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The regularization method for an obstacle problem

Hongci Huang[1,⋆], **Weimin Han**[2,⋆⋆], **Jinshi Zhou**[2]

[1] Department of Mathematics, Hong Kong Baptist College, Kowloon, Hong Kong
[2] Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA

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Summary. We give a relatively complete analysis for the regularization method, which is usually used in solving non-differentiable minimization problems. The model problem considered in the paper is an obstacle problem. In addition to the usual convergence result and a-priori error estimates, we provide a-posteriori error estimates which are highly desired for practical implementation of the regularization method.

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1. An obstacle problem

The purpose of the paper is to give a relatively complete analysis of the regularization method for solving non-differentiable minimization problems. In addition to the usual convergence analysis and a-priori error estimates, we will also provide a-posteriori error estimates.

The model problem to be solved is an obstacle problem considered in [15]. Let Ω be a Lipschitz domain. Let $g \in H^{1/2}(\partial\Omega)$ be non-negative. Denote the energy functional

$$E(v) = \int_\Omega \left(\frac{1}{2} |\nabla v|^2 + v \right) dx \,. \tag{1.1}$$

Then the obstacle problem is to find

$$u \in H^1_g(\Omega): \; E(u) = \inf\{E(v) : v \in H^1_g(\Omega),\; v \geq 0 \text{ in } \Omega\} \tag{1.2}$$

where, $H^1_g(\Omega) = \left\{ v \in H^1(\Omega) : v = g \text{ on } \partial\Omega \right\}$. In this form, the problem is equivelent to an elliptic variational inequality of the first kind. The existence of a unique solution of the problem follows from the standard result on the unique solvability of variational inequalities of the first kind ([7]), see also the proof of Theorem 1.1 below.

To develop the regularization method, we write the obstacle problem in another form.

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Correspondence to: W. Han

Theorem 1.1. *Denote*

$$\tilde{E}(v) = \int_\Omega \left(\frac{1}{2} |\nabla v|^2 + |v| \right) dx \ . \tag{1.3}$$

Then the problem (1.2) is equivalent to the problem of finding

$$u \in H^1_g(\Omega), \text{ such that } \tilde{E}(u) = \inf\{\tilde{E}(v) : v \in H^1_g(\Omega)\} \ . \tag{1.4}$$

Proof. We first show that the problem (1.4) has a unique solution. Since $g \in H^{1/2}(\partial\Omega)$, it is the trace of an $H^1(\Omega)$ function. In this paper, we will use the same letter g to denote an $H^1(\Omega)$ function whose trace on the boundary is the given function $g \in H^{1/2}(\partial\Omega)$. Let us introduce a change of variables

$$v_0 = v - g, \quad \text{for } v \in H^1(\Omega) \ .$$

Then a solution of the problem (1.4) is

$$u = u_0 + g \tag{1.5}$$

where, $u_0 \in H^1_0(\Omega)$ minimizes the functional

$$\int_\Omega \left(\frac{1}{2} |\nabla v_0|^2 + \nabla v_0 \, \nabla g + |v_0 + g| \right) dx$$

among all the functions $v_0 \in H^1_0(\Omega)$. This problem is equivalent to an elliptic variational inequality of the second kind,

$$u_0 \in H^1_0(\Omega) : \ a(u_0, v_0 - u_0) + j(v_0) - j(u_0) \geq l(v_0 - u_0), \ \forall \, v_0 \in H^1_0(\Omega) \tag{1.6}$$

where,

$$a(u_0, v_0) = \int_\Omega \nabla u_0 \nabla v_0 \, dx$$

$$j(v_0) = \int_\Omega |v_0 + g| \, dx$$

$$l(v_0) = -\int_\Omega \nabla g \, \nabla v_0 \, dx \ .$$

Obviously, a is a continuous, $H^1_0(\Omega)$-elliptic bilinear form, $j : H^1_0(\Omega) \to \mathbb{R}$ is proper, convex and continuous, l is a continuous linear form on $H^1_0(\Omega)$. So the variational inequality (1.6), and thus the problem (1.4), has a unique solution ([7]).

To prove that the problems (1.2) and (1.4) are equivalent, we need the following result (cf. [6]).

If $v \in H^1(\Omega)$, then $|v| \in H^1(\Omega)$, and

$$\nabla |v| = \begin{cases} \nabla v, & \text{if } v > 0 \\ 0, & \text{if } v = 0 \\ -\nabla v, & \text{if } v < 0 \ . \end{cases}$$

A simple consequence is the equality,

$$\|\nabla |v|\|_{L^2(\Omega)} = \|\nabla v\|_{L^2(\Omega)} \ . \tag{1.7}$$

For the solution u of (1.4), we have $|u| \in H^1(\Omega)$, $|u| = g$ on $\partial\Omega$, and by (1.7),

$$\tilde{E}(|u|) = \tilde{E}(u) \, .$$

By the uniqueness of a solution of the problem (1.4), we thus have $u = |u| \geq 0$ in Ω. Hence, the solution of (1.4) is also the unique solution of the problem (1.2). □

From (1.5), we see that the solution u of the problem (1.4) satisfies

$$\text{(1.8)} \qquad u \in H^1_g(\Omega): \; a(u, v-u) + \int_\Omega (|v| - |u|)\, dx \geq 0, \quad \forall\, v \in H^1_g(\Omega) \, .$$

This relation will be used in Sect. 3.

A major difficulty in solving the problem (1.4) numerically is the treatment of the non-differentiable term $\int_\Omega |v|\, dx$. In practice, there are several approaches to circumvent the difficulty. One approach is to introduce a Lagrange multiplier for the non-differentiable term, and the problem (1.4) (and its discretization) is solved by an iterative procedure, for detail, see, e.g., [7], [10]. In this paper, we will give a detailed analysis of another approach, namely, the regularization method. The idea of the regularization method is to approximate the non-differnetiable term by a sequence of differentiable ones. The regularization method has been widely used in applications (cf. [7], [8], [12], [14]). An approximating differentiable sequence in the regularization method depends on a small parameter $\varepsilon > 0$. The convergence is obtained when ε goes to 0. However, as $\varepsilon \to 0$, the conditioning of a regularized problem deteriorates. So, there is a tradeoff in the selection of the regularization parameter. Theoretically, to get more accurate approximations, we need to use smaller ε. On the other hand, if ε is too small, the numerical solution of the regularized problem cannot be computed accurately. Thus, it is highly desirable to have a-posteriori error estimates which can give us computable error bounds once we have solutions of regularized problems. We can use the a-posteriori error estimates in devising a stopping criterion in actual computations: if the estimated error is within the given error tolerance, we accept the solution of the regularized problem as the exact solution; and if the estimated error is large, then we need to use a smaller value for the regularization parameter ε. An adaptive algorithm can be developed based on the a-posteriori error analysis. We will discuss such an adaptive algorithm elsewhere.

We remark that for a direct finite dimensional discretization of the problem (1.4), an efficient solution method is the nonlinear SOR method; for detail, cf. [7], [8].

In [13], the regularization technique is combined with the finite element method in solving a general class of free boundary problems. In particular, for the problem (1.4) considered in this paper, the regularization sequence in [13] corresponds to the choice 1 in the next section. Whereas in [13] no specific form of the non-differentiable term is needed, here we are satisfied with the explicit formulation of the problem (1.4), for the main purpose of the paper is to show how to derive a-posteriori error estimates for solutions of the regularized problems. The same technique presented in this paper can be used to derive a-posteriori error estimates for the regularization method for other variational inequality problems.

In the next section, we will introduce various forms of the regularization method, mention some convergence results and a-priori error estimates. In Sect. 3, we provide a-posteriori error estimates for the regularization methods. In the last section, we consider the regularization method for the discretizations of the obstacle problem, and present in particular the corresponding a-posteriori error estimates for the solutions of the regularized discrete problems.

2. The regularization method

As mentioned in the last section, in a regularization method, we approximate the non-differentiable term $j(v_0)$ by a sequence of differentiable ones, $j_\varepsilon(v_0) = \int_\Omega \phi_\varepsilon(v_0+g)\,dx$. The regularized problem is

$$
\begin{aligned}
u_{0,\varepsilon} \in H_0^1(\Omega):\ & a(u_{0,\varepsilon}, v_0 - u_{0,\varepsilon}) + j_\varepsilon(v_0) - j_\varepsilon(u_{0,\varepsilon}) \\
& \geq l(v_0 - u_{0,\varepsilon}),\ \forall\, v_0 \in H_0^1(\Omega)
\end{aligned}
\tag{2.1}
$$

or

$$
u_\varepsilon \in H_g^1(\Omega):\ a(u_\varepsilon, v - u_\varepsilon) + \int_\Omega (\phi_\varepsilon(v) - \phi_\varepsilon(u_\varepsilon))\,dx \geq 0, \quad \forall\, v \in H_g^1(\Omega)\,. \tag{2.2}
$$

The relation between the solutions of the two problems is $u_\varepsilon = u_{0,\varepsilon} + g$. We expect, under certain conditions, $u_{0,\varepsilon} \to u_0$ (equivalently, $u_\varepsilon \to u$) as $\varepsilon \to 0$. See the discussion below.

For a given non-differentiable term, there are many ways to construct sequences of differentiable approximations. Let us list five natural choices of a regularizing sequence for the obstacle problem considered in this paper (the first two choices are taken from [12]).

Choice 1. $j_\varepsilon(v_0) = \int_\Omega \phi_\varepsilon^1(v_0 + g)\,dx$, where,

$$
\phi_\varepsilon^1(t) = \begin{cases} t - \dfrac{\varepsilon}{2}, & \text{if } t \geq \varepsilon \\ \dfrac{1}{2\varepsilon}\, t^2, & \text{if } |t| \leq \varepsilon \\ -t - \dfrac{\varepsilon}{2}, & \text{if } t \leq -\varepsilon \end{cases}\,.
$$

Choice 2. $j_\varepsilon(v_0) = \int_\Omega \phi_\varepsilon^2(v_0 + g)\,dx$, where,

$$
\phi_\varepsilon^2(t) = \frac{\varepsilon}{\varepsilon + 1} \left(\frac{|t|}{\varepsilon} \right)^{\varepsilon+1}\,.
$$

Choice 3. $j_\varepsilon(v_0) = \int_\Omega \phi_\varepsilon^3(v_0 + g)\,dx$, where,

$$
\phi_\varepsilon^3(t) = \frac{|t|^{\varepsilon+1}}{\varepsilon + 1}\,.
$$

Choice 4. $j_\varepsilon(v_0) = \int_\Omega \phi_\varepsilon^4(v_0 + g)\,dx$, where,

$$
\phi_\varepsilon^4(t) = \sqrt{t^2 + \varepsilon^2}\,.
$$

Choice 5. $j_\varepsilon(v_0) = \int_\Omega \phi_\varepsilon^5(v_0 + g)\,dx$, where,

$$\phi_\varepsilon^5(t) = \begin{cases} t, & \text{if } t \geq \varepsilon \\ \frac{1}{2}\left(\frac{t^2}{\varepsilon} + \varepsilon\right), & \text{if } |t| \leq \varepsilon \\ -t, & \text{if } t \leq -\varepsilon \ . \end{cases}$$

Let us first consider convergence of the regularization method. We follow a framework given in [8].

Lemma 2.1. *Let V be a Hilbert space, $a : V \times V \to \mathbb{R}$ a continuous, V-elliptic bilinear form, $j : V \to \mathbb{R}$ proper, non-negative, convex, weakly continuous, f a linear continuous form on V. Assume $j_\varepsilon : V \to \mathbb{R}$ is proper, non-negative, convex and weakly l.s.c. (lower semi-continuous). Assume further that*

$$j_\varepsilon(v) \to j(v), \ \forall\, v \in V \tag{2.3}$$

$$u_\varepsilon \to u \text{ weakly in } V \Longrightarrow j(u) \leq \liminf_{\varepsilon \to 0} j_\varepsilon(u_\varepsilon) \ . \tag{2.4}$$

Let $u, u_\varepsilon \in V$ be the solutions of the variational inequalities

$$a(u, v-u) + j(v) - j(u) \geq f(v-u), \quad \forall\, v \in V \tag{2.5}$$

and

$$a(u_\varepsilon, v-u_\varepsilon) + j_\varepsilon(v) - j_\varepsilon(u_\varepsilon) \geq f(v-u_\varepsilon), \quad \forall\, v \in V \tag{2.6}$$

respectively. Then, $u_\varepsilon \to u$ in V, as $\varepsilon \to 0$.

The proof is standard, so we will only give a sketch. From the assumptions made, both problems have unique solutions. Using the natural assumption that $j_\varepsilon(v) \geq 0$ (we can achieve this by adjusting f), we see that $\{u_\varepsilon\}$ is bounded in V. So a subsequence, still denoted by $\{u_\varepsilon\}$, converges weakly to u in V. The inequality

$$a(u_\varepsilon, v-u_\varepsilon) + j_\varepsilon(v) - j_\varepsilon(u_\varepsilon) \geq f(v-u_\varepsilon), \quad \forall\, v \in V$$

implies

$$a(u_\varepsilon, v) + j_\varepsilon(v) - f(v-u_\varepsilon) \geq a(u_\varepsilon, u_\varepsilon) + j_\varepsilon(u_\varepsilon), \quad \forall\, v \in V \ .$$

Then use the assumptions (2.3) and (2.4) to obtain

$$a(u, v) + j(v) - f(v-u) \geq a(u, u) + j(u), \quad \forall\, v \in V$$

i.e., the limit u is a solution of the problem (2.5). Since a solution of (2.5) is unique, the whole sequence $\{u_\varepsilon\}$ converges weakly to u in V. To show the strong convergence, we take $v = u_\varepsilon$ in (2.5) and $v = u$ in (2.6), and add the two resulting inequalities to obtain

$$a(u-u_\varepsilon, u-u_\varepsilon) \leq [j_\varepsilon(u) - j(u)] + [j(u_\varepsilon) - j(u)] + [j(u) - j_\varepsilon(u_\varepsilon)] \ . \tag{2.7}$$

It is then easy to see $u - u_\varepsilon \to 0$ in V as $\varepsilon \to 0$.

To verify the conditions (2.3) and (2.4), we have

Lemma 2.2. *Assume $j(v) = \int_\Omega \phi(v)\, dx$, $j_\varepsilon(v) = \int_\Omega \phi_\varepsilon(v)\, dx$, and j is weakly l.s.c. If*

$$\phi_\varepsilon(t) \to \phi(t) \ \text{ uniformly in } t, \text{ as } \varepsilon \to 0 \tag{2.8}$$

then both (2.3) and (2.4) are true.

The proof of (2.3) is immediate, and the proof of (2.4) can be made from the relation

$$j_\varepsilon(u_\varepsilon) - j(u) = [j_\varepsilon(u_\varepsilon) - j(u_\varepsilon)] + [j(u_\varepsilon) - j(u)] \ .$$

We also notice that a sufficient condition for (2.8) is

$$|\phi_\varepsilon(t) - \phi(t)| \le c\,\varepsilon, \quad \forall\, t \in \mathbb{R} \ . \tag{2.9}$$

For the choices 1, 4, and 5, the inequality (2.9) holds. So the regularization method based on each of the three choices is convergent.

For the choices 2 and 3, the condition (2.8) is not satified. However, we can prove (2.3) and (2.4) of Lemma 2.1 directly. Weak convergence in $H^1(\Omega)$ implies a.e. pointwise convergence. So the inequality (2.4) follows from Fatou's Lemma. For a fixed $\lambda > 0$, denote

$$f(\varepsilon) = \frac{1}{\lambda + 1}\,\lambda^{\varepsilon+1} \ .$$

It can be proved easily that there exists an $\varepsilon_0 > 0$, such that

$$f(\varepsilon) \le \max\left\{\lambda,\ \frac{1}{2}\,\lambda^2\right\}, \quad \text{if } \varepsilon \le \varepsilon_0 \ .$$

Thus, for the choice 3, (2.3) follows from the Lebesgue Dominated Convergence Theorem. Similar argument applies to the choice 2.

In conclusion, we have shown the convergence of the regularization method using any one of the five choices for the regularization sequence.

As for a-priori error estimates, we have, from (2.7),

$$a(u - u_\varepsilon, u - u_\varepsilon) \le [j(u_\varepsilon) - j_\varepsilon(u_\varepsilon)] + [j_\varepsilon(u) - j(u)] \ . \tag{2.10}$$

Thus, since the inequality (2.9) is true for the choices 1, 4, and 5, for a solution of the corresponding regularized problem, we have

$$\|u_0 - u_{0,\varepsilon}\|_{H^1(\Omega)} \le c\,\sqrt{\varepsilon} \tag{2.11}$$

or, equivalently,

$$\|u - u_\varepsilon\|_{H^1(\Omega)} \le c\,\sqrt{\varepsilon} \ . \tag{2.12}$$

We notice that for the choices 2 and 3, the above a-priori error estimates are not available.

3. A-posteriori error estimates

As far as practical computation is concerned, a convergence result and an a-priori error estimate are not enough for a complete numerical analysis with the regularization method. As explained in Sect. 1, a-posteriori error estimates are much needed, which will provide a quantitative error bound once a solution of the regularized problem is computed. We will use the duality theory from the convex analysis (cf. [4]) to derive the a-posteriori error estimates. The idea was used in analyzing a particular regularization method for a simplified friction problem ([9]) and a holonomic elastic-plastic problem ([10]). Here, we will first give an analysis in the framework of a general regularization method, and then discuss the applications to the regularization method with various choices of the regularization sequence. We notice that similar ideas can be employed in analyzing modelling errors, see e.g., [11].

Now we present some needed results from convex analysis (cf. [4]).

Let V, Q be two normed spaces, V^*, Q^* their dual spaces. Assume there exists a linear continuous operator $\Lambda \in \mathscr{L}(V,Q)$, with transpose $\Lambda^* \in \mathscr{L}(Q^*,V^*)$. Let J be a function mapping $V \times Q$ into $\overline{\mathbb{R}}$ – the extended real line. Consider the minimization problem:

$$\inf_{v\in V} J(v,\Lambda v)\ . \tag{3.1}$$

Define the conjugate function of J by:

$$J^*(v^*,q^*) = \sup_{v\in V, q\in Q} \left[\langle v,v^*\rangle + \langle q,q^*\rangle - J(v,q)\right]\ . \tag{3.2}$$

Theorem 3.1. *Assume*

(1) V is a reflexive Banach space, Q a normed space.

(2) $J : V \times Q \to \overline{\mathbb{R}}$ is a proper, l.s.c., strictly convex function.

(3) $\exists\, u_0 \in V$, such that $J(u_0,\Lambda u_0) < \infty$ and $q \mapsto J(u_0,q)$ is continuous at Λu_0.

(4) $J(v,\Lambda v) \to +\infty$, as $\|v\| \to \infty, v \in V$.

Then problem (3.1) has a unique solution $u \in V$, and

$$-\,J(u,\Lambda u) \le J^*(\Lambda^* q^*, -q^*), \quad \forall\, q^* \in Q^*\ . \tag{3.3}$$

In computing conjugate functions, we need the following result. Let Ω be an open set of $\mathbb{R}^n$, $g : \Omega \times \mathbb{R}^l \to \mathbb{R}$. Assume $\forall\, \xi \in \mathbb{R}^l$, $x \mapsto g(x,\xi)$ is a measurable function, and for a.e. $x \in \Omega$, $\xi \mapsto g(x,\xi)$ is a continuous function. Then the conjugate of the function

$$G(v) = \int_\Omega g(x,v(x))\, dx$$

(assuming G is well-defined over some function space V) is

$$G^*(v^*) = \int_\Omega g^*\left(x, v^*(x)\right)\, dx, \quad \forall\, v^* \in V^*,$$

where, V^* is the dual of V, and

$$g^*(x,y) = \sup_{\xi\in\mathbb{R}^l} [\, y\cdot\xi - g(x,\xi)\,]\,.$$

For the obstacle problem (1.4) considered in this paper, we take (n is the dimension of the domain Ω)

$$\begin{aligned}
&V = H^1(\Omega)\\
&Q = Q^* = \left(L^2(\Omega)\right)^n \times L^2(\Omega)\\
&\Lambda v = (\nabla v, v)\\
&J(v,\Lambda v) = F(v) + G(\Lambda v)\\
&F(v) = \begin{cases} 0, & \text{if } v = g \text{ on } \partial\Omega\\ \infty, & \text{otherwise}\end{cases}\\
&G(q) = \int_\Omega \left(\frac{1}{2}\,|q_1|^2 + |q_2|\right)\, dx
\end{aligned}$$

where and in what follows, we use the notation $q = (q_1, q_2)$ for $q \in Q$, with $q_1 \in \left(L^2(\Omega)\right)^n$ and $q_2 \in L^2(\Omega)$. A similar notation is used for $q^* \in Q^*$.

With the above notations, the obstacle problem (1.4) can be rewritten in the form of (3.1). To apply Theorem 3.1, we first compute the conjugate of the functional J. We have

$$\begin{aligned}
F^*(\Lambda^* q^*) &= \sup_{v \in V} \{\langle \Lambda v, q^* \rangle - F(v)\} \\
&= \sup_{v \in H^1_g(\Omega)} \int_\Omega (\nabla v\, q_1^* + v\, q_2^*)\, dx \\
&= \int_\Omega (\nabla g\, q_1^* + g\, q_2^*)\, dx + \sup_{v \in H^1_0(\Omega)} \int_\Omega (\nabla v\, q_1^* + v\, q_2^*)\, dx \\
&= \begin{cases} \displaystyle\int_\Omega (\nabla g\, q_1^* + g\, q_2^*)\, dx, & \text{if } -\mathrm{div} q_1^* + q_2^* = 0 \text{ in } \Omega \\ \infty, & \text{otherwise} \end{cases} .
\end{aligned}$$

and

$$\begin{aligned}
G^*(-q^*) &= \sup_{q \in Q} \{\langle q, -q^* \rangle - G(q)\} \\
&= \sup_{q \in Q} \int_\Omega \left(-q_1^* q_1 - q_2^* q_2 - \frac{1}{2} |q_1|^2 - |q_2| \right) dx \\
&= \begin{cases} \displaystyle\int_\Omega \frac{1}{2} |q_1^*|^2\, dx, & \text{if } |q_2^*| \le 1 \text{ in } \Omega \\ \infty, & \text{otherwise} \end{cases}
\end{aligned}$$

Hence,

$$\begin{aligned}
&J^*(\Lambda^* q^*, -q^*) \\
&= \begin{cases} \displaystyle\int_\Omega \left(\frac{1}{2} |q_1^*|^2 + \nabla g\, q_1^* + g\, q_2^* \right) dx, & \text{if } -\mathrm{div} q_1^* + q_2^* = 0 \text{ and } |q_2^*| \le 1 \text{ in } \Omega \\ \infty, & \text{otherwise} \end{cases}
\end{aligned} \tag{3.4}$$

Now let us consider the energy difference

$$J(u_\varepsilon, \Lambda u_\varepsilon) - J(u, \Lambda u) = \int_\Omega \left(\frac{1}{2} |\nabla u_\varepsilon|^2 - \frac{1}{2} |\nabla u|^2 + |u_\varepsilon| - |u| \right) dx .$$

Using (1.8) with $v = u_\varepsilon$, we find that

$$J(u_\varepsilon, \Lambda u_\varepsilon) - J(u, \Lambda u) \ge \frac{1}{2} \|\nabla(u_\varepsilon - u)\|^2_{L^2(\Omega)} . \tag{3.5}$$

On the other hand, applying Theorem 3.1 and using (3.4), we have

$$\begin{aligned}
J(u_\varepsilon, \Lambda u_\varepsilon) - J(u, \Lambda u) \le \int_\Omega \left(\frac{1}{2} |\nabla u_\varepsilon|^2 + |u_\varepsilon| + \frac{1}{2} |q_1^*|^2 + \nabla g\, q_1^* + g\, q_2^* \right) dx \\
\forall\, q^* = (q_1^*, q_2^*) \in Q^*, \text{ such that } -\mathrm{div} q_1^* + q_2^* = 0,\ |q_2^*| \le 1 \text{ in } \Omega .
\end{aligned}$$

Hence, we have the important inequality for a-posteriori error analysis:

$$\tag{3.6} \begin{aligned} \frac{1}{2}\,\|\nabla(u_\varepsilon - u)\|^2_{L^2(\Omega)} \le \int_\Omega \left(\frac{1}{2}\,|\nabla u_\varepsilon|^2 + |u_\varepsilon| + \frac{1}{2}\,|q_1^*|^2 + \nabla g\, q_1^* + g\, q_2^*\right)\, dx \\ \forall\, q^* = (q_1^*, q_2^*) \in Q^*,\ \text{such that}\ -\operatorname{div} q_1^* + q_2^* = 0,\ |q_2^*| \le 1 \text{ in } \Omega\ . \end{aligned}$$

Let us then study the regularized problem (2.2). Since ϕ_ε is differentiable, the variational inequality (2.2) is equivalent to the relation

$$\tag{3.7} u_\varepsilon \in H^1_g(\Omega):\ a(u_\varepsilon, v) + \int_\Omega (\phi_\varepsilon)'(u_\varepsilon)\, v = 0, \quad \forall\, v \in H^1_0(\Omega)\ .$$

Thus, u_ε is the weak solution of the elliptic boundary value problem

$$\tag{3.8} \begin{cases} -\Delta u_\varepsilon + (\phi_\varepsilon)'(u_\varepsilon) = 0, & \text{in } \Omega \\ u_\varepsilon = g, & \text{on } \partial\Omega\ . \end{cases}$$

From (3.7), we observe that if the regularizing function ϕ satifies the inequality

$$\tag{3.9} |(\phi_\varepsilon)'(t)| \le 1, \quad \forall\, t \in \mathbb{R}$$

then, a natural selection of an auxiliary field q^* in the basic inequality (3.6) is

$$\tag{3.10} q_1^* = -\nabla u_\varepsilon, \quad q_2^* = -(\phi_\varepsilon)'(u_\varepsilon)\ .$$

And then, an a-posteriori error estimate is obtained from (3.6),

$$\frac{1}{2}\,\|\nabla(u_\varepsilon - u)\|^2_{L^2(\Omega)} \le \int_\Omega \big(\nabla u_\varepsilon\, \nabla(u_\varepsilon - g) + |u_\varepsilon| - g\,(\phi_\varepsilon)'(u_\varepsilon)\big)\, dx\ .$$

Taking $v = u_\varepsilon - g \in H^1_0(\Omega)$ in (3.7), we find that

$$\int_\Omega \big(\nabla u_\varepsilon\, \nabla(u_\varepsilon - g) + (\phi_\varepsilon)'(u_\varepsilon)\,(u_\varepsilon - g)\big)\, dx = 0\ .$$

Therefore, we can write the a-posteriori error estimate in the form of

$$\tag{3.11} \frac{1}{2}\,\|\nabla(u_\varepsilon - u)\|^2_{L^2(\Omega)} \le \int_\Omega \big(|u_\varepsilon| - u_\varepsilon\,(\phi_\varepsilon)'(u_\varepsilon)\big)\, dx\ .$$

The regularizing functions in the choices 1, 4, and 5 indeed satisfies the inequality (3.9). Hence, we have the following a-posteriori error estimates.

For the choice 1, we have

$$(\phi^1_\varepsilon)'(t) = \begin{cases} 1, & \text{if } t \ge \varepsilon \\ \dfrac{1}{\varepsilon}\, t, & \text{if } |t| \le \varepsilon \\ -1, & \text{if } t \le -\varepsilon\ . \end{cases}$$

Thus, the a-posteriori error estimate is

$$\tag{3.12} \frac{1}{2}\,\|\nabla(u_\varepsilon - u)\|^2_{L^2(\Omega)} \le \int_{|u_\varepsilon| \le \varepsilon} |u_\varepsilon|\, \left(1 - \frac{|u_\varepsilon|}{\varepsilon}\right)\, dx\ .$$

For the choice 4, we have

$$(\phi_\varepsilon^4)'(t) = \frac{t}{\sqrt{t^2+\varepsilon^2}}\ .$$

Thus, the a-posteriori error estimate is

$$\frac{1}{2}\,\|\nabla(u_\varepsilon - u)\|^2_{L^2(\Omega)} \le \int_\Omega \frac{|u_\varepsilon|\,\varepsilon^2}{u_\varepsilon^2+\varepsilon^2+|u_\varepsilon|\sqrt{u_\varepsilon^2+\varepsilon^2}}\,dx\ . \tag{3.13}$$

For the choice 5, we have

$$(\phi_\varepsilon^5)'(t) = \begin{cases} 1, & \text{if } t \ge \varepsilon \\ \frac{1}{\varepsilon}\,t, & \text{if } |t| \le \varepsilon \\ -1, & \text{if } t \le -\varepsilon\ . \end{cases}$$

So we have the same form of a-posteriori error estimate as that given in (3.12).

For the choices 2 and 3, the regularizing functions do not satisfy the (3.9). It is still possible to construct an admissible field q^* from u_ε to produce a good error bound, but the procedure will be complicated. From the practical point of view, the computation of the a-posteriori error bound will then cost quite a bit of time. So it is doubtful if these choices of the regularizing function will be preferable than the other choices (choices 1, 4, and 5).

4. A-posteriori error estimates for regularized discrete problems

In actual computations, the obstacle problem (1.4) is first discretized, e.g. by the finite element method, and then the discretized problem is solved using, e.g. the regularization method.

Let V_h be a finite element space approximating $H^1(\Omega)$, let S_h be the finite element subspace of V_h consisting all the functions in V_h which are zero on the boundary of the domain. We have $S_h \subset H_0^1(\Omega)$. For simplicity of exposition, assume the boundary condition function g can be represented exactly by a function from V_h. Then, a finite element solution $u_h \in V_h$ for the obstacle problem (1.8) is determined from the following problem,

$$\begin{aligned} &u_h = g \text{ on } \partial\Omega,\ \text{such that} \\ &a(u_h, v_h - u_h) + \int_\Omega (|v_h| - |u_h|)\,dx \ge 0, \quad \forall\, v_h \in V_h,\ v_h = g \text{ on } \partial\Omega\ . \end{aligned} \tag{4.1}$$

Or, if we write $u_h = u_{0,h} + g$, then $u_{0,h}$ is the solution of the problem (cf. (1.6))

$$\begin{aligned} &u_{0,h} \in S_h : a(u_{0,h}, v_{0,h} - u_{0,h}) + j(v_{0,h}) - j(u_{0,h}) \ge l(v_{0,h} - u_{0,h}), \\ &\qquad \forall\, v_{0,h} \in S_h\ . \end{aligned} \tag{4.2}$$

Convergence of the finite element approximations can be proved similarly as in [7] and [8]. For a-priori error estimates, one can proceed similarly as in the just-mentioned references or in [10] to prove the following useful inequality (if we assume $\Delta u \in L^2(\Omega)$):

$$\begin{aligned} \|u_{0,h} - u_0\|_{H^1(\Omega)} \le c \inf_{v_{0,h}\in S_h} \big\{ &\|v_{0,h} - u_0\|_{H^1(\Omega)} \\ &+ \|v_{0,h} - u_0\|_{L^1(\Omega)} + \|\Delta u\|_{L^2(\Omega)} \|v_{0,h} - u_0\|_{L^2(\Omega)} \big\} \end{aligned} \tag{4.3}$$

where, c is an absolute constant, i.e., it does not depend on u_0. From (4.3), the usual finite element approximation theory (cf. [1], [3]) can be applied to give optimal order a-priori error estimates.

Like for the continuous problems, a major difficulty in solving the discretized problems (4.1) and/or (4.2) lies in the fact that there is a non-differentiable term. So let us replace the problem (4.1) or (4.2) by a sequence of regularized problems,

$$\begin{aligned} &u_{h,\varepsilon} \in V_h,\ u_{h,\varepsilon} = g \text{ on } \partial\Omega, \text{ such that} \\ &a(u_{h,\varepsilon}, v_h - u_{h,\varepsilon}) + \int_\Omega (\phi_\varepsilon(v_h) - \phi_\varepsilon(u_{h,\varepsilon}))\,dx \geq 0, \quad \forall\, v_h \in V_h,\ v_h = g \text{ on } \partial\Omega \end{aligned} \tag{4.4}$$

or

$$\begin{aligned} u_{0,h,\varepsilon} \in S_h : a(u_{0,h,\varepsilon}, v_{0,h} - u_{0,h,\varepsilon}) + j_\varepsilon(v_{0,h}) - j_\varepsilon(u_{0,h,\varepsilon}) \\ \geq l(v_{0,h} - u_{0,h,\varepsilon}),\ \forall\, v_{0,h} \in S_h\ . \end{aligned} \tag{4.5}$$

We can similarly prove that problems (4.4) and (4.5) are uniquely solvable, and under the assumptions (2.3) and (2.4), their solutions converge to the corresponding solutions of the problems (4.1) and (4.2). Finally, we can apply the duality theory on the discrete problems to prove the following a-posteriori error estimates.

For the choices 1 and 5,

$$\frac{1}{2}\,\|\nabla(u_{h,\varepsilon} - u_h)\|^2_{L^2(\Omega)} \leq \int_{|u_{h,\varepsilon}|\leq\varepsilon} |u_{h,\varepsilon}| \left(1 - \frac{|u_{h,\varepsilon}|}{\varepsilon}\right) dx\ . \tag{4.6}$$

For the choice 4, we have

$$\frac{1}{2}\,\|\nabla(u_{h,\varepsilon} - u_h)\|^2_{L^2(\Omega)} \leq \int_\Omega \frac{|u_{h,\varepsilon}|\,\varepsilon^2}{u^2_{h,\varepsilon} + \varepsilon^2 + |u_{h,\varepsilon}|\sqrt{u^2_{h,\varepsilon} + \varepsilon^2}}\,dx\ . \tag{4.7}$$

A complete adaptive algorithm can be developed for solving the obstacle problem (1.4) by combining the a-posteriori error estimates for the regularization method and a-posteriori error estimates for the finite element solutions, for the latter, one can consult [2], [5], and references therein.

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