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Stability analysis of stationary variational and hemivariational inequalities with applications

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ABSTRACT

In this paper, we provide a comprehensive stability analysis for stationary variational inequalities, hemivariational inequalities, and variational-hemivariational inequalities. With contact mechanics as application background, stability is analyzed for solutions with respect to combined or separate perturbations in constitutive relations, external forces, constraints, and non-smooth contact boundary conditions of the inequality problems. The stability result is first proved for a general variational-hemivariational inequality. Then, stability results are obtained for various variational inequalities and hemivariational inequalities as special cases. Finally, we illustrate applications of the theoretical results for the stability analysis of model problems in contact mechanics.

1. Introduction

In many complicated physical processes and engineering applications, mathematical models of the problems are formulated as inequalities instead of the more commonly seen equations. Two types of inequality problems have been studied: variational inequalities and hemivariational inequalities. Variational inequalities refer to those inequality problems with a convex structure. They have been studied extensively for over half a century since 1960s, both theoretically and numerically. Some representative references include [1–4] on mathematical theories and [5–8] on numerical solutions. Since the early 1980s, hemivariational inequalities have been introduced, analyzed and applied to a variety of engineering problems involving non-monotone and possibly multi-valued constitutive or interface laws for deformable bodies. Studies of hemivariational inequalities can be found in the comprehensive references [9–15]. The inequality problems from applications can only be solved by numerical methods. The book [16] is devoted to the finite element approximations of

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hemivariational inequalities, where convergence of the numerical methods is discussed but no error estimates are derived. In the recent years, there have been much effort from numerous researchers to derive error estimates for numerical solutions of hemivariational inequalities. In particular, optimal order error estimates have been derived for linear finite element solutions of various hemivariational inequalities, starting with [17] for a stationary variational-hemivariational inequality modeling stationary frictional contact, followed by [18] for a hyperbolic hemivariational inequality arising in dynamic frictional contact, along with several more publications. More recently, general frameworks are presented for error analysis, on internal numerical approximations of hemivariational inequalities in [19], on internal numerical approximations of variational-hemivariational inequalities in [20], and on convergence and error analysis for both internal and external approximations of elliptic variational-hemivariational inequalities in [21].

In addition to solution existence and uniqueness of an inequality problem, the data continuous dependence, or the stability, is an important property, since in applications, one cannot expect to know the problem data exactly. The stability is especially significant from the view-point of numerical approximations since a numerical solution is meaningful only if the problem being solved is stable with respect to the data (for some ill-posed problems, the regularization technique may be employed for numerical treatment). The stability is also useful in optimal control of the hemivariational inequalities [22,23]. In the literature, only partial stability results are available. In this paper, we perform a more systematic stability analysis to include continuous dependence of the solution on other data as well. For inequality problems arising in contact mechanics, our general result provides stability of the solution with respect to constitutive relations, external forces, constraints, and non-smooth contact boundary conditions; in comparison, in existing references on hemivariational inequalities only the stability of the solution with respect to non-smooth contact boundary conditions (e.g. [15,24]) or the stability of the solution with respect to the external forces and a proportionality constant of a constraint set (e.g. [22,23]) is shown. Our general stability result in this paper will be useful in the study of general optimal control problems for inequality problems.

The rest of the paper is organized as follows. In Section 2 we review some preliminary material needed in the study of inequality problems. In Section 3, we introduce a variational-hemivariational inequality, state and prove a general result on its stability. The stability result on the variational-hemivariational inequality leads to corresponding ones on hemivariational inequalities and variational inequalities under simplified conditions. In Section 4, we present stability results for hemivariational inequalities and variational inequalities constraints, and in Section 5, we present stability results for hemivariational inequalities and variational inequalities without constraints, as consequences of the general result shown in Section 3. In Section 6 we illustrate the application of the stability results on two contact problems.

2. Preliminaries

Only real spaces are used in this paper. For a normed space X, we denote by $\|\cdot\|_X$ its norm, by X^* its topological dual, and by $\langle\cdot,\cdot\rangle_{X^*\times X}$ the duality pairing of X and X^*. When no confusion may arise, we simply write $\langle\cdot,\cdot\rangle$ instead of $\langle\cdot,\cdot\rangle_{X^*\times X}$. Strong convergence is indicated by the symbol $\to$, whereas weak convergence by $\rightharpoonup$. The space of all linear continuous operators from one normed space X to another normed space Y is denoted by $\mathcal{L}(X,Y)$.

An operator $A\colon X\to X^*$ is said to be pseudomonotone if it is bounded and $u_n\rightharpoonup u$ in X together with $\limsup\,\langle Au_n,u_n-u\rangle_{X^*\times X}\le 0$ imply

$$\langle Au,u-v\rangle_{X^*\times X}\le \liminf\,\langle Au_n,u_n-v\rangle_{X^*\times X}\quad \forall\, v\in X.$$

A function $\varphi\colon X\to\mathbb{R}\cup\{+\infty\}$ is said to be lower semicontinuous (l.s.c.) if for any sequence $\{x_n\}\subset X$ and any $x\in X$, $x_n\to x$ in X implies $\varphi(x)\le\liminf\varphi(x_n)$. For a convex function φ, the set

$$\widetilde{\partial}\varphi(x):=\{x^*\in X^*\mid \varphi(v)-\varphi(x)\ge\langle x^*,v-x\rangle_{X^*\times X}\ \forall\, v\in X\}$$

is called the subdifferential of φ at $x \in X$. If $\widetilde{\partial}\varphi(x)$ is non-empty, any element $x^* \in \widetilde{\partial}\varphi(x)$ is called a subgradient of φ at x.

Assume $\psi\colon X \to \mathbb{R}$ is locally Lipschitz continuous. The generalized (Clarke) directional derivative of ψ at $x \in X$ in the direction $v \in X$ is defined by

$$\psi^0(x;v) := \limsup_{y\to x,\, \lambda\downarrow 0} \frac{\psi(y+\lambda v)-\psi(y)}{\lambda}.$$

The generalized subdifferential of ψ at x is a subset of the dual space X^* given by

$$\partial\psi(x) := \left\{\xi \in X^* \mid \psi^0(x;v) \ge \langle \xi, v\rangle_{X^*\times X}\ \forall\, v \in X\right\}.$$

The function $(x,v) \mapsto \psi^0(x;v)$ is upper semicontinuous on $X \times X$ [25]; in other words,

$$x_n \to x \text{ and } v_n \to v \text{ in } X \quad \Longrightarrow \quad \limsup_{n\to\infty} \psi^0(x_n;v_n) \le \psi^0(x;v). \tag{2.1}$$

Details on properties of convex functions can be found in [26], whereas that of the subdifferential in the Clarke sense can be found in the books [10,13,25,27].

The following two properties of convex functions will be useful later in this paper.

Lemma 2.1 (*[26, p. 13]*). *A l.s.c. convex function $\varphi : X \to \mathbb{R}$ on a Banach space X is continuous.*

Lemma 2.2 (*[28, Lemma 11.3.5], [27, Prop. 5.2.25]*). *Let X be a normed space and let $\varphi : X \to \overline{\mathbb{R}}$ be proper, convex and l.s.c. Then there exist a continuous linear functional $\ell_\varphi \in X^*$ and a constant $c \in \mathbb{R}$ such that*

$$\varphi(x) \ge \ell_\varphi(x) + c \quad \forall\, x \in X.$$

Consequently, there exist two constants $\overline{c}$ and $\tilde{c}$ such that

$$\varphi(x) \ge \overline{c} + \tilde{c}\,\|x\|_X \quad \forall\, x \in X. \tag{2.2}$$

3. Stability result on a general variational-hemivariational inequality

The aim of this section is to provide a stability result on a general variational-hemivariational inequality. We will first introduce the variational-hemivariational inequality, then a family of perturbed variational-hemivariational inequalities, and finally we state and prove the convergence of solutions of the perturbed inequalities to the solution of the variational-hemivariational inequality. In the context of applications in contact mechanics, the perturbations are with respect to the material constitutive relations, external forces, constraints, and non-smooth contact boundary conditions.

The assumptions to be made on the data for the abstract variational-hemivariational inequality and its perturbations involve positive constants m_A, c_φ and c_j, as well as non-negative constants c_0, c_1, α_φ and α_j. These constants are independent of the perturbation parameter $\varepsilon > 0$; in other words, the corresponding properties will be assumed valid uniformly with respect to the perturbation parameter.

3.1. The variational-hemivariational inequality

We will need the following data and assumptions in the study of the abstract variational-hemivariational inequality.

(H_V) V is a reflexive Banach space.

(H_K) $K \subset V$ is non-empty, closed and convex.

(H_A) $A\colon V \to V^*$ is pseudomonotone and strongly monotone with the constant $m_A > 0$:

$$\langle Av_1 - Av_2, v_1 - v_2\rangle \ge m_A \|v_1 - v_2\|_V^2 \quad \forall\, v_1, v_2 \in V, \tag{3.1}$$

(H_φ) V_φ is a Banach space and $\gamma_\varphi \in \mathcal{L}(V, V_\varphi)$ with its norm bounded by c_φ. $\varphi\colon V_\varphi \times V_\varphi \to \mathbb{R}$ is such that $\varphi(z,\cdot)\colon V_\varphi \to \mathbb{R}$ is convex and l.s.c. for all $z \in V_\varphi$, and

$$\begin{aligned}&\varphi(z_1, z_4) - \varphi(z_1, z_3) + \varphi(z_2, z_3) - \varphi(z_2, z_4)\\ &\quad \le \alpha_\varphi \|z_1 - z_2\|_{V_\varphi} \|z_3 - z_4\|_{V_\varphi} \quad \forall\, z_1, z_2, z_3, z_4 \in V_\varphi.\end{aligned} \tag{3.2}$$

(H_j) V_j is a Banach space and $\gamma_j \in \mathcal{L}(V, V_j)$ with its norm bounded by c_j. $j\colon V_j \to \mathbb{R}$ is locally Lipschitz continuous and

$$\|\partial j(z)\|_{V_j^*} \le c_0 + c_1 \|z\|_{V_j} \quad \forall\, z \in V_j, \tag{3.3}$$

$$j^0(z_1; z_2 - z_1) + j^0(z_2; z_1 - z_2) \le \alpha_j \|z_1 - z_2\|_{V_j}^2 \quad \forall\, z_1, z_2 \in V_j. \tag{3.4}$$

(H_s)

$$\alpha_\varphi c_\varphi^2 + \alpha_j c_j^2 < m_A. \tag{3.5}$$

(H_f)

$$f \in V^*. \tag{3.6}$$

The abstract variational-hemivariational inequality is as follows.

Problem (P). *Find an element $u \in K$ such that*

$$\begin{aligned}&\langle Au, v - u\rangle + \varphi(\gamma_\varphi u, \gamma_\varphi v) - \varphi(\gamma_\varphi u, \gamma_\varphi u)\\ &\quad + j^0(\gamma_j u; \gamma_j v - \gamma_j u) \ge \langle f, v - u\rangle \quad \forall\, v \in K.\end{aligned} \tag{3.7}$$

Note that from (H_φ) and (H_j), we have the inequalities

$$\|\gamma_\varphi v\|_{V_\varphi} \le c_\varphi \|v\|_V \quad \forall\, v \in V, \tag{3.8}$$

$$\|\gamma_j v\|_{V_j} \le c_j \|v\|_V \quad \forall\, v \in V. \tag{3.9}$$

In the statement of Problem (P), the function $\varphi(z,\cdot)$ is assumed to be convex for any $z \in V_\varphi$ whereas the function j is allowed to be nonconvex. Thus, (3.7) represents a variational-hemivariational inequality. The spaces V_φ and V_j are introduced to facilitate error analysis of numerical solutions of Problem (P) [19–21] as well as for stability analysis in this paper. For applications in contact mechanics, the functionals $\varphi(\cdot,\cdot)$ and $j(\cdot)$ are integrals over the contact boundary Γ_3. In such a situation, V_φ and V_j can be chosen to be $L^2(\Gamma_3)^d$ and/or $L^2(\Gamma_3)$, and V is a subspace of $H^1(\Omega)^d$. The operators $\gamma_\varphi \in \mathcal{L}(V, V_\varphi)$ and $\gamma_j \in \mathcal{L}(V, V_j)$ are trace operators, and are in fact compact. For a locally Lipschitz function $j\colon V_j \to \mathbb{R}$, the inequality (3.4) is equivalent to

$$\langle \xi_1 - \xi_2, z_1 - z_2\rangle_{V_j^* \times V_j} \ge -\alpha_j \|z_1 - z_2\|_{V_j}^2 \quad \forall\, z_1, z_2 \in V_j,\ \xi_1 \in \partial j(z_1),\ \xi_2 \in \partial j(z_2), \tag{3.10}$$

known as a relaxed monotonicity condition. Note that if $j\colon V_j \to \mathbb{R}$ is convex, (3.10) is satisfied with $\alpha_j = 0$.

By slightly modifying the proof in [24], we have the following existence and uniqueness result.

Theorem 3.1. *Under assumptions (H_V), (H_K), (H_A), (H_φ), (H_j), (H_s) and (H_f), Problem (P) has a unique solution $u \in K$.*

The following Minty-type lemma for variational-hemivariational inequalities is shown in [21] where it is applied in convergence analysis of numerical solutions. In this paper, we will need it in proving the general stability result stated in Theorem 3.4. Recall that the operator A is said to be radially continuous if the function $t \mapsto \langle A(u+t\,v), v\rangle$ is continuous on $[0,1]$ for any $u, v \in V$.

Lemma 3.2. *Assume $K \subset V$ is convex, $A\colon V \to V^*$ is monotone and radially continuous, and for all $z \in V_\varphi$, $\varphi(z,\cdot)$ is convex on V_φ. Then $u \in K$ is a solution of Problem* (P) *if and only if it satisfies*

$$\begin{aligned}\langle Av, v-u\rangle + \varphi(\gamma_\varphi u, \gamma_\varphi v) - \varphi(\gamma_\varphi u, \gamma_\varphi u)&\\ + j^0(\gamma_j u; \gamma_j v - \gamma_j u) \ge \langle f, v-u\rangle \quad \forall\, v \in K.&\end{aligned} \tag{3.11}$$

3.2. The perturbed variational-hemivariational inequalities

Denote by $\varepsilon > 0$ a small perturbation parameter. To describe the perturbed problems, we keep (H_V) and (H_s) from Section 3.1, and introduce perturbed versions of other data and assumptions for each $\varepsilon > 0$.

(H_{K_ε}) $K_\varepsilon \subset V$ is non-empty, closed and convex.

(H_{A_ε}) $A_\varepsilon\colon V \to V^*$ is pseudomonotone and uniformly strongly monotone with the constant $m_A > 0$:

$$\langle A_\varepsilon v_1 - A_\varepsilon v_2, v_1 - v_2\rangle \ge m_A \|v_1 - v_2\|_V^2 \quad \forall\, v_1, v_2 \in V. \tag{3.12}$$

$(H_{\varphi_\varepsilon})$ V_φ is a Banach space and $\gamma_\varphi \in \mathcal{L}(V, V_\varphi)$ with its norm bounded by c_φ. $\varphi_\varepsilon\colon V_\varphi \times V_\varphi \to \mathbb{R}$ is such that $\varphi_\varepsilon(z,\cdot)\colon V_\varphi \to \mathbb{R}$ is convex and l.s.c. for all $z \in V_\varphi$, and

$$\begin{aligned}\varphi_\varepsilon(z_1, z_4) - \varphi_\varepsilon(z_1, z_3) + \varphi_\varepsilon(z_2, z_3) - \varphi_\varepsilon(z_2, z_4)&\\ \le \alpha_\varphi \|z_1 - z_2\|_{V_\varphi} \|z_3 - z_4\|_{V_\varphi} \quad \forall\, z_1, z_2, z_3, z_4 \in V_\varphi.&\end{aligned} \tag{3.13}$$

(H_{j_ε}) V_j is a Banach space and $\gamma_j \in \mathcal{L}(V, V_j)$ with its norm bounded by c_j. $j_\varepsilon\colon V_j \to \mathbb{R}$ is locally Lipschitz and

$$\|\partial j_\varepsilon(z)\|_{V_j^*} \le c_0 + c_1 \|z\|_{V_j} \quad \forall\, z \in V_j, \tag{3.14}$$

$$j_\varepsilon^0(z_1; z_2 - z_1) + j_\varepsilon^0(z_2; z_1 - z_2) \le \alpha_j \|z_1 - z_2\|_{V_j}^2 \quad \forall\, z_1, z_2 \in V_j. \tag{3.15}$$

(H_{f_ε})

$$f_\varepsilon \in V^*. \tag{3.16}$$

Then, the perturbed variational-hemivariational inequality is the following.

Problem (P_ε). *Find an element $u_\varepsilon \in K_\varepsilon$ such that*

$$\begin{aligned}\langle A_\varepsilon u_\varepsilon, v - u_\varepsilon\rangle + \varphi_\varepsilon(\gamma_\varphi u_\varepsilon, \gamma_\varphi v) - \varphi_\varepsilon(\gamma_\varphi u_\varepsilon, \gamma_\varphi u_\varepsilon)&\\ + j_\varepsilon^0(\gamma_j u_\varepsilon; \gamma_j v - \gamma_j u_\varepsilon) \ge \langle f_\varepsilon, v - u_\varepsilon\rangle \quad \forall\, v \in K_\varepsilon.&\end{aligned} \tag{3.17}$$

Similar to Theorem 3.1, we have the existence and uniqueness result for the perturbed variational-hemivariational inequality.

Theorem 3.3. *Under assumptions (H_V), (H_{K_ε}), (H_{A_ε}), $(H_{\varphi_\varepsilon})$, (H_{j_ε}), (H_s) and (H_{f_ε}), Problem* (P_ε) *has a unique solution $u_\varepsilon \in K_\varepsilon$.*

3.3. A general stability result

We explore a general stability result that under certain approximation conditions on the data,

$$u_\varepsilon \to u \quad \text{in } V \text{ as } \varepsilon \to 0.$$

For this purpose, we will assume convergence of the data defined as follows.

$(H_{A_\varepsilon \to A})$: For any $v \in V$, $A_\varepsilon v \to Av$ in V^*.

Note that since $\{A_\varepsilon\}_{\varepsilon>0} \subset \mathcal{L}(V, V^*)$, by the principle of uniform boundedness (cf. [28, p. 75]), we know that if $(H_{A_\varepsilon \to A})$ holds, then $\{A_\varepsilon\}$ is uniformly bounded in $\mathcal{L}(V, V^*)$.

$(H_{\varphi_\varepsilon \to \varphi})$: There exists a non-negative valued function $b_\varphi(\varepsilon)$ with $b_\varphi(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$|\varphi_\varepsilon(z_1, z_2) - \varphi_\varepsilon(z_1, z_1) - \varphi(z_1, z_2) + \varphi(z_1, z_1)| \le b_\varphi(\varepsilon) \left(1 + \|z_1\|_{V_\varphi}\right) \|z_1 - z_2\|_{V_\varphi} \quad \forall\, z_1, z_2 \in V_\varphi. \tag{3.18}$$

$(H_{j_\varepsilon \to j})$: There exists a non-negative valued function $b_j(\varepsilon)$ with $b_j(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$\left|j_\varepsilon^0(z_1; z_2) - j^0(z_1; z_2)\right| \le b_j(\varepsilon) \left(1 + \|z_1\|_{V_j}\right) \|z_2\|_{V_j} \quad \forall\, z_1, z_2 \in V_j. \tag{3.19}$$

$(H_{f_\varepsilon \to f})$: $\|f_\varepsilon - f\|_{V^*} \to 0$ as $\varepsilon \to 0$.

$(H_{K_\varepsilon \to K})$: (i) If $v_\varepsilon \in K_\varepsilon$ and $v_\varepsilon \rightharpoonup v$ in V, then $v \in K$.

(ii) For any $v \in K$, there exist $v_\varepsilon \in K_\varepsilon$ such that $v_\varepsilon \to v$ in V.

(H_c) $\gamma_\varphi \in \mathcal{L}(V, V_\varphi)$ and $\gamma_j \in \mathcal{L}(V, V_j)$ are compact.

We comment that in applications of contact mechanics, (H_c) is automatically satisfied, cf. the paragraph between (3.9) and (3.10).

Theorem 3.4. *Keep the assumptions stated in Theorems 3.1 and 3.3. Assume $(H_{A_\varepsilon \to A})$, $(H_{\varphi_\varepsilon \to \varphi})$, $(H_{j_\varepsilon \to j})$, $(H_{f_\varepsilon \to f})$, $(H_{K_\varepsilon \to K})$ and (H_c). Then for the solution u of Problem (P) and the solution u_ε of Problem (P$_\varepsilon$), we have the convergence:*

$$u_\varepsilon \to u \text{ in } V \text{ as } \varepsilon \to 0. \tag{3.20}$$

Proof. The proof consists of three steps.

Step 1. We prove that the set $\{\|u_\varepsilon\|_V\}$ is uniformly bounded.

Since K is nonempty, there exists an element $u_0 \in K$. By the assumption $(H_{K_\varepsilon \to K})$, there exist $u_{0,\varepsilon} \in K_\varepsilon$ with the property

$$u_{0,\varepsilon} \to u_0 \quad \text{in } V.$$

In particular, this implies the set $\{\|u_{0,\varepsilon}\|_V\}$ is uniformly bounded.

Take $v = u_{0,\varepsilon}$ in (3.17),

$$\begin{aligned} &\langle A_\varepsilon u_\varepsilon, u_{0,\varepsilon} - u_\varepsilon\rangle + \varphi_\varepsilon(\gamma_\varphi u_\varepsilon, \gamma_\varphi u_{0,\varepsilon}) - \varphi_\varepsilon(\gamma_\varphi u_\varepsilon, \gamma_\varphi u_\varepsilon) \\ &\quad + j_\varepsilon^0(\gamma_j u_\varepsilon; \gamma_j u_{0,\varepsilon} - \gamma_j u_\varepsilon) \ge \langle f_\varepsilon, u_{0,\varepsilon} - u_\varepsilon\rangle. \end{aligned}$$

Then using (3.12),

$$m_A \|u_\varepsilon - u_{0,\varepsilon}\|_V^2 \le \langle A_\varepsilon u_\varepsilon, u_\varepsilon - u_{0,\varepsilon}\rangle - \langle A_\varepsilon u_{0,\varepsilon}, u_\varepsilon - u_{0,\varepsilon}\rangle.$$

Thus,

$$\begin{aligned} m_A \|u_\varepsilon - u_{0,\varepsilon}\|_V^2 &\le \varphi_\varepsilon(\gamma_\varphi u_\varepsilon, \gamma_\varphi u_{0,\varepsilon}) - \varphi_\varepsilon(\gamma_\varphi u_\varepsilon, \gamma_\varphi u_\varepsilon) + j_\varepsilon^0(\gamma_j u_\varepsilon; \gamma_j u_{0,\varepsilon} - \gamma_j u_\varepsilon) \\ &\quad + \langle f_\varepsilon - A_\varepsilon u_{0,\varepsilon}, u_\varepsilon - u_{0,\varepsilon}\rangle. \end{aligned} \tag{3.21}$$

By (3.13) with $z_1 = z_3 = \gamma_\varphi u_\varepsilon$ and $z_2 = z_4 = \gamma_\varphi u_{0,\varepsilon}$, we have

$$\varphi_\varepsilon(\gamma_\varphi u_\varepsilon, \gamma_\varphi u_{0,\varepsilon}) - \varphi_\varepsilon(\gamma_\varphi u_\varepsilon, \gamma_\varphi u_\varepsilon) + \varphi_\varepsilon(\gamma_\varphi u_{0,\varepsilon}, \gamma_\varphi u_\varepsilon) - \varphi_\varepsilon(\gamma_\varphi u_{0,\varepsilon}, \gamma_\varphi u_{0,\varepsilon}) \le \alpha_\varphi c_\varphi^2 \|u_\varepsilon - u_{0,\varepsilon}\|_V^2.$$

So,

$$\varphi_\varepsilon(\gamma_\varphi u_\varepsilon, \gamma_\varphi u_{0,\varepsilon}) - \varphi_\varepsilon(\gamma_\varphi u_\varepsilon, \gamma_\varphi u_\varepsilon) \le \alpha_\varphi c_\varphi^2 \|u_\varepsilon - u_{0,\varepsilon}\|_V^2 + \varphi_\varepsilon(\gamma_\varphi u_{0,\varepsilon}, \gamma_\varphi u_{0,\varepsilon}) - \varphi_\varepsilon(\gamma_\varphi u_{0,\varepsilon}, \gamma_\varphi u_\varepsilon). \tag{3.22}$$

By (3.18) with $z_1 = \gamma_\varphi u_{0,\varepsilon}$ and $z_2 = \gamma_\varphi u_\varepsilon$,

$$\begin{aligned}\varphi_\varepsilon(\gamma_\varphi u_{0,\varepsilon}, \gamma_\varphi u_{0,\varepsilon}) - \varphi_\varepsilon(\gamma_\varphi u_{0,\varepsilon}, \gamma_\varphi u_\varepsilon) \le\ & b_\varphi(\varepsilon)\left(1 + \|\gamma_\varphi u_{0,\varepsilon}\|_{V_\varphi}\right) \|\gamma_\varphi(u_\varepsilon - u_{0,\varepsilon})\|_{V_\varphi} \\ & + \varphi(\gamma_\varphi u_{0,\varepsilon}, \gamma_\varphi u_{0,\varepsilon}) - \varphi(\gamma_\varphi u_{0,\varepsilon}, \gamma_\varphi u_\varepsilon).\end{aligned} \tag{3.23}$$

By (3.2) with $z_1 = z_4 = \gamma_\varphi u_{0,\varepsilon}$, $z_2 = \gamma_\varphi u_0$ and $z_3 = \gamma_\varphi u_\varepsilon$,

$$\begin{aligned}\varphi(\gamma_\varphi u_{0,\varepsilon}, \gamma_\varphi u_{0,\varepsilon}) - \varphi(\gamma_\varphi u_{0,\varepsilon}, \gamma_\varphi u_\varepsilon) \le\ & \alpha_\varphi \|\gamma_\varphi(u_{0,\varepsilon} - u_0)\|_{V_\varphi} \|\gamma_\varphi(u_{0,\varepsilon} - u_\varepsilon)\|_{V_\varphi} \\ & + \varphi(\gamma_\varphi u_0, \gamma_\varphi u_{0,\varepsilon}) - \varphi(\gamma_\varphi u_0, \gamma_\varphi u_\varepsilon).\end{aligned}$$

Applying Lemma 2.2, we have two constants c_3 and c_4, depending only on φ and u_0, such that

$$\varphi(\gamma_\varphi u_0, z) \ge c_3 + c_4 \|z\|_{V_\varphi} \quad \forall\, z \in V_\varphi.$$

Then,

$$\begin{aligned}\varphi(\gamma_\varphi u_{0,\varepsilon}, \gamma_\varphi u_{0,\varepsilon}) - \varphi(\gamma_\varphi u_{0,\varepsilon}, \gamma_\varphi u_\varepsilon) \le\ & \alpha_\varphi \|\gamma_\varphi(u_{0,\varepsilon} - u_0)\|_{V_\varphi} \|\gamma_\varphi(u_{0,\varepsilon} - u_\varepsilon)\|_{V_\varphi} \\ & + \varphi(\gamma_\varphi u_0, \gamma_\varphi u_{0,\varepsilon}) - \left(c_3 + c_4 \|\gamma_\varphi u_\varepsilon\|_{V_\varphi}\right).\end{aligned} \tag{3.24}$$

Combining (3.22)–(3.24), we have

$$\begin{aligned}\varphi_\varepsilon(\gamma_\varphi u_\varepsilon, \gamma_\varphi u_{0,\varepsilon}) - \varphi_\varepsilon(\gamma_\varphi u_\varepsilon, \gamma_\varphi u_\varepsilon) \le\ & \alpha_\varphi c_\varphi^2 \|u_\varepsilon - u_{0,\varepsilon}\|_V^2 + \alpha_\varphi \|\gamma_\varphi(u_{0,\varepsilon} - u_0)\|_{V_\varphi} \|\gamma_\varphi(u_{0,\varepsilon} - u_\varepsilon)\|_{V_\varphi} \\ & + b_\varphi(\varepsilon)\left(1 + \|\gamma_\varphi u_{0,\varepsilon}\|_{V_\varphi}\right) \|\gamma_\varphi(u_\varepsilon - u_{0,\varepsilon})\|_{V_\varphi} \\ & + \varphi(\gamma_\varphi u_0, \gamma_\varphi u_{0,\varepsilon}) - \left(c_3 + c_4 \|\gamma_\varphi u_\varepsilon\|_{V_\varphi}\right).\end{aligned} \tag{3.25}$$

By (3.15),

$$j_\varepsilon^0(\gamma_j u_\varepsilon; \gamma_j u_{0,\varepsilon} - \gamma_j u_\varepsilon) + j_\varepsilon^0(\gamma_j u_{0,\varepsilon}; \gamma_j u_\varepsilon - \gamma_j u_{0,\varepsilon}) \le \alpha_j \|\gamma_j(u_\varepsilon - u_{0,\varepsilon})\|_{V_j}^2$$

and so

$$j_\varepsilon^0(\gamma_j u_\varepsilon; \gamma_j u_{0,\varepsilon} - \gamma_j u_\varepsilon) \le \alpha_j c_j^2 \|u_\varepsilon - u_{0,\varepsilon}\|_V^2 - j_\varepsilon^0(\gamma_j u_{0,\varepsilon}; \gamma_j u_\varepsilon - \gamma_j u_{0,\varepsilon}).$$

Further, use the bound (3.14),

$$j_\varepsilon^0(\gamma_j u_\varepsilon; \gamma_j u_{0,\varepsilon} - \gamma_j u_\varepsilon) \le \alpha_j c_j^2 \|u_\varepsilon - u_{0,\varepsilon}\|_V^2 + c\left(1 + \|u_{0,\varepsilon}\|_V\right) \|\gamma_j(u_\varepsilon - u_{0,\varepsilon})\|_{V_j}.$$

Summarizing, we deduce from (3.21) that

$$\begin{aligned}m_A \|u_\varepsilon - u_{0,\varepsilon}\|_V^2 \le\ & \left(\alpha_\varphi c_\varphi^2 + \alpha_j c_j^2\right) \|u_\varepsilon - u_{0,\varepsilon}\|_V^2 + \alpha_\varphi \|\gamma_\varphi(u_{0,\varepsilon} - u_0)\|_{V_\varphi} \|\gamma_\varphi(u_{0,\varepsilon} - u_\varepsilon)\|_{V_\varphi} \\ & + b_\varphi(\varepsilon)\left(1 + \|\gamma_\varphi u_{0,\varepsilon}\|_{V_\varphi}\right) \|\gamma_\varphi(u_\varepsilon - u_{0,\varepsilon})\|_{V_\varphi} + \varphi(\gamma_\varphi u_0, \gamma_\varphi u_{0,\varepsilon}) \\ & - \left(c_3 + c_4 \|\gamma_\varphi u_\varepsilon\|_{V_\varphi}\right) + c\left(1 + \|u_{0,\varepsilon}\|_V\right) \|\gamma_j(u_\varepsilon - u_{0,\varepsilon})\|_{V_j} \\ & + \|f_\varepsilon - A_\varepsilon u_{0,\varepsilon}\|_{V^*} \|u_\varepsilon - u_{0,\varepsilon}\|_V.\end{aligned} \tag{3.26}$$

Recalling the smallness condition (3.5), we use the modified Cauchy–Schwarz inequality and elementary manipulations to find

$$\begin{aligned}&\alpha_\varphi\|\gamma_\varphi(u_{0,\varepsilon}-u_0)\|_{V_\varphi}\|\gamma_\varphi(u_{0,\varepsilon}-u_\varepsilon)\|_{V_\varphi}+b_\varphi(\varepsilon)\left(1+\|\gamma_\varphi u_{0,\varepsilon}\|_{V_\varphi}\right)\|\gamma_\varphi(u_\varepsilon-u_{0,\varepsilon})\|_{V_\varphi}\\&\quad-\left(c_3+c_4\|\gamma_\varphi u_\varepsilon\|_{V_\varphi}\right)+c\left(1+\|u_{0,\varepsilon}\|_V\right)\|\gamma_j(u_\varepsilon-u_{0,\varepsilon})\|_{V_j}+\|f_\varepsilon-A_\varepsilon u_{0,\varepsilon}\|_{V^*}\|u_\varepsilon-u_{0,\varepsilon}\|_V\\&\qquad\le\frac{1}{2}\left(m_A-\alpha_\varphi c_\varphi^2-\alpha_j c_j^2\right)\|u_\varepsilon-u_{0,\varepsilon}\|_V^2\\&\qquad\quad+c\left(1+\|u_{0,\varepsilon}-u_0\|_V^2+b_\varphi(\varepsilon)^4+\|u_{0,\varepsilon}\|_V^2+\|u_{0,\varepsilon}\|_V^4+\|f_\varepsilon-A_\varepsilon u_{0,\varepsilon}\|_{V^*}^2\right).\end{aligned}$$

So from (3.26), we have

$$\begin{aligned}&\frac{1}{2}\left(m_A-\alpha_\varphi c_\varphi^2-\alpha_j c_j^2\right)\|u_\varepsilon-u_{0,\varepsilon}\|_V^2\\&\quad\le c\left(1+\|u_{0,\varepsilon}-u_0\|_V^2+b_\varphi(\varepsilon)^4+\|u_{0,\varepsilon}\|_V^2+\|u_{0,\varepsilon}\|_V^4+\|f_\varepsilon-A_\varepsilon u_{0,\varepsilon}\|_{V^*}^2\right)+\varphi(\gamma_\varphi u_0,\gamma_\varphi u_{0,\varepsilon}).\end{aligned}$$

Using the boundedness of $b_\varphi(\varepsilon)$, $\|u_{0,\varepsilon}\|_V$, $\|f_\varepsilon\|_{V^*}$, uniform boundedness of the operators $A_\varepsilon\in\mathcal{L}(V,V^*)$, and noting that $\varphi(\gamma_\varphi u_0,\gamma_\varphi u_{0,\varepsilon})\to\varphi(\gamma_\varphi u_0,\gamma_\varphi u_0)$ (cf. Lemma 2.1), we conclude that $\{\|u_\varepsilon-u_{0,\varepsilon}\|_V\}$, and then $\{\|u_\varepsilon\|_V\}$, is uniformly bounded.

Step 2. We prove the weak convergence:

$$u_\varepsilon\rightharpoonup u\quad\text{in } V \text{ as } \varepsilon\to 0.$$

Since $\{\|u_\varepsilon\|_V\}$ is uniformly bounded, and γ_φ and γ_j are compact, there exist a subsequence, still denoted by $\{u_\varepsilon\}$, and an element $w\in V$ such that as $\varepsilon\to 0$,

$$\begin{aligned}&u_\varepsilon\rightharpoonup w\quad\text{in } V,\\&\gamma_\varphi u_\varepsilon\to\gamma_\varphi w\quad\text{in } V_\varphi,\\&\gamma_j u_\varepsilon\to\gamma_j w\quad\text{in } V_j.\end{aligned}$$

By $(H_{K_\varepsilon\to K})$ (i), $w\in K$.

Now fix an arbitrary $v\in K$. Then by $(H_{K_\varepsilon\to K})$ (ii), there exist $v_\varepsilon\in K_\varepsilon$ such that

$$v_\varepsilon\to v\quad\text{in } V \text{ as } \varepsilon\to 0.$$

We start with the following inequality (cf. Lemma 3.2)

$$\begin{aligned}&\langle A_\varepsilon v_\varepsilon,v_\varepsilon-u_\varepsilon\rangle+\varphi_\varepsilon(\gamma_\varphi u_\varepsilon,\gamma_\varphi v_\varepsilon)-\varphi_\varepsilon(\gamma_\varphi u_\varepsilon,\gamma_\varphi u_\varepsilon)\\&\quad+j_\varepsilon^0(\gamma_j u_\varepsilon;\gamma_j v_\varepsilon-\gamma_j u_\varepsilon)\ge\langle f_\varepsilon,v_\varepsilon-u_\varepsilon\rangle.\end{aligned}\tag{3.27}$$

Write

$$\langle A_\varepsilon v_\varepsilon,v_\varepsilon-u_\varepsilon\rangle=\langle A_\varepsilon(v_\varepsilon-v),v_\varepsilon-u_\varepsilon\rangle+\langle(A_\varepsilon-A)\,v,v_\varepsilon-u_\varepsilon\rangle+\langle Av,v_\varepsilon-u_\varepsilon\rangle.$$

We have

$$|\langle A_\varepsilon(v_\varepsilon-v),v_\varepsilon-u_\varepsilon\rangle|\le\|A_\varepsilon\|\,\|v_\varepsilon-v\|_V\|v_\varepsilon-u_\varepsilon\|_V\to 0,$$

since $\|A_\varepsilon\|$ and $\|v_\varepsilon-u_\varepsilon\|_V$ are uniformly bounded, and $\|v_\varepsilon-v\|_V\to 0$. Similarly,

$$|\langle(A_\varepsilon-A)\,v,v_\varepsilon-u_\varepsilon\rangle|\le\|\,(A_\varepsilon-A)\,v\|_{V^*}\|v_\varepsilon-u_\varepsilon\|_V\to 0$$

and

$$\langle Av,v_\varepsilon-u_\varepsilon\rangle\to\langle Av,v-w\rangle.$$

Hence,

$$\langle A_\varepsilon v_\varepsilon, v_\varepsilon - u_\varepsilon\rangle \to \langle Av, v - w\rangle \quad \text{as } \varepsilon \to 0. \tag{3.28}$$

By $(H_{\varphi_\varepsilon\to\varphi})$,

$$\varphi_\varepsilon(\gamma_\varphi u_\varepsilon, \gamma_\varphi v_\varepsilon) - \varphi_\varepsilon(\gamma_\varphi u_\varepsilon, \gamma_\varphi u_\varepsilon) \le \varphi(\gamma_\varphi u_\varepsilon, \gamma_\varphi v_\varepsilon) - \varphi(\gamma_\varphi u_\varepsilon, \gamma_\varphi u_\varepsilon) \\ + b_\varphi(\varepsilon)\left(1 + \|\gamma_\varphi u_\varepsilon\|_{V_\varphi}\right)\|\gamma_\varphi(u_\varepsilon - v_\varepsilon)\|_{V_\varphi}.$$

Since $\|\gamma_\varphi u_\varepsilon\|_{V_\varphi}$ and $\|\gamma_\varphi(u_\varepsilon - v_\varepsilon)\|_{V_\varphi}$ are uniformly bounded, $b_\varphi(\varepsilon) \to 0$ as $\varepsilon \to 0$, we have

$$b_\varphi(\varepsilon)\left(1 + \|\gamma_\varphi u_\varepsilon\|_{V_\varphi}\right)\|\gamma_\varphi(u_\varepsilon - v_\varepsilon)\|_{V_\varphi} \to 0 \quad \text{as } \varepsilon \to 0.$$

By (3.2) with $z_1 = z_3 = \gamma_\varphi u_\varepsilon$, $z_2 = \gamma_\varphi w$ and $z_4 = \gamma_\varphi v_\varepsilon$,

$$\varphi(\gamma_\varphi u_\varepsilon, \gamma_\varphi v_\varepsilon) - \varphi(\gamma_\varphi u_\varepsilon, \gamma_\varphi u_\varepsilon) \le \varphi(\gamma_\varphi w, \gamma_\varphi v_\varepsilon) - \varphi(\gamma_\varphi w, \gamma_\varphi u_\varepsilon) \\ + \alpha_\varphi\|\gamma_\varphi(u_\varepsilon - w)\|_{V_\varphi}\|\gamma_\varphi(u_\varepsilon - v_\varepsilon)\|_{V_\varphi}.$$

Here, $\|\gamma_\varphi(u_\varepsilon - v_\varepsilon)\|_{V_\varphi}$ is uniformly bounded and $\|\gamma_\varphi(u_\varepsilon - w)\|_{V_\varphi} \to 0$. Thus,

$$\alpha_\varphi\|\gamma_\varphi(u_\varepsilon - w)\|_{V_\varphi}\|\gamma_\varphi(u_\varepsilon - v_\varepsilon)\|_{V_\varphi} \to 0 \quad \text{as } \varepsilon \to 0.$$

By (H_φ) and Lemma 2.1, $\varphi(\gamma_\varphi w, \cdot)$ is continuous on V_φ. Then as $\varepsilon \to 0$,

$$\varphi(\gamma_\varphi w, \gamma_\varphi v_\varepsilon) \to \varphi(\gamma_\varphi w, \gamma_\varphi v),$$
$$\varphi(\gamma_\varphi w, \gamma_\varphi u_\varepsilon) \to \varphi(\gamma_\varphi w, \gamma_\varphi w).$$

By $(H_{j_\varepsilon\to j})$,

$$j_\varepsilon^0(\gamma_j u_\varepsilon; \gamma_j v_\varepsilon - \gamma_j u_\varepsilon) \le j^0(\gamma_j u_\varepsilon; \gamma_j v_\varepsilon - \gamma_j u_\varepsilon) + b_j(\varepsilon)\left(1 + \|\gamma_j u_\varepsilon\|_{V_j}\right)\|\gamma_j(v_\varepsilon - u_\varepsilon)\|_{V_j}.$$

Due to the uniform boundedness of $\|\gamma_j u_\varepsilon\|_{V_j}$ and $\|\gamma_j(v_\varepsilon - u_\varepsilon)\|_{V_j}$, and condition on $b_j(\varepsilon)$,

$$b_j(\varepsilon)\left(1 + \|\gamma_j u_\varepsilon\|_{V_j}\right)\|\gamma_j(v_\varepsilon - u_\varepsilon)\|_{V_j} \to 0 \quad \text{as } \varepsilon \to 0.$$

Moreover,

$$j^0(\gamma_j w; \gamma_j v - \gamma_j w) \ge \limsup_{\varepsilon\to 0} j^0(\gamma_j u_\varepsilon; \gamma_j v_\varepsilon - \gamma_j u_\varepsilon).$$

Therefore, taking the upper limit in (3.27), we obtain

$$\langle Av, v - w\rangle + \varphi(\gamma_\varphi w, \gamma_\varphi v) - \varphi(\gamma_\varphi w, \gamma_\varphi w) + j^0(\gamma_j w; \gamma_j v - \gamma_j w) \ge \langle f, v - w\rangle.$$

This inequality is valid for any $v \in K$. By Lemma 3.2, $w \in K$ is a solution of Problem (P). Since the solution u to Problem (P) is unique, we have $w = u$.

Step 3. We prove the strong convergence:

$$u_\varepsilon \to u \quad \text{as } \varepsilon \to 0.$$

Apply the condition (3.12),

$$m_A\|u_\varepsilon - u\|_V^2 \le \langle A_\varepsilon u_\varepsilon - A_\varepsilon u, u_\varepsilon - u\rangle.$$

Thus,

$$m_A\|u_\varepsilon - u\|_V^2 \le \langle A_\varepsilon u_\varepsilon, u_\varepsilon - u\rangle + \langle (A - A_\varepsilon)u, u_\varepsilon - u\rangle - \langle Au, u_\varepsilon - u\rangle. \tag{3.29}$$

First, we have, as $\varepsilon \to 0$,

$$|\langle (A - A_\varepsilon)u, u_\varepsilon - u\rangle| \le \|(A - A_\varepsilon)u\|_{V^*}\|u_\varepsilon - u\|_V \to 0, \tag{3.30}$$

$$|\langle Au, u_\varepsilon - u\rangle| \to 0. \tag{3.31}$$

By $(H_{K_\varepsilon \to K})$, there exists $w_\varepsilon \in K_\varepsilon$ such that

$$w_\varepsilon \to u \quad \text{in } V.$$

Write

$$\langle A_\varepsilon u_\varepsilon, u_\varepsilon - u\rangle = \langle A_\varepsilon u_\varepsilon, u_\varepsilon - w_\varepsilon\rangle + \langle A_\varepsilon u_\varepsilon, w_\varepsilon - u\rangle. \tag{3.32}$$

Note that

$$|\langle A_\varepsilon u_\varepsilon, w_\varepsilon - u\rangle| \le \|A_\varepsilon\|\,\|u_\varepsilon\|_V\|w_\varepsilon - u\|_V \to 0. \tag{3.33}$$

By (3.17) with $v = w_\varepsilon$,

$$\langle A_\varepsilon u_\varepsilon, u_\varepsilon - w_\varepsilon\rangle \le \varphi_\varepsilon(\gamma_\varphi u_\varepsilon, \gamma_\varphi w_\varepsilon) - \varphi_\varepsilon(\gamma_\varphi u_\varepsilon, \gamma_\varphi u_\varepsilon) + j_\varepsilon^0(\gamma_j u_\varepsilon; \gamma_j w_\varepsilon - \gamma_j u_\varepsilon) - \langle f_\varepsilon, w_\varepsilon - u_\varepsilon\rangle. \tag{3.34}$$

By $(H_{\varphi_\varepsilon \to \varphi})$ with $z_1 = \gamma_\varphi u_\varepsilon$ and $z_2 = \gamma_\varphi w_\varepsilon$,

$$\begin{aligned}\varphi_\varepsilon(\gamma_\varphi u_\varepsilon, \gamma_\varphi w_\varepsilon) - \varphi_\varepsilon(\gamma_\varphi u_\varepsilon, \gamma_\varphi u_\varepsilon) &\le \varphi(\gamma_\varphi u_\varepsilon, \gamma_\varphi w_\varepsilon) - \varphi(\gamma_\varphi u_\varepsilon, \gamma_\varphi u_\varepsilon)\\ &\quad + b_\varphi(\varepsilon)\left(1 + \|\gamma_\varphi u_\varepsilon\|_{V_\varphi}\right)\|\gamma_\varphi(u_\varepsilon - w_\varepsilon)\|_{V_\varphi}.\end{aligned}$$

Note that since $u_\varepsilon \rightharpoonup u$ in V, $w_\varepsilon \to u$ in V, and $\gamma_\varphi \colon V \to V_\varphi$ is compact, we have

$$\|\gamma_\varphi(u_\varepsilon - w_\varepsilon)\|_{V_\varphi} \to 0 \quad \text{as } \varepsilon \to 0$$

and since $\{\|u_\varepsilon\|_V\}_{\varepsilon>0}$ is uniformly bounded, $b_\varphi(\varepsilon) \to 0$ as $\varepsilon \to 0$, it follows that

$$b_\varphi(\varepsilon)\left(1 + \|\gamma_\varphi u_\varepsilon\|_{V_\varphi}\right)\|\gamma_\varphi(u_\varepsilon - w_\varepsilon)\|_{V_\varphi} \to 0 \quad \text{as } \varepsilon \to 0.$$

By (3.2) with $z_1 = z_3 = \gamma_\varphi u_\varepsilon$, $z_2 = \gamma_\varphi u$ and $z_4 = \gamma_\varphi w_\varepsilon$,

$$\begin{aligned}\varphi(\gamma_\varphi u_\varepsilon, \gamma_\varphi w_\varepsilon) - \varphi(\gamma_\varphi u_\varepsilon, \gamma_\varphi u_\varepsilon) &\le \varphi(\gamma_\varphi u, \gamma_\varphi w_\varepsilon) - \varphi(\gamma_\varphi u, \gamma_\varphi u_\varepsilon)\\ &\quad + \alpha_\varphi\|\gamma_\varphi(u_\varepsilon - u)\|_{V_\varphi}\|\gamma_\varphi(u_\varepsilon - w_\varepsilon)\|_{V_\varphi},\end{aligned}$$

where

$$\alpha_\varphi\|\gamma_\varphi(u_\varepsilon - u)\|_{V_\varphi}\|\gamma_\varphi(u_\varepsilon - w_\varepsilon)\|_{V_\varphi} \to 0 \quad \text{as } \varepsilon \to 0.$$

By the continuity of $\varphi(\gamma_\varphi u, \cdot)$ on V_φ,

$$\varphi(\gamma_\varphi u, \gamma_\varphi w_\varepsilon) - \varphi(\gamma_\varphi u, \gamma_\varphi u_\varepsilon) \to \varphi(\gamma_\varphi u, \gamma_\varphi u) - \varphi(\gamma_\varphi u, \gamma_\varphi u) = 0.$$

From $(H_{j_\varepsilon \to j})$,

$$j_\varepsilon^0(\gamma_j u_\varepsilon; \gamma_j w_\varepsilon - \gamma_j u_\varepsilon) \le j^0(\gamma_j u_\varepsilon; \gamma_j w_\varepsilon - \gamma_j u_\varepsilon) + b_j(\varepsilon)\left(1 + \|\gamma_j u_\varepsilon\|_{V_j}\right)\|\gamma_j(u_\varepsilon - w_\varepsilon)\|_{V_j}.$$

Due to the uniform boundedness of $\|\gamma_j u_\varepsilon\|_{V_j}$, $\|\gamma_j(u_\varepsilon - w_\varepsilon)\|_{V_j} \to 0$, and $b_j(\varepsilon) \to 0$, we have

$$b_j(\varepsilon)\left(1 + \|\gamma_j u_\varepsilon\|_{V_j}\right)\|\gamma_j(u_\varepsilon - w_\varepsilon)\|_{V_j} \to 0 \quad \text{as } \varepsilon \to 0.$$

Moreover, by (2.1),

$$\limsup_{\varepsilon \to 0} j^0(\gamma_j u_\varepsilon; \gamma_j w_\varepsilon - \gamma_j u_\varepsilon) \le j^0(\gamma_j u; \gamma_j u - \gamma_j u) = 0.$$

Finally,

$$\langle f_\varepsilon, w_\varepsilon - u_\varepsilon\rangle = \langle f_\varepsilon - f, w_\varepsilon - u_\varepsilon\rangle + \langle f, w_\varepsilon - u_\varepsilon\rangle \to 0.$$

Summarizing, from (3.34), we deduce that

$$\limsup_{\varepsilon\to 0}\langle A_\varepsilon u_\varepsilon, u_\varepsilon - w_\varepsilon\rangle \le 0. \tag{3.35}$$

Therefore, from (3.29)–(3.33) and (3.35), we conclude that

$$m_A \|u_\varepsilon - u\|_V^2 \to 0,$$

i.e., we have the strong convergence $u_\varepsilon \to u$ as $\varepsilon \to 0$. ■

4. Stability results for special constrained inequality problems

Problem (P) contains as special cases various problems studied in the literature. In this section, we apply Theorem 3.4 to deduce stability results for several special inequality problems with $K \neq V$. The inclusion $u \in K$ represents a constraint on the solution u. The unconstrained special cases are discussed in the next section.

Special case 1. When $\varphi(z_1, z_2) \equiv \varphi(z_2)$ is a function of the second argument z_2 only, the problem (3.7) has the form

$$u \in K, \quad \langle Au, v - u\rangle + \varphi(\gamma_\varphi v) - \varphi(\gamma_\varphi u) + j^0(\gamma_j u; \gamma_j v - \gamma_j u) \ge \langle f, v - u\rangle \quad \forall\, v \in K. \tag{4.1}$$

The condition (3.2) is trivially satisfied since the left side of (3.2) is identically zero. We introduce the following assumptions to replace (H_φ) and (H_s):

(H'_φ) V_φ is a Banach space and $\gamma_\varphi \in \mathcal{L}(V, V_\varphi)$ with its norm bounded by c_φ. φ: $V_\varphi \to \mathbb{R}$ is convex and l.s.c.

(H'_s)

$$\alpha_j c_j^2 < m_A. \tag{4.2}$$

Then by Theorem 3.1, under the assumptions (H_V), (H_K), (H_A), (H'_φ), (H_j), (H'_s) and (H_f), the inequality (4.1) has a unique solution $u \in K$.

The perturbed inequality problem is

$$u_\varepsilon \in K_\varepsilon, \quad \langle A_\varepsilon u_\varepsilon, v - u_\varepsilon\rangle + \varphi_\varepsilon(\gamma_\varphi v) - \varphi_\varepsilon(\gamma_\varphi u_\varepsilon) + j^0_\varepsilon(\gamma_j u_\varepsilon; \gamma_j v - \gamma_j u_\varepsilon) \ge \langle f_\varepsilon, v - u_\varepsilon\rangle \quad \forall\, v \in K_\varepsilon. \tag{4.3}$$

The assumption $(H_{\varphi_\varepsilon})$ is to be replaced by

$(H'_{\varphi_\varepsilon})$ V_φ is a Banach space and $\gamma_\varphi \in \mathcal{L}(V, V_\varphi)$ with its norm bounded by c_φ. φ_ε: $V_\varphi \to \mathbb{R}$ is convex and l.s.c.

Then by Theorem 3.3, under the assumptions (H_V), (H_{K_ε}), (H_{A_ε}), $(H'_{\varphi_\varepsilon})$, (H_{j_ε}), (H'_s) and (H_{f_ε}), the inequality (4.3) has a unique solution $u \in K$.

For stability analysis, we replace $(H_{\varphi_\varepsilon\to\varphi})$ by

$(H'_{\varphi_\varepsilon\to\varphi})$: There exists a non-negative valued function $b_\varphi(\varepsilon)$ with $b_\varphi(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$|\varphi_\varepsilon(z_2) - \varphi_\varepsilon(z_1) - \varphi(z_2) + \varphi(z_1)| \le b_\varphi(\varepsilon)\, \|z_1 - z_2\|_{V_\varphi} \quad \forall\, z_1, z_2 \in V_\varphi. \tag{4.4}$$

Then, by applying Theorem 3.4, we conclude the convergence

$$u_\varepsilon \to u \text{ in } V \text{ as } \varepsilon \to 0$$

for the solutions u of (4.1) and u_ε of (4.3), under the additional assumptions $(H_{A_\varepsilon\to A})$, $(H'_{\varphi_\varepsilon\to\varphi})$, $(H_{j_\varepsilon\to j})$, $(H_{f_\varepsilon\to f})$, $(H_{K_\varepsilon\to K})$ and (H_c).

Special case 2. When $\varphi\equiv 0$, we have a "pure" hemivariational inequality from (3.7) [19]:

$$u\in K,\quad \langle Au,v-u\rangle + j^0(\gamma_j u;\gamma_j v-\gamma_j u)\ge \langle f,v-u\rangle\quad \forall\, v\in K. \tag{4.5}$$

Then by Theorem 3.1, under the assumptions (H_V), (H_K), (H_A), (H_j), (H'_s) and (H_f), the inequality (4.5) has a unique solution $u\in K$.

The perturbed inequality problem is

$$u_\varepsilon\in K_\varepsilon,\quad \langle A_\varepsilon u_\varepsilon,v-u_\varepsilon\rangle + j^0_\varepsilon(\gamma_j u_\varepsilon;\gamma_j v-\gamma_j u_\varepsilon)\ge \langle f_\varepsilon,v-u_\varepsilon\rangle\quad \forall\, v\in K_\varepsilon. \tag{4.6}$$

By Theorem 3.3, under the assumptions (H_V), (H_{K_ε}), (H_{A_ε}), (H_{j_ε}), (H'_s) and (H_{f_ε}), the inequality (4.6) has a unique solution $u_\varepsilon\in K_\varepsilon$.

For stability, we apply Theorem 3.4 to conclude

$$u_\varepsilon\to u \text{ in } V \text{ as } \varepsilon\to 0$$

for the solutions u of (4.5) and u_ε of (4.6), under the additional assumptions $(H_{A_\varepsilon\to A})$, $(H_{j_\varepsilon\to j})$, $(H_{f_\varepsilon\to f})$, $(H_{K_\varepsilon\to K})$ and (H'_c). Here,

(H'_c) $\gamma_j\in\mathcal{L}(V,V_j)$ is compact.

Special case 3. When $j\equiv 0$, we have a quasi-variational inequality from (3.7) [8]:

$$u\in K,\quad \langle Au,v-u\rangle + \varphi(\gamma_\varphi u,\gamma_\varphi v)-\varphi(\gamma_\varphi u,\gamma_\varphi u)\ge \langle f,v-u\rangle\quad \forall\, v\in K. \tag{4.7}$$

The smallest condition (H_s) is modified to

(H''_s)
$$\alpha_\varphi c_\varphi^2 < m_A. \tag{4.8}$$

By Theorem 3.1, under the assumptions (H_V), (H_K), (H_A), (H_φ), (H''_s) and (H_f), the inequality (4.7) has a unique solution $u\in K$.

The perturbed quasi-variational inequality is

$$u_\varepsilon\in K_\varepsilon,\quad \langle A_\varepsilon u_\varepsilon,v-u_\varepsilon\rangle + \varphi_\varepsilon(\gamma_\varphi u_\varepsilon,\gamma_\varphi v)-\varphi_\varepsilon(\gamma_\varphi u_\varepsilon,\gamma_\varphi u_\varepsilon)\ge \langle f_\varepsilon,v-u_\varepsilon\rangle\quad \forall\, v\in K_\varepsilon. \tag{4.9}$$

By Theorem 3.3, under the assumptions (H_V), (H_{K_ε}), (H_{A_ε}), $(H_{\varphi_\varepsilon})$, (H''_s) and (H_{f_ε}), the inequality (4.9) has a unique solution $u_\varepsilon\in K_\varepsilon$.

For stability, we apply Theorem 3.4 to conclude

$$u_\varepsilon\to u \text{ in } V \text{ as } \varepsilon\to 0$$

for the solutions u of (4.7) and u_ε of (4.9), under the additional assumptions $(H_{A_\varepsilon\to A})$, $(H_{\varphi_\varepsilon\to\varphi})$, $(H_{f_\varepsilon\to f})$, $(H_{K_\varepsilon\to K})$ and (H''_c). Here,

(H''_c) $\gamma_\varphi\in\mathcal{L}(V,V_\varphi)$ is compact.

Special case 4. When $j\equiv 0$ and $\varphi(z_1,z_2)\equiv\varphi(z_2)$, we have the variational inequality [1,29]

$$u\in K,\quad \langle Au,v-u\rangle + \varphi(\gamma_\varphi v)-\varphi(\gamma_\varphi u)\ge \langle f,v-u\rangle\quad \forall\, v\in K. \tag{4.10}$$

By Theorem 3.1, under the assumptions (H_V), (H_K), (H_A), (H'_φ), (H''_s) and (H_f), the inequality (4.10) has a unique solution $u\in K$.

The perturbed inequality problem is

$$u_\varepsilon \in K_\varepsilon, \quad \langle A_\varepsilon u_\varepsilon, v - u_\varepsilon\rangle + \varphi_\varepsilon(\gamma_\varphi v) - \varphi_\varepsilon(\gamma_\varphi u_\varepsilon) \geq \langle f_\varepsilon, v - u_\varepsilon\rangle \quad \forall\, v \in K_\varepsilon. \tag{4.11}$$

By Theorem 3.3, under the assumptions (H_V), (H_{K_ε}), (H_{A_ε}), $(H'_{\varphi_\varepsilon})$, (H''_s) and (H_{f_ε}), the inequality (4.11) has a unique solution $u_\varepsilon \in K_\varepsilon$.

For stability, we apply Theorem 3.4 to conclude

$$u_\varepsilon \to u \text{ in } V \text{ as } \varepsilon \to 0$$

for the solutions u of (4.10) and u_ε of (4.11), under the additional assumptions $(H_{A_\varepsilon\to A})$, $(H'_{\varphi_\varepsilon\to\varphi})$, $(H_{f_\varepsilon\to f})$, $(H_{K_\varepsilon\to K})$ and (H''_c).

Special case 5. When $j \equiv 0$ and $\varphi \equiv 0$, we have the variational inequality of the first kind [1]

$$u \in K, \quad \langle Au, v - u\rangle \geq \langle f, v - u\rangle \quad \forall\, v \in K. \tag{4.12}$$

By Theorem 3.1, under the assumptions (H_V), (H_K), (H_A) and (H_f), the variational inequality (4.12) has a unique solution $u \in K$.

The perturbed inequality problem is

$$u_\varepsilon \in K_\varepsilon, \quad \langle A_\varepsilon u_\varepsilon, v - u_\varepsilon\rangle \geq \langle f_\varepsilon, v - u_\varepsilon\rangle \quad \forall\, v \in K_\varepsilon. \tag{4.13}$$

By Theorem 3.3, under the assumptions (H_V), (H_{K_ε}), (H_{A_ε}) and (H_{f_ε}), the inequality (4.13) has a unique solution $u_\varepsilon \in K_\varepsilon$.

For stability, we apply Theorem 3.4 to conclude

$$u_\varepsilon \to u \text{ in } V \text{ as } \varepsilon \to 0$$

for the solutions u of (4.12) and u_ε of (4.13), under the additional assumptions $(H_{A_\varepsilon\to A})$, $(H_{f_\varepsilon\to f})$ and $(H_{K_\varepsilon\to K})$.

5. Stability results for special inequality problems without constraints

When $K = V$, we have the special cases of unconstrained problems. The general variational -hemivariational inequality (3.7) is reduced to

$$\begin{aligned} u \in V, \qquad & \langle Au, v - u\rangle + \varphi(\gamma_\varphi u, \gamma_\varphi v) - \varphi(\gamma_\varphi u, \gamma_\varphi u) \\ & \quad + j^0(\gamma_j u; \gamma_j v - \gamma_j u) \geq \langle f, v - u\rangle \quad \forall\, v \in V \end{aligned} \tag{5.1}$$

with the corresponding perturbed problem

$$\begin{aligned} u_\varepsilon \in V, \qquad & \langle A_\varepsilon u_\varepsilon, v - u_\varepsilon\rangle + \varphi_\varepsilon(\gamma_\varphi u_\varepsilon, \gamma_\varphi v) - \varphi_\varepsilon(\gamma_\varphi u_\varepsilon, \gamma_\varphi u_\varepsilon) \\ & \quad + j_\varepsilon^0(\gamma_j u_\varepsilon; \gamma_j v - \gamma_j u_\varepsilon) \geq \langle f_\varepsilon, v - u_\varepsilon\rangle \quad \forall\, v \in V. \end{aligned} \tag{5.2}$$

We can apply Theorems 3.1, 3.3 and 3.4 and conclude that under assumptions (H_V), (H_A), (H_φ), (H_j), (H_s) and (H_f), the problem (5.1) has a unique solution $u \in V$; under assumptions (H_V), (H_{A_ε}), $(H_{\varphi_\varepsilon})$, (H_{j_ε}), (H_s) and (H_{f_ε}), the problem (5.2) has a unique solution $u_\varepsilon \in K_\varepsilon$; and under additional assumptions $(H_{A_\varepsilon\to A})$, $(H_{\varphi_\varepsilon\to\varphi})$, $(H_{j_\varepsilon\to j})$, $(H_{f_\varepsilon\to f})$ and (H_c), we have the convergence result

$$u_\varepsilon \to u \text{ in } V \text{ as } \varepsilon \to 0.$$

The unconstrained counterparts of the five special cases in Section 4 are as follows.

Special case 1′. When $K = V$ and $\varphi(z_1, z_2) \equiv \varphi(z_2)$ is a function of the second argument z_2 only, the problem is

$$u \in V, \quad \langle Au, v-u\rangle + \varphi(\gamma_\varphi v) - \varphi(\gamma_\varphi u) + j^0(\gamma_j u; \gamma_j v - \gamma_j u) \geq \langle f, v-u\rangle \quad \forall\, v \in V. \tag{5.3}$$

The corresponding perturbed problem is

$$u_\varepsilon \in V, \quad \langle A_\varepsilon u_\varepsilon, v-u_\varepsilon\rangle + \varphi_\varepsilon(\gamma_\varphi v) - \varphi_\varepsilon(\gamma_\varphi u_\varepsilon) + j_\varepsilon^0(\gamma_j u_\varepsilon; \gamma_j v - \gamma_j u_\varepsilon) \geq \langle f_\varepsilon, v-u_\varepsilon\rangle \quad \forall\, v \in V. \tag{5.4}$$

Then, applying Theorems 3.1, 3.3 and 3.4, we know that the problem (5.3) has a unique solution $u \in V$ under assumptions (H_V), (H_A), (H'_φ), (H_j), (H'_s) and (H_f); the problem (5.4) has a unique solution $u_\varepsilon \in V$ under assumptions (H_V), (H_{A_ε}), $(H'_{\varphi_\varepsilon})$, (H_{j_ε}), (H'_s) and (H_{f_ε}); and

$$u_\varepsilon \to u \text{ in } V \text{ as } \varepsilon \to 0$$

under additional assumptions $(H_{A_\varepsilon \to A})$, $(H'_{\varphi_\varepsilon \to \varphi})$, $(H_{j_\varepsilon \to j})$, $(H_{f_\varepsilon \to f})$ and (H_c).

Special case 2′. When $K = V$ and $\varphi \equiv 0$, we have the simplest form hemivariational inequality [10]:

$$u \in V, \quad \langle Au, v\rangle + j^0(\gamma_j u; \gamma_j v) \geq \langle f, v\rangle \quad \forall\, v \in V. \tag{5.5}$$

The corresponding perturbed problem is

$$u_\varepsilon \in V, \quad \langle A_\varepsilon u_\varepsilon, v-u_\varepsilon\rangle + j_\varepsilon^0(\gamma_j u_\varepsilon; \gamma_j v - \gamma_j u_\varepsilon) \geq \langle f_\varepsilon, v-u_\varepsilon\rangle \quad \forall\, v \in V. \tag{5.6}$$

By Theorems 3.1, 3.3 and 3.4, the problem (5.5) has a unique solution $u \in V$ under assumptions (H_V), (H_A), (H_j), (H'_s) and (H_f); the problem (5.6) has a unique solution $u_\varepsilon \in V$ under assumptions (H_V), (H_{A_ε}), (H_{j_ε}), (H'_s) and (H_{f_ε}); and

$$u_\varepsilon \to u \text{ in } V \text{ as } \varepsilon \to 0$$

under additional assumptions $(H_{A_\varepsilon \to A})$, $(H_{j_\varepsilon \to j})$, $(H_{f_\varepsilon \to f})$ and (H'_c).

Special case 3′. When $K = V$ and $j \equiv 0$, we have a quasi-variational inequality [29]:

$$u \in V, \quad \langle Au, v-u\rangle + \varphi(\gamma_\varphi u, \gamma_\varphi v) - \varphi(\gamma_\varphi u, \gamma_\varphi u) \geq \langle f, v-u\rangle \quad \forall\, v \in V. \tag{5.7}$$

The corresponding perturbed problem is

$$u_\varepsilon \in V, \quad \langle A_\varepsilon u_\varepsilon, v-u_\varepsilon\rangle + \varphi_\varepsilon(\gamma_\varphi u_\varepsilon, \gamma_\varphi v) - \varphi_\varepsilon(\gamma_\varphi u_\varepsilon, \gamma_\varphi u_\varepsilon) \geq \langle f_\varepsilon, v-u_\varepsilon\rangle \quad \forall\, v \in V. \tag{5.8}$$

By Theorems 3.1, 3.3 and 3.4, the problem (5.7) has a unique solution $u \in V$ under assumptions (H_V), (H_A), (H_φ), (H''_s) and (H_f); the problem (5.8) has a unique solution $u_\varepsilon \in V$ under assumptions (H_V), (H_{A_ε}), $(H_{\varphi_\varepsilon})$, (H_{j_ε}), (H''_s) and (H_{f_ε}); and

$$u_\varepsilon \to u \text{ in } V \text{ as } \varepsilon \to 0$$

under additional assumptions $(H_{A_\varepsilon \to A})$, $(H'_{\varphi_\varepsilon \to \varphi})$, $(H_{f_\varepsilon \to f})$ and (H''_c).

Special case 4′. When $K = V$, $j \equiv 0$ and $\varphi(z_1, z_2) \equiv \varphi(z_2)$, we have the variational inequality [1,8]

$$u \in V, \quad \langle Au, v-u\rangle + \varphi(\gamma_\varphi v) - \varphi(\gamma_\varphi u) \geq \langle f, v-u\rangle \quad \forall\, v \in V. \tag{5.9}$$

The corresponding perturbed problem is

$$u_\varepsilon \in V, \quad \langle A_\varepsilon u_\varepsilon, v-u_\varepsilon\rangle + \varphi_\varepsilon(\gamma_\varphi v) - \varphi_\varepsilon(\gamma_\varphi u_\varepsilon) \geq \langle f_\varepsilon, v-u_\varepsilon\rangle \quad \forall\, v \in V. \tag{5.10}$$

By Theorems 3.1, 3.3 and 3.4, we know that the problem (5.9) has a unique solution $u \in V$ under assumptions (H_V), (H_A), (H'_φ), (H''_s) and (H_f); the problem (5.10) has a unique solution $u_\varepsilon \in V$ under assumptions (H_V), (H_{A_ε}), $(H'_{\varphi_\varepsilon})$, (H''_s) and (H_{f_ε}); and

$$u_\varepsilon \to u \text{ in } V \text{ as } \varepsilon \to 0$$

under additional assumptions $(H_{A_\varepsilon \to A})$, $(H'_{\varphi_\varepsilon \to \varphi})$, $(H_{f_\varepsilon \to f})$ and (H''_c).

Special case 5′. When $K = V$, $j \equiv 0$ and $\varphi \equiv 0$, we have the variational equation:

$$u \in V, \quad \langle Au, v\rangle = \langle f, v\rangle \quad \forall\, v \in V. \tag{5.11}$$

The corresponding perturbed problem is

$$u_\varepsilon \in V, \quad \langle A_\varepsilon u_\varepsilon, v\rangle = \langle f_\varepsilon, v - u_\varepsilon\rangle \quad \forall\, v \in V. \tag{5.12}$$

By Theorems 3.1, 3.3 and 3.4, we know that the problem (5.11) has a unique solution $u \in V$ under assumptions (H_V), (H_A) and (H_f); the problem (5.12) has a unique solution $u_\varepsilon \in V$ under assumptions (H_V), (H_{A_ε}) and (H_{f_ε}); and

$$u_\varepsilon \to u \text{ in } V \text{ as } \varepsilon \to 0$$

under additional assumptions $(H_{A_\varepsilon \to A})$ and $(H_{f_\varepsilon \to f})$.

6. Applications in sample contact problems

In this section, we take stability analysis of two static contact problems as examples to illustrate the application of the theoretical results presented in previous sections. The physical setting of a contact problem is as follows: the reference configuration of a deformable body is an open, bounded, connected set $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3 in applications) with a Lipschitz boundary $\Gamma = \partial\Omega$ partitioned into three disjoint and measurable parts Γ_1, Γ_2 and Γ_3 such that $\operatorname{meas}(\Gamma_1) > 0$. The body is fixed on Γ_1, is in contact on Γ_3 with a foundation, and is in equilibrium under the action of a volume force of density $\boldsymbol{f}_0$ in Ω and a surface traction of density $\boldsymbol{f}_2$ on Γ_2. The material of the deformable body is assumed to be elastic.

To describe the contact problems, we use $\boldsymbol{u}\colon \Omega \to \mathbb{R}^d$ for the displacement field, $\boldsymbol{\varepsilon}(\boldsymbol{u}) := \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right)/2$ for the linearized strain tensor, and $\boldsymbol{\sigma}\colon \Omega \to \mathbb{S}^d$ for the stress field. Here, the symbol $\mathbb{S}^d$ denotes the space of second order symmetric tensors on $\mathbb{R}^d$. We use "$\cdot$" and "$\|\cdot\|$" for the canonical inner product and norm on the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$.

Let $\boldsymbol{\nu}$ be the unit outward normal vector on the boundary Γ, which is defined a.e. For a vector field $\boldsymbol{v}$, $v_\nu := \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau := \boldsymbol{v} - v_\nu \boldsymbol{\nu}$ are the normal and tangential components of $\boldsymbol{v}$ on Γ. For the stress field $\boldsymbol{\sigma}$, $\sigma_\nu := (\boldsymbol{\sigma\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau := \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}$ are its normal and tangential components on the boundary. For the stress and strain fields, we will use the Hilbert space $Q = L^2(\Omega; \mathbb{S}^d)$ with the canonical inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_Q := \int_\Omega \sigma_{ij}(\boldsymbol{x})\, \tau_{ij}(\boldsymbol{x})\, dx, \quad \boldsymbol{\sigma}, \boldsymbol{\tau} \in Q.$$

The function space for the displacement field is

$$V = \left\{ \boldsymbol{v} = (v_i) \in H^1(\Omega; \mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_1 \right\}.$$

Since $\operatorname{meas}(\Gamma_1) > 0$, an application of Korn's inequality shows that V is a Hilbert space with the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_V := \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \quad \boldsymbol{u}, \boldsymbol{v} \in V.$$

We will use the same symbol $\boldsymbol{v}$ for the trace of a function $\boldsymbol{v} \in H^1(\Omega; \mathbb{R}^d)$ on Γ.

6.1. A contact problem with unilateral constraint

The equations and conditions for this contact problem are

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{6.1}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega, \tag{6.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{6.3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2, \tag{6.4}$$

supplemented by the following contact conditions [24]:

$$u_\nu \le g,\ \sigma_\nu + \xi_\nu \le 0,\ (u_\nu - g)(\sigma_\nu + \xi_\nu) = 0,\ \xi_\nu \in \partial j_\nu(u_\nu) \quad \text{on } \Gamma_3, \tag{6.5}$$

$$\|\boldsymbol{\sigma}_\tau\| \le F_b(u_\nu),\ -\boldsymbol{\sigma}_\tau = F_b(u_\nu)\,\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \text{ if } \boldsymbol{u}_\tau \ne \boldsymbol{0} \quad \text{on } \Gamma_3. \tag{6.6}$$

In these equations and conditions, (6.1) is the elastic constitutive law, (6.2) represents the equilibrium equation, (6.3) is the displacement boundary condition, and (6.4) describes the traction boundary condition. In (6.1), $\mathcal{F}\colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is the elasticity operator and is assumed to have the following properties:

$$\begin{cases} \text{(a) there exists } L_{\mathcal{F}} > 0 \text{ such that for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega, \\ \qquad \|\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|; \\ \text{(b) there exists } m_{\mathcal{F}} > 0 \text{ such that for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega, \\ \qquad (\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2; \\ \text{(c) } \mathcal{F}(\cdot, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d; \\ \text{(d) } \mathcal{F}(\boldsymbol{x}, \boldsymbol{0}) = \boldsymbol{0} \text{ for a.e. } \boldsymbol{x} \in \Omega. \end{cases} \tag{6.7}$$

The densities of the body force and the surface traction are assumed to satisfy

$$\boldsymbol{f}_0 \in L^2(\Omega; \mathbb{R}^d), \quad \boldsymbol{f}_2 \in L^2(\Gamma_2; \mathbb{R}^d). \tag{6.8}$$

We define $\boldsymbol{f} \in V^*$ by the relation

$$\langle \boldsymbol{f}, \boldsymbol{v} \rangle_{V^* \times V} = (\boldsymbol{f}_0, \boldsymbol{v})_{L^2(\Omega;\mathbb{R}^d)} + (\boldsymbol{f}_2, \boldsymbol{v})_{L^2(\Gamma_2;\mathbb{R}^d)} \quad \forall\, \boldsymbol{v} \in V. \tag{6.9}$$

In the normal contact condition (6.5), the relation $u_\nu \le g$ restricts the allowed penetration, where g represents the thickness of the elastic layer. We assume $g\colon \Gamma_3 \to \mathbb{R}$ satisfies

$$g \in L^2(\Gamma_3), \quad g(\boldsymbol{x}) \ge 0 \quad \text{a.e. on } \Gamma_3. \tag{6.10}$$

The contact condition (6.5) represents a combination of the Signorini contact condition for contact with a rigid foundation and the normal compliance condition for contact with a deformable foundation. Details on the normal compliance and Signorini contact conditions can be found in [13,30]. The tangential contact condition (6.6) describes a version of Coulomb's law of dry friction. The friction bound $F_b\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ may depend on the normal displacement u_ν, and we assume

$$\begin{cases} \text{(a) There exists } L_{F_b} > 0 \text{ such that} \\ \qquad |F_b(\boldsymbol{x}, r_1) - F_b(\boldsymbol{x}, r_2)| \le L_{F_b} |r_1 - r_2| \quad \forall\, r_1, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3; \\ \text{(b) } F_b(\cdot, r) \text{ is measurable on } \Gamma_3, \text{ for all } r \in \mathbb{R}; \\ \text{(c) } F_b(\boldsymbol{x}, r) = 0 \text{ for } r \le 0,\ F_b(\boldsymbol{x}, r) \ge 0 \text{ for } r \ge 0, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{6.11}$$

For the potential function $j_\nu\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$, we assume

$$\begin{cases} \text{(a) } j_\nu(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R} \text{ and there} \\ \qquad \text{exists } \overline{e} \in L^2(\Gamma_3) \text{ such that } j_\nu(\cdot, \overline{e}(\cdot)) \in L^1(\Gamma_3); \\ \text{(b) } j_\nu(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_3; \\ \text{(c) } |\partial j_\nu(\boldsymbol{x}, r)| \le \overline{c}_0 + \overline{c}_1 |r| \text{ for a.e. } \boldsymbol{x} \in \Gamma_3\ \forall\, r \in \mathbb{R} \text{ with } \overline{c}_0,\ \overline{c}_1 \ge 0; \\ \text{(d) } j_\nu^0(\boldsymbol{x}, r_1; r_2 - r_1) + j_\nu^0(\boldsymbol{x}, r_2; r_1 - r_2) \le \alpha_{j_\nu} |r_1 - r_2|^2 \\ \qquad \text{for a.e. } \boldsymbol{x} \in \Gamma_3, \text{ all } r_1, r_2 \in \mathbb{R} \text{ with } \alpha_{j_\nu} \ge 0. \end{cases} \tag{6.12}$$

The displacement will be sought from the following subset of the space V:

$$K := \{\boldsymbol{v} \in V \mid v_\nu \le g \text{ on } \Gamma_3\}.$$

The weak formulation of the contact problem is the following.

PROBLEM (P$_1$). *Find a displacement field $\boldsymbol{u} \in K$ such that*

$$(\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u}))_Q + \int_{\Gamma_3} F_b(u_\nu)\,(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_\tau\|)\,ds$$
$$+ \int_{\Gamma_3} j_\nu^0(u_\nu; v_\nu - u_\nu)\,ds \ge \langle \boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u}\rangle_{V^*\times V} \quad \forall\, \boldsymbol{v} \in K. \tag{6.13}$$

We can apply the results of the previous sections on Problem (P$_1$). Let $V_\varphi = L^2(\Gamma_3)^d$ with γ_φ the trace operator from V to V_φ, $V_j = L^2(\Gamma_3)$ with $\gamma_j \boldsymbol{v} = v_\nu$ for $\boldsymbol{v} \in V$. Define

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = (\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q,$$
$$\varphi(\gamma_\varphi \boldsymbol{u}, \gamma_\varphi \boldsymbol{v}) = \int_{\Gamma_3} F_b(u_\nu)\,\|\boldsymbol{v}_\tau\|\,ds,$$
$$j(\gamma_j \boldsymbol{v}) = \int_{\Gamma_3} j_\nu(v_\nu)\,ds.$$

Then (3.2) is satisfied with $\alpha_\varphi = L_{F_b}$:

$$\begin{aligned}\varphi(\boldsymbol{z}_1, \boldsymbol{z}_4) - \varphi(\boldsymbol{z}_1, \boldsymbol{z}_3) + \varphi(\boldsymbol{z}_2, \boldsymbol{z}_3) - \varphi(\boldsymbol{z}_2, \boldsymbol{z}_4) &= \int_{\Gamma_3} (F_b(z_{1,\nu}) - F_b(z_{2,\nu}))\,(\|\boldsymbol{z}_{4,\tau}\| - \|\boldsymbol{z}_{3,\tau}\|)\,ds \\ &\le L_{F_b} \int_{\Gamma_3} |z_{1,\nu} - z_{2,\nu}|\,\|\boldsymbol{z}_{3,\tau} - \boldsymbol{z}_{4,\tau}\|\,ds \\ &\le L_{F_b} \|\boldsymbol{z}_1 - \boldsymbol{z}_2\|_{L^2(\Gamma_3)^d} \|\boldsymbol{z}_3 - \boldsymbol{z}_4\|_{L^2(\Gamma_3)^d}.\end{aligned}$$

Moreover, $\alpha_j = \alpha_{j_\nu}$. Applying Theorem 3.1, we know that Problem (P$_1$) has a unique solution $\boldsymbol{u} \in K$ under the stated assumptions, and (3.5) takes the form

$$L_{F_b} \lambda_{1,V}^{-1} + \alpha_{j_\nu} \lambda_{1\nu,V}^{-1} < m_{\mathcal{F}}, \tag{6.14}$$

where $\lambda_{1,V} > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in V, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx = \lambda \int_{\Gamma_3} \boldsymbol{u}\cdot\boldsymbol{v}\,ds \quad \forall\, \boldsymbol{v} \in V,$$

and $\lambda_{1\nu,V} > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in V, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx = \lambda \int_{\Gamma_3} u_\nu v_\nu ds \quad \forall\, \boldsymbol{v} \in V.$$

The perturbation of Problem (P$_1$) is the following.

PROBLEM (P$_{1,\varepsilon}$). *Find a displacement field $\boldsymbol{u}_\varepsilon \in K_\varepsilon$ such that*

$$(\mathcal{F}_\varepsilon(\boldsymbol{\varepsilon}(\boldsymbol{u}_\varepsilon)), \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u}_\varepsilon))_Q + \int_{\Gamma_3} F_{b,\varepsilon}(u_\nu)\,(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_{\varepsilon,\tau}\|)\,ds$$
$$+ \int_{\Gamma_3} j_{\varepsilon,\nu}^0(u_{\varepsilon,\nu}; v_\nu - u_{\varepsilon,\nu})\,ds \ge \langle \boldsymbol{f}_\varepsilon, \boldsymbol{v}-\boldsymbol{u}_\varepsilon\rangle_{V^*\times V} \quad \forall\, \boldsymbol{v} \in K_\varepsilon. \tag{6.15}$$

We assume (6.7) with $\mathcal{F}$ replaced by $\mathcal{F}_\varepsilon$, (6.11) with F_b replaced by $F_{b,\varepsilon}$, (6.12) with j_ν replaced by $j_{\varepsilon,\nu}$, (6.8) with $\boldsymbol{f}_0$ and $\boldsymbol{f}_2$ replaced by $\boldsymbol{f}_{0,\varepsilon}$ and $\boldsymbol{f}_{2,\varepsilon}$, and (6.10) with g replaced by g_ε. The linear functional $\boldsymbol{f}_\varepsilon$ and the constraint set K_ε are defined by

$$\langle \boldsymbol{f}_\varepsilon, \boldsymbol{v}\rangle_{V^*\times V} = (\boldsymbol{f}_{0,\varepsilon}, \boldsymbol{v})_{L^2(\Omega;\mathbb{R}^d)} + (\boldsymbol{f}_{2,\varepsilon}, \boldsymbol{v})_{L^2(\Gamma_2;\mathbb{R}^d)} \quad \forall\, \boldsymbol{v} \in V,$$
$$K_\varepsilon = \{\boldsymbol{v} \in V \mid v_\nu \le g_\varepsilon \text{ on } \Gamma_3\}.$$

Then Problem (P$_{1,\varepsilon}$) has a unique solution $\boldsymbol{u}_\varepsilon \in K_\varepsilon$ under the condition (6.14).

Applying Theorem 3.4, we have the stability result

$$\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|_V \to 0 \quad \text{as } \varepsilon \to 0$$

under the assumptions $(H_{A_\varepsilon \to A})$, $(H_{\varphi_\varepsilon \to \varphi})$, $(H_{j_\varepsilon \to j})$, $(H_{f_\varepsilon \to f})$ and $(H_{K_\varepsilon \to K})$ adapted to the notation of Problem (P_1) and Problem $(\mathrm{P}_{1,\varepsilon})$.

As an example of perturbations of the constraint set K, consider

$$K_\varepsilon := \{\boldsymbol{v} \in V \mid v_\nu \le (1+\delta_\varepsilon)\, g \text{ on } \Gamma_3\},$$

i.e., $g_\varepsilon = (1+\delta_\varepsilon)\, g$, where $\delta_\varepsilon = \delta_\varepsilon(\boldsymbol{x})$ is Lipschitz continuous in $\boldsymbol{x} \in \Omega$, for some constant $c > 0$,

$$|\delta_\varepsilon(\boldsymbol{x})| \le c\,\varepsilon, \quad \boldsymbol{x} \in \Omega,$$

and

$$\|\nabla \delta_\varepsilon\|_{L^2(\Omega)^d} \to 0 \quad \text{as } \varepsilon \to 0.$$

In the verification of condition $(H_{K_\varepsilon \to K})$ (i), we note that if $\{\boldsymbol{v}_\varepsilon\} \subset K_\varepsilon$ and $\boldsymbol{v}_\varepsilon \rightharpoonup \boldsymbol{v}$ in V, then $\boldsymbol{v}_\varepsilon \to \boldsymbol{v}$ in $L^2(\Gamma_3)^d$ and for a subsequence, $\boldsymbol{v}_\varepsilon(\boldsymbol{x}) \to \boldsymbol{v}(\boldsymbol{x})$ a.e. $\boldsymbol{x} \in \Gamma_3$. Thus, from

$$v_{\varepsilon\,\nu}(\boldsymbol{x}) \le (1+\delta_\varepsilon(\boldsymbol{x}))\, g(\boldsymbol{x}), \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3$$

we have

$$v_\nu(\boldsymbol{x}) \le g(\boldsymbol{x}), \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3.$$

Therefore, the limit $\boldsymbol{v} \in K$. To verify the condition $(H_{K_\varepsilon \to K})$ (ii), for $\boldsymbol{v} \in K$, we can simply choose $\boldsymbol{v}_\varepsilon = \boldsymbol{v}/(1+\delta_\varepsilon)$. Thus, $(H_{K_\varepsilon \to K})$ is satisfied.

As an example of perturbations of A, consider

$$\langle A_\varepsilon \boldsymbol{u}, \boldsymbol{v}\rangle = (\mathcal{F}_\varepsilon(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q.$$

Assume

$$\mathcal{F}_\varepsilon(\boldsymbol{\varepsilon}(\boldsymbol{v})) \rightharpoonup \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{v})) \quad \text{in } Q \text{ for } \boldsymbol{v} \in V.$$

Then, $(H_{A_\varepsilon \to A})$ is satisfied.

As an example of perturbations of φ, consider

$$\varphi_\varepsilon(\boldsymbol{z}_1, \boldsymbol{z}_2) = \int_{\Gamma_3} F_{b,\varepsilon}(z_{1,\nu})\, \|\boldsymbol{z}_{2,\tau}\|\, ds$$

and assume, for some constant $c > 0$,

$$|F_{b,\varepsilon}(t) - F_b(t)| \le c\,\varepsilon\,(1+|t|)\,.$$

Then from

$$\varphi_\varepsilon(\boldsymbol{z}_1, \boldsymbol{z}_2) - \varphi_\varepsilon(\boldsymbol{z}_1, \boldsymbol{z}_1) - \varphi(\boldsymbol{z}_1, \boldsymbol{z}_2) + \varphi(\boldsymbol{z}_1, \boldsymbol{z}_1) = \int_{\Gamma_3} (F_{b,\varepsilon}(z_{1,\nu}) - F_b(z_{1,\nu}))\,(\|\boldsymbol{z}_{2,\tau}\| - \|\boldsymbol{z}_{1,\tau}\|)\, ds,$$

we conclude

$$\begin{aligned} |\varphi_\varepsilon(\boldsymbol{z}_1, \boldsymbol{z}_2) - \varphi_\varepsilon(\boldsymbol{z}_1, \boldsymbol{z}_1) - \varphi(\boldsymbol{z}_1, \boldsymbol{z}_2) + \varphi(\boldsymbol{z}_1, \boldsymbol{z}_1)| &\le \int_{\Gamma_3} |F_{b,\varepsilon}(z_{1,\nu}) - F_b(z_{1,\nu})|\, \|\boldsymbol{z}_{1,\tau} - \boldsymbol{z}_{2,\tau}\|\, ds \\ &\le \int_{\Gamma_3} c\,\varepsilon\,(1+|z_{1,\nu}|)\, \|\boldsymbol{z}_{1,\tau} - \boldsymbol{z}_{2,\tau}\|\, ds \\ &\le \tilde{c}\,\varepsilon\,\big(1+\|\boldsymbol{z}_1\|_{V_\varphi}\big)\, \|\boldsymbol{z}_1 - \boldsymbol{z}_2\|_{V_\varphi} \end{aligned}$$

for possibly a different constant $\tilde{c} > 0$ in the last upper bound. Thus, (3.18) is satisfied with $b_\varphi(\varepsilon) = \tilde{c}\,\varepsilon$.

For a concrete choice of j_ν, we consider [18]

$$j_\nu(\boldsymbol{x};t) = S(\boldsymbol{x}) \int_0^{|t|} \mu(r)\,dr,$$

where $S(\boldsymbol{x})\colon \overline{\Gamma_3} \to \mathbb{R}_+$ is a continuous function of $\boldsymbol{x} \in \overline{\Gamma_3}$, $\mu\colon [0,\infty) \to \mathbb{R}$ is continuous and

$$|\mu(r)| \le \bar{c}\,(1+r) \quad \forall\, r \ge 0, \tag{6.16}$$

$$\mu(r_1) - \mu(r_2) \ge -\lambda\,(r_1 - r_2)\,, \quad \lambda > 0,\ \forall\, r_1 > r_2 \ge 0. \tag{6.17}$$

Then,

$$|\tau| \le S\,\bar{c}\,(1+|t|) \quad \forall\, t \in \mathbb{R},\ \tau \in \partial j_\nu(t),$$
$$(\tau_1 - \tau_2)\,(t_1 - t_2) \ge -S\,\lambda\,|t_1 - t_2|^2 \quad \forall\, t_i \in \mathbb{R},\ \tau_i \in \partial j_\nu(t_i),\ i = 1,2.$$

Following from [18, Lemma 3.2],

$$j_\nu^0(t_1;t_2) = \begin{cases} \operatorname{sgn}(t_1)\,S\,\mu(|t_1|)\,t_2 & \text{if } t_1 \ne 0, \\ S\,\mu(0)\,|t_2| & \text{if } t_1 = 0, \end{cases} \tag{6.18}$$

and

$$j^0(\gamma_j \boldsymbol{v}; \gamma_j \boldsymbol{w}) = \int_{\Gamma_3} j_\nu^0(v_\nu; w_\nu)\,ds, \quad \boldsymbol{v}, \boldsymbol{w} \in V. \tag{6.19}$$

As an example of perturbations of j, let

$$j_{\nu,\varepsilon}(\boldsymbol{x};t) = S_\varepsilon(\boldsymbol{x}) \int_0^{|t|} \mu(r)\,dr,$$

where $S_\varepsilon(\boldsymbol{x})\colon \overline{\Gamma_3} \to \mathbb{R}_+$ is a continuous function of $\boldsymbol{x} \in \overline{\Gamma_3}$, and

$$\max_{\overline{\Gamma_3}} |S_\varepsilon - S| \to 0 \quad \text{as } \varepsilon \to 0. \tag{6.20}$$

Then by (6.18),

$$\left| j_{\nu,\varepsilon}^0(t_1; t_2 - t_1) - j_\nu^0(t_1; t_2 - t_1) \right| \le |S_\varepsilon - S|\,\mu(|t_1|)\,|t_2 - t_1| \quad \forall\, t_1, t_2 \in \mathbb{R},$$

and combined with (6.19),

$$\left| j_\varepsilon^0(\gamma_j \boldsymbol{v}; \gamma_j \boldsymbol{w}) - j^0(\gamma_j \boldsymbol{v}; \gamma_j \boldsymbol{w}) \right| \le c \max_{\overline{\Gamma_3}} |S_\varepsilon - S| \left(1 + \|v_\nu\|_{L^2(\Gamma_3)}\right) \|w_\nu\|_{L^2(\Gamma_3)}, \quad \boldsymbol{v}, \boldsymbol{w} \in V.$$

Thus, $(H_{j_\varepsilon \to j})$ is satisfied in lieu of (6.20).

Finally, to satisfy the condition $(H_{f_\varepsilon \to f})$, we only need to assume

$$\|\boldsymbol{f}_{0,\varepsilon} - \boldsymbol{f}_0\|_{L^2(\Omega;\mathbb{R}^d)} + \|\boldsymbol{f}_{2,\varepsilon} - \boldsymbol{f}_2\|_{L^2(\Gamma_2;\mathbb{R}^d)} \to 0 \quad \text{as } \varepsilon \to 0.$$

6.2. A contact problem without unilateral constraint

For this contact problem, the equations and conditions (6.1)–(6.4) are supplemented by the following contact conditions [17]:

$$-\sigma_\nu \in \partial j_\nu(u_\nu), \quad \|\boldsymbol{\sigma}_\tau\| \le F_b(u_\nu), \quad -\boldsymbol{\sigma}_\tau = F_b(u_\nu)\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \text{ if } \boldsymbol{u}_\tau \ne \boldsymbol{0} \quad \text{on } \Gamma_3. \tag{6.21}$$

The potential function j_ν is assumed to satisfy (6.12), whereas the friction bound F_b is assumed to satisfy (6.11).

The contact condition in (6.21) does not involve the unilateral constraint and it can be viewed as a limiting case of the condition (6.5) as $g \to \infty$. The weak formulation of the corresponding contact problem is the following.

PROBLEM (P_2). *Find a displacement field $\boldsymbol{u} \in V$ such that*

$$(\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u}))_Q + \int_{\Gamma_3} F_b(u_\nu)\,(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_\tau\|)\,ds + \int_{\Gamma_3} j_\nu^0(u_\nu; v_\nu - u_\nu)\,ds \ge \langle \boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u}\rangle_{V^*\times V} \quad \forall\, \boldsymbol{v} \in V.$$

This problem can be viewed as a special case of Problem (P_1) where $K = V$. Thus, for the stability analysis of Problem (P_2), the discussion in Section 6.1 carries over verbatim with any reference to K or K_ε removed.

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