma_3}\left[j^0(u;-w_h^I)+j^0(u_h;u_I-u_h)\right]ds.\end{aligned} \tag{4.42}$$

By (2.1) on the sub-additivity for the generalized directional derivative,

$$\begin{aligned}j^0(u;-w_h^I)&\le j^0(u;w_h-w_h^I)+j^0(u;u_h-u_I),\\ j^0(u_h;u_I-u_h)&\le j^0(u_h;u_I-u)+j^0(u_h;u-u_h),\\ j^0(u;u_h-u_I)&\le j^0(u;u_h-u)+j^0(u;u-u_I).\end{aligned}$$

Hence,

$$\begin{aligned}j^0(u;-w_h^I)+j^0(u_h;u_I-u_h)\le&\left[j^0(u_h;u-u_h)+j^0(u;u_h-u)\right]\\ &+j^0(u_h;u_I-u)+j^0(u;u-u_I)+j^0(u;w_h-w_h^I).\end{aligned}$$

By (3.13) and (4.19), we have

$$\int_{\Gamma_3}\left[j^0(u_h;u-u_h)+j^0(u;u_h-u)\right]ds\le\alpha_j\|u-u_h\|_{L^2(\Gamma_3)}^2\le\alpha_j\tilde{c}_\gamma^2|u-u_h|_{2,h}^2. \tag{4.43}$$

From (3.12) and (2.2), we have

$$j^0(u_h;u_I-u)\le(c_0+c_1|u_h|)|u-u_I|, \tag{4.44}$$

$$j^0(u;u-u_I)\le(c_0+c_1|u|)|u-u_I|, \tag{4.45}$$

$$j^0(u;w_h-w_h^I)\le(c_0+c_1|u|)|w_h-w_h^I|. \tag{4.46}$$

In view of (4.19) and the boundedness of $|u_h|_{2,h}$ in Lemma 4.6, we obtain the boundedness of $\|u_h\|_{L^2(\Gamma_3)}$. We derive from (4.42)–(4.46) and (4.2) that

$$\tilde{m}_A|w_h|_{2,h}^2 \le \alpha_j \tilde{c}_\gamma^2 |u-u_h|_{2,h}^2 + c\left(h\,|u|_3 + h^2\|f\|_{0,\Omega} + \|f-f_h\|_{V_h^*}\right)|w_h|_{2,h} + c\left(\|u-u_I\|_{L^2(\Gamma_3)} + \|w_h - w_h^I\|_{L^2(\Gamma_3)}\right).$$

The quantity $\|w_h - w_h^I\|_{L^2(\Gamma_3)}$ is bounded as follows:

$$\begin{aligned}\|w_h - w_h^I\|_{L^2(\Gamma_3)} &= \Big(\sum_{\gamma\subset\Gamma_3} \|w_h - w_h^I\|_{0,\gamma}^2\Big)^{1/2}\\ &\lesssim \Big(\sum_{K:\partial K\cap\Gamma_3\neq\emptyset} [h_K^{-1}\|w_h - w_h^I\|_{0,K}^2 + h_K|w_h - w_h^I|_{1,K}^2]\Big)^{1/2}\\ &\lesssim h^{3/2}|w_h|_{2,h}.\end{aligned}$$

Noting that

$$|u-u_h|_{2,h}^2 \le |u-u_I|_{2,h}^2 + |w_h|_{2,h}^2 + 2|u-u_I|_{2,h}|w_h|_{2,h},$$

we obtain

$$(\tilde{m}_A - \alpha_j\tilde{c}_\gamma^2)|w_h|_{2,h}^2 \lesssim |w_h|_{2,h}\Big(h|u|_3 + h^2\|f\|_{0,\Omega} + |u-u_I|_{2,h} + h^{3/2} + \|f-f_h\|_{V_h^*}\Big) + |u-u_I|_{2,h}^2 + \|u-u_I\|_{L^2(\Gamma_3)}.$$

Then from (4.21), (4.18) and (2.3),

$$|w_h|_{2,h} \lesssim h|u|_{3,\Omega} + h^2\|f\|_{0,\Omega} + h^{3/2} + \|f-f_h\|_{V_h^*} + \|u-u_I\|_{L^2(\Gamma_3)}^{1/2}.$$

Hence, by the triangle inequality

$$|u-u_h|_{2,h} \le |u-u_I|_{2,h} + |w_h|_{2,h},$$

we obtain the inequality (4.27). □

Finally, we define the approximation of the right hand side by

$$\langle f_h, v_h\rangle = (f, \Pi_h v_h)\,, \tag{4.47}$$

and derive an optimal order error estimate for the resulting VEM.

Theorem 4.3 *Assume the conditions stated in Theorem 4.2, and let the right hand side of (4.8) be defined by (4.47). Then we have the optimal order error estimate*

$$|u-u_h|_{2,h} \lesssim h. \tag{4.48}$$

Proof Write

$$(f, v_h) - \langle f_h, v_h\rangle = (f, v_h - \Pi_h v_h) = \sum_{K\in\mathcal{T}_h}(f, v_h - \Pi^K v_h).$$

As in [18], we have

$$|(f, v_h) - \langle f_h, v_h\rangle| \lesssim h^2\|f\|_{0,\Omega}\Big(\sum_{K\in\mathcal{T}_h}|v_h - \Pi^K v_h|_{2,K}^2\Big)^{1/2} \lesssim h^2\|f\|_{0,\Omega}|v_h|_{2,h}. \tag{4.49}$$

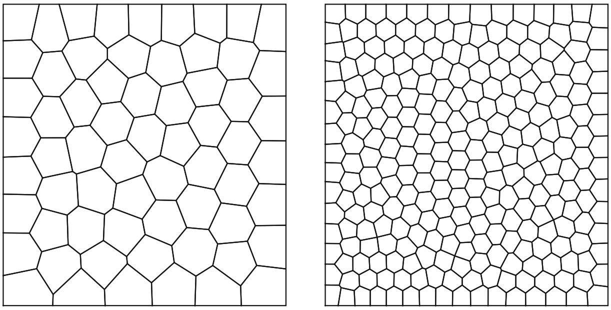

Fig. 1 Polygonal meshes with $N = 64$ (left) and $N = 256$ (right)

On the other hand,

$$\begin{aligned}\|u - u_I\|_{L^2(\Gamma_3)} &= \Big(\sum_{\gamma\subset\Gamma_3} \|u - u_I\|_{0,\gamma}^2\Big)^{1/2} \\ &\le \Big(\sum_{K:\partial K\cap\Gamma_3\neq\emptyset} [h_K^{-1}\|u - u_I\|_{0,K}^2 + h_K|u - u_I|_{1,K}^2]\Big)^{1/2} \\ &\lesssim h^{5/2}\Big(\sum_{K:\partial K\cap\Gamma_3\neq\emptyset} |u|_{3,K}^2\Big)^{1/2} \le h^{5/2}|u|_{3,\Omega}. \end{aligned} \tag{4.50}$$

Using (4.50) and (4.49) in (4.27), we obtain the error bound (4.48). □

5 Numerical Results

We use an efficient algorithm (the double bundle method) developed in [33] to solve the discrete problems. A description of the solution algorithm can be found in [24].

Let $\Omega = (-1, 1) \times (-1, 1)$, $\nu = 0.3$. A generic point in $\overline{\Omega}$ is denoted as $\boldsymbol{x} = (x, y)^T$. The Dirichlet boundary is $\Gamma_1 = (-1, 1) \times \{1\}$, and the free boundary is $\Gamma_2 = \{\{-1\} \times (-1, 1)\} \cup \{\{1\} \times (-1, 1)\}$. The friction boundary is $\Gamma_3 = (-1, 1) \times \{-1\}$. The right hand side function is chosen to be

$$f(\boldsymbol{x}) = 24\,(1 - x^2)^2 + 24\,(1 - y^2)^2 + 32\,(3x^2 - 1)\,(3y^2 - 1).$$

Let

$$\mathcal{N}(u) = \begin{cases} 0 \ \text{ if } u \le 0, \\ 5 \ \text{ if } u \in (0, 0.1], \\ 10 - 50\,u \ \text{ if } \ u \in (0.1, 0.15), \\ 20\,u - 0.5 \ \text{ if } \ u \ge 0.15. \end{cases} \tag{5.1}$$

By (3.9), we obtain

$$j(u) = \begin{cases} 0 \ \text{ if } u \le 0, \\ 5\,u \ \text{ if } u \in (0, 0.1], \\ 10\,u - 25\,u^2 - 0.25 \ \text{ if } u \in (0.1, 0.15), \\ 10\,u^2 - 0.5\,u + 0.5375 \ \text{ if } u \ge 0.15. \end{cases} \tag{5.2}$$

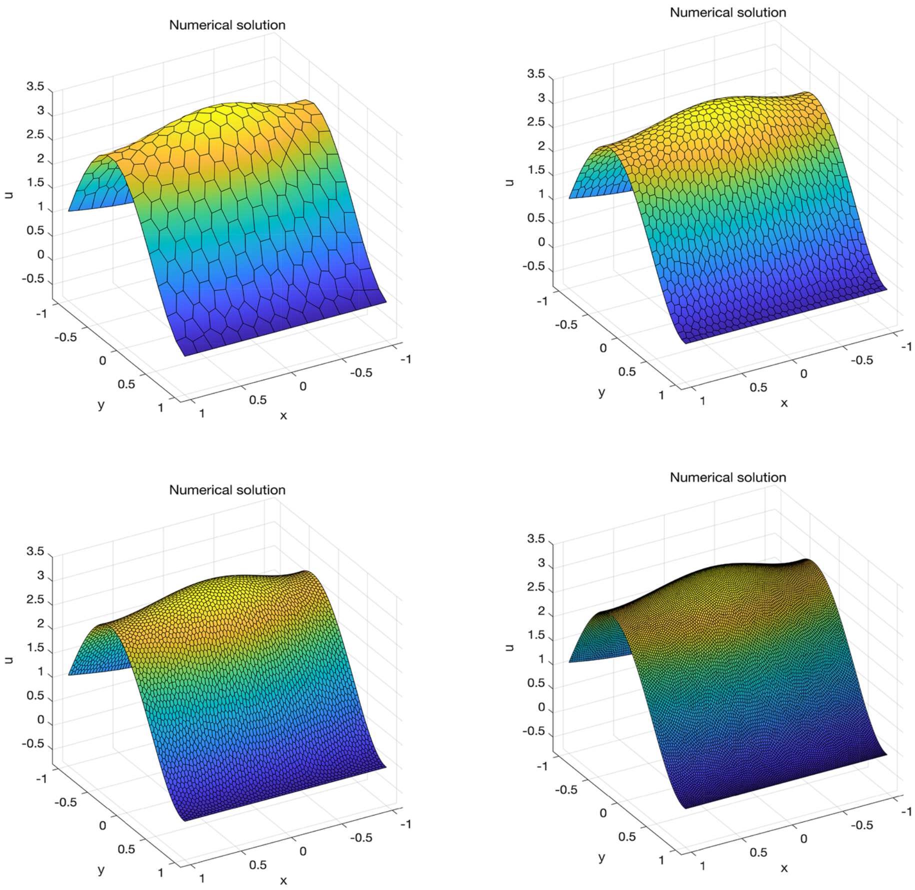

Fig. 2 The numerical solution for different number of elements: $N = 256$ (upper left), $N = 1024$ (upper right), $N = 4096$ (bottom left) and $N = 16384$ (bottom right)

Let $j(u) = f_0(u) - \tilde{f}_0(u)$, where $f_0(u)$ and $\tilde{f}_0(u)$ are defined as follows:

$$f_0(u) = \begin{cases} 25\,u^2 \;\text{ if } u \le 0, \\ 25\,u^2 + 5\,u \;\text{ if } u \in (0, 0.1], \\ 10\,u - 0.25 \;\text{ if } u \in (0.1, 0.15), \\ 35\,u^2 - 0.5\,u + 0.5375 \;\text{ if } u \ge 0.15, \end{cases}$$

$$\tilde{f}_0(u) = 25\,u^2.$$

The integral $\int_{\Gamma_3} j(u_h)\,ds$ is calculated with the trapezoidal rule.

We use the code PolyMesher ([38]) to generate the polygonal meshes and then solve the discrete problem. Meshes with element numbers $N = 64$ and $N = 256$ are displayed in Fig. 1.

The numerical solutions on $\mathcal{V}_h$ corresponding to several meshes with $N = 256$, $N = 1024$, $N = 4096$, $N = 16384$ are displayed in Fig. 2, respectively. A convergence trend is evident for the numerical solutions as N increases.

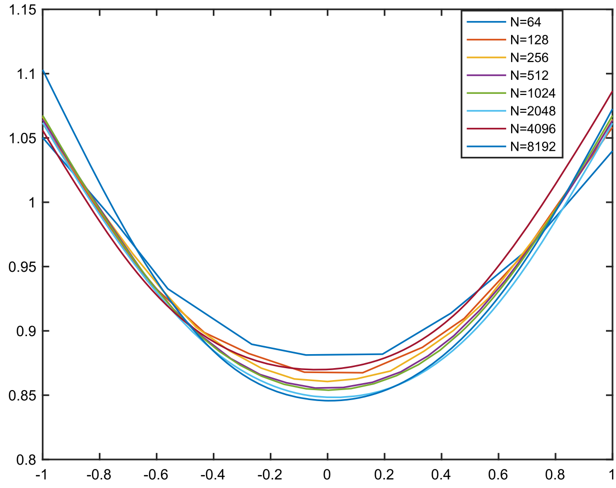

Fig. 3 The numerical solution of normal direction for different meshes

Table 1 Numerical errors on square meshes for lowest-order VEM

h	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
Error	1.388	0.733	0.371	0.182

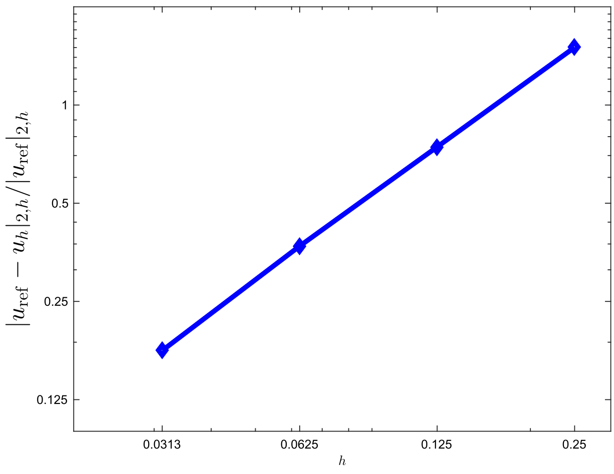

Fig. 4 Relative errors in energy norm

For the numerical solutions on the boundary $[-1, 1] \times \{-1\}$, a similar convergence trend is clearly observed (cf. Fig. 3).

In Table 1 and Fig. 4, we report relative errors of the numerical solutions in the energy norm on square meshes:

$$\text{error:} = \left(\frac{a_h(u - u_h, u - u_h)}{a_h(u, u)} \right)^{\frac{1}{2}}. \tag{5.3}$$

Since the true solution u in (5.3) is not available, we use the numerical solution with a fine mesh as the “reference” solution u_{ref} in computing the solution errors. Specifically, the “reference” solution u_{ref} is set as the numerical solution with $h = 1/64$. Note that the error bound (4.27) predicts an optimal first order convergence of the numerical solutions measured in the energy norm, under the regularity assumptions (3.18). Observe that the numerical convergence orders from the results in Table 1 and Fig. 4 are close to 1, matching the theoretical prediction.

Acknowledgements The authors would like to thank Profs. K. Joki and O. Montonen for providing their algorithms freely at the website http://napsu.karmitsa.fi/nsosoftware/, which are very helpful for solving the discrete problems given in this paper. The authors also thank the two anonymous referees for valuable comments and suggestions which helped to improve an early version of the paper.

Author Contributions All authors contributed equally to this manuscript.

Funding The work of Prof. Weimin Han was partially supported by Simons Foundation Collaboration Grants, No. 850737. The work of Prof. Jianguo Huang was partially supported by NSFC (Grant No. 12071289).

Code Availability The codes during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of interest The authors have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this manuscript.

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