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Numerical analysis of an evolutionary variational–hemivariational inequality with application to a dynamic contact problem

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ABSTRACT

In this paper, we consider the numerical solution of an evolutionary variational–hemivariational inequality arising in a dynamic contact problem. The material is assumed to be viscoelastic with short memory. The contact is featured by a normal damped response in the normal direction and by the Tresca friction law in the tangential direction. The linear finite elements are used to discretize the spatial variable. Optimal order error estimates are derived for the discrete velocity and discrete displacement under suitable solution regularity assumptions.

1. Introduction

Variational inequalities and hemivariational inequalities are powerful tools in the study of nonsmooth problems in many applications. While variational inequalities deal with nonsmooth problems with a convex structure, hemivariational inequalities allow the presence of nonconvex terms and are thus more challenging to be analyzed and approximated numerically. Hemivariational inequalities started with the pioneering work by Panagiotopoulos in early 1980s [1] and have since attracted steady attention from the research communities in mathematics, physical sciences and engineering. Hemivariational inequalities have been shown very useful across a variety of subjects, and in the context of mechanics, they are especially useful for problems involving non-monotone, non-smooth and multivalued constitutive laws, forces, and boundary conditions. The number of publications on hemivariational inequalities is growing rapidly, e.g., mathematical theory and applications of hemivariational inequalities can be found in several monographs, [2–6]; the first two references also include results of numerical simulations. Relatively fewer publications can be found on numerical approximations of hemivariational inequalities. The monograph [7] discusses finite element approximations of hemivariational inequalities, including their convergence; however, no error estimates are provided. Recently, optimal order error estimates are derived for numerical solutions of hemivariational inequalities. The first paper along this direction is [8] where optimal order error estimates are derived for the linear finite element solutions for certain stationary hemivariational and variational–hemivariational inequalities. This paper is followed by several papers on optimal order

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error estimates of the linear finite element solutions for various hemivariational inequalities of different form, e.g., [9] for the numerical solution of a hyperbolic hemivariational inequality for a dynamic contact problem, [10] for the numerical solution of an evolutionary variational–hemivariational inequality in a quasistatic frictionless contact problem with a viscoelastic material, and [11] for the numerical solution of elliptic hemivariational inequalities arising in the study of semipermeable media. A general framework is presented on convergence analysis and error estimation for internal approximations of elliptic hemivariational inequalities and for variational–hemivariational inequalities in [12] and [13], respectively. In [14], a comprehensive convergence analysis and error estimation are given for both internal and external approximations of stationary variational–hemivariational inequalities and hemivariational inequalities. Several papers can be found on numerical analysis of history-dependent hemivariational inequalities with applications in studies of quasistatic contact problems: [15] for a quasistatic history-dependent variational–hemivariational inequality, [16,17] for quasistatic history-dependent hemivariational inequalities, and [18] for a first order evolutionary hemivariational inequality.

In this paper, we study the numerical solution of a first order evolutionary variational–hemivariational inequality involving a history-dependent operator. Such an inequality problem arises in dynamic contact problems. The paper is organized as follows. In Section 2, we introduce the evolutionary variational–hemivariational inequality. In Section 3, we study a fully discrete method for solving the variational–hemivariational inequality and derive an error estimate. In Section 4, we consider a dynamic frictional contact problem and apply the theoretical results to the study of the contact problem; in particular, we present an optimal order error estimate for the discrete velocity and discrete displacement from the linear finite element approximations of the contact problem.

2. An evolutionary variational–hemivariational inequality

To describe and study the variational–hemivariational inequality, we need to recall the definitions of the convex and the Clarke subdifferentials. Let X be a Banach space, with its dual space X^*. The symbol 2^{X^*} stands for the set of all subsets of X^*. Denote by $\langle\cdot,\cdot\rangle$ the duality pairing between X^* and X. In this paper, all the spaces are real.

Definition 2.1. Let $\varphi\colon X\to\mathbb{R}\cup\{+\infty\}$ be a proper, convex and lower semicontinuous function. The mapping $\partial_c\varphi\colon X\to 2^{X^*}$ defined by

$$\partial_c\varphi(x):=\{x^*\in X^*\mid \varphi(v)-\varphi(x)\ge\langle x^*,v-x\rangle\ \forall\,v\in X\}$$

is called the (convex) subdifferential of φ. An element $x^*\in\partial_c\varphi(x)$ (if it is non-empty) is called a subgradient of φ at x.

Definition 2.2. Let $\psi\colon X\to\mathbb{R}$ be a locally Lipschitz functional. The generalized (Clarke) directional derivative of ψ at $x\in X$ in the direction $v\in X$ is defined by

$$\psi^0(x;v):=\limsup_{y\to x,\ \lambda\downarrow 0}\frac{\psi(y+\lambda v)-\psi(y)}{\lambda}.$$

The generalized gradient (subdifferential) of ψ at x is defined by

$$\partial\psi(x):=\left\{\zeta\in X^*\mid \psi^0(x;v)\ge\langle\zeta,v\rangle\ \forall\,v\in X\right\}.$$

Properties of the subdifferential mappings, both in the convex and Clarke sense, can be found in several books, e.g. [2,3,5,19,20]. In particular, we will need the sub-additivity property:

$$\psi^0(x;v_1+v_2)\le\psi^0(x;v_1)+\psi^0(x;v_2)\quad\forall\,x,v_1,v_2\in X. \tag{2.1}$$

Let $V\subset H\subset V^*$ be an evolution triple of function spaces, i.e., V is a separable reflexive Banach space, H is a separable Hilbert space, and the embedding $V\subset H$ is continuous and dense. The dual space of H is identified with H itself. The duality pairing between V^* and V is denoted by $\langle\cdot,\cdot\rangle$, and the inner product in H is denoted by $(\cdot,\cdot)$. For a given $T>0$, the time interval of interest is $[0,T]$. Denote $\mathcal{V}=L^2(0,T;V)$, $\mathcal{V}^*=L^2(0,T;V^*)$, and $\mathcal{W}=\{v\in\mathcal{V}\mid\dot v\in\mathcal{V}^*\}$. A dot above a variable represents the time derivative of the variable. Note that $\mathcal{W}\subset C(0,T;H)$. Introduce two more real Banach spaces V_φ and V_j. Let $\gamma_\varphi\in\mathcal{L}(V,V_\varphi)$ and $\gamma_j\in\mathcal{L}(V,V_j)$ be given, and denote by c_φ and c_j upper bounds of the operator norms of γ_φ and γ_j:

$$\|\gamma_\varphi v\|_{V_\varphi}\le c_\varphi\|v\|_V\quad\forall\,v\in V, \tag{2.2}$$

$$\|\gamma_j v\|_{V_j}\le c_j\|v\|_V\quad\forall\,v\in V. \tag{2.3}$$

Now we introduce an evolutionary variational–hemivariational inequality to be considered in this paper.

Problem 2.3. *Find $w\in\mathcal{W}$ such that*

$$\langle\dot w(t)+Aw(t)+\mathcal{S}w(t),v-w(t)\rangle+j^0(\gamma_j w(t);\gamma_j v-\gamma_j w(t))$$

$$+\ \varphi(\gamma_\varphi v)-\varphi(\gamma_\varphi w(t))\ge\langle f(t),v-w(t)\rangle\quad\forall\,v\in V,\ \text{a.e. } t\in(0,T), \tag{2.4}$$

$$w(0)=w_0. \tag{2.5}$$

In the study of Problem 2.3 we make the following assumptions, with positive constants $m_A, c_{\mathcal{S}}$ and non-negative constants $a_0, a_1, c_{0j}, c_{1j}, \alpha_j, c_{0\varphi}, c_{1\varphi}$. For the operator $A: V \to V^*$,

$$\begin{cases} \text{(a) } A \text{ is demicontinuous, i.e. } u_n \to u \text{ in } V \implies Au_n \rightharpoonup Au \text{ in } V^*; \\ \text{(b) } \|Av\|_{V^*} \le a_0 + a_1\|v\|_V \ \forall\, v \in V; \\ \text{(c) } \langle Av_1 - Av_2, v_1 - v_2\rangle \ge m_A\|v_1 - v_2\|_V^2 \ \forall\, v_1, v_2 \in V. \end{cases} \tag{2.6}$$

For the functional $j: V_j \to \mathbb{R}$,

$$\begin{cases} \text{(a) } j(\cdot) \text{ is locally Lipschitz on } V_j; \\ \text{(b) } \|\partial j(z)\|_{V_j^*} \le c_{0j} + c_{1j}\|z\|_{V_j} \ \forall\, z \in V_j; \\ \text{(c) } j^0(z_1; z_2 - z_1) + j^0(z_2; z_1 - z_2) \le \alpha_j\|z_1 - z_2\|_{V_j}^2 \ \forall\, z_1, z_2 \in V_j. \end{cases} \tag{2.7}$$

For the functional $\varphi: V_\varphi \to \mathbb{R}$,

$$\begin{cases} \text{(a) } \varphi \text{ is convex and l.s.c. on } V_\varphi; \\ \text{(b) } \|\partial_c\varphi(z)\|_{V_\varphi^*} \le c_{0\varphi} + c_{1\varphi}\|z\|_{V_\varphi} \ \forall\, z \in V_\varphi. \end{cases} \tag{2.8}$$

The operator $\mathcal{S}: \mathcal{V} \to \mathcal{V}^*$ is assumed to be a history-dependent operator [6], i.e.,

$$\|\mathcal{S}v_1(t) - \mathcal{S}v_2(t)\|_{V^*} \le c_{\mathcal{S}} \int_0^t \|v_1(s) - v_2(s)\|_V ds \quad \forall\, v_1, v_2 \in \mathcal{V}, \text{ a.e. } t \in (0, T). \tag{2.9}$$

$$\alpha_j c_j^2 < m_A. \tag{2.10}$$

$$f \in \mathcal{V}^*, \quad w_0 \in V. \tag{2.11}$$

We remark that (2.7)(b) means

$$\|\xi\|_{V_j^*} \le c_{0j} + c_{1j}\|z\|_{V_j} \quad \forall\, z \in V_j,\ \forall\, \xi \in \partial j(z).$$

A similar remark can be made on the meaning of (2.8)(b).

By adapting the proof of Theorem 98 in [6], we have the following existence and uniqueness result.

Theorem 2.4. *Under the assumptions (2.6)–(2.11), Problem 2.3 has a unique solution.*

Remark 2.5. In [6], Theorem 98 is proved by applying Theorem 97. In the proof of Theorem 97, a condition labeled (7.4) is used to prove the coercivity of an operator, cf. [6, pp. 184–185]. However, in the context of Theorem 97, the condition (7.4) can be weakened to $m_\psi < m_A$ for the coercivity of the operator. The condition $m_\psi < m_A$ takes the form (2.10) in the context of Theorem 2.4 of this paper.

3. Numerical analysis of a fully discrete scheme

For definiteness and without loss of generality, we will consider the case where $\mathcal{S}$ is of the form

$$(\mathcal{S}v)(t) = G\Big(\int_0^t q(t,s)\,v(s)\,ds + a_S\Big) \quad \forall\, v \in C(0, T; V),\ t \in (0, T), \tag{3.1}$$

where $G \in \mathcal{L}(V; V^*)$, $q \in C([0, T]^2; \mathcal{L}(V))$, $a_S \in V$. This is valid for applications in contact problems, cf. [6] and also the example in Section 4.

In a fully discrete scheme, we discretize with respect to both the temporal variable and the spatial variable. For simplicity in exposition, we use a uniform partition of the time interval $[0, T]$, and we comment that the discussion can be directly extended to the case of general non-uniform partitions. Thus, given a positive integer N, let $k = T/N$ be the time step-size, and denote the node points by $t_n = nk$, $0 \le n \le N$. For a continuous function $v(t)$ with values in a function space, we write $v_n = v(t_n)$, $0 \le n \le N$. For the spatial discretization, we use a family of finite-dimensional subspaces $\{V^h\}_h$ of V, indexed by a discretization parameter $h > 0$. In the example in Section 4, we use finite element spaces corresponding to a regular family of finite element partitions.

We use the backward divided difference to approximate the time derivative in (2.4) and use the left-point rule to approximate the integral in the definition of the operator $\mathcal{S}$ in (3.1), i.e., we use the operator $\mathcal{S}_n^k$ defined by

$$\mathcal{S}_n^k w^{hk} := G\Big(k\sum_{i=0}^{n-1} q(t_n, t_i)w_i^{hk} + a_S\Big), \quad w^{hk} = \{w_i^{hk}\}_{i=0}^N \subset V^h \tag{3.2}$$

to approximate the history-dependent operator $\mathcal{S}$. We assume

$$f \in C(0, T; V^*). \tag{3.3}$$

Then the pointwise values $f_n = f(t_n)$, $0 \le n \le N$, are well-defined.

The fully discrete scheme for Problem 2.3 is the following.

Problem 3.1. *Find* $w^{hk} = \{w_n^{hk}\}_{n=0}^N \subset V^h$ *such that for* $1 \le n \le N$,

$$\left\langle \frac{w_n^{hk} - w_{n-1}^{hk}}{k} + A(w_n^{hk}) + \mathcal{S}_n^k w^{hk}, v^h - w_n^{hk} \right\rangle + j^0(\gamma_j w_n^{hk}; \gamma_j v^h - \gamma_j w_n^{hk})$$
$$+ \varphi(\gamma_\varphi v^h) - \varphi(\gamma_\varphi w_n^{hk}) \ge \langle f_n, v^h - w_n^{hk} \rangle \quad \forall\, v^h \in V^h, \tag{3.4}$$

and

$$w_0^{hk} = w_0^h, \tag{3.5}$$

where $w_0^h \in V^h$ *is an approximation of* w_0, $w_0^h \to w_0$ *as* $h \to 0$.

The focus of this section is to derive an error bound for the numerical solution defined by Problem 3.1. As a first step, we show a boundedness result for the numerical solutions. Note that for $\mathcal{S}_n^k$ defined by (3.2), there is a constant $c > 0$ such that

$$\|\mathcal{S}_n^k w^{hk}\|_{V^*} \le c \left(k \sum_{i=0}^{n-1} \|w_i^{hk}\|_V + 1 \right). \tag{3.6}$$

As a preparation, we need a property of the function φ satisfying (2.8).

Lemma 3.2. *Assume* $\varphi: V_\varphi \to \mathbb{R}$ *satisfies* (2.8). *Then*

$$\varphi(z_1) - \varphi(z_2) \le \left(c_{0\varphi} + c_{1\varphi} \|z_1\|_{V_\varphi} \right) \|z_1 - z_2\|_{V_\varphi} \quad \forall\, z_1, z_2 \in V_\varphi. \tag{3.7}$$

Proof. By definition of the convex subdifferential, for any $z_1, z_2 \in V_\varphi$,

$$\varphi(z_2) - \varphi(z_1) \ge \langle \xi, z_2 - z_1 \rangle_{V_\varphi^* \times V_\varphi} \quad \forall\, \xi \in \partial_c \varphi(z_1).$$

Thus,

$$\varphi(z_1) - \varphi(z_2) \le \langle \xi, z_1 - z_2 \rangle_{V_\varphi^* \times V_\varphi} \le \|\xi\|_{V_\varphi^*} \|z_1 - z_2\|_{V_\varphi}.$$

From (2.8)(b),

$$\|\xi\|_{V_\varphi^*} \le c_{0\varphi} + c_{1\varphi} \|z_1\|_{V_\varphi}.$$

Hence, (3.7) holds. ∎

On several occasions, we will use the modified Cauchy–Schwarz inequality

$$a\, b \le \epsilon\, a^2 + c\, b^2 \quad \forall\, a, b \in \mathbb{R}, \tag{3.8}$$

where $\epsilon > 0$ is an arbitrary positive number and the constant $c > 0$ depends on ϵ, indeed, we may simply take $c = 1/(4\epsilon)$. We will also use the following form of a discrete Gronwall inequality (e.g. [21, Section 7.4]).

Lemma 3.3. *For a fixed* T *and a positive integer* N, *let* $k = T/N$. *Assume* $\{g_n\}_{n=0}^N$ *and* $\{e_n\}_{n=0}^N$ *are two sequences of non-negative numbers satisfying*

$$e_n \le c\, g_n + c\, k \sum_{i=1}^{n-1} e_i, \quad n = 0, \ldots, N$$

for a constant $c > 0$. *Then, for another constant* $c > 0$ *independent of* N,

$$\max_{0 \le n \le N} e_n \le c \max_{0 \le n \le N} g_n. \tag{3.9}$$

Proposition 3.4. *Assume* (2.6)–(2.11) *and* (3.1)–(3.3). *Then there is a constant* $c > 0$ *such that*

$$\max_{0 \le n \le N} \|w_n^{hk}\|_H^2 + \sum_{n=1}^N \|w_n^{hk} - w_{n-1}^{hk}\|_H^2 + k \sum_{n=1}^N \|w_n^{hk}\|^2 \le c. \tag{3.10}$$

Proof. From the condition (2.6)(b), we obtain the inequality

$$\langle Av, v \rangle \ge m_A \|v\|_V^2 + \langle A0, v \rangle \ge m_A \|v\|_V^2 - a_0 \|v\|_V \quad \forall\, v \in V.$$

From the conditions (2.7)(b) and (2.7)(c), we obtain the inequality

$$j^0(z;-z)\le \alpha_j\|z\|_{V_j}^2+c\,\|z\|_{V_j}\quad \forall\, z\in V_j.$$

We take $v^h=0$ in (3.4),

$$\left\langle \frac{w_n^{hk}-w_{n-1}^{hk}}{k}+A(w_n^{hk})+\mathcal{S}_n^k w^{hk},\, w_n^{hk}\right\rangle \le j^0(\gamma_j w_n^{hk};-\gamma_j w_n^{hk})+\varphi(0)-\varphi(\gamma_\varphi w_n^{hk})+\langle f_n, w_n^{hk}\rangle.$$

Notice that

$$\left\langle w_n^{hk}-w_{n-1}^{hk}, w_n^{hk}\right\rangle=\frac{1}{2}\left(\|w_n^{hk}\|_H^2-\|w_{n-1}^{hk}\|_H^2+\|w_n^{hk}-w_{n-1}^{hk}\|_H^2\right).$$

So we have the inequality

$$\begin{aligned}\frac{1}{2\,k}&\left(\|w_n^{hk}\|_H^2-\|w_{n-1}^{hk}\|_H^2+\|w_n^{hk}-w_{n-1}^{hk}\|_H^2\right)+m_A\|w_n^{hk}\|_V^2-a\,\|w_n^{hk}\|_V\\ &\le \alpha_j\|\gamma_j w_n^{hk}\|_{V_\varphi}^2+c\,\|\gamma_j w_n^{hk}\|_{V_\varphi}+\varphi(0)-\varphi(\gamma_\varphi w_n^{hk})+\|f_n\|_{V^*}\|w_n^{hk}\|_V\\ &\quad-\left\langle \mathcal{S}_n^k w^{hk}, w_n^{hk}\right\rangle.\end{aligned}\tag{3.11}$$

Apply (3.7) to get

$$\varphi(0)-\varphi(\gamma_\varphi w_n^{hk})\le c_{0\varphi}\|\gamma_\varphi w_n^{hk}\|_{V_\varphi}.\tag{3.12}$$

By (3.6), we have the bound

$$\left|\left\langle \mathcal{S}_n^k w^{hk}, w_n^{hk}\right\rangle\right|\le \|\mathcal{S}_n^k w^{hk}\|_{V^*}\|w_n^{hk}\|_V\le c\left(k\sum_{i=0}^{n-1}\|w_i^{hk}\|_V+1\right)\|w_n^{hk}\|_V.$$

Then from (3.11) we get

$$\begin{aligned}\frac{1}{2\,k}&\left(\|w_n^{hk}\|_H^2-\|w_{n-1}^{hk}\|_H^2+\|w_n^{hk}-w_{n-1}^{hk}\|_H^2\right)+m_A\|w_n^{hk}\|_V^2\\ &\le \alpha_j c_j^2\|w_n^{hk}\|_V^2+c\left(\|w_n^{hk}\|_V+1\right)+c\,\|w_n^{hk}\|_V k\sum_{i=0}^{n-1}\|w_i^{hk}\|_V.\end{aligned}\tag{3.13}$$

By (3.8), for any $\epsilon>0$, we have a constant c depending on ϵ such that

$$\frac{1}{2\,k}\left(\|w_n^{hk}\|_H^2-\|w_{n-1}^{hk}\|_H^2+\|w_n^{hk}-w_{n-1}^{hk}\|_H^2\right)+\left(m_A-\alpha_j c_j^2-\epsilon\right)\|w_n^{hk}\|_V^2\le c+c\,k\sum_{i=1}^{n-1}\|w_i^{hk}\|_V^2.$$

Here, c depends on $\max_n\|f_n\|_{V^*}$ and an upper bound of $\|w_0^h\|_V$, and as an intermediate step of the derivation, we used

$$\left(k\sum_{i=0}^{n-1}\|w_i^{hk}\|_V\right)^2\le k^2 n\sum_{i=0}^{n-1}\|w_i^{hk}\|_V^2\le c\,k\sum_{i=1}^{n-1}\|w_i^{hk}\|_V^2+c\,k\,\|w_0^h\|_V^2.$$

Since $m_A-\alpha_j c_j^2>0$, we can choose $\epsilon=(m_A-\alpha_j c_j^2)/2$ to obtain

$$\|w_n^{hk}\|_H^2-\|w_{n-1}^{hk}\|_H^2+\|w_n^{hk}-w_{n-1}^{hk}\|_H^2+\left(m_A-\alpha_j c_j^2\right)k\,\|w_n^{hk}\|_V^2\le c\,k+c\,k^2\sum_{i=1}^{n-1}\|w_i^{hk}\|_V^2.$$

Replace n by l in the above inequality and sum over l from 1 to n,

$$\begin{aligned}\|w_n^{hk}\|_H^2&+\sum_{l=1}^{n}\|w_l^{hk}-w_{l-1}^{hk}\|_H^2+\left(m_A-\alpha_j c_j^2\right)k\sum_{l=1}^{n}\|w_l^{hk}\|_V^2\\ &\le c+c\,k\sum_{l=1}^{n}k\sum_{i=1}^{l-1}\|w_i^{hk}\|_V^2.\end{aligned}\tag{3.14}$$

From (3.14), we have

$$k\sum_{l=1}^{n}\|w_l^{hk}\|_V^2\le c+c\,k\sum_{l=1}^{n}k\sum_{i=1}^{l-1}\|w_i^{hk}\|_V^2=c+c\,k\sum_{l=0}^{n-1}k\sum_{i=1}^{l}\|w_i^{hk}\|_V^2.$$

Then apply Lemma 3.3 to get

$$k\sum_{l=1}^{n}\|w_l^{hk}\|_V^2 \le c.$$

By (3.14) again,

$$\|w_n^{hk}\|_H^2+\sum_{l=1}^{n}\|w_l^{hk}-w_{l-1}^{hk}\|_H^2 \le c + c\,k\sum_{l=1}^{n}k\sum_{i=1}^{l-1}\|w_i^{hk}\|_V^2 \le c.$$

Thus, (3.10) holds. ■

We now turn to error analysis. For this purpose, we will assume the operator A is Lipschitz continuous,

$$\|Au-Av\|_{V^*}\le L_A\|u-v\|_V \quad \forall\, u,v\in V. \tag{3.15}$$

Also assume the smoothness

$$w\in H^1(0,T;V)\cap H^2(0,T;V^*),\quad q\in C^1([0,T]^2;\mathcal{L}(V)). \tag{3.16}$$

This implies in particular,

$$w\in C(0,T;V).$$

To simplify the notation, we denote the error

$$e_n = w_n - w_n^{hk},\quad 0\le n\le N. \tag{3.17}$$

Applying the condition (2.6)(c), we have

$$m_A\|e_n\|_V^2 \le \left\langle A(w_n)-A(w_n^{hk}),e_n\right\rangle.$$

Then for any $v_n^h\in V^h$,

$$\begin{aligned} m_A\|e_n\|_V^2 \le{}& \left\langle A(w_n)-A(w_n^{hk}),w_n-v_n^h\right\rangle+\left\langle A(w_n),v_n^h-w_n\right\rangle\\ &+\ \langle A(w_n),e_n\rangle+\left\langle A(w_n^{hk}),w_n^{hk}-v_n^h\right\rangle. \end{aligned}\tag{3.18}$$

By (2.4) at $t=t_n$ with $v=w_n^{hk}$,

$$\langle A(w_n),e_n\rangle \le \langle f_n-\dot w_n-\mathcal{S}_n w,e_n\rangle + j^0(\gamma_j w_n;-\gamma_j e_n)+\varphi(\gamma_\varphi w_n^{hk})-\varphi(\gamma_\varphi w_n). \tag{3.19}$$

By (3.4),

$$\begin{aligned} \left\langle A(w_n^{hk}),w_n^{hk}-v_n^h\right\rangle \le{}& \left\langle \frac{w_n^{hk}-w_{n-1}^{hk}}{k}+\mathcal{S}_n^k w^{hk}-f_n,\,v_n^h-w_n^{hk}\right\rangle\\ &+j^0(\gamma_j w_n^{hk};\gamma_j v_n^h-\gamma_j w_n^{hk})+\varphi(\gamma_\varphi v_n^h)-\varphi(\gamma_\varphi w_n^{hk}). \end{aligned}\tag{3.20}$$

Use (3.19) and (3.20) in (3.18) to obtain

$$\begin{aligned} m_A\|e_n\|_V^2 \le{}& -\frac{1}{k}\langle e_n-e_{n-1},e_n\rangle+\left\langle A(w_n)-A(w_n^{hk}),w_n-v_n^h\right\rangle+\frac{1}{k}\langle e_n-e_{n-1},w_n-v_n^h\rangle\\ &+\ R_n(v_n^h)+I_1+I_2+I_3, \end{aligned}\tag{3.21}$$

where

$$R_n(v)=\langle \dot w_n + A(w_n)+\mathcal{S}_n w-f_n, v-w_n\rangle + j^0(\gamma_j w_n;\gamma_j v-\gamma_j w_n)+\varphi(\gamma_\varphi v)-\varphi(\gamma_\varphi w_n) \tag{3.22}$$

is a residual type term, and

$$I_1=\langle E_n, w_n^{hk}-v_n^h\rangle, \tag{3.23}$$

$$I_2=\left\langle \mathcal{S}_n w-\mathcal{S}_n^k w^{hk}, w_n^{hk}-v_n^h\right\rangle, \tag{3.24}$$

$$I_3=j^0(\gamma_j w_n;-\gamma_j e_n)+j^0(\gamma_j w_n^{hk};\gamma_j v_n^h-\gamma_j w_n^{hk})-j^0(\gamma_j w_n;\gamma_j v_n^h-\gamma_j w_n), \tag{3.25}$$

with

$$E_n=\dot w_n-\frac{w_n-w_{n-1}}{k},\quad 1\le n\le N. \tag{3.26}$$

Let us bound the terms on the right side of (3.21). Let $\epsilon>0$ be an arbitrary small number to be chosen. In the following, the constant $c>0$ may depend on ϵ. First,

$$\langle e_n-e_{n-1},e_n\rangle=\frac12\left(\|e_n\|_H^2-\|e_{n-1}\|_H^2+\|e_n-e_{n-1}\|_H^2\right)\ge\frac12\left(\|e_n\|_H^2-\|e_{n-1}\|_H^2\right).$$

Thus,

$$-\frac{1}{k}\,\langle e_n - e_{n-1}, e_n\rangle \le -\frac{1}{2\,k}\left(\|e_n\|_H^2 - \|e_{n-1}\|_H^2\right). \tag{3.27}$$

By the Lipschitz continuity of A (cf. (3.15)),

$$\left\langle A(w_n) - A(w_n^{hk}),\, w_n - v_n^h\right\rangle \le L_A \|e_n\|_V \|w_n - v_n^h\|_V.$$

Applying (3.8),

$$\left\langle A(w_n) - A(w_n^{hk}),\, w_n - v_n^h\right\rangle \le \epsilon\, \|e_n\|_V^2 + c\, \|w_n - v_n^h\|_V^2. \tag{3.28}$$

For I_1, note that

$$\langle E_n, w_n^{hk} - v_n^h\rangle \le \|E_n\|_{V^*} \|w_n^{hk} - v_n^h\|_V \le \|E_n\|_{V^*}\left(\|e_n\|_V + \|w_n - v_n^h\|_V\right).$$

Then by (3.8),

$$\langle E_n, w_n^{hk} - v_n^h\rangle \le \epsilon\, \|e_n\|_V^2 + c\, \|E_n\|_{V^*}^2 + c\, \|w_n - v_n^h\|_V^2. \tag{3.29}$$

For I_2, we start with

$$\begin{aligned} |I_2| &\le \|\mathcal{S}_n w - \mathcal{S}_n^k w^{hk}\|_{V^*} \|w_n^{hk} - v_n^h\|_V \\ &\le \left(\|\mathcal{S}_n w - \mathcal{S}_n^k w\|_{V^*} + \|\mathcal{S}_n^k w - \mathcal{S}_n^k w^{hk}\|_{V^*}\right)\left(\|e_n\|_V + \|w_n - v_n^h\|_V\right). \end{aligned}$$

Now

$$\|\mathcal{S}_n w - \mathcal{S}_n^k w\|_{V^*} \le c\, \left\| \int_0^{t_n} q(t_n, s)\, w(s)\, ds - k \sum_{i=0}^{n-1} q(t_n, t_i)\, w_i \right\|_V.$$

From

$$\begin{aligned} \int_0^{t_n} q(t_n, s)\, w(s)\, ds - k \sum_{i=0}^{n-1} q(t_n, t_i)\, w_i &= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} [q(t_n, s)\, w(s) - q(t_n, t_i)\, w_i]\, ds \\ &= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \frac{d}{d\tau} [q(t_n, \tau)\, w(\tau)]\, d\tau\, ds, \end{aligned}$$

we find that, using the assumption (3.16) on q,

$$\|\mathcal{S}_n w - \mathcal{S}_n^k w\|_{V^*} \le c\, k \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} [\|w(s)\|_V + \|\dot{w}(s)\|_V]\, ds.$$

Then,

$$\|\mathcal{S}_n w - \mathcal{S}_n^k w\|_{V^*}^2 \le c\, k^2 \|w\|_{H^1(0,T;V)}^2,$$

i.e.,

$$\|\mathcal{S}_n w - \mathcal{S}_n^k w\|_{V^*} \le c\, k\, \|w\|_{H^1(0,T;V)}.$$

From the definition of $\mathcal{S}^k$, and assumptions on G and q, we have

$$\|\mathcal{S}_n^k w - \mathcal{S}_n^k w^{hk}\|_{V^*} \le c\, k \sum_{i=0}^{n-1} \|e_i\|_V.$$

Hence,

$$|I_2| \le \epsilon\, \|e_n\|_V^2 + c\, k \sum_{i=0}^{n-1} \|e_i\|_V^2 + c\, k^2 \|w\|_{H^1(0,T;V)}^2 + c\, \|w_n - v_n^h\|_V^2. \tag{3.30}$$

To bound I_3, we first apply the sub-additivity property (2.1),

$$j^0(\gamma_j w_n^{hk}; \gamma_j v_n^h - \gamma_j w_n^{hk}) \le j^0(\gamma_j w_n^{hk}; \gamma_j e_n) + j^0(\gamma_j w_n^{hk}; \gamma_j v_n^h - \gamma_j w_n),$$

and then apply the condition (3.5)(c),

$$j^0(\gamma_j w_n; -\gamma_j e_n) + j^0(\gamma_j w_n^{hk}; \gamma_j e_n) \le \alpha_j \|\gamma_j e_n\|_{V_j}^2.$$

Consequently,

$$I_3 \le \alpha_j \|\gamma_j e_n\|_{V_j}^2 + j^0(\gamma_j w_n^{hk}; \gamma_j v_n^h - \gamma_j w_n) - j^0(\gamma_j w_n; \gamma_j v_n^h - \gamma_j w_n).$$

By the condition (2.7)(b),

$$\left|j^0(\gamma_j w_n^{hk}; \gamma_j v_n^h - \gamma_j w_n)\right| \le \left(c_0 + c_1 \|\gamma_j w_n^{hk}\|_{V_j}\right) \|\gamma_j(w_n - v_n^h)\|_{V_j},$$
$$\left|j^0(\gamma_j w_n; \gamma_j v_n^h - \gamma_j w_n)\right| \le \left(c_0 + c_1 \|\gamma_j w_n\|_{V_j}\right) \|\gamma_j(w_n - v_n^h)\|_{V_j}.$$

Since $\|\gamma_j w_n\|_{V_j} \le c_j \|w\|_{C(0,T;V)}$ is uniformly bounded, we conclude that there is a constant c such that

$$I_3 \le \alpha_j c_j^2 \|e_n\|_{V_j}^2 + c\left(1 + \|\gamma_j w_n^{hk}\|_{V_j}\right) \|\gamma_j(w_n - v_n^h)\|_{V_j}. \tag{3.31}$$

By applying (3.27)–(3.31) in (3.21), we have

$$\begin{aligned}
&\frac{1}{2\,k}\left(\|e_n\|_H^2 - \|e_{n-1}\|_H^2\right) + m_A \|e_n\|_V^2 \\
&\quad \le \left(\alpha_j c_j^2 + 3\,\epsilon\right) \|e_n\|_V^2 + \frac{1}{k}\langle e_n - e_{n-1}, w_n - v_n^h\rangle + R_n(v_n^h) \\
&\qquad + c\,k \sum_{i=0}^{n-1} \|e_i\|_V^2 + c\,k^2 \|w\|_{H^1(0,T;V)}^2 + c\,\|w_n - v_n^h\|_V^2 + c\,\|E_n\|_{V^*}^2 \\
&\qquad + c\left(1 + \|\gamma_j w_n^{hk}\|_{V_j}\right) \|\gamma_j(w_n - v^h)\|_{V_j}.
\end{aligned} \tag{3.32}$$

Recalling the condition $\alpha_j c_j^2 < m_A$ from (2.10), by choosing $\epsilon > 0$ small enough, we derive from (3.32) the following inequality with any $v_n^h \in V^h$,

$$\begin{aligned}
\|e_n\|_H^2 - \|e_{n-1}\|_H^2 + k\,\|e_n\|_V^2 &\le c\,k\left(\|w_n - v_n^h\|_V^2 + |R_n(v_n^h)| + \|E_n\|_{V^*}^2\right) \\
&\quad + c\,k\left(1 + \|\gamma_j w_n^{hk}\|_{V_j}\right) \|\gamma_j(w_n - v_n^h)\|_{V_j} + c\left\langle e_n - e_{n-1}, w_n - v_n^h\right\rangle \\
&\quad + c\,k^3 \|w\|_{H^1(0,T;V)}^2 + c\,k^2 \sum_{i=0}^{n-1} \|e_i\|_V^2.
\end{aligned} \tag{3.33}$$

We replace n by l in (3.33) and make a summation over l from 1 to n,

$$\begin{aligned}
\|e_n\|_H^2 - \|e_0\|_H^2 + k \sum_{l=1}^{n} \|e_l\|_V^2 &\le c\,k \sum_{l=1}^{n} \left(\|w_l - v_l^h\|_V^2 + |R_l(v_l^h)| + \|E_l\|_{V^*}^2\right) \\
&\quad + c\,k \sum_{l=1}^{n} \left(1 + \|\gamma_j w_l^{hk}\|_{V_j}\right) \|\gamma_j(w_l - v_l^h)\|_{V_j} \\
&\quad + c \sum_{l=1}^{n} \left\langle e_l - e_{l-1}, w_l - v_l^h\right\rangle + c\,k \|e_0\|_V^2 \\
&\quad + c\,k^2 \|w\|_{H^1(0,T;V)}^2 + c\,k \sum_{l=0}^{n-1} k \sum_{i=1}^{l} \|e_i\|_V^2.
\end{aligned} \tag{3.34}$$

For the term E_n defined by (3.26), we can write

$$E_n = \frac{1}{k} \int_{t_{n-1}}^{t_n} (t - t_{n-1})\, \ddot{w}(t)\, dt.$$

Thus, we have the upper bound

$$\|E_n\|_{V^*}^2 \le \frac{1}{k^2} \int_{t_{n-1}}^{t_n} (t - t_{n-1})^2 dt \int_{t_{n-1}}^{t_n} \|\ddot{w}(t)\|_{V^*}^2 dt = \frac{k}{3} \int_{t_{n-1}}^{t_n} \|\ddot{w}(t)\|_{V^*}^2 dt.$$

Using this inequality, we have

$$k \sum_{l=1}^{n} \|E_l\|_{V^*}^2 \le \frac{k^2}{3} \|\ddot{w}\|_{L^2(0,T;V^*)}^2. \tag{3.35}$$

Now

$$\sum_{l=1}^{n}\left(1+\|\gamma_j w_l^{hk}\|_{V_j}\right)\|\gamma_j(w_l-v_l^h)\|_{V_j}$$
$$\leq\left[\sum_{l=1}^{n}\left(1+\|\gamma_j w_l^{hk}\|_{V_j}\right)^2\right]^{1/2}\left[\sum_{l=1}^{n}\|\gamma_j(w_l-v_l^h)\|_{V_j}^2\right]^{1/2}.$$

By Proposition 3.4, $k\sum_{l=1}^{n}\left(1+\|\gamma_j w_l^{hk}\|_{V_j}\right)^2$ is uniformly bounded. Thus,

$$k\sum_{l=1}^{n}\left(1+\|\gamma_j w_l^{hk}\|_{V_j}\right)\|\gamma_j(w_l-v_l^h)\|_{V_j}\leq c\left[k\sum_{l=1}^{n}\|\gamma_j(w_l-v_l^h)\|_{V_j}^2\right]^{1/2}. \tag{3.36}$$

Write

$$\sum_{l=1}^{n}\langle e_l-e_{l-1},w_l-v_l^h\rangle=\sum_{l=1}^{n}\langle e_l,w_l-v_l^h\rangle-\sum_{l=0}^{n-1}\langle e_l,w_{l+1}-v_{l+1}^h\rangle$$
$$=\langle e_n,w_n-v_n^h\rangle+\sum_{l=1}^{n-1}\langle e_l,(w_l-v_l^h)-(w_{l+1}-v_{l+1}^h)\rangle$$
$$-\ \langle e_0,w_1-v_1^h\rangle.$$

The terms on the right side can be bounded as follows. For the first term,

$$\left|\langle e_n,w_n-v_n^h\rangle\right|\leq\|e_n\|_H\|w_n-v_n^h\|_H\leq\frac{1}{2}\left(\|e_n\|_H^2+\|w_n-v_n^h\|_H^2\right).$$

For the second term,

$$\left|\langle e_l,(w_l-v_l^h)-(w_{l+1}-v_{l+1}^h)\rangle\right|$$
$$\leq k\,\|e_l\|_H\left\|\frac{(w_l-v_l^h)-(w_{l+1}-v_{l+1}^h)}{k}\right\|_H$$
$$\leq\frac{k}{2}\left(\|e_l\|_H^2+k^{-2}\left\|(w_l-v_l^h)-(w_{l+1}-v_{l+1}^h)\right\|_H^2\right).$$

For the last term,

$$\left|\langle e_0,w_1-v_1^h\rangle\right|\leq\|e_0\|_H\|w_1-v_1^h\|_H\leq\frac{1}{2}\left(\|e_0\|_H^2+\|w_1-v_1^h\|_H^2\right).$$

Hence, from (3.34),

$$\begin{aligned}\|e_n\|_H^2+k\sum_{l=1}^{n}\|e_l\|_V^2\leq{}& c\,k\sum_{l=1}^{n}\left(\|w_l-v_l^h\|_V^2+|R_l(v_l^h)|\right)+c\,k^2\|w\|_{H^2(0,T;V^*)}^2\\&+c\left[k\sum_{l=1}^{n}\|\gamma_j(w_l-v_l^h)\|_{V_j}^2\right]^{1/2}\\&+c\,k^{-1}\sum_{l=1}^{n-1}\left\|(w_l-v_l^h)-(w_{l+1}-v_{l+1}^h)\right\|_H^2\\&+c\left(\|e_0\|_H^2+k\,\|e_0\|_V^2+\|w_1-v_1^h\|_H^2+\|w_n-v_n^h\|_H^2\right)\\&+c\,k^2\|w\|_{H^1(0,T;V)}^2+c\,k\sum_{l=0}^{n-1}\left(\|e_l\|_H^2+k\sum_{i=1}^{l}\|e_i\|_V^2\right).\end{aligned} \tag{3.37}$$

Applying Lemma 3.3, we deduce from (3.37) that

$$\begin{aligned}\max_{1\leq n\leq N}\|e_n\|_H^2+k\sum_{n=1}^{N}\|e_n\|_V^2\leq{}& c\,k^2\left(\|w\|_{H^2(0,T;V^*)}^2+\|w\|_{H^1(0,T;V)}^2\right)\\&+c\left(\|e_0\|_H^2+k\,\|e_0\|_V^2\right)+c\max_{1\leq n\leq N}\tilde{E}_n,\end{aligned} \tag{3.38}$$

where

$$\tilde{E}_n = \inf_{v_l^h \in V^h,\, 1\le l\le n} \Bigg\{ k\sum_{l=1}^{n}\left(\|w_l - v_l^h\|_V^2 + |R_l(v_l^h)|\right) + \Big[k\sum_{l=1}^{n}\|\gamma_j(w_l - v_l^h)\|_{V_j}^2\Big]^{1/2} \\ + k^{-1}\sum_{l=1}^{n-1}\left\|(w_l - v_l^h) - (w_{l+1} - v_{l+1}^h)\right\|_H^2 \\ + \|w_1 - v_1^h\|_H^2 + \|w_n - v_n^h\|_H^2 \Bigg\}, \tag{3.39}$$

in which R_l is defined by (3.22).

We summarize the above result in the following theorem.

Theorem 3.5. *Assume* (2.6)–(2.11), (3.1)–(3.3), (3.15) and (3.16). *Then we have the inequality*

$$\max_{1\le n\le N}\|w_n - w_n^{hk}\|_H^2 + k\sum_{n=1}^{N}\|w_n - w_n^{hk}\|_V^2 \\ \le c\,k^2\left(\|w\|_{H^2(0,T;V^*)}^2 + \|w\|_{H^1(0,T;V)}^2\right) \\ + c\left(\|w_0 - w_0^h\|_H^2 + k\,\|w_0 - w_0^h\|_V^2\right) + c\max_{1\le n\le N}\tilde{E}_n, \tag{3.40}$$

where $\tilde{E}_n$ is defined by (3.39).

4. Application to a dynamic frictional contact problem

We consider the application of the results from the previous sections on a dynamic contact problem over a time interval $[0, T]$. The configuration of the material before the contact Ω is an open bounded Lipschitz domain in $\mathbb{R}^d$, with $d = 2$ or 3 for applications. The main unknown of the problem is the displacement field $\boldsymbol{u}: \Omega \times [0, T] \to \mathbb{R}^d$. Denote by $\boldsymbol{\sigma}: \Omega \times [0, T] \to \mathbb{S}^d$ the stress field. The relations of the classical formulation of the problem are described as follows.

We assume the material is viscoelastic with short memory. The constitutive equation is

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \quad \text{in } \Omega, \tag{4.1}$$

where $\mathcal{A}$ represents the viscosity operator and $\mathcal{B}$ represents the elasticity operator. In the linear case, the constitutive law (4.1) becomes the well-known Kelvin–Voigt law

$$\sigma_{ij} = a_{ijkl}\varepsilon_{kl}(\dot{\boldsymbol{u}}) + b_{ijkl}\varepsilon_{kl}(\boldsymbol{u}), \tag{4.2}$$

$\{a_{ijkl}\}$ being the components of the viscosity tensor $\mathcal{A}$, and $\{b_{ijkl}\}$ the components of the elasticity tensor $\mathcal{B}$. Analysis and numerical approximations of quasistatic contact problems for viscoelastic materials of the form (4.1) can be found in [21].

The motion equation is

$$\rho\,\ddot{\boldsymbol{u}}(t) = \operatorname{Div}\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) \quad \text{in } \Omega, \tag{4.3}$$

where ρ is the mass density of mass, assumed to be constant. Without loss of generality, in what follows, we set $\rho \equiv 1$.

The boundary $\Gamma = \partial\Omega$ of the domain Ω is decomposed to three measurable subsets Γ_1, Γ_2 and Γ_3, with meas$(\Gamma_1) > 0$, meas$(\Gamma_3) > 0$. Since Γ is Lipschitz continuous, the unit outward normal vector $\boldsymbol{\nu}$ exists a.e. on Γ. For a vector $\boldsymbol{v}$ defined on Γ, its normal component is $v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}$ and its tangential component is $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}$. For a tensor $\boldsymbol{\sigma}$ defined on Γ, its normal component is $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$ and its tangential component is $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$. On Γ_1, we impose the clamped boundary condition,

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on } \Gamma_1. \tag{4.4}$$

On Γ_2, we assume the surface traction boundary condition

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on } \Gamma_2. \tag{4.5}$$

The set Γ_2 is allowed to be empty, and if this is the case, then the boundary condition (4.5) is voided. On Γ_3, we apply a contact condition with normal damped response, in which ∂j_ν denotes the Clarke subdifferential of a given function j_ν:

$$-\sigma_\nu(t) \in \partial j_\nu(\dot{u}_\nu(t)) \quad \text{on } \Gamma_3, \tag{4.6}$$

$$\|\boldsymbol{\sigma}_\tau(t)\| \le F_b, \quad -\boldsymbol{\sigma}_\tau(t) = F_b\frac{\dot{\boldsymbol{u}}_\tau(t)}{\|\dot{\boldsymbol{u}}_\tau(t)\|} \text{ if } \dot{\boldsymbol{u}}_\tau(t) \ne \boldsymbol{0} \quad \text{on } \Gamma_3. \tag{4.7}$$

Such a contact condition is used to model the situation where the foundation is covered with a thin lubricant layer such as oil. Condition (4.7) represents the Tresca friction law with the friction bound F_b, assumed to be independent of the process variables.

Finally, with given vectors $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$, we impose the initial conditions

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{w}_0 \quad \text{in } \Omega. \tag{4.8}$$

We will study the contact problem through its weak formulation. For this purpose, we will make various assumptions on the problem data. Let the constants $L_{\mathcal{A}}, m_{\mathcal{A}}, L_{\mathcal{B}} > 0$, and $\bar{c}_0, \alpha_{j_\nu} \geq 0$. For the viscosity operator $\mathcal{A}\colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d$, we assume

$$\begin{cases} \text{(a) } \|\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega; \\ \text{(b) } (\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega; \\ \text{(c) } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega,\ \forall\, \boldsymbol{\varepsilon} \in \mathbb{S}^d. \end{cases} \tag{4.9}$$

For the elasticity operator $\mathcal{B}\colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d$, we assume

$$\begin{cases} \text{(a) } \|\mathcal{B}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{B}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{B}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega; \\ \text{(b) } \boldsymbol{x} \mapsto \mathcal{B}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega,\ \forall\, \boldsymbol{\varepsilon} \in \mathbb{S}^d. \end{cases} \tag{4.10}$$

For the potential function $j_\nu\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$, we assume

$$\begin{cases} \text{(a) } j_\nu(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R} \text{ and there} \\ \quad \text{exists } \bar{e} \in L^2(\Gamma_3) \text{ such that } j_\nu(\cdot, \bar{e}(\cdot)) \in L^1(\Gamma_3); \\ \text{(b) } j_\nu(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_3; \\ \text{(c) } |\partial j_\nu(\boldsymbol{x}, r)| \leq \bar{c}_0 \ \forall\, r \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3; \\ \text{(d) } j_\nu^0(\boldsymbol{x}, r_1; r_2 - r_1) + j_\nu^0(\boldsymbol{x}, r_2; r_1 - r_2) \leq \alpha_{j_\nu} |r_1 - r_2|^2 \ \forall\, r_1, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{4.11}$$

For the friction bound F_b, we assume

$$F_b \in L^2(\Gamma_3), \quad F_b(\boldsymbol{x}) \geq 0 \quad \text{a.e. on } \Gamma_3. \tag{4.12}$$

For the densities of body forces and surface tractions, we assume

$$\boldsymbol{f}_0 \in L^2(0, T; L^2(\Omega)^d), \quad \boldsymbol{f}_2 \in L^2(0, T; L^2(\Gamma_2)^d), \tag{4.13}$$

and for the initial data,

$$\boldsymbol{u}_0 \in V, \quad \boldsymbol{w}_0 \in V. \tag{4.14}$$

The spaces V and H are defined by

$$V = \left\{ \boldsymbol{v} \in H^1(\Omega)^d \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1 \right\}, \quad H = L^2(\Omega)^d. \tag{4.15}$$

Since meas$(\Gamma_1) > 0$, Korn's inequality holds ([22, p. 79]):

$$\|\boldsymbol{v}\|_{H^1(\Omega)^d} \leq c\, \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{L^2(\Omega)^{d\times d}} \quad \forall\, \boldsymbol{v} \in V.$$

This allows us to use the norm $\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{L^2(\Omega)^{d\times d}}$ over the space V, and this norm is equivalent to the standard norm $\|\boldsymbol{v}\|_{H^1(\Omega)^d}$.

By a standard derivation [21], we can derive the following weak formulation of the contact problem.

Problem 4.1. *Find a displacement field* $\boldsymbol{u}\colon [0, T] \to V$ *such that for all* $t \in [0, T]$,

$$\begin{aligned} &\int_\Omega \ddot{\boldsymbol{u}}(t) \cdot (\boldsymbol{v} - \dot{\boldsymbol{u}}(t))\, dx + \int_\Omega (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))\, dx \\ &\quad + \int_{\Gamma_3} F_b\, (\|\boldsymbol{v}_\tau\| - \|\dot{\boldsymbol{u}}_\tau(t)\|)\, da + \int_{\Gamma_3} j_\nu^0(\dot{u}_\nu(t); v_\nu - \dot{u}_\nu(t))\, da \\ &\quad\quad \geq \int_\Omega \boldsymbol{f}_0(t) \cdot (\boldsymbol{v} - \dot{\boldsymbol{u}}(t))\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot (\boldsymbol{v} - \dot{\boldsymbol{u}}(t))\, da \quad \forall\, \boldsymbol{v} \in V, \end{aligned} \tag{4.16}$$

and

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{w}_0. \tag{4.17}$$

To fit in the framework of Section 2, we use the spaces $V_\varphi = L^2(\Gamma_3)^d$ and $V_j = L^2(\Gamma_3)$, and let $\gamma_\varphi \in \mathcal{L}(V, V_\varphi)$ be the tangential trace operator and $\gamma_j \in \mathcal{L}(V, V_j)$ the normal trace operator, i.e., for $\boldsymbol{v} \in V$, $\gamma_\varphi(\boldsymbol{v}) = \boldsymbol{v}_\tau$ and $\gamma_j(\boldsymbol{v}) = v_\nu$. Denote by $\lambda_{1\nu} > 0$ the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in V, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx = \lambda \int_{\Gamma_3} u_\nu v_\nu da \quad \forall \boldsymbol{v} \in V.$$

We can take $c_j = \lambda_{1\nu}^{-1/2}$ for the constant c_j in (2.3).

Define operators $A: V \to V^*$ and $\mathcal{S}: \mathcal{V} \to \mathcal{V}^*$ by

$$\langle A\boldsymbol{w}, \boldsymbol{v}\rangle = \int_\Omega \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx, \quad \boldsymbol{w}, \boldsymbol{v} \in V, \tag{4.18}$$

$$\langle \mathcal{S}\boldsymbol{w}(t), \boldsymbol{v}\rangle = \int_\Omega \mathcal{B}\Big(\int_0^t \boldsymbol{\varepsilon}(\boldsymbol{w}(s))\,ds + \boldsymbol{u}_0\Big) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx, \quad \boldsymbol{w} \in \mathcal{V},\ \boldsymbol{v} \in V,\ \text{a.e. } t \in (0,T). \tag{4.19}$$

Moreover, define functions $j: V_j \to \mathbb{R}$, $\varphi: V_\varphi \to \mathbb{R}$ and $\boldsymbol{f}: (0,T) \to V^*$ by

$$j(z) = \int_{\Gamma_3} j_\nu(z)\,da, \quad z \in V_j, \tag{4.20}$$

$$\varphi(\boldsymbol{z}) = \int_{\Gamma_3} F_b\,\|\boldsymbol{z}\|\,da, \quad \boldsymbol{z} \in V_\varphi, \tag{4.21}$$

$$\langle \boldsymbol{f}(t), \boldsymbol{v}\rangle = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\,dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\,da, \quad \boldsymbol{v} \in V,\ \text{a.e. } t \in (0,T). \tag{4.22}$$

In terms of the velocity $\boldsymbol{w} = \dot{\boldsymbol{u}}$, we introduce an auxiliary problem.

Problem 4.2. *Find $\boldsymbol{w} \in \mathcal{W}$ such that for a.e. $t \in (0,T)$,*

$$\langle \dot{\boldsymbol{w}}(t) + A\boldsymbol{w}(t) + \mathcal{S}\boldsymbol{w}(t), \boldsymbol{v} - \boldsymbol{w}(t)\rangle + j^0(w_\nu(t); v_\nu - w_\nu(t)) + \varphi(\boldsymbol{v}_\tau) - \varphi(\boldsymbol{w}_\tau(t)) \\ \geq \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{w}(t)\rangle \quad \forall \boldsymbol{v} \in V, \tag{4.23}$$

and

$$\boldsymbol{w}(0) = \boldsymbol{w}_0. \tag{4.24}$$

We apply Theorem 2.4 to Problem 4.2. For this purpose, we examine the conditions (2.6)–(2.11). From (4.9)(a), (c), it is easy to know that A is Lipschitz continuous and

$$\|A(\boldsymbol{u}_1) - A(\boldsymbol{u}_2)\|_{V^*} \leq L_{\mathcal{A}}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \quad \forall \boldsymbol{u}_1, \boldsymbol{u}_2 \in V. \tag{4.25}$$

In particular, A is demicontinuous and

$$\|A\boldsymbol{u}\|_{V^*} \leq \|A\boldsymbol{0}\|_{V^*} + L_{\mathcal{A}}\|\boldsymbol{u}\|_V \quad \forall \boldsymbol{u} \in V,$$

i.e. (2.6)(c) holds with $a_0 = \|A\boldsymbol{0}\|_{V^*}$ and $a_1 = L_{\mathcal{A}}$. Similarly, (4.9)(b) implies (2.6)(c) with $m_A = m_{\mathcal{A}}$. The assumption (4.11) implies that the function j defined by (4.20) satisfies the condition (2.7) with $c_{1j} = 0$. The assumption (4.12) ensures that the function φ defined by (4.21) satisfies the condition (2.8) with $c_{1\varphi} = 0$. The assumption (4.10) implies that the operator $\mathcal{S}$ defined by (4.19) satisfies the condition (2.9) with $c_{\mathcal{S}} = L_{\mathcal{B}}$. Indeed, for $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \mathcal{V}$, $\boldsymbol{v} \in V$ and $t \in (0,T)$, it is easy to see that

$$|\langle \mathcal{S}\boldsymbol{w}_1(t) - \mathcal{S}\boldsymbol{w}_2(t), \boldsymbol{v}\rangle| \leq L_{\mathcal{B}} \int_0^t \|\boldsymbol{w}_1(s) - \boldsymbol{w}_2(s)\|_V\,ds\,\|\boldsymbol{v}\|_V.$$

The condition (2.10) follows from (4.26), and the condition (2.11) is a consequence of assumptions (4.13) and (4.14) and definition (4.22). Hence, all the conditions in Theorem 2.4 are satisfied and we conclude that Problem 4.2 has a unique solution $\boldsymbol{w} \in \mathcal{W}$.

The unique solvability of Problem 4.1 is given by the following result.

Theorem 4.3. *Assume (4.9)–(4.14) and*

$$\alpha_{j_\nu}\lambda_{1\nu}^{-1} < m_{\mathcal{A}}. \tag{4.26}$$

Then Problem 4.1 has a unique solution with regularity

$$\boldsymbol{u} \in H^1(0,T;V), \quad \dot{\boldsymbol{u}} \in \mathcal{W}, \quad \ddot{\boldsymbol{u}} \in \mathcal{V}^*. \tag{4.27}$$

Proof. Define the displacement function $\boldsymbol{u}: [0,T] \to V$ by

$$\boldsymbol{u}(t) = \int_0^t \boldsymbol{w}(s)\,ds + \boldsymbol{u}_0, \quad t \in [0,T]. \tag{4.28}$$

Then from Problem 4.2 for $\boldsymbol{w}$ and the inequality

$$j^0(v_\nu; w_\nu) \le \int_{\Gamma_3} j_\nu^0(v_\nu; w_\nu)\, da \quad \forall\, \boldsymbol{v}, \boldsymbol{w} \in V,$$

it follows that $\boldsymbol{u}$ is a solution to Problem 4.1. It is easy to see that the regularity (4.27) holds.

The uniqueness of a solution $\boldsymbol{u}$ is shown by a standard argument (cf. [6,12]), and is hence omitted. □

We now proceed with the discretization of Problem 4.1 using the finite element method for the spatial variable. For simplicity, assume Ω is a polygonal/polyhedral domain and express the three parts of the boundary, Γ_k, $1 \le k \le 3$, as unions of closed flat components with disjoint interiors:

$$\overline{\Gamma_k} = \cup_{i=1}^{i_k} \Gamma_{k,i}, \quad 1 \le k \le 3.$$

Let $\{\mathcal{T}^h\}$ be a regular family of partitions of $\overline{\Omega}$ into triangles/tetrahedrons that are compatible with the partition of the boundary $\partial\Omega$ into $\Gamma_{k,i}$, $1 \le i \le i_k$, $1 \le k \le 3$, in the sense that if the intersection of one side/face of an element with one set $\Gamma_{k,i}$ has a positive measure with respect to $\Gamma_{k,i}$, then the side/face lies entirely in $\Gamma_{k,i}$. Then construct a linear element space corresponding to $\mathcal{T}^h$,

$$V^h = \left\{ \boldsymbol{v}^h \in C(\overline{\Omega})^d \mid \boldsymbol{v}^h|_T \in \mathbb{P}_1(T)^d \text{ for } T \in \mathcal{T}^h,\ \boldsymbol{v}^h = \boldsymbol{0} \text{ on } \Gamma_1 \right\}. \tag{4.29}$$

Similar to (3.3), instead of (4.13), we assume

$$\boldsymbol{f}_0 \in C(0,T; L^2(\Omega;\mathbb{R}^d)), \quad \boldsymbol{f}_2 \in C(0,T; L^2(\Gamma_2;\mathbb{R}^d)). \tag{4.30}$$

Let $\boldsymbol{w}_0^h \in V^h$ be an approximation of the initial value $\boldsymbol{w}_0$. We use $(\cdot,\cdot)$ to denote the standard inner product in $L^2(\Omega)^d$. Consider the following fully discrete scheme for solving Problem 4.2.

Problem 4.4. Find $\boldsymbol{w}^{hk} = \left\{\boldsymbol{w}_n^{hk}\right\}_{n=0}^N \subset V^h$ such that for $1 \le n \le N$,

$$\left(\frac{\boldsymbol{w}_n^{hk} - \boldsymbol{w}_{n-1}^{hk}}{k}, \boldsymbol{v}^h - \boldsymbol{w}_n^{hk} \right) + \langle A\boldsymbol{w}_n^{hk} + \mathcal{S}_n^k \boldsymbol{w}^{hk}, \boldsymbol{v}^h - \boldsymbol{w}_n^{hk} \rangle + \int_{\Gamma_3} j_\nu^0(w_{n,\nu}^{hk}; v_\nu^h - w_{n,\nu}^{hk})\, da$$
$$+ \varphi(\boldsymbol{v}_\tau^h) - \varphi(\boldsymbol{w}_{n,\tau}^{hk}) \ge \langle \boldsymbol{f}_n, \boldsymbol{v}^h - \boldsymbol{w}_n^{hk} \rangle \quad \forall\, \boldsymbol{v}^h \in V^h, \tag{4.31}$$

and

$$\boldsymbol{w}_0^{hk} = \boldsymbol{w}_0^h. \tag{4.32}$$

Note that the Lipschitz condition (3.15) follows from (4.25). Corresponding to we assume the solution regularity

$$\boldsymbol{w} \in H^1(0,T;V) \cap H^2(0,T;V^*). \tag{4.33}$$

It is easy to observe that the derivation of the error bound (3.40) can be adapted straightforward for the numerical solution defined by Problem 4.4, and we have

$$\max_{1\le n\le N} \|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_H^2 + k \sum_{n=1}^N \|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_V^2$$
$$\le c\, k^2 \left(\|\boldsymbol{w}\|_{H^2(0,T;V^*)}^2 + \|\boldsymbol{w}\|_{H^1(0,T;V)}^2 \right)$$
$$+ c\left(\|\boldsymbol{w}_0 - \boldsymbol{w}_0^h\|_H^2 + k\, \|\boldsymbol{w}_0 - \boldsymbol{w}_0^h\|_V^2 \right) + c \max_{1\le n\le N} \tilde{E}_n, \tag{4.34}$$

where

$$\tilde{E}_n = \inf_{\boldsymbol{v}_l^h \in V^h,\, 1\le l\le n} \Bigg\{ k \sum_{l=1}^n \left(\|\boldsymbol{w}_l - \boldsymbol{v}_l^h\|_V^2 + |R_l(\boldsymbol{v}_l^h)| \right) + \left[k \sum_{l=1}^n \|w_{l,\nu} - v_{l,\nu}^h\|_{V_j}^2 \right]^{1/2}$$
$$+ k^{-1} \sum_{l=1}^{n-1} \left\| (\boldsymbol{w}_l - \boldsymbol{v}_l^h) - (\boldsymbol{w}_{l+1} - \boldsymbol{v}_{l+1}^h) \right\|_H^2$$
$$+ \|\boldsymbol{w}_1 - \boldsymbol{v}_1^h\|_H^2 + \|\boldsymbol{w}_n - \boldsymbol{v}_n^h\|_H^2 \Bigg\} \tag{4.35}$$

in which,

$$R_n(\boldsymbol{v}) = \langle \dot{\boldsymbol{w}}_n + A(\boldsymbol{w}_n) + \mathcal{S}_n \boldsymbol{w} - \boldsymbol{f}_n, \boldsymbol{v} - \boldsymbol{w}_n \rangle$$
$$+ \int_{\Gamma_3} j_\nu^0(w_{n,\nu}; v_\nu - w_{n,\nu})\, da + \varphi(\boldsymbol{v}_\tau) - \varphi(\boldsymbol{w}_{n,\tau}). \tag{4.36}$$

For error estimation, we further assume the solution regularity

$$\boldsymbol{u} \in C^1(0,T;H^2(\Omega)^d), \quad \dot{\boldsymbol{u}}|_{\Gamma_{3,i}} \in C(0,T;H^2(\Gamma_{3,i})^d),\ 1 \le i \le i_3, \quad \boldsymbol{\sigma\nu} \in C(0,T;L^2(\Gamma)^d). \tag{4.37}$$

Recall that $\boldsymbol{\sigma}$ is defined by (4.1). Observe that (4.33) implies

$$\boldsymbol{u} \in H^2(0,T;V) \cap H^3(0,T;V^*).$$

We will first bound the term $R_n(\boldsymbol{v})$ defined by (4.36). Introduce a subspace of V,

$$\tilde{V} := \left\{ \boldsymbol{w} \in C^\infty(\overline{\Omega};\mathbb{R}^d) \mid \boldsymbol{w} = \boldsymbol{0} \text{ on } \Gamma_1 \cup \Gamma_3 \right\}. \tag{4.38}$$

Take $\boldsymbol{v} \in \tilde{V}$ in the inequality (4.23) to obtain

$$\langle \dot{\boldsymbol{w}}(t), \boldsymbol{v} \rangle + \int_\Omega \boldsymbol{\sigma}(t)\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx = \int_\Omega \boldsymbol{f}_0(t)\cdot\boldsymbol{v}\,dx + \int_{\Gamma_2} \boldsymbol{f}_2(t)\cdot\boldsymbol{v}\,da \quad \forall\,\boldsymbol{v} \in \tilde{V}. \tag{4.39}$$

We derive from (4.39) that for a.e. $t \in (0,T)$,

$$\dot{\boldsymbol{w}}(t) - \operatorname{Div}\boldsymbol{\sigma}(t) = \boldsymbol{f}_0(t) \quad \text{a.e. in } \Omega \tag{4.40}$$

and

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{a.e. on } \Gamma_2. \tag{4.41}$$

Now we multiply Eq. (4.40) by an arbitrary function $\boldsymbol{v} \in V$ and integrate over Ω:

$$\langle \dot{\boldsymbol{w}}(t), \boldsymbol{v} \rangle - \int_\Omega \operatorname{Div}\boldsymbol{\sigma}(t)\cdot\boldsymbol{v}\,dx = \int_\Omega \boldsymbol{f}_0(t)\cdot\boldsymbol{v}\,dx.$$

Perform an integration by parts on the second integral and use (4.41) to get

$$\begin{aligned} &\langle \dot{\boldsymbol{w}}(t), \boldsymbol{v} \rangle + \int_\Omega \boldsymbol{\sigma}(t)\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx - \int_{\Gamma_3} \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot\boldsymbol{v}\,da \\ &\quad = \int_\Omega \boldsymbol{f}_0(t)\cdot\boldsymbol{v}\,dx + \int_{\Gamma_2} \boldsymbol{f}_2(t)\cdot\boldsymbol{v}\,da \quad \forall\,\boldsymbol{v} \in V,\ t \in (0,T). \end{aligned} \tag{4.42}$$

Thus, the term $R_n(\boldsymbol{v})$ can be simplified to

$$R_n(\boldsymbol{v}) = \int_{\Gamma_3} \left[\boldsymbol{\sigma}_n\boldsymbol{\nu}\cdot(\boldsymbol{v} - \boldsymbol{w}_n) + j_\nu^0(w_{n,\nu}; v_\nu - w_{n,\nu})\right] da + \varphi(\boldsymbol{v}_\tau) - \varphi(\boldsymbol{w}_{n,\tau}), \quad \boldsymbol{v} \in V. \tag{4.43}$$

Therefore,

$$|R_n(\boldsymbol{v})| \le c\,\|\boldsymbol{v} - \boldsymbol{w}_n\|_{L^2(\Gamma_3)^d} \quad \forall\,\boldsymbol{v} \in V \tag{4.44}$$

and

$$\begin{aligned} \tilde{E}_n \le c \inf_{\boldsymbol{v}_l^h \in V^h,\ 1 \le l \le n} \Bigg\{ & k\sum_{l=1}^n \|\boldsymbol{w}_l - \boldsymbol{v}_l^h\|_V^2 + \left[k\sum_{l=1}^n \|\boldsymbol{w}_l - \boldsymbol{v}_l^h\|_{L^2(\Gamma_3)^d}^2 \right]^{1/2} \\ & + k^{-1}\sum_{l=1}^{n-1} \left\|(\boldsymbol{w}_l - \boldsymbol{v}_l^h) - (\boldsymbol{w}_{l+1} - \boldsymbol{v}_{l+1}^h)\right\|_H^2 \\ & + \|\boldsymbol{w}_1 - \boldsymbol{v}_1^h\|_H^2 + \|\boldsymbol{w}_n - \boldsymbol{v}_n^h\|_H^2 \Bigg\}. \end{aligned} \tag{4.45}$$

We apply the finite element interpolation error estimates (cf. [23,24]). Take $\boldsymbol{v}_l^h \in V^h$ to be the finite element interpolant of $\boldsymbol{w}_l$. Then

$$\|\boldsymbol{w}_l - \boldsymbol{v}_l^h\|_V \le c\,h\,\|\boldsymbol{w}_l\|_{H^2(\Omega)^d}, \quad 0 \le l \le N. \tag{4.46}$$

Notice that $\boldsymbol{v}_l^h$ interpolates $\boldsymbol{w}_l$ on Γ_3, and thanks to the assumption (4.37),

$$\|\boldsymbol{w}_l - \boldsymbol{v}_l^h\|_{L^2(\Gamma_3)^d}^2 \le c\,h^4 \sum_{i=1}^{i_3} \|\boldsymbol{w}_l\|_{H^2(\Gamma_{3,i})}^2, \quad 1 \le l \le N.$$

Therefore,

$$\left[k\sum_{l=1}^n \|\boldsymbol{w}_l - \boldsymbol{v}_l^h\|_{L^2(\Gamma_3)^d}^2 \right]^{1/2} \le c\,h^4 \sum_{i=1}^{i_3} \|\boldsymbol{w}\|_{C(0,T;H^2(\Gamma_{3,i}))}^2. \tag{4.47}$$

Also notice that $(\boldsymbol{v}_l^h - \boldsymbol{v}_{l+1}^h)$ is the finite element interpolant of $(\boldsymbol{w}_l - \boldsymbol{w}_{l+1})$. Then,

$$\left\|(\boldsymbol{w}_l - \boldsymbol{v}_l^h) - (\boldsymbol{w}_{l+1} - \boldsymbol{v}_{l+1}^h)\right\|_H^2 \le c\, h^2 \|\boldsymbol{w}_l - \boldsymbol{w}_{l+1}\|_V^2 \le c\, h^2 k \int_{t_l}^{t_{l+1}} \|\dot{\boldsymbol{w}}(t)\|_V^2 dt,$$

and

$$k^{-1} \sum_{l=1}^{n-1} \left\|(\boldsymbol{w}_l - \boldsymbol{v}_l^h) - (\boldsymbol{w}_{l+1} - \boldsymbol{v}_{l+1}^h)\right\|_H^2 \le c\, h^2 \|\boldsymbol{w}\|^2_{H^1(0,T;H^2(\Omega)^d)}, \quad 1 \le n \le N. \tag{4.48}$$

Finally,

$$\max_{0\le n\le N} \|\boldsymbol{w}_n - \boldsymbol{v}_n^h\|_H \le c\, h^2 \|\boldsymbol{w}\|_{C(0,T;H^2(\Omega)^d)}. \tag{4.49}$$

An application of (4.46)–(4.49) in (4.34) with (4.45) leads to the optimal order error estimate

$$\max_{1\le n\le N} \|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_H^2 + k \sum_{n=1}^{N} \|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_V^2 \le c\left(k^2 + h^2\right). \tag{4.50}$$

For the discrete displacement,

$$\boldsymbol{u}_n^{hk} = \boldsymbol{u}_0^h + k \sum_{i=0}^{n-1} \boldsymbol{w}_i,$$

where $\boldsymbol{u}_0^h \in V^h$ is the finite element interpolant of $\boldsymbol{u}_0$. The displacement error can be represented as

$$\boldsymbol{u}_n - \boldsymbol{u}_n^{hk} = \boldsymbol{u}_0 - \boldsymbol{u}_0^h + k \sum_{i=0}^{n-1} \left(\boldsymbol{w}_i - \boldsymbol{w}_i^{hk}\right) + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (\boldsymbol{w}(t) - \boldsymbol{w}_i)\, dt.$$

Then,

$$\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V \le \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V + k \sum_{i=0}^{n-1} \|\boldsymbol{w}_i - \boldsymbol{w}_i^{hk}\|_V + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \|\boldsymbol{w}(t) - \boldsymbol{w}_i\|_V\, dt,$$

and so,

$$\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V \le k \sum_{i=0}^{n-1} \|\boldsymbol{w}_i - \boldsymbol{w}_i^{hk}\|_V + c\left(h\, \|\boldsymbol{u}_0\|_{H^2(\Omega;\mathbb{R}^d)} + k\, \|\ddot{\boldsymbol{u}}\|_{L^1(0,T;V)}\right).$$

Apply the Cauchy–Schwarz inequality and (4.50),

$$\sum_{i=1}^{n} \|\boldsymbol{w}_i - \boldsymbol{w}_i^{hk}\|_V \le c\left[k \sum_{i=1}^{n} \|\boldsymbol{w}_i - \boldsymbol{w}_i^{hk}\|_V^2\right]^{1/2} \le c\,(k+h).$$

Therefore, we have the optimal order error estimate for the displacement:

$$\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V \le c\,(k+h). \tag{4.51}$$

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