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Research paper

On variational–hemivariational inequalities in Banach spaces

Weimin Han [a,*,1,2], M.Z. Nashed [b,1]

[a] Department of Mathematics, University of Iowa, Iowa City, IA 52242-1410, USA
[b] Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA

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ABSTRACT

This paper is devoted to a well-posedness analysis of elliptic variational–hemivariational inequalities in Banach spaces. The differential operator associated with the variational–hemivariational inequality is assumed to be strongly monotone of a general order, in contrast to that in the majority of existing references on this subject where the differential operator is assumed to be strongly monotone of order 2. Moreover, the solution existence is proved with an approach more accessible to applied mathematicians and engineers, instead of through an abstract surjectivity result for pseudomonotone operators in existing references. Equivalent minimization principles are established for certain variational–hemivariational inequalities, which are valuable for developing efficient numerical algorithms. The theoretical results are applied to the analysis of a mixed hemivariational inequality in the study of a generalized Newtonian fluid flow problem involving a nonsmooth slip boundary condition of friction type. Existence and uniqueness of both the velocity and pressure unknowns are shown for the mixed hemivariational inequalities.

1. Introduction

Similar to variational inequalities, hemivariational inequalities play an important role in modeling and studying nonlinear, nonsmooth problems in applications. Variational inequalities refer to inequality problems in which non-smooth terms have a convex structure, whereas hemivariational inequalities are inequality problems in which non-smooth terms are allowed to be non-convex. Rigorous mathematical analysis on variational inequalities began in the 1960s, and the theory of variational inequalities forms a pretty mature area. Interest in hemivariational inequalities was started by Panagiotopoulos in the early 1980s [1], in responding to the need of modeling and solving engineering problems involving non-smooth, non-monotone or set-valued relations among physical quantities. Recent years have witnessed explosive growth in the literature on modeling, analysis, numerical approximation and simulations, and applications of hemivariational inequalities, or more generally, of variational–hemivariational inequalities. In this paper, the two terms "hemivariational inequalities" and "variational–hemivariational inequalities" are used interchangeably. Variational–hemivariational inequalities have the features of both variational inequalities and hemivariational inequalities, i.e., both convex and non-convex non-smooth terms are present in such problems. For two recent representative references, we

* Corresponding author.
E-mail addresses: weimin-han@uiowa.edu (W. Han), zuhair.nashed@ucf.edu (M.Z. Nashed).

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refer to [2] on theoretical analysis and to [3] on numerical analysis of variational–hemivariational inequalities. The latter reference employs the finite element method for the spatial discretization; other numerical methods have also been used for the spatial discretization, cf. [4–6] on the analysis of the virtual element method for solving such problems.

In general, variational–hemivariational inequalities do not correspond to minimization principles. Nevertheless, in [7], minimization principles are shown for some particular variational–hemivariational inequalities. A further step is taken in [8] where solution existence and uniqueness are proved for general variational–hemivariational inequalities starting with the results in [7]. The "standard" technique to prove the solution existence for variational–hemivariational inequalities is through applications of abstract surjectivity results for pseudomonotone operators (e.g., [2] and the references therein). The alternative approach presented in [8] on the study of variational–hemivariational inequalities avoids referencing to pseudomonotone operators and the related abstract surjectivity results, and is more accessible for applied mathematicians and engineers. Additional references in this regard include [9–11]. The well-posedness theories developed in the majority of the literature on variational–hemivariational inequalities are for the case of a strongly monotone operator of order 2 (e.g., [2,8,12,13]) and are thus essentially in the setting of Hilbert spaces. In this paper, we extend the alternative approach to study variational–hemivariational inequalities with a strongly monotone operator of a general order $p > 1$ which is genuinely in a Banach space setting. As an example of applications of the theoretical results, we study a generalized Newtonian fluid flow problem subject to a non-smooth slip boundary condition.

Early references on mathematical models for viscous incompressible Newtonian fluid flows involving nonsmooth slip or leak boundary conditions are [14,15]. Further studies of such problems can be found in numerous references, e.g., [16,17] on variational inequalities governed by the Stokes equations, and [18,19] on variational inequalities governed by the Navier–Stokes equations. In these references, the slip and leak boundary conditions are expressed by monotone relations between physical quantities, and thus the mathematical formulations of the problems are in the form of variational inequalities. When the nonsmooth boundary conditions involve non-monotone relations between physical quantities, the mathematical formulations become hemivariational inequalities, cf. [20–23]. Mixed finite element methods have been studied for the numerical solution of a stationary hemivariational inequality of the Stokes equations with the slip boundary condition in [24], and for that of the Navier–Stokes equations in [25]. Recently, hemivariational inequalities arising in non-Newtonian or generalized Newtonian fluid flow problems have been studied in several papers, cf. [26–28]. In a non-Newtonian fluid, the viscosity depends on the unknown solution. Non-Newtonian fluids are found in various industrial and engineering applications, cf. [29,30] and more recently [31].

Let us briefly summarize the main novelty of this paper. We provide a well-posedness analysis of elliptic variational–hemivariational inequalities in Banach spaces, starting with minimization principles for a special variational–hemivariational inequality. With applications in mind, we provide additional theoretical results that are not explored by other researchers (Theorems 4.1 and 4.4). These results are new and are presented in a way more accessible to applied mathematicians and engineers. The minimization principles are of independent importance, especially when the numerical solution of variational–hemivariational inequalities is concerned. In the study of the variational–hemivariational inequalities for the generalized Newtonian fluid flow problem with a non-smooth slip boundary condition of friction type, we provide not only the existence and uniqueness of the velocity variable, but also that of the pressure variable that is not available in the existing literature.

The rest of the paper is organized as follows. In Section 2, we review definitions and basic properties of the generalized directional derivative and generalized subdifferential in the sense of Clarke, and provide a detailed discussion of operators of strong monotonicity of a general order $p > 1$ in a Banach space. In Section 3, we consider an elliptic variational–hemivariational inequality in a reflexive Banach space, and prove the solution existence and uniqueness of the problem through the study of an equivalent minimization principle. In Section 4, we introduce variants of the results in Section 3 which are more relevant to applications, and extend the results to more general elliptic variational–hemivariational inequalities in a reflexive Banach space. In Section 5, we apply the theory developed in previous sections to the study of the steady incompressible generalized Newtonian fluid flow problem subject to non-smooth slip boundary conditions of friction type.

2. Preliminaries

This section consists of two parts. The first part provides a brief review of the generalized directional derivative and generalized subdifferential in the sense of Clarke. The second part discusses the strong convexity for functionals on a Banach space.

We first briefly recall the notions and basic properties of the generalized directional derivative and generalized subdifferential in the sense of Clarke [32,33]. Let V be a real Banach space. Denote by V^* the dual space of V, and by $\langle\cdot,\cdot\rangle$ the duality pairing between V^* and V. For a locally Lipschitz continuous functional $\Psi : V \to \mathbb{R}$ defined on V, the generalized (Clarke) directional derivative of Ψ at $u \in V$ in the direction $v \in V$ is

$$\Psi^0(u; v) := \limsup_{w\to u,\, \lambda\downarrow 0} \frac{\Psi(w + \lambda v) - \Psi(w)}{\lambda}.$$

Then, the generalized subdifferential of Ψ at $u \in V$ is defined as a subset of V^*:

$$\partial\Psi(u) := \left\{u^* \in V^* \mid \Psi^0(u; v) \ge \langle u^*, v\rangle\ \forall\, v \in V\right\}.$$

Some basic properties needed later are stated below. Details of the generalized directional derivative and the generalized subdifferential can be found in [33].

Proposition 2.1. *Let V be a real Banach space.*

(i) *For a locally Lipschitz continuous functional $\Psi : V \to \mathbb{R}$,*

$$\Psi^0(u; v) = \max\left\{\langle u^*, v\rangle \mid u^* \in \partial\Psi(u)\right\} \quad \forall\, u, v \in V. \tag{2.1}$$

(ii) *Let $\Psi, \Psi_1, \Psi_2 : V \to \mathbb{R}$ be locally Lipschitz continuous. Then $\partial(\lambda\,\Psi)(u) = \lambda\,\partial\Psi(u)$ for all $\lambda \in \mathbb{R}$ and all $u \in V$. Moreover, the inclusion*

$$\partial(\Psi_1 + \Psi_2)(u) \subset \partial\Psi_1(u) + \partial\Psi_2(u) \quad \forall\, u \in V \tag{2.2}$$

holds. This inclusion relation is equivalent to the inequality

$$(\Psi_1 + \Psi_2)^0(u; v) \le \Psi_1^0(u; v) + \Psi_2^0(u; v) \quad \forall\, u, v \in V. \tag{2.3}$$

(iii) *If $\Psi : V \to \mathbb{R}$ is locally Lipschitz continuous and convex, then the subdifferential $\partial\Psi(u)$ at any $u \in V$ in the sense of Clarke coincides with the convex subdifferential $\partial\Psi(u)$.*

Because of Proposition 2.1(iii), we use the same symbol ∂ to denote the subdifferential operator in the sense of Clarke as well as that in convex analysis.

Next, we discuss about strong convexity of functionals, strong monotonicity of operators, both of a general order $p \in (0, \infty)$. The presentation of this part extends that in [34] where notions of strong convexity and strong monotonicity of order 2 are discussed. For convenience, we provide detailed arguments of the results.

Definition 2.2. An operator $A : V \to V^*$ is said to be strongly monotone of order p if there exists a constant $m_A > 0$ such that

$$\langle Av_1 - Av_2, v_1 - v_2\rangle \ge m_A \|v_1 - v_2\|_V^p \quad \forall\, v_1, v_2 \in V. \tag{2.4}$$

Definition 2.3. A functional $\Phi : V \to \mathbb{R}$ is said to be strongly convex of order p if there exists a constant $m_\Phi > 0$ such that

$$\Phi(\lambda\, u + (1-\lambda)\, v) \le \lambda\,\Phi(u) + (1-\lambda)\,\Phi(v) - m_\Phi \lambda\,(1-\lambda)\,\|u - v\|_V^p \quad \forall\, u, v \in V,\ \forall\,\lambda \in [0, 1]. \tag{2.5}$$

Proposition 2.4. *Assume (2.5). Then*

$$\Phi(v) - \Phi(u) \ge \langle u^*, v - u\rangle + m_\Phi \|v - u\|_V^p \quad \forall\, u, v \in V,\ \forall\, u^* \in \partial\Phi(u). \tag{2.6}$$

Proof. We switch u and v in (2.5) to obtain

$$\Phi(u) - \Phi(v) + \frac{1}{\lambda}\left[\Phi(u + \lambda\,(v-u)) - \Phi(u)\right] \le -m_\Phi\,(1-\lambda)\,\|v-u\|_V^p \quad \forall\,\lambda \in (0, 1].$$

The condition (2.5) implies that Φ is convex and is bounded above on a non-empty open set in V; thus, Φ is locally Lipschitz continuous on V (cf. [35, Corollary 2.4, p. 12]). Take the upper limit as $\lambda \to 0+$ in the previous inequality,

$$\Phi(u) - \Phi(v) + \Phi^0(u; v-u) \le -m_\Phi \|v-u\|_V^p,$$

i.e.,

$$\Phi(v) - \Phi(u) \ge \Phi^0(u; v-u) + m_\Phi \|v-u\|_V^p. \tag{2.7}$$

Thanks to the property (2.1), we deduce the inequality (2.6) from (2.7). ■

Corollary 2.5. *A functional strongly convex of order $p > 1$ is coercive.*

Proof. Assume $\Phi : V \to \mathbb{R}$ is strongly convex. Choose an element $u_0 \in V$ and $u_0^* \in \partial\Phi(u_0)$. Then by Proposition 2.4,

$$\Phi(v) \ge m_\Phi \|v - u_0\|_V^p + \langle u_0^*, v - u_0\rangle + \Phi(u_0) \quad \forall\, v \in V.$$

Hence,

$$\begin{aligned}\Phi(v) &\geq m_\Phi \,|\, \|v\|_V - \|u_0\|_V|^p - \|u_0^*\|_{V^*} (\|v\|_V + \|u_0\|_V) + \Phi(u_0) \\ &\geq \frac{1}{2} m_\Phi \|v\|_V^p - c_1 \|v\|_V - c_2\end{aligned}$$

for some constants $c_1, c_2 \in \mathbb{R}$. So $\Phi(\cdot)$ is coercive on V. ■

Proposition 2.6. *Assume (2.6). Then we have a modified version of (2.5):*

$$\Phi(\lambda u + (1-\lambda) v) \leq \lambda \Phi(u) + (1-\lambda) \Phi(v) - m_\Phi c_p \lambda (1-\lambda) \|u - v\|_V^p \quad \forall u, v \in V, \ \forall \lambda \in [0, 1]. \tag{2.8}$$

where

$$c_p = \begin{cases} 2^{2-p} \text{ if } p \in (0, 1] \cup (2, \infty), \\ 1 \text{ if } p \in (1, 2]. \end{cases} \tag{2.9}$$

Proof. Let $\xi \in \partial\Phi(\lambda u + (1-\lambda) v)$. Then from (2.6),

$$\begin{aligned}\Phi(u) - \Phi(\lambda u + (1-\lambda) v) &\geq (1-\lambda) \langle \xi, u - v\rangle + m_\Phi (1-\lambda)^p \|u - v\|_V^p, \\ \Phi(v) - \Phi(\lambda u + (1-\lambda) v) &\geq \lambda \langle \xi, v - u\rangle + m_\Phi \lambda^p \|u - v\|_V^p.\end{aligned}$$

We multiply the first inequality by λ, multiply the second inequality by $(1-\lambda)$, and add the two resulting inequalities to obtain

$$\lambda \Phi(u) + (1-\lambda) \Phi(v) - \Phi(\lambda u + (1-\lambda) v) \geq m_\Phi \lambda (1-\lambda) g_p(\lambda) \|v - u\|_V^p, \tag{2.10}$$

where

$$g_p(\lambda) = \lambda^{p-1} + (1-\lambda)^{p-1}, \quad \lambda \in (0, 1).$$

Through elementary calculations, it can be shown that

$$\inf \left\{ g_p(\lambda) \mid 0 < \lambda < 1 \right\} = c_p.$$

Then (2.8) follows from (2.10) for $\lambda \in (0, 1)$ and the observation that (2.8) is obvious for $\lambda = 0, 1$. ■

Proposition 2.7. *Assume (2.6). Then*

$$\langle u^* - v^*, u - v\rangle \geq 2 m_\Phi \|u - v\|_V^p \quad \forall u, v \in V, \ \forall u^* \in \partial\Phi(u), \ v^* \in \partial\Phi(v). \tag{2.11}$$

Proof. For any $u, v \in V$ and any $u^* \in \partial\Phi(u)$, $v^* \in \partial\Phi(v)$, we have from (2.6) that

$$\begin{aligned}\Phi(v) - \Phi(u) &\geq \langle u^*, v - u\rangle + m_\Phi \|v - u\|_V^p, \\ \Phi(u) - \Phi(v) &\geq \langle v^*, u - v\rangle + m_\Phi \|u - v\|_V^p.\end{aligned}$$

Adding these two inequalities, we obtain (2.11). ■

Proposition 2.8. *Assume (2.11). Then we have a modified version of (2.6):*

$$\Phi(v) - \Phi(u) \geq \langle u^*, v - u\rangle + \frac{2}{p} m_\Phi \|v - u\|_V^p \quad \forall u, v \in V, \ \forall u^* \in \partial\Phi(u). \tag{2.12}$$

Proof. For fixed $u, v \in V$, consider the function

$$g(\lambda) = \Phi(\lambda v + (1-\lambda) u), \quad 0 \leq \lambda \leq 1.$$

Assumption (2.11) implies the convexity of g. By [36, Theorem 2.3.4],

$$\Phi(v) - \Phi(u) = g(1) - g(0) = \langle u^*, v - u\rangle + \int_0^1 \langle \xi_\lambda - \xi_0, v - u\rangle \, d\lambda,$$

where $u^* = \xi_0 \in \partial\Phi(u)$ and $\xi_\lambda \in \partial\Phi(\lambda v + (1-\lambda) u)$. Denote $w = \lambda v + (1-\lambda) u$. Then,

$$w - u = \lambda (v - u)$$

and

$$\langle \xi_\lambda - \xi_0, v - u\rangle = \frac{1}{\lambda} \langle \xi_\lambda - \xi_0, w - u\rangle.$$

Apply the condition (2.11),

$$\langle \xi_\lambda - \xi_0, w - u\rangle \geq 2\, m_\Phi \|w-u\|_V^p = 2\, m_\Phi \lambda^p \|v-u\|_V^p.$$

Hence,

$$\begin{aligned}\Phi(v) - \Phi(u) &\geq \langle \xi_0, v-u\rangle + \int_0^1 2\, m_\Phi \lambda^{p-1} \|v-u\|_V^p d\lambda \\ &= \langle \xi_0, v-u\rangle + \frac{2}{p}\, m_\Phi \|v-u\|_V^p.\end{aligned}$$

Therefore, (2.12) holds. ■

We end the section with a summarizing result on the strong convexity of a general order p.

Theorem 2.9. *Let V be a real Banach space, $\Phi: V \to \mathbb{R}$, and $p \in (0,\infty)$. If (2.11) holds, then Φ is strongly convex of order p over V.*

Proof. By Proposition 2.8, the assumption (2.11) implies (2.12). By Proposition 2.6, the following inequality holds for any $u, v \in V$ and any $\lambda \in [0,1]$,

$$\Phi(\lambda\, u + (1-\lambda)\, v) \leq \lambda\, \Phi(u) + (1-\lambda)\, \Phi(v) - \frac{2}{p}\, m_\Phi c_p \lambda\, (1-\lambda)\, \|u-v\|_V^p.$$

Thus, Φ is strongly convex of order p over V. ■

3. A variational–hemivariational inequality in a reflexive banach space and a minimization principle

In this and the next sections, we will make the following assumption.

$H(K)$: V is a real reflexive Banach space, $K \subset V$ is non-empty, closed and convex.

From now on, the range of the order p is limited to $(1,\infty)$. Consider the following variational–hemivariational inequality.

Problem 3.1. Find an element $u \in K$ such that

$$\langle Au, v-u\rangle + \Phi(v) - \Phi(u) + \Psi^0(u; v-u) \geq \langle f, v-u\rangle \quad \forall\, v \in K. \tag{3.1}$$

We will consider the case where A is a potential operator; note that this is a dominant case for applications in physical sciences and engineering. Recall that A is a potential operator if $A = F_A'$ is the Gâteaux derivative of a functional $F_A: V \to \mathbb{R}$. The functional F_A is known as a potential of A, and we will use the symbol F_A for the potential of A. Given a potential operator A, its potential functional F_A is not unique and the difference between any two potential functionals is a constant. The formula

$$F_A(v) = \int_0^1 \langle A(t\, v), v\rangle\, dt \tag{3.2}$$

provides one potential functional. Discussions of potential operators can be found in [37, Section 41.3].

The assumptions on the data are as follows.

$H(A)$: $A: V \to V^*$ is a locally Lipschitz potential operator and is strongly monotone of order $p > 1$ over V:

$$\langle Av_1 - Av_2, v_1 - v_2\rangle \geq m_A \|v_1 - v_2\|_V^p \quad \forall\, v_1, v_2 \in V. \tag{3.3}$$

$H(\Phi)$: $\Phi: V \to \mathbb{R}$ is convex and continuous.
$H(\Psi)$: $\Psi: V \to \mathbb{R}$ is locally Lipschitz, and for a constant $m_\Psi \geq 0$,

$$\Psi^0(v_1; v_2 - v_1) + \Psi^0(v_2; v_1 - v_2) \leq m_\Psi \|v_1 - v_2\|_V^p \quad \forall\, v_1, v_2 \in V. \tag{3.4}$$

$H(f)$: $f \in V^*$.
$H(s)$: $m_\Psi < m_A$.

Let us make some comments on the assumption $H(\Phi)$. In the literature, the convex function Φ is usually assumed to be proper and l.s.c. from V to $\mathbb{R} \cup \{+\infty\}$. By redefining the constraint set K to reflect the effective domain of Φ, we may restrict our attention to the case where the symbol Φ represents a real-valued function on V. By [35, Corollary 2.5, p. 13], a real-valued l.s.c. convex function on a Banach space is continuous. Moreover, from [35, Corollary 2.4, p. 12], we know that the convex function $\Phi: V \to \mathbb{R}$ is continuous if and only if it is locally Lipschitz continuous, and if and only if it is bounded above on a non-empty open set in V.

By an argument similar to that found on page 124 in [2], it can be shown that (3.4) is equivalent to

$$\langle \eta_1 - \eta_2, v_1 - v_2 \rangle \geq -m_\Psi \|v_1 - v_2\|_V^p \quad \forall\, v_i \in V,\ \ \eta_i \in \partial\Psi(v_i),\ \ i = 1, 2. \tag{3.5}$$

In the literature, a condition of the form $H(s)$ is called a smallness condition.

Corresponding to Problem 3.1, we introduce a minimization problem.

Problem 3.2. Find an element $u \in K$ such that

$$E(u) = \inf \{E(v) \mid v \in K\}, \tag{3.6}$$

where the energy functional

$$E(v) = F_A(v) + \Phi(v) + \Psi(v) - \langle f, v\rangle. \tag{3.7}$$

We first provide a result on the local Lipschitz continuity of the potential F_A. For $r > 0$, we denote by $B_r(0)$ the closed ball in V with radius r centered at 0.

Lemma 3.3. *If $A: V \to V^*$ is locally Lipschitz continuous, then the potential $F_A: V \to \mathbb{R}$ is locally Lipschitz continuous.*

Proof. Fix a positive number $r > 0$. Then there is a number $c_r > 0$ such that

$$\|Au - Av\|_{V^*} \leq c_r \|u - v\|_V \quad \forall\, u, v \in B_r(0).$$

For any $u, v \in B_r(0)$,

$$\begin{aligned} F_A(u) - F_A(v) &= \int_0^1 \frac{d}{ds} F_A(v + s(u - v))\, ds \\ &= \int_0^1 \langle A(v + s(u - v)), u - v\rangle\, ds \\ &= \int_0^1 \langle A(v + s(u - v)) - A(0), u - v\rangle\, ds + \langle A(0), u - v\rangle. \end{aligned}$$

Then,

$$|F_A(u) - F_A(v)| \leq (c_r r + \|A(0)\|_{V^*})\, \|u - v\|_V.$$

Hence, $F_A: V \to \mathbb{R}$ is locally Lipschitz continuous. ■

Theorem 3.4. *Assume $H(K)$, $H(A)$, $H(\Phi)$, $H(\Psi)$, $H(f)$ and $H(s)$. Then Problem 3.2 and Problem 3.1 are equivalent, and both admit the same unique solution $u \in K$. The solution u depends Hölder-continuously on f; more precisely, if $u_1, u_2 \in V$ are the solutions of Problem 3.1 corresponding to $f = f_1 \in V^*$ and $f = f_2 \in V^*$, then*

$$\|u_1 - u_2\|_V \leq (m_A - m_\Psi)^{-1/(p-1)} \|f_1 - f_2\|_{V^*}^{1/(p-1)}. \tag{3.8}$$

Proof. First, we show that Problem 3.2 has a unique solution. Under the stated assumptions, we know that the functional $E: V \to \mathbb{R}$ is locally Lipschitz continuous; the local Lipschitz continuity property of F_A follows from Lemma 3.3, while that for Φ follows from $H(\Phi)$ and properties of convex functions ([35, Chapter I, Section 2.3]). Applying Proposition 2.1, we have

$$\partial E(v) \subset Av + \partial\Phi(v) + \partial\Psi(v) - f.$$

So for $v_i \in V$, $\zeta_i \in \partial E(v_i)$, $i = 1, 2$, we can write

$$\zeta_i = Av_i + \xi_i + \eta_i - f, \quad i = 1, 2,$$

where $\xi_i \in \partial\Phi(v_i)$, $\eta_i \in \partial\Psi(v_i)$. Then

$$\langle \zeta_1 - \zeta_2, v_1 - v_2\rangle = \langle Av_1 - Av_2, v_1 - v_2\rangle + \langle \xi_1 - \xi_2, v_1 - v_2\rangle + \langle \eta_1 - \eta_2, v_1 - v_2\rangle.$$

Since Φ is convex, $\langle \xi_1 - \xi_2, v_1 - v_2\rangle \geq 0$. Applying (3.3) and (3.5), we have

$$\langle \zeta_1 - \zeta_2, v_1 - v_2\rangle \geq m_A \|v_1 - v_2\|_V^p + 0 - m_\Psi \|v_1 - v_2\|_V^p = (m_A - m_\Psi)\, \|v_1 - v_2\|_V^p. \tag{3.9}$$

Note that by $H(s)$, $m_A - m_\Psi > 0$. Thus, by Theorem 2.9, the functional E is strongly convex of order p. Moreover, under the stated assumptions on the data, E is continuous. Hence, by a standard result on convex minimization (cf. e.g. [38, Section 3.3]), Problem 3.2 has a unique solution.

Denote by

$$I_K(v) = \begin{cases} 0, & v \in K, \\ +\infty, & v \notin K \end{cases}$$

the indicator functional of K. It is known that $I_K : V \to \overline{\mathbb{R}}$ is proper, convex and l.s.c. The solution $u \in K$ of Problem 3.2 satisfies

$$0 \in \partial\,(E(u) + I_K(u)) \subset Au + \partial\Phi(u) + \partial I_K(u) + \partial\Psi(u) - f.$$

Then,

$$\langle Au, v - u\rangle + \Phi(v) - \Phi(u) + \Psi^0(u; v - u) \geq \langle f, v - u\rangle \quad \forall\, v \in K.$$

Thus, the solution $u \in K$ of Problem 3.2 is also a solution of Problem 3.1. Suppose $\tilde{u} \in K$ is another solution of Problem 3.1. Then,

$$\langle A\tilde{u}, v - \tilde{u}\rangle + \Phi(v) - \Phi(\tilde{u}) + \Psi^0(\tilde{u}; v - \tilde{u}) \geq \langle f, v - \tilde{u}\rangle \quad \forall\, v \in K. \tag{3.10}$$

Take $v = \tilde{u}$ in (3.1), take $v = u$ in (3.10), and add the two inequalities to obtain

$$\langle A\tilde{u} - Au, \tilde{u} - u\rangle \leq \Psi^0(\tilde{u}; u - \tilde{u}) + \Psi^0(u; \tilde{u} - u).$$

By $H(A)$ and $H(\Psi)$,

$$m_A \|\tilde{u} - u\|_V^p \leq m_\Psi \|\tilde{u} - u\|_V^p.$$

Since $m_A > m_\Psi$, the above inequality is possible only if $\|\tilde{u} - u\|_V = 0$, i.e., $\tilde{u} = u$. So a solution of Problem 3.1 is unique.

In conclusion, under the stated assumptions, both Problem 3.2 and Problem 3.1 have $u \in K$ as the unique solution, and therefore, the two problems are equivalent.

Finally, let us prove (3.8). Take $v = u_2$ in the defining inequality (3.1) for u_1 to obtain

$$\langle Au_1, u_2 - u_1\rangle + \Phi(u_2) - \Phi(u_1) + \Psi^0(u_1; u_2 - u_1) \geq \langle f_1, u_2 - u_1\rangle.$$

Similarly,

$$\langle Au_2, u_1 - u_2\rangle + \Phi(u_1) - \Phi(u_2) + \Psi^0(u_2; u_1 - u_2) \geq \langle f_2, u_1 - u_2\rangle.$$

Add the two inequalities,

$$\langle Au_1 - Au_2, u_1 - u_2\rangle \leq \Psi^0(u_1; u_2 - u_1) + \Psi^0(u_2; u_1 - u_2) + \langle f_1 - f_2, u_1 - u_2\rangle. \tag{3.11}$$

Apply the conditions (3.3) and (3.4) to the above inequality,

$$m_A \|u_1 - u_2\|_V^p \leq m_\Psi \|u_1 - u_2\|_V^p + \|f_1 - f_2\|_{V^*} \|u_1 - u_2\|_V.$$

So

$$(m_A - m_\Psi)\, \|u_1 - u_2\|_V^{p-1} \leq \|f_1 - f_2\|_{V^*}$$

from which we deduce (3.8). ■

We comment that in [28], a solution existence and uniqueness result for Problem 3.1 is proved under somewhat different assumptions: instead of $H(A)$, it is assumed that $A : V \to V^*$ is pseudomonotone and strongly monotone of order $p > 1$, and an additional assumption is made on the growth of the subdifferential $\partial\Psi$. The proof there is achieved through an application of an abstract surjectivity result for pseudomonotone operators. For applications, we usually need a variant of Theorem 3.4, cf. Theorem 4.1 next section.

4. Variant and generalization

In some applications (cf. Section 5), we need a variant of the theory developed in Section 3. This need is based on the observation that in the assumption (3.4) on Ψ, the exponent p is typically 2, even for problems where the operator A is strongly monotone of a general order $p > 1$. In other words, Theorem 3.4 is not directly applicable because of a mismatch between the two exponents in (3.3) and (3.4) when the norm in the original space V is used. To deal with this issue, in addition to the reflexive Banach space V, we introduce another reflexive Banach space V_1 such that V is continuously embedded in V_1. The strong convexity assumption on the operator $A : V \to V^*$ will be made over the space V_1 instead of V. In this section, the duality pairing between V^* and V will be indicated by the full notation $\langle \cdot, \cdot \rangle_{V^* \times V}$. Also, to avoid potential confusion, we use $p_1 \in (1, \infty)$ to denote the order of the strong convexity of $A : V \to V^*$ over V_1. Specifically, we replace $H(K)$, $H(A)$, $H(\Psi)$ and $H(f)$ by the following:

$H(K)_1$: V and V_1 are real reflexive Banach spaces with V continuously embedded in V_1, and $K \subset V$ is non-empty, closed and convex.

$H(A)_1$: $A : V \to V^*$ is a locally Lipschitz potential operator and is strongly monotone of order $p_1 > 1$ over V_1:

$$\langle Av_1 - Av_2, v_1 - v_2\rangle_{V^*\times V} \ge m_A \|v_1 - v_2\|_{V_1}^{p_1} \quad \forall\, v_1, v_2 \in V. \tag{4.1}$$

$H(\Psi)_1$: $\Psi : V \to \mathbb{R}$ is locally Lipschitz, and for a constant $m_\Psi \ge 0$,

$$\Psi^0(v_1; v_2 - v_1) + \Psi^0(v_2; v_1 - v_2) \le m_\Psi \|v_1 - v_2\|_{V_1}^{p_1} \quad \forall\, v_1, v_2 \in V. \tag{4.2}$$

$H(f)_1$: $f \in V_1^*$.

A modification of Theorem 3.4 is the next result.

Theorem 4.1. *Assume $H(K)_1$, $H(A)_1$, $H(\Phi)$, $H(\Psi)_1$, $H(f)_1$ and $H(s)$. Then Problems 3.2 and 3.1 are equivalent, and both admit the same unique solution $u \in K$. For $f_1, f_2 \in V_1^*$ and the corresponding solutions $u_1, u_2 \in V$ of Problem 3.1,*

$$\|u_1 - u_2\|_{V_1} \le (m_A - m_\Psi)^{-1/(p_1-1)} \|f_1 - f_2\|_{V_1^*}^{1/(p_1-1)}.$$

Since V is continuously embedded in V_1, we may view V_1^* as a subspace of V^*. The proof of Theorem 3.4 can be adapted for the proof of Theorem 4.1 when we replace (3.9) by

$$\langle \zeta_1 - \zeta_2, v_1 - v_2\rangle_{V^*\times V} \ge (m_A - m_\Psi)\, \|v_1 - v_2\|_{V_1}^{p_1}.$$

We comment that Theorem 3.4 may be viewed as a special case of Theorem 4.1 with $V_1 = V$ and $p_1 = p$. In our application problems to be discussed in Section 5, we apply Theorem 3.4 with $V \subset W^{1,p}(\Omega; \mathbb{R}^d)$ and $p \ge 2$; and when we apply Theorem 4.1, we use in addition $V_1 \subset H^1(\Omega; \mathbb{R}^d)$ and $p_1 = 2$.

We now consider a problem more general than Problem 3.1.

Problem 4.2. Find an element $u \in K$ such that

$$\langle Au, v - u\rangle_{V^*\times V} + \Phi(u, v) - \Phi(u, u) + \Psi^0(u; v - u) \ge \langle f, v - u\rangle_{V^*\times V} \quad \forall\, v \in K. \tag{4.3}$$

Note that in (4.3), the functional Φ has two arguments. In the study of Problem 4.2, we modify the condition $H(\Phi)$ on Φ to one of the following two forms.

$H(\Phi)'$: $\Phi : V \times V \to \mathbb{R}$ is convex and continuous with respect to its second argument, and for a constant $m_\Phi > 0$,

$$\Phi(u_1, v_2) + \Phi(u_2, v_1) - \Phi(u_1, v_1) - \Phi(u_2, v_2) \le m_\Phi \|u_1 - u_2\|_V^{p-1} \|v_1 - v_2\|_V \quad \forall\, u_1, u_2, v_1, v_2 \in V. \tag{4.4}$$

$H(\Phi)_1'$: $\Phi : V \times V \to \mathbb{R}$ is convex and continuous with respect to its second argument, and for a constant $m_\Phi > 0$,

$$\Phi(u_1, v_2) + \Phi(u_2, v_1) - \Phi(u_1, v_1) - \Phi(u_2, v_2) \le m_\Phi \|u_1 - u_2\|_{V_1}^{p_1-1} \|v_1 - v_2\|_{V_1} \quad \forall\, u_1, u_2, v_1, v_2 \in V. \tag{4.5}$$

Moreover, we modify $H(s)$ as follows.

$H(s)'$: $m_\Psi + m_\Phi < m_A$.

A generalization of Theorem 3.4 is the next result.

Theorem 4.3. *Assume $H(K)$, $H(A)$, $H(\Phi)'$, $H(\Psi)$, $H(f)$, and $H(s)'$. Then Problem 4.2 has a unique solution $u \in K$. For $f_1, f_2 \in V^*$ and the corresponding solutions $u_1, u_2 \in V$ of Problem 4.2,*

$$\|u_1 - u_2\|_V \le (m_A - m_\Phi - m_\Psi)^{-1/(p-1)} \|f_1 - f_2\|_{V^*}^{1/(p-1)}. \tag{4.6}$$

Proof. For any $w \in K$, we apply Theorem 3.4 to know that there is a unique element $u \in K$ such that

$$\langle Au, v - u\rangle_{V^*\times V} + \Phi(w, v) - \Phi(w, u) + \Psi^0(u; v - u) \ge \langle f, v - u\rangle_{V^*\times V} \quad \forall\, v \in K.$$

This defines a mapping $P : K \to K$ by the formula $u = P(w)$.

Let us show that the mapping P is a contraction on K. For this purpose, let w_1 and w_2 be any elements in K, and denote $u_1 = P(w_1)$ and $u_2 = P(w_2)$. Then

$$\langle Au_1, v - u_1\rangle_{V^*\times V} + \Phi(w_1, v) - \Phi(w_1, u_1) + \Psi^0(u_1; v - u_1) \ge \langle f, v - u_1\rangle_{V^*\times V} \quad \forall\, v \in K,$$
$$\langle Au_2, v - u_2\rangle_{V^*\times V} + \Phi(w_2, v) - \Phi(w_2, u_2) + \Psi^0(u_2; v - u_2) \ge \langle f, v - u_2\rangle_{V^*\times V} \quad \forall\, v \in K.$$

Take $v = u_2$ in the first inequality, take $v = u_1$ in the second inequality, and add the two inequalities to obtain

$$\begin{aligned}\langle Au_1 - Au_2, u_1 - u_2\rangle_{V^*\times V} \le\ & \Phi(w_1, u_2) + \Phi(w_2, u_1) - \Phi(w_1, u_1) - \Phi(w_2, u_2)\\ & + \Psi^0(u_1; u_2 - u_1) + \Psi^0(u_2; u_1 - u_2).\end{aligned}$$

Apply assumptions $H(A)$, $H(\Phi)'$ and $H(\Psi)$,

$$m_A \|u_1 - u_2\|_V^p \le m_\Phi \|w_1 - w_2\|_V^{p-1} \|u_1 - u_2\|_V + m_\Psi \|u_1 - u_2\|_V^p.$$

Then, by $H(s)'$,

$$\|u_1 - u_2\|_V \le \alpha \|w_1 - w_2\|_V, \quad \alpha := (m_\Phi/(m_A - m_\Psi))^{1/(p-1)} < 1.$$

Hence the mapping $P: K \to K$ is a contraction. By Banach fixed-point theorem, the operator P has a unique fixed-point $u \in K$. It is easy to see that the fixed-point $u \in K$ is the unique solution of Problem 4.2.

The proof of (4.6) is similar to that of (3.8). We replace (3.11) by

$$\langle Au_1 - Au_2, u_1 - u_2\rangle_{V^*\times V} \le \Phi(u_1, u_2) + \Phi(u_2, u_1) - \Phi(u_1, u_1) - \Phi(u_2, u_2) + \Psi^0(u_1; u_2 - u_1) + \Psi^0(u_2; u_1 - u_2) + \langle f_1 - f_2, u_1 - u_2\rangle_{V^*\times V}.$$

Then we apply the conditions (3.3), (4.4) and (3.4) to obtain (4.6). ■

A generalization of Theorem 4.1 is the next result and its proof is similar to that of Theorem 4.3.

Theorem 4.4. *Assume $H(K)_1$, $H(A)_1$, $H(\Phi)'_1$, $H(\Psi)_1$, $H(f)_1$ and $H(s)'$. Then Problem 4.2 has a unique solution $u \in K$. For $f_1, f_2 \in V_1^*$ and the corresponding solutions $u_1, u_2 \in V$ of Problem 4.2,*

$$\|u_1 - u_2\|_{V_1} \le (m_A - m_\Phi - m_\Psi)^{-1/(p_1-1)} \|f_1 - f_2\|_{V_1^*}^{1/(p_1-1)}.$$

5. Application to a steady incompressible generalized newtonian fluid flow problem

As an application of the theory developed in previous sections, we consider a model of steady incompressible generalized Newtonian fluids subject to a general slip boundary condition. We consider the fluid flow in a Lipschitz domain Ω in $\mathbb{R}^d$; for applications, we let $d \le 3$. The boundary $\partial\Omega$ of the domain Ω is split into two non-overlapping parts: $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2}$ with Γ_1 and Γ_2 relatively open, meas$(\Gamma_1) > 0$, meas$(\Gamma_2) > 0$, and $\Gamma_1 \cap \Gamma_2 = \emptyset$. Since the boundary $\partial\Omega$ is Lipschitz continuous, the unit outward normal $\boldsymbol{n} = (n_1, \ldots, n_d)^T$ exists a.e. on $\partial\Omega$. For a vector-valued function $\boldsymbol{u}$ on the boundary, we denote by $u_n = \boldsymbol{u}\cdot\boldsymbol{n}$ and $\boldsymbol{u}_\tau = \boldsymbol{u} - u_n\boldsymbol{n}$ the normal component and the tangential component, respectively. With the flow velocity field $\boldsymbol{u}$, we define the deformation rate tensor $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)/2$ which takes on values in the space $\mathbb{S}^d$ of second-order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of real symmetric matrices of order d. We adopt the summation convention over a repeated index. The indices i and j are between 1 and d. The canonical inner products and norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i, \quad |\boldsymbol{v}|_{\mathbb{R}^d} = (\boldsymbol{v}\cdot\boldsymbol{v})^{1/2} \quad \forall\, \boldsymbol{u} = (u_i),\ \boldsymbol{v} = (v_i) \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad |\boldsymbol{\sigma}|_{\mathbb{S}^d} = (\boldsymbol{\sigma}:\boldsymbol{\sigma})^{1/2} \quad \forall\, \boldsymbol{\sigma} = (\sigma_{ij}),\ \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d.$$

For $\boldsymbol{\sigma} \in \mathbb{S}^d$, let $\sigma_n = \boldsymbol{n}\cdot\boldsymbol{\sigma}\boldsymbol{n}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{n} - \sigma_n\boldsymbol{n}$ be its normal component and the tangential component on the boundary $\partial\Omega$.

We recall that in an incompressible non-Newtonian fluid, the stress is expressed as [39]

$$\boldsymbol{\sigma} = -\pi\,\boldsymbol{I} + \boldsymbol{S}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \tag{5.1}$$

where $\boldsymbol{u}$ is the velocity field, π is the pressure, $\boldsymbol{I}$ is the identity matrix of order d, and $\boldsymbol{S}(\boldsymbol{\varepsilon}(\boldsymbol{u}))$ is the extra stress tensor, $\boldsymbol{S}: \mathbb{S}^d \to \mathbb{S}^d$. In our study below, the constitutive function can be allowed to depend on the spatial location $\boldsymbol{x}$: $\boldsymbol{S} = \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{\varepsilon}(\boldsymbol{u}))$. However, for simplicity in writing, we focus on the case where $\boldsymbol{S} = \boldsymbol{S}(\boldsymbol{\varepsilon}(\boldsymbol{u}))$ is independent of $\boldsymbol{x}$.

The classical formulation of the fluid problem we consider is the following [28].

Problem 5.1. Find a velocity $\boldsymbol{u}: \Omega \to \mathbb{R}^p$ and a pressure $\pi: \Omega \to \mathbb{R}$ such that

$$-\mathrm{Div}\,\boldsymbol{S} + \nabla\pi = \boldsymbol{f} \quad \text{in } \Omega, \tag{5.2}$$
$$\mathrm{div}\,\boldsymbol{u} = \boldsymbol{0} \quad \text{in } \Omega, \tag{5.3}$$
$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{5.4}$$
$$u_\nu = 0,\ -\boldsymbol{\sigma}_\tau(\boldsymbol{u}) \in \partial\psi_\tau(\boldsymbol{u}_\tau) \quad \text{on } \Gamma_2. \tag{5.5}$$

In the conservation law (5.2) for the stationary flow, $\boldsymbol{f}$ represents an external body force density function. The incompressibility of the fluid is described by (5.3). The boundary condition (5.4) means that the fluid adheres to part of the boundary, Γ_1. The boundary condition (5.5) on Γ_2 consists of the impermeability along the normal direction: $u_\nu = 0$, and a nonlinear slip relation of the friction type along the tangential direction: $-\boldsymbol{\sigma}_\tau(\boldsymbol{u}) \in \partial\psi_\tau(\boldsymbol{u}_\tau)$. When

$$\psi_\tau(\boldsymbol{z}) = \frac{1}{2}\, c_0 |\boldsymbol{z}|^2_{\mathbb{R}^d} \quad \forall\, \boldsymbol{z} \in \mathbb{R}^d$$

is a quadratic function, $c_0 > 0$ being a constant or a positive-valued function on Γ_2, the slip condition reduces to the classical Navier condition [18]

$$-\boldsymbol{\sigma}_\tau(\boldsymbol{u}) = c_0\boldsymbol{u}_\tau.$$

The slip condition with

$$\psi_\tau(\boldsymbol{z}) = c_0|\boldsymbol{z}|_{\mathbb{R}^d} \quad \forall\, \boldsymbol{z} \in \mathbb{R}^d$$

is used in the study of variational inequalities for the Stokes and Navier–Stokes equations (e.g. [19]); note that the above function is non-smooth and convex. When we allow the function ψ_τ to be non-smooth and non-convex, the weak formulation of Problem 5.1 will be a hemivariational inequality (in a reduced form where the pressure variable is eliminated) or a mixed hemivariational inequality in which both the velocity and pressure variables are present.

Let us introduce assumptions on the data. In the following, $p \ge 2$ is a given number, and $p' \in (1, \infty)$ is the conjugate exponent of p, defined by the relation $1/p + 1/p' = 1$. We assume $\boldsymbol{S}$ can be generated from a potential functional $U : \mathbb{S}^d \to \mathbb{R}$:

$$\boldsymbol{S}(\boldsymbol{\varepsilon}) = \frac{\partial U(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}}, \quad \text{i.e.,} \quad S_{ij}(\boldsymbol{\varepsilon}) = \frac{\partial U(\boldsymbol{\varepsilon})}{\partial \varepsilon_{ij}}, \ 1 \le i, j \le d. \tag{5.6}$$

In addition, we assume

$$|\boldsymbol{S}(\boldsymbol{\varepsilon})|_{\mathbb{S}^d} \le c\left(1 + |\boldsymbol{\varepsilon}|^{p-1}_{\mathbb{S}^d}\right) \quad \forall\, \boldsymbol{\varepsilon} \in \mathbb{S}^d, \tag{5.7}$$

$$\|\boldsymbol{S}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1)) - \boldsymbol{S}(\boldsymbol{\varepsilon}(\boldsymbol{u}_2))\|_{L^{p'}(\Omega;\mathbb{S}^d)} \le b(\|\boldsymbol{u}_1\|_V, \|\boldsymbol{u}_2\|_V)\, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \quad \forall\, \boldsymbol{u}_1, \boldsymbol{u}_2 \in V, \tag{5.8}$$

where the function $b(\cdot,\cdot)$ is bounded over bounded ranges of its arguments, and either

$$(\boldsymbol{S}(\boldsymbol{\varepsilon}_1) - \boldsymbol{S}(\boldsymbol{\varepsilon}_2)) : (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_S|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^p_{\mathbb{S}^d} \quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d \tag{5.9}$$

or

$$(\boldsymbol{S}(\boldsymbol{\varepsilon}_1) - \boldsymbol{S}(\boldsymbol{\varepsilon}_2)) : (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_S|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2_{\mathbb{S}^d} \quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d. \tag{5.10}$$

On $\psi_\tau : \mathbb{R}^d \to \mathbb{R}$, we assume

$$\psi_\tau \text{ is locally Lipschitz continuous on } \mathbb{R}^d, \tag{5.11}$$

$$|\partial\psi_\tau(\boldsymbol{\xi})|_{\mathbb{R}^d} \le c\left(1 + |\boldsymbol{\xi}|^{p-1}_{\mathbb{R}^d}\right) \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^d, \tag{5.12}$$

and either

$$\psi^0_\tau(\boldsymbol{\xi}_1; \boldsymbol{\xi}_2 - \boldsymbol{\xi}_1) + \psi^0_\tau(\boldsymbol{\xi}_2; \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \le M_\psi|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^p_{\mathbb{R}^d} \quad \forall\, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^d \tag{5.13}$$

or

$$\psi^0_\tau(\boldsymbol{\xi}_1; \boldsymbol{\xi}_2 - \boldsymbol{\xi}_1) + \psi^0_\tau(\boldsymbol{\xi}_2; \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \le M_\psi|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2_{\mathbb{R}^d} \quad \forall\, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^d. \tag{5.14}$$

On the source function, we assume

$$\boldsymbol{f} \in V^*. \tag{5.15}$$

In the study of weak formulations of Problem 5.1, we need to introduce some function spaces. Let

$$V = \left\{\boldsymbol{v} \in W^{1,p}(\Omega;\mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1,\ v_\nu = 0 \text{ on } \Gamma_2\right\}, \tag{5.16}$$

$$\tilde{V} = \{\boldsymbol{v} \in V \mid \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\}, \tag{5.17}$$

$$Q = \left\{q \in L^{p'}(\Omega) \mid (q, 1)_\Omega = 0\right\}, \tag{5.18}$$

where $(q, 1)_\Omega$ stands for the integral of q over Ω. We will also use the following subspace of the space V:

$$V_1 = \left\{\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1,\ v_\nu = 0 \text{ on } \Gamma_2\right\}. \tag{5.19}$$

There exists a constant $c > 0$, depending on Ω only, such that

$$\|\boldsymbol{v}\|_{V_1} \le c\, \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V.$$

Following a standard procedure, we can derive the next weak formulation of Problem 5.1.

Problem 5.2. Find a velocity $\boldsymbol{u} \in V$ and a pressure $\pi \in Q$ such that

$$\int_\Omega \boldsymbol{S}(\boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx - \int_\Omega \pi \operatorname{div} \boldsymbol{v}\, dx + \int_{\Gamma_2} \psi^0_\tau(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\, da \ge \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{V^*\times V} \quad \forall\, \boldsymbol{v} \in V, \tag{5.20}$$

$$\int_\Omega q \operatorname{div} \boldsymbol{u}\, dx = 0 \quad \forall\, q \in Q. \tag{5.21}$$

We can eliminate the constraint (5.21) to get a reduced weak formulation.

Problem 5.3. Find a velocity $\boldsymbol{u} \in \tilde{V}$ such that

$$\int_\Omega \boldsymbol{S}(\boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx + \int_{\Gamma_2} \psi_\tau^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\, da \ge \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{V^*\times V} \quad \forall\, \boldsymbol{v} \in \tilde{V}. \tag{5.22}$$

We denote by $c_p > 0$ (the best) constant of the inequality

$$\int_{\Gamma_2} |\boldsymbol{v}_\tau|_{\mathbb{R}^d}^p da \le c_p \int_\Omega |\boldsymbol{\varepsilon}(\boldsymbol{v})|_{\mathbb{S}^d}^p dx \quad \forall\, \boldsymbol{v} \in V. \tag{5.23}$$

Theorem 5.4. *Assume (5.6), (5.7), (5.8), (5.11), (5.12), and (5.15). Also assume either* (5.9) and (5.13), *or* (5.10) and (5.14). *Then under the smallness condition $c_p M_\psi < m_S$, Problem 5.3 has a unique solution $\boldsymbol{u} \in \tilde{V}$, which is also the unique minimizer of the energy functional*

$$E(\boldsymbol{v}) = \int_\Omega U(\boldsymbol{\varepsilon}(\boldsymbol{v}))\, dx + \int_{\Gamma_2} \psi_\tau(\boldsymbol{v}_\tau)\, da - \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{V^*\times V} \tag{5.24}$$

over $\tilde{V}$.

Proof. We apply Theorem 3.4 or Theorem 4.1 with the operator $A: V \to V^*$ defined by

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle_{V^*\times V} = \int_\Omega \boldsymbol{S}(\boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \tag{5.25}$$

the functionals

$$\Phi \equiv 0, \quad \Psi(\boldsymbol{v}) = \int_{\Gamma_2} \psi_\tau(\boldsymbol{v}_\tau)\, da,$$

and $f \in V^*$ defined by $\langle \boldsymbol{f}, \boldsymbol{v}\rangle_{V^*\times V}$. Then A is well defined thanks to the assumption (5.7), and is locally Lipschitz continuous thanks to the assumption (5.8). The assumption (5.9) implies

$$\langle A\boldsymbol{v}_1 - A\boldsymbol{v}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2\rangle_{V^*\times V} \ge m_S \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V^p \quad \forall\, \boldsymbol{v}_1, \boldsymbol{v}_2 \in V$$

whereas (5.10) implies

$$\langle A\boldsymbol{v}_1 - A\boldsymbol{v}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2\rangle_{V^*\times V} \ge m_S \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{V_1}^2 \quad \forall\, \boldsymbol{v}_1, \boldsymbol{v}_2 \in V_1.$$

So $H(A)$ or $H(A)_1$ is satisfied with $m_A = m_S$. Since $\Phi \equiv 0$, $H(\Phi)$ is trivial. From a slight variant of [40, Theorem 4.20], by assumptions (5.11) and (5.12), the functional Ψ is well-defined, is locally Lipschitz continuous on V, and moreover,

$$\Psi^0(\boldsymbol{u}; \boldsymbol{v}) \le \int_{\Gamma_2} \psi_\tau^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\, da \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V. \tag{5.26}$$

Then, for any $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$, by (5.26),

$$\Psi^0(\boldsymbol{v}_1; \boldsymbol{v}_2 - \boldsymbol{v}_1) + \Psi^0(\boldsymbol{v}_2; \boldsymbol{v}_1 - \boldsymbol{v}_2) \le \int_{\Gamma_2} \left[\psi_\tau^0(\boldsymbol{v}_{1,\tau}; \boldsymbol{v}_{2,\tau} - \boldsymbol{v}_{1,\tau}) + \psi_\tau^0(\boldsymbol{v}_{2,\tau}; \boldsymbol{v}_{1,\tau} - \boldsymbol{v}_{2,\tau})\right] da.$$

So it follows from (5.13) that

$$\Psi^0(\boldsymbol{v}_1; \boldsymbol{v}_2 - \boldsymbol{v}_1) + \Psi^0(\boldsymbol{v}_2; \boldsymbol{v}_1 - \boldsymbol{v}_2) \le M_\psi \|\boldsymbol{v}_{1,\tau} - \boldsymbol{v}_{2,\tau}\|_{L^p(\Gamma_2;\mathbb{R}^d)}^p \le M_\psi c_p \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V^p,$$

or from (5.14),

$$\Psi^0(\boldsymbol{v}_1; \boldsymbol{v}_2 - \boldsymbol{v}_1) + \Psi^0(\boldsymbol{v}_2; \boldsymbol{v}_1 - \boldsymbol{v}_2) \le M_\psi \|\boldsymbol{v}_{1,\tau} - \boldsymbol{v}_{2,\tau}\|_{L^2(\Gamma_2;\mathbb{R}^d)}^2 \le M_\psi c_2 \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{V_1}^2.$$

Thus, $H(\Psi)$ or $H(\Psi)_1$ holds true with $m_\Psi = M_\psi c_p$. Then by Theorem 3.4 or Theorem 4.1, there is a unique element $\boldsymbol{u} \in \tilde{V}$ satisfying

$$\int_\Omega \boldsymbol{S}(\boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx + \Psi^0(\boldsymbol{u}; \boldsymbol{v}) \ge \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{V^*\times V} \quad \forall\, \boldsymbol{v} \in \tilde{V}.$$

Because of (5.26), $\boldsymbol{u}$ is also a solution of Problem 5.3.

The uniqueness of a solution of Problem 5.3 can be proved similarly as that of Problem 3.1 in the proof of Theorem 3.4, with Ψ^0 replaced by the integral of ψ^0 over Γ_2; we omit the detailed argument here. ■

To recover the pressure variable from Problem 5.3, we recall two results below, the first being a special case of [41, Lemma 2.2.2, Chapter II], whereas the second being a part of [41, Lemma 2.1.1, Chapter II].

Lemma 5.5. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $d \geq 2$, let $p \in (1, \infty)$ and let p' be its conjugate. If $\boldsymbol{\ell} \in W^{-1,p'}(\Omega)^d$ satisfies*

$$\langle \boldsymbol{\ell}, \boldsymbol{v} \rangle_{V^* \times V} = 0 \quad \forall\, \boldsymbol{v} \in C_0^\infty(\Omega)^d,\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega,$$

then there exists a unique $\pi \in Q$ such that

$$\boldsymbol{\ell} = \nabla \pi \text{ in the sense of distributions.}$$

Moreover, for two positive constants c_1, c_2, depending only on Ω and p,

$$c_1 \|\boldsymbol{\ell}\|_{V^*} \leq \|\pi\|_Q \leq c_2 \|\boldsymbol{\ell}\|_{V^*}.$$

Lemma 5.6. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $d \geq 2$, let $p \in (1, \infty)$ and let p' be its conjugate. Then for each $g \in Q$, there exists at least one element $\boldsymbol{v} \in W_0^{1,p}(\Omega)^d$ such that*

$$\operatorname{div} \boldsymbol{v} = g \text{ in } \Omega, \quad \|\boldsymbol{v}\|_{W^{1,p}(\Omega)^d} \leq c\, \|g\|_Q,$$

where $c > 0$ is a constant depending only on Ω and p.

Now we are ready to prove a unique solvability result for Problem 5.2.

Theorem 5.7. *Under the assumptions stated in Theorem 5.4, Problem 5.2 has a unique solution $(\boldsymbol{u}, \pi) \in V \times Q$; in addition, $\boldsymbol{u} \in \tilde{V}$ and it is the unique minimizer of the energy functional $E(\cdot)$ defined in (5.24) over $\tilde{V}$.*

Proof. By Theorem 5.4, Problem 5.3 has a unique solution $\boldsymbol{u} \in \tilde{V}$, which is also the unique minimizer of the energy functional of $E(\cdot)$ over $\tilde{V}$. From (5.22), we deduce that

$$\int_\Omega \boldsymbol{S}(\boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{V^* \times V} \quad \forall\, \boldsymbol{v} \in C_0^\infty(\Omega)^d,\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega. \tag{5.27}$$

Applying Lemma 5.5, we know that there is a unique element $\pi \in Q$ such that

$$\int_\Omega \boldsymbol{S}(\boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx - \int_\Omega \pi \operatorname{div} \boldsymbol{v}\, dx = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{V^* \times V} \quad \forall\, \boldsymbol{v} \in C_0^\infty(\Omega)^d.$$

By a density argument, we have

$$\int_\Omega \boldsymbol{S}(\boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx - \int_\Omega \pi \operatorname{div} \boldsymbol{v}\, dx = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{V^* \times V} \quad \forall\, \boldsymbol{v} \in W_0^{1,p}(\Omega)^d. \tag{5.28}$$

Now for any $\boldsymbol{v} \in V$,

$$\int_\Omega \operatorname{div} \boldsymbol{v}\, dx = \int_{\partial\Omega} \boldsymbol{v} \cdot \boldsymbol{n}\, da = 0.$$

Thus, $\operatorname{div} \boldsymbol{v} \in Q$. Applying Lemma 5.6, we have the existence of an element $\boldsymbol{v}_1 \in W_0^{1,p}(\Omega)^d$ such that

$$\operatorname{div} \boldsymbol{v}_1 = \operatorname{div} \boldsymbol{v} \quad \text{in } \Omega.$$

Then, $(\boldsymbol{v} - \boldsymbol{v}_1) \in \tilde{V}$. Use $(\boldsymbol{v} - \boldsymbol{v}_1)$ as the test function in (5.22) to obtain

$$\int_\Omega \boldsymbol{S}(\boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{v}_1)\, dx + \int_{\Gamma_2} \psi_\tau^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\, da \geq \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{v}_1 \rangle_{V^* \times V}. \tag{5.29}$$

By (5.28),

$$\int_\Omega \boldsymbol{S}(\boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{v}_1)\, dx - \int_\Omega \pi \operatorname{div} \boldsymbol{v}_1\, dx = \langle \boldsymbol{f}, \boldsymbol{v}_1 \rangle_{V^* \times V}.$$

Hence, from (5.29),

$$\int_\Omega \boldsymbol{S}(\boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx - \int_\Omega \pi \operatorname{div} \boldsymbol{v}_1\, dx + \int_{\Gamma_2} \psi_\tau^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\, da \geq \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{V^* \times V}.$$

Since $\operatorname{div} \boldsymbol{v}_1 = \operatorname{div} \boldsymbol{v}$ in Ω, we recover (5.20) from the above inequality. So $(\boldsymbol{u}, \pi) \in V \times Q$ is a solution of Problem 5.2.

It is easy to see that if $(\boldsymbol{u}, \pi) \in V \times Q$ is a solution of Problem 5.2, then $\boldsymbol{u}$ solves Problem 5.3 which has a unique solution. From the first paragraph of the proof, we know that $\pi \in Q$ is uniquely determined from $\boldsymbol{u}$. In conclusion, $(\boldsymbol{u}, \pi) \in V \times Q$ is the unique solution of Problem 5.2. ■

In the rest of the section, we comment on the assumptions made on $\boldsymbol{S}$ and ψ_τ.

Let $U:\mathbb{S}^d \to \mathbb{R}$ be a potential of the operator $\boldsymbol{S}$, cf. (5.6), and let us introduce conditions on U that imply the assumptions on $\boldsymbol{S}$. Assume, for some $p \geq 2$ and constants $c_1, c_2 > 0$,

$$\begin{aligned}
&U(\mathbf{0}) = 0,\\
&\frac{\partial U(\mathbf{0})}{\partial \varepsilon_{ij}} = 0, \quad 1 \leq i,j \leq d,\\
&\frac{\partial^2 U(\boldsymbol{\varepsilon})}{\partial \varepsilon_{ij}\partial \varepsilon_{kl}}\, \eta_{ij}\eta_{kl} \geq c_1 \left(1 + |\boldsymbol{\varepsilon}|_{\mathbb{S}^d}\right)^{p-2} |\boldsymbol{\eta}|^2_{\mathbb{S}^d} \quad \forall\, \boldsymbol{\varepsilon}, \boldsymbol{\eta} \in \mathbb{S}^d,\\
&\left|\frac{\partial^2 U(\boldsymbol{\varepsilon})}{\partial \varepsilon_{ij}\partial \varepsilon_{kl}}\right| \leq c_2 \left(1 + |\boldsymbol{\varepsilon}|_{\mathbb{S}^d}\right)^{p-2} \quad \forall\, \boldsymbol{\varepsilon} \in \mathbb{S}^d,\ 1 \leq i,j,k,l \leq d.
\end{aligned}$$

Then, it is shown in [39] that there exist constants $c_3, c_4, c_5 > 0$ such that

$$|\boldsymbol{S}(\boldsymbol{\varepsilon})|_{\mathbb{S}^d} \leq c_3 \left(1 + |\boldsymbol{\varepsilon}|_{\mathbb{S}^d}\right)^{p-1}, \tag{5.30}$$

$$(\boldsymbol{S}(\boldsymbol{\varepsilon}_1) - \boldsymbol{S}(\boldsymbol{\varepsilon}_2)) : (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq \max\left\{c_4|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2_{\mathbb{S}^d}, c_5|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^p_{\mathbb{S}^d}\right\} \quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d. \tag{5.31}$$

In other words, (5.7), (5.9) and (5.10) are satisfied. In addition, from

$$S_{ij}(\boldsymbol{\varepsilon}_1) - S_{ij}(\boldsymbol{\varepsilon}_2) = \int_0^1 \frac{\partial^2 U}{\partial \varepsilon_{ij}\partial \varepsilon_{kl}}(\boldsymbol{\varepsilon}_2 + s(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2))(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2)_{kl}\, ds, \quad 1 \leq i,j \leq d,$$

we find

$$|\boldsymbol{S}(\boldsymbol{\varepsilon}_1) - \boldsymbol{S}(\boldsymbol{\varepsilon}_2)|_{\mathbb{S}^d} \leq c \left(1 + |\boldsymbol{\varepsilon}_1|_{\mathbb{S}^d} + |\boldsymbol{\varepsilon}_2|_{\mathbb{S}^d}\right)^{p-2} |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|_{\mathbb{S}^d}.$$

Hence, (5.8) holds:

$$\|\boldsymbol{S}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1)) - \boldsymbol{S}(\boldsymbol{\varepsilon}(\boldsymbol{u}_2))\|_{L^{p'}(\Omega;\mathbb{S}^d)} \leq c\,(1 + \|\boldsymbol{u}_1\|_V + \|\boldsymbol{u}_2\|_V)^{p-2}\, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V.$$

If

$$\boldsymbol{S}(\boldsymbol{\varepsilon}) = 2\,\nu(|\boldsymbol{\varepsilon}|^2_{\mathbb{S}^d})\,\boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \in \mathbb{S}^d, \tag{5.32}$$

the non-Newtonian fluid is called a generalized Newtonian fluid. In the special case where the viscosity coefficient $\nu > 0$ is a constant, we recover from (5.32) the Stokes' law:

$$\boldsymbol{S}(\boldsymbol{\varepsilon}) = 2\,\nu\,\boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \in \mathbb{S}^d. \tag{5.33}$$

For the generalized Newtonian fluid (5.32), we can define a potential function

$$U(\boldsymbol{\varepsilon}) = \int_0^{|\boldsymbol{\varepsilon}|^2_{\mathbb{S}^d}} \nu(s)\, ds, \quad \boldsymbol{\varepsilon} \in \mathbb{S}^d, \tag{5.34}$$

Next, we consider three concrete examples of (5.32) found in the literature. The following two inequalities will be useful:

$$(1+s)^{p-2} \geq \frac{1}{2}\left(1 + s^{p-2}\right) \quad \forall\, s \geq 0,\ p \geq 2, \tag{5.35}$$

$$\left(|\boldsymbol{\varepsilon}_1|^{p-2}\boldsymbol{\varepsilon}_1 - |\boldsymbol{\varepsilon}_2|^{p-2}\boldsymbol{\varepsilon}_2\right) : (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq 2^{2-p}p^{-1}|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^p_{\mathbb{S}^d} \quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d,\ p \geq 2. \tag{5.36}$$

The first inequality can be found, e.g., in [39]. The second inequality follows, e.g., from [42] or [43, Lemma 3].

Lemma 5.8. *For $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, define*

$$\Delta_p(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2) = \left(\left(1 + |\boldsymbol{\varepsilon}_1|_{\mathbb{S}^d}\right)^{p-2} \boldsymbol{\varepsilon}_1 - \left(1 + |\boldsymbol{\varepsilon}_2|_{\mathbb{S}^d}\right)^{p-2} \boldsymbol{\varepsilon}_2\right) : (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2). \tag{5.37}$$

Then, for $p \geq 2$,

$$\Delta_p(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2) \geq 2^{-1}|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2_{\mathbb{S}^d} + \min\{2^{-1}(p-1)^{-1}, 8^{-1}12^{-p/2}\}|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^p_{\mathbb{S}^d}. \tag{5.38}$$

Proof. Write

$$\begin{aligned}
\Delta_p(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2) &= \int_0^1 \frac{d}{ds}\left[\left(1 + |\boldsymbol{\varepsilon}_2 + s(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2)|_{\mathbb{S}^d}\right)^{p-2} (\boldsymbol{\varepsilon}_2 + s(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2))\right] ds : (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2)\\
&= (p-2)\int_0^1 \frac{\left(1 + |\boldsymbol{\varepsilon}_2 + s(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2)|_{\mathbb{S}^d}\right)^{p-3}}{|\boldsymbol{\varepsilon}_2 + s(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2)|_{\mathbb{S}^d}}\, |\,(\boldsymbol{\varepsilon}_2 + s(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2)) : (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2)|^2 ds
\end{aligned}$$

$$+\int_0^1 \left(1+|\boldsymbol{\varepsilon}_2+s(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)|_{\mathbb{S}^d}\right)^{p-2} ds\,|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2|^2_{\mathbb{S}^d}.$$

Then,

$$\Delta_p(\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2)\ge\int_0^1 \left(1+|\boldsymbol{\varepsilon}_2+s(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)|_{\mathbb{S}^d}\right)^{p-2} ds\,|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2|^2_{\mathbb{S}^d}. \tag{5.39}$$

It is easy to see that

$$\Delta_p(\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2)\ge|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2|^2_{\mathbb{S}^d}. \tag{5.40}$$

Further, we use (5.39) to derive a second lower bound for $\Delta_p(\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2)$. According to (5.35),

$$\left(1+|\boldsymbol{\varepsilon}_2+s(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)|_{\mathbb{S}^d}\right)^{p-2}\ge\frac{1}{2}\left(1+|\boldsymbol{\varepsilon}_2+s(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)|^{p-2}_{\mathbb{S}^d}\right).$$

Thus,

$$\Delta_p(\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2)\ge\frac{1}{2}\,|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2|^2_{\mathbb{S}^d}+\frac{1}{2}\int_0^1 |\boldsymbol{\varepsilon}_2+s(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)|^{p-2}_{\mathbb{S}^d}ds\,|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2|^2_{\mathbb{S}^d}.$$

To proceed further, we distinguish two cases.

If $|\boldsymbol{\varepsilon}_2|_{\mathbb{S}^d}\ge|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2|_{\mathbb{S}^d}$, then

$$|\boldsymbol{\varepsilon}_2+s(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)|_{\mathbb{S}^d}\ge|\boldsymbol{\varepsilon}_2|_{\mathbb{S}^d}-s\,|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2|_{\mathbb{S}^d}\ge(1-s)\,|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2|_{\mathbb{S}^d}.$$

Hence,

$$\int_0^1 |\boldsymbol{\varepsilon}_2+s(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)|^{p-2}_{\mathbb{S}^d}ds\ge\int_0^1(1-s)^{p-2}ds\,|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2|^{p-2}_{\mathbb{S}^d}=\frac{1}{p-1}\,|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2|^{p-2}_{\mathbb{S}^d}.$$

If $|\boldsymbol{\varepsilon}_2|_{\mathbb{S}^d}<|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2|_{\mathbb{S}^d}$, write

$$\int_0^1 |\boldsymbol{\varepsilon}_2+s(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)|^{p-2}_{\mathbb{S}^d}ds=\int_0^1\frac{|\boldsymbol{\varepsilon}_2+s(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)|^p_{\mathbb{S}^d}}{|\boldsymbol{\varepsilon}_2+s(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)|^2_{\mathbb{S}^d}}\,ds.$$

For the denominator of the integrand,

$$|\boldsymbol{\varepsilon}_2+s(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)|^2_{\mathbb{S}^d}\le(|\boldsymbol{\varepsilon}_2|_{\mathbb{S}^d}+|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2|_{\mathbb{S}^d})^2\le4\,|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2|^2_{\mathbb{S}^d}.$$

For the numerator of the integrand, from

$$\int_0^1 |\boldsymbol{\varepsilon}_2+s(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)|^2_{\mathbb{S}^d}ds\le\left(\int_0^1 |\boldsymbol{\varepsilon}_2+s(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)|^p_{\mathbb{S}^d}ds\right)^{2/p},$$

we have

$$\begin{aligned}\int_0^1 |\boldsymbol{\varepsilon}_2+s(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)|^p_{\mathbb{S}^d}ds&\ge\left(\int_0^1 |\boldsymbol{\varepsilon}_2+s(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)|^2_{\mathbb{S}^d}ds\right)^{p/2}\\&=\left(|\boldsymbol{\varepsilon}_2|^2_{\mathbb{S}^d}+\boldsymbol{\varepsilon}_2:(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)+\frac{1}{3}\,|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2|^2_{\mathbb{S}^d}\right)^{p/2}\\&=3^{-p/2}\left(|\boldsymbol{\varepsilon}_1|^2_{\mathbb{S}^d}+\boldsymbol{\varepsilon}_1:\boldsymbol{\varepsilon}_2+|\boldsymbol{\varepsilon}_2|^2_{\mathbb{S}^d}\right)^{p/2}\\&\ge6^{-p/2}\left(|\boldsymbol{\varepsilon}_1|^2_{\mathbb{S}^d}+|\boldsymbol{\varepsilon}_2|^2_{\mathbb{S}^d}\right)^{p/2}.\end{aligned}$$

Since $|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2|^2_{\mathbb{S}^d}\le2\left(|\boldsymbol{\varepsilon}_1|^2_{\mathbb{S}^d}+|\boldsymbol{\varepsilon}_2|^2_{\mathbb{S}^d}\right)$,

$$\int_0^1 |\boldsymbol{\varepsilon}_2+s(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)|^p_{\mathbb{S}^d}ds\ge12^{-p/2}|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2|^p_{\mathbb{S}^d}.$$

Thus,

$$\int_0^1 |\boldsymbol{\varepsilon}_2+s(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)|^{p-2}_{\mathbb{S}^d}ds\ge\frac{12^{-p/2}|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2|^p_{\mathbb{S}^d}}{4\,|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2|^2_{\mathbb{S}^d}}=4^{-1}12^{-p/2}|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2|^{p-2}_{\mathbb{S}^d}.$$

Summarizing the two cases, we have (5.38). ■

Similarly, we can prove the next result.

Lemma 5.9. *For* $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, *define*

$$\tilde{\Delta}_p(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2) = \left(\left(1 + |\boldsymbol{\varepsilon}_1|^2_{\mathbb{S}^d}\right)^{p/2-1} \boldsymbol{\varepsilon}_1 - \left(1 + |\boldsymbol{\varepsilon}_2|^2_{\mathbb{S}^d}\right)^{p/2-1} \boldsymbol{\varepsilon}_2\right) : (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2). \tag{5.41}$$

Then, for $p \geq 2$,

$$\tilde{\Delta}_p(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2) \geq 2^{-1}|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2_{\mathbb{S}^d} + \min\{2^{-1}(p-1)^{-1}, 8^{-1}12^{-p/2}\}|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^p_{\mathbb{S}^d}. \tag{5.42}$$

Example 5.10. In the first example,

$$\boldsymbol{S}(\boldsymbol{\varepsilon}) = 2\,\nu_\infty \boldsymbol{\varepsilon} + 2\,\nu_0 |\boldsymbol{\varepsilon}|^{p-2}_{\mathbb{S}^d} \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \in \mathbb{S}^d. \tag{5.43}$$

Assume $\nu_\infty, \nu_0 > 0$. We have

$$\nu(s) = \nu_\infty + \nu_0 s^{p/2-1}, \quad s \geq 0,$$
$$U(\boldsymbol{\varepsilon}) = \nu_\infty |\boldsymbol{\varepsilon}|^2_{\mathbb{S}^d} + \frac{2\,\nu_0}{p}\,|\boldsymbol{\varepsilon}|^p_{\mathbb{S}^d}, \quad \boldsymbol{\varepsilon} \in \mathbb{S}^d.$$

It is straightforward to verify (5.7) and (5.8). Consider

$$(\boldsymbol{S}(\boldsymbol{\varepsilon}_1) - \boldsymbol{S}(\boldsymbol{\varepsilon}_2)) : (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) = 2\,\nu_\infty |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2_{\mathbb{S}^d} + 2\,\nu_0 \left(|\boldsymbol{\varepsilon}_1|^{p-2}_{\mathbb{S}^d}\boldsymbol{\varepsilon}_1 - |\boldsymbol{\varepsilon}_2|^{p-2}_{\mathbb{S}^d}\boldsymbol{\varepsilon}_2\right) : (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2).$$

Apply the inequality (5.36) to obtain

$$(\boldsymbol{S}(\boldsymbol{\varepsilon}_1) - \boldsymbol{S}(\boldsymbol{\varepsilon}_2)) : (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq 2\,\nu_\infty |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2_{\mathbb{S}^d} + 2^{3-p}p^{-1}\nu_0 |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^p_{\mathbb{S}^d}.$$

Hence, (5.9) holds with $m_S = 2^{3-p}p^{-1}\nu_0$; (5.10) also holds with $m_S = 2\,\nu_\infty$. □

Example 5.11. In the second example,

$$\boldsymbol{S}(\boldsymbol{\varepsilon}) = 2\,\nu_\infty \boldsymbol{\varepsilon} + 2\,\nu_0 \left(1 + |\boldsymbol{\varepsilon}|_{\mathbb{S}^d}\right)^{p-2} \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \in \mathbb{S}^d. \tag{5.44}$$

Assume $\nu_\infty, \nu_0 > 0$. We have

$$\nu(s) = \nu_\infty + \nu_0 (1 + s^{1/2})^{p-2}, \quad s \geq 0,$$
$$U(\boldsymbol{\varepsilon}) = \nu_\infty |\boldsymbol{\varepsilon}|^2_{\mathbb{S}^d} + 2\,\nu_0 \int_0^{|\boldsymbol{\varepsilon}|_{\mathbb{S}^d}} s\,(1+s)^{p-2} ds, \quad \boldsymbol{\varepsilon} \in \mathbb{S}^d.$$

For $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, by the definition (5.44),

$$(\boldsymbol{S}(\boldsymbol{\varepsilon}_1) - \boldsymbol{S}(\boldsymbol{\varepsilon}_2)) : (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) = 2\,\nu_\infty |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2_{\mathbb{S}^d} + 2\,\nu_0 \Delta_p(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2),$$

where $\Delta_p(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2)$ is bounded below by Lemma 5.8.

We conclude that (5.9) holds with $m_S = \nu_0 \min\{2^{-1}(p-1)^{-1}, 8^{-1}12^{-p/2}\}$, and (5.10) holds with $m_S = 2\,\nu_\infty + \nu_0$. □

Example 5.12. In the third example,

$$\boldsymbol{S}(\boldsymbol{\varepsilon}) = 2\,\nu_\infty \boldsymbol{\varepsilon} + 2\,\nu_0 \left(1 + |\boldsymbol{\varepsilon}|^2_{\mathbb{S}^d}\right)^{p/2-1} \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \in \mathbb{S}^d. \tag{5.45}$$

Assume $\nu_\infty, \nu_0 > 0$. We have

$$\nu(s) = \nu_\infty + \nu_0 (1 + s)^{p/2-1}, \quad s \geq 0,$$
$$U(\boldsymbol{\varepsilon}) = \nu_\infty |\boldsymbol{\varepsilon}|^2_{\mathbb{S}^d} + \nu_0 \int_0^{|\boldsymbol{\varepsilon}|^2_{\mathbb{S}^d}} (1+s)^{p/2-1} ds, \quad \boldsymbol{\varepsilon} \in \mathbb{S}^d.$$

Analysis of this example is similar to that of Example 5.11, and we apply Lemma 5.9 instead of Lemma 5.8. Detail is omitted. □

Finally, we examine one example of ψ_τ, written as ψ for simplicity.

Example 5.13. Following [27], for a constant $a \in [0, 1)$, consider

$$\psi(\boldsymbol{z}) = (a-1)\,e^{-|\boldsymbol{z}|_{\mathbb{R}^d}} + a\,|\boldsymbol{z}|_{\mathbb{R}^d}, \quad \boldsymbol{z} \in \mathbb{R}^d. \tag{5.46}$$

Then, (5.11) is obvious. Denote by $B(\boldsymbol{0}, 1) \subset \mathbb{R}^d$ the unit closed ball centered at the origin. Since

$$\partial\psi(\boldsymbol{z}) = \begin{cases} B(\boldsymbol{0}, 1) \text{if } \boldsymbol{z} = \boldsymbol{0}, \\ \left((1-a)\,e^{-|\boldsymbol{z}|_{\mathbb{R}^d}} + a\right)\boldsymbol{z}/|\boldsymbol{z}|_{\mathbb{R}^d} \quad \text{if } \boldsymbol{z} \neq \boldsymbol{0}, \end{cases}$$

it is easy to see that (5.12) is valid; indeed, $\partial\psi(\boldsymbol{z})$ is bounded. Now we provide a detailed derivation of (5.14) for the reader's convenience. Note that (5.14) is equivalent to

$$\left(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\right)\cdot(\boldsymbol{z}_1-\boldsymbol{z}_2)\ge -M_\psi|\boldsymbol{z}_1-\boldsymbol{z}_2|_{\mathbb{R}^d}^2 \quad \forall\,\boldsymbol{z}_1,\boldsymbol{z}_2\in\mathbb{R}^d,\ \boldsymbol{\xi}_1\in\partial\psi(\boldsymbol{z}_1),\ \boldsymbol{\xi}_2\in\partial\psi(\boldsymbol{z}_2). \tag{5.47}$$

We distinguish two cases.

Case 1: $\boldsymbol{z}_1\neq\boldsymbol{0}$, $\boldsymbol{z}_2\neq\boldsymbol{0}$. Then,

$$\begin{aligned}\left(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\right)\cdot(\boldsymbol{z}_1-\boldsymbol{z}_2)&=\int_0^1\frac{d}{ds}\psi(\boldsymbol{z}_2+s\,(\boldsymbol{z}_1-\boldsymbol{z}_2))\,ds\,(\boldsymbol{z}_1-\boldsymbol{z}_2)\\&=(\boldsymbol{z}_1-\boldsymbol{z}_2)^T\int_0^1 G(\boldsymbol{z}_2+s\,(\boldsymbol{z}_1-\boldsymbol{z}_2))\,ds\,(\boldsymbol{z}_1-\boldsymbol{z}_2)\,,\end{aligned}$$

where G is the Hessian matrix of ψ. Through elementary calculations, we find that

$$G(\boldsymbol{z})=\left((1-a)\,e^{-|\boldsymbol{z}|_{\mathbb{R}^d}}+a\right)|\boldsymbol{z}|_{\mathbb{R}^d}^{-1}\boldsymbol{I}+\left((a-1)\,e^{-|\boldsymbol{z}|_{\mathbb{R}^d}}\,|\boldsymbol{z}|_{\mathbb{R}^d}^{-2}-\left((1-a)\,e^{-|\boldsymbol{z}|_{\mathbb{R}^d}}+a\right)|\boldsymbol{z}|_{\mathbb{R}^d}^{-3}\right)\boldsymbol{z}\,\boldsymbol{z}^T.$$

It can be shown that $G(\boldsymbol{z})$ has two distinct eigenvalues

$$(a-1)\,e^{-|\boldsymbol{z}|_{\mathbb{R}^d}},\quad \left((1-a)\,e^{-|\boldsymbol{z}|_{\mathbb{R}^d}}+a\right)|\boldsymbol{z}|_{\mathbb{R}^d}^{-1}.$$

The first eigenvalue is simple and the second eigenvalue has multiplicity $(d-1)$. Hence,

$$(\boldsymbol{z}_1-\boldsymbol{z}_2)^T\,G(\boldsymbol{z}_2+s\,(\boldsymbol{z}_1-\boldsymbol{z}_2))\cdot(\boldsymbol{z}_1-\boldsymbol{z}_2)\ge-(1-a)\,e^{-|\boldsymbol{z}|_{\mathbb{R}^d}}\,|\boldsymbol{z}_1-\boldsymbol{z}_2|_{\mathbb{R}^d}^2.$$

Therefore,

$$\left(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\right)\cdot(\boldsymbol{z}_1-\boldsymbol{z}_2)\ge-(1-a)\,|\boldsymbol{z}_1-\boldsymbol{z}_2|_{\mathbb{R}^d}^2.$$

Case 2: $\boldsymbol{z}_1\neq\boldsymbol{0}$, $\boldsymbol{z}_2=\boldsymbol{0}$. Let $\boldsymbol{\xi}_2\in\partial\psi(\boldsymbol{z}_2)=B(\boldsymbol{0},1)$ be arbitrary. Then,

$$\begin{aligned}\left(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\right)\cdot(\boldsymbol{z}_1-\boldsymbol{z}_2)&=\left(\left((1-a)\,e^{-|\boldsymbol{z}_1|_{\mathbb{R}^d}}+a\right)\boldsymbol{z}_1/|\boldsymbol{z}_1|_{\mathbb{R}^d}-\boldsymbol{\xi}_2\right)\cdot\boldsymbol{z}_1\\&=\left((1-a)\,e^{-|\boldsymbol{z}_1|_{\mathbb{R}^d}}+a\right)|\boldsymbol{z}_1|_{\mathbb{R}^d}-\boldsymbol{\xi}_2\cdot\boldsymbol{z}_1\\&\ge\left((1-a)\,e^{-|\boldsymbol{z}_1|_{\mathbb{R}^d}}+a\right)|\boldsymbol{z}_1|_{\mathbb{R}^d}-|\boldsymbol{z}_1|_{\mathbb{R}^d}\\&=-(1-a)\,|\boldsymbol{z}_1|_{\mathbb{R}^d}\left(1-e^{-|\boldsymbol{z}_1|_{\mathbb{R}^d}}\right)\\&\ge-(1-a)\,|\boldsymbol{z}_1|_{\mathbb{R}^d}^2.\end{aligned}$$

Summarizing, we see that (5.47) holds with $M_\psi=1-a$. We comment that for a non-convex non-smooth function ψ_τ, the natural choice of the exponent p in the condition (5.13) is $p=2$. □

Declaration of competing interest

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Data availability

No data was used for the research described in the article.

References

[1] Panagiotopoulos PD. Nonconvex energy functions, hemivariational inequalities and substationary principles. Acta Mech 1983;42:160–83.
[2] Sofonea M, Migórski S. Variational–hemivariational Inequalities with Applications. Boca Raton, FL: CRC Press; 2018.
[3] Han W, Sofonea M. Numerical analysis of hemivariational inequalities in contact mechanics. Acta Numer 2019;28:175–286.
[4] Feng F, Han W, Huang J. Virtual element method for elliptic hemivariational inequalities. J Sci Comput 2019;81:2388–412.
[5] Ling M, Wang F, Han W. The nonconforming virtual element method for a stationary Stokes hemivariational inequality with slip boundary condition. J Sci Comput 2020;85:56.
[6] Wu B, Wang F, Han W. Virtual element method for a frictional contact problem with normal compliance. Commun Nonlinear Sci Numer Simul 2022;107:106125.
[7] Han W. Minimization principles for elliptic hemivariational inequalities. Nonlinear Anal RWA 2020;54:103114.
[8] Han W. A revisit of elliptic variational–hemivariational inequalities. Numer Funct Anal Optim 2021;42:371–95.
[9] Sofonea M, Han W. Minimization arguments in analysis of variational-hemivartiational inequalities. Z Angew Math Phys 2022;73:6.
[10] Han W, Matei A. Minimax principles for elliptic mixed hemivariational-variational inequalities. Nonlinear Anal RWA 2022;64:103448.
[11] Han W, Matei A. Well-posedness of a general class of elliptic mixed hemivariational-variational inequalities. Nonlinear Anal RWA 2022;66:103553.
[12] Han W, Migórski S, Sofonea M. A class of variational–hemivariational inequalities with applications to frictional contact problems. SIAM J Math Anal 2014;46:3891–912.

[13] Migórski S, Ochal A, Sofonea M. A class of variational–hemivariational inequalities in reflexive Banach spaces. J Elasticity 2017;127:151–78.
[14] Fujita H. Flow problems with unilateral boundary conditions. Lecons: College de France; 1993.
[15] Fujita H. A mathematical analysis of motions of viscous incompressible fluid under leak or slip boundary conditions. RIMS Kôkyûroku 1994;888:199–216.
[16] Le Roux C. Steady Stokes flows with threshold slip boundary conditions. Math Models Methods Appl Sci 2005;15:1141–68.
[17] Saito N. On the Stokes equation with the leak or slip boundary conditions of friction type: regularity of solutions. Publ Res Inst Math Sci 2004;40:345–83.
[18] Le Roux C, Tani A. Steady solutions of the Navier–Stokes equations with threshold slip boundary conditions. Math Methods Appl Sci 2007;30:595–624.
[19] Li Y, Li K. Existence of the solution to stationary Navier–Stokes equations with nonlinear slip boundary conditions. J Math Anal Appl 2011;381:1–9.
[20] Fang C, Han W. Well-posedness and optimal control of a hemivariational inequality for nonstationary Stokes fluid flow. Discrete Contin Dyn Syst 2016;36:5369–86.
[21] Fang C, Han W, Migórski S, Sofonea M. A class of hemivariational inequalities for nonstationary Navier–Stokes equations. Nonlinear Anal RWA 2016;31:257–76.
[22] Migórski S, Ochal A. Hemivariational inequalities for stationary Navier–Stokes equations. J Math Anal Appl 2005;306:197–217.
[23] Migórski S, Ochal A. Navier–Stokes problems modeled by evolution hemivariational inequalities. Discrete Contin Dyn Syst 2007;(Supplement):731–40.
[24] Fang C, Czuprynski K, Han W, Cheng XL, Dai X. Finite element method for a stationary Stokes hemivariational inequality with slip boundary condition. IMA J Numer Anal 2020;40:2696–716.
[25] Han W, Czuprynski K, Jing F. Mixed finite element method for a hemivariational inequality of stationary Navier–Stokes equations. J Sci Comput 2021;89:8.
[26] Dudek S, Kalita P, Migórski S. Stationary flow of non-Newtonian fluid with nonmonotone frictional boundary conditions. ZAMP 2015;66:2625–46.
[27] Dudek S, Kalita P, Migórski S. Steady flow of generalized Newtonian fluid with multivalued rheology and nonmonotone friction law. Comput Math Appl 2017;74:1813–25.
[28] Migórski S, Dudek S. Steady flow with unilateral and leak/slip boundary conditions by the Stokes variational–hemivariational inequality. Appl Anal 2022;101:2949–65.
[29] Schowalter WR. Mechanics of Non-Newtonian Fluids. Oxford: Pergamon Press; 1978.
[30] Rajagopal KR. Mechanics of non-Newtonian fluid. In: Galdi GP, Nečas J, editors. Recent Developments in Theoretical Fluid Mechanics. Pitman research notes in mathematics, vol. 291, Essex: Longman Scientific & Technical; 1993, p. 129–62.
[31] Wu W-T, Massoudi M, editors. Recent Advances in Mechanics of Non-Newtonian Fluids. Basel: MDPI; 2020.
[32] Clarke FH. Generalized gradients and applications. Trans Amer Math Soc 1975;205:247–62.
[33] Clarke FH. Optimization and Nonsmooth Analysis. New York: Wiley; 1983.
[34] Fan L, Liu S, Gao S. Generalized monotonicity and convexity of non-differentiable functions. J Math Anal Appl 2003;279:276–89.
[35] Ekel I, Temam R. Convex Analysis and Variational Problems. Amsterdam: North-Holland; 1976.
[36] Hiriart-Urruty J-B, Lemaréchal C. Convex Analysis and Minimizarion Algorithms, I: Fundamentals. Berlin: Springer-Verlag; 1993.
[37] Zeidler E. Nonlinear Functional Analysis and its Applications. III: Variational Methods and Optimization. New York: Springer-Verlag; 1986.
[38] Atkinson K, Han W. Theoretical Numerical Analysis: A Functional Analysis Framework. 3rd ed.. New York: Springer; 2009.
[39] Málek J, Nečas J, Rokyta M, Ružička M. Weak and Measure-valued Solutions to Evolutionary PDEs. Springer; 1996.
[40] Migórski S, Ochal A, Sofonea M. Nonlinear Inclusions and Hemivariational Inequalities. Advances in mechanics and mathematics, vol. 26, New York: Springer; 2013.
[41] Sohr H. The Navier–Stokes Equations: An Elementary Functional Analytic Approach. Switzerland: Birkhäuser-Verlag; 2001.
[42] DiBenedetto E. Degenerate Parabolic Equations. New York: Springer-Verlag; 1993.
[43] Matei A, Micu S, Niţă C. Optimal control for antiplane frictional contact problems involving nonlinear elastic materials of Hencky type. Math Mech Solids 2018;23:308–28.