Numer. Math. (2000) 87: 283–315
Digital Object Identifier (DOI) 10.1007/s002110000194

Numerische
Mathematik

Convergence of approximations to the primal problem in plasticity under conditions of minimal regularity

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Received June 8, 1998 / Published online July 12, 2000 –

Summary. This work considers semi- and fully discrete approximations to the primal problem in elastoplasticity. The unknowns are displacement and internal variables, and the problem takes the form of an evolution variational inequality. Strong convergence of time-discrete, as well as spatially and fully discrete approximations, is established without making any assumptions of regularity over and above those established in the proof of well-posedness of this problem.

Mathematics Subject Classification (1991): 65M60

1 Introduction

In a recent work [4], Han and Reddy have revisited the problem of obtaining estimates for the rate of convergence of approximations to semi- and fully discrete problems of elastoplasticity. In that work the authors have taken as their point of departure the dual formulation studied in the extended work [3], which gives a detailed account of the variational basis of hardening problems in plasticity, and investigates the theoretical basis of various approximation schemes. The work [3] makes reference to two alternative forms of the problem: the primal form, in which the flow law is formulated in terms

* The work of this author was supported by NSF under Grant DMS-9874015.
** The work of this author was supported by a grant from the Foundation for Research Development of South Africa.
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of the dissipation function, and for which case the unknown variables are displacement, plastic strain and internal variables; and the dual form, in which the flow law is written in terms of the yield function, with the result that the unknown variables are displacement and generalized stress.

Convergence of semi- and fully discrete finite element approximations was proved in [3] under conditions in which not much attention was given to the degree of smoothness expected of the solutions. Thus, for the primal problem convergence was established under the assumption that the solutions satisfy particular regularity requirements; for example, for the primal problem in which linear kinematic and isotropic hardening are present, it is required that the displacement $\boldsymbol{u}$ belong to $W^{1,2}(0,T;H^2(\Omega)^3)$, and that the internal variables $\boldsymbol{\xi}$ belong to $W^{1,2}(0,T;H^1(\Omega)^n)$, for suitable n. On the other hand, it is shown that a unique solution $(\boldsymbol{u}, \boldsymbol{\xi})$ to the primal problem exists with $\boldsymbol{u} \in W^{1,2}(0,T;H^1(\Omega)^3)$ and $\boldsymbol{\xi} \in W^{1,2}(0,T;L^2(\Omega)^n)$. Similar disparities are present in the case of the dual problem.

In [4], the authors have proved convergence of approximations to the dual problem under the minimum conditions of regularity, that is, the regularity established in the existence proof. The purpose of this contribution is to achieve the same goal in respect of the primal problem.

By way of background, it is worth mentioning some works that have preceded the analyses in [3] and [4]. An early contribution in this area was that of Johnson [8], who considered a formulation with stress as the primary variable, and who derived error estimates for the fully discrete problem. About the same time, independent work by Korneev and others was carried out on such problems (see, for example, [11] and the monograph [10]). Related work may also be found in [6]. Johnson [9] subsequently analyzed fully discrete finite element approximations of the problem with hardening, in the context of a mixed formulation in which both stress and velocity are the variables. With regard to the primal problem, the first detailed analysis of convergence of finite element approximations was provided by Han, Reddy and Schroeder [5]. The work [3] served the purpose of synthesising and generalizing the results referred to above.

This paper is organized as follows. In Sect. 2 the primal problem is formulated, both in classical and variational forms. Section 3 is concerned with time-discrete approximations of an abstract problem, of which the primal problem is an example. The main result in this section establishes strong convergence of such time-discrete approximations, under conditions of minimal regularity of the solution. Then, in Sect. 4, strong convergence of spatially and fully discrete approximations is established, again under conditions of minimal regularity. Finally in Sect. 5, we apply the convergence results proved for the abstract problem to conclude the convergence of various schemes for solving the primal problem of elastoplasticity.

2 Formulation of the primal problem

Consider the initial-boundary value problem for quasistatic behavior of an elastoplastic body which occupies a bounded domain $\Omega \subset \mathbb{R}^d$ ($d \leq 3$ for practical applications) with Lipschitz boundary Γ. We assume that deformations are sufficiently small to warrant adoption of the small strain assumption. The plastic behavior of the material is assumed to be describable within the classical framework of a convex, elastic domain coupled with the normality law. The material is assumed to undergo kinematic or isotropic hardening or a combination of both.

Suppose that the system is initially at rest, and that it is initially undeformed and unstressed. A time-dependent field of body force $\boldsymbol{f}(\boldsymbol{x}, t)$ is given, with $\boldsymbol{f}(\boldsymbol{x}, 0) = \boldsymbol{0}$. Then the problem is governed by the following set of equations in Ω:

the equilibrium equation

$$\operatorname{div} \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0}, \tag{2.1}$$

the additive decomposition of strain

$$\boldsymbol{\epsilon} = \boldsymbol{e} + \boldsymbol{p}, \tag{2.2}$$

and the strain-displacement relation

$$\boldsymbol{\epsilon}(\boldsymbol{u}) = \tfrac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T). \tag{2.3}$$

Here $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{\epsilon}$ is the strain tensor, $\boldsymbol{u}$ the displacement vector, $\boldsymbol{p}$ the plastic strain tensor and $\boldsymbol{e}$ the elastic strain. All the tensors encountered here are symmetric. The plastic deformation is assumed to be incompressible so that

$$\operatorname{tr} \boldsymbol{p} = 0 \ \text{ or } \sum_{i=1}^{d} p_{ii} = 0. \tag{2.4}$$

For simplicity, and with little loss in generality, we assume that

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma, \tag{2.5}$$

while the initial conditions are assumed to be

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{p}(\boldsymbol{x}, 0) = \boldsymbol{0}. \tag{2.6}$$

2.1 Constitutive relations

There is a linear relation between stress and the elastic strain, that is,

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{e} = \boldsymbol{C}(\boldsymbol{\epsilon}(\boldsymbol{u}) - \boldsymbol{p}), \tag{2.7}$$

where $\boldsymbol{C}$ is the elasticity modulus.

The features of hardening behavior are captured through the introduction of a set of scalar and/or tensorial internal variables, denoted collectively here by an m-dimensional unknown variable $\boldsymbol{\xi}$. We also introduce a stress-like variable $\boldsymbol{\chi}$ conjugate to the internal variable $\boldsymbol{\xi}$, in the sense that

$$\boldsymbol{\chi} = -\boldsymbol{H}\boldsymbol{\xi}, \tag{2.8}$$

where $\boldsymbol{H}$ is a hardening modulus. The ordered pairs $\boldsymbol{\Sigma} = (\boldsymbol{\sigma}, \boldsymbol{\chi})$ and $\boldsymbol{P} = (\boldsymbol{p}, \boldsymbol{\xi})$ are referred to respectively as the *generalized stress* and *generalized plastic strain*.

The generalized stress takes values only in a closed convex set K; the interior of K contains the origin and is called the elastic region while its boundary is known as the yield surface. Then the *flow law* or *normality law* governing the evolution of the plastic strain and internal variables takes the form

$$\dot{\boldsymbol{P}} \equiv (\dot{\boldsymbol{p}}, \dot{\boldsymbol{\xi}}) \in N_K(\boldsymbol{\Sigma}), \tag{2.9}$$

where $N_K(\boldsymbol{\Sigma}) = \{\boldsymbol{M} \mid \boldsymbol{M} : (\boldsymbol{T} - \boldsymbol{\Sigma}) \le 0 \ \ \forall\, \boldsymbol{T} \in K\}$ denotes the normal cone to K at $\boldsymbol{\Sigma}$.

The primal formulation is based on an alternative description of the flow law which uses the support function of K, defined by

$$D(\dot{\boldsymbol{P}}) = \sup\{\boldsymbol{T} : \dot{\boldsymbol{P}} \mid \boldsymbol{T} \in K\}. \tag{2.10}$$

The function D is nonnegative and may take on the value $+\infty$. From the theory of convex analysis, the flow law (2.9) is equivalent to the relation (cf. [3])

$$\boldsymbol{\Sigma} \in \partial D(\dot{\boldsymbol{P}}), \tag{2.11}$$

where $\partial D(\dot{\boldsymbol{P}})$ denotes the subdifferential of D at $\dot{\boldsymbol{P}}$, defined by

$$D(\boldsymbol{q}, \boldsymbol{\eta}) \ge D(\dot{\boldsymbol{p}}, \dot{\boldsymbol{\xi}}) + \boldsymbol{\sigma} : (\boldsymbol{q} - \dot{\boldsymbol{p}}) + \boldsymbol{\chi} : (\boldsymbol{\eta} - \dot{\boldsymbol{\xi}}) \quad \forall\, (\boldsymbol{q}, \boldsymbol{\eta}). \tag{2.12}$$

In the context of plasticity, the function D is a measure of the rate of irreversible or plastic work, and is known as the *dissipation function*.

2.2 Primal variational formulation of the problem

We first introduce some spaces and functionals that are required for the primal formulation. The space V of displacements is defined by

$$V = [H_0^1(\Omega)]^d.$$

Let

$$Q = \{\boldsymbol{q} = (q_{ij})_{d\times d} \mid q_{ji} = q_{ij},\ q_{ij} \in L^2(\Omega)\},$$

with the usual inner product and norm of the space $[L^2(\Omega)]^{d\times d}$. Then the space Q_0 of plastic strains is a closed subspace of Q defined by

$$Q_0 = \{\boldsymbol{q} \in Q \mid \operatorname{tr} \boldsymbol{q} = 0 \text{ a.e. in } \Omega\}.$$

We will use $M = [L^2(\Omega)]^m$ for the space of internal variables $\boldsymbol{\xi}$. The product space $Z = V \times Q_0 \times M$ is a Hilbert space with the inner product

$$(\boldsymbol{w}, \boldsymbol{z})_Z = (\boldsymbol{u}, \boldsymbol{v})_V + (\boldsymbol{p}, \boldsymbol{q})_Q + (\boldsymbol{\xi}, \boldsymbol{\eta})_M$$

and norm

$$\|\boldsymbol{z}\|_Z = (\boldsymbol{z}, \boldsymbol{z})_Z^{1/2},$$

where $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{p}, \boldsymbol{\xi})$ and $\boldsymbol{z} = (\boldsymbol{v}, \boldsymbol{q}, \boldsymbol{\eta})$.

Corresponding to the set $K_p = \operatorname{dom} D$, the effective domain of D, we define

(2.13) $$Z_p = \{\boldsymbol{z} = (\boldsymbol{v}, \boldsymbol{q}, \boldsymbol{\eta}) \in Z \mid (\boldsymbol{q}, \boldsymbol{\eta}) \in K_p \text{ a.e. in } \Omega\},$$

which is a non-empty, closed, convex cone in Z.

Over the space Z, we introduce the bilinear form

(2.14) $$a(\boldsymbol{w}, \boldsymbol{z}) = \int_\Omega [\boldsymbol{C}(\boldsymbol{\epsilon}(\boldsymbol{u}) - \boldsymbol{p}) : (\boldsymbol{\epsilon}(\boldsymbol{v}) - \boldsymbol{q}) + \boldsymbol{\xi} \cdot \boldsymbol{H}\boldsymbol{\eta}]\ dx,$$

the linear functional

(2.15) $$\langle \ell(t), \boldsymbol{z} \rangle = \int_\Omega \boldsymbol{f}(t) \cdot \boldsymbol{v}\, dx$$

and the functional

(2.16) $$j(\boldsymbol{z}) = \int_\Omega D(\boldsymbol{q}, \boldsymbol{\eta})\, dx,$$

where as before $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{p}, \boldsymbol{\xi})$ and $\boldsymbol{z} = (\boldsymbol{v}, \boldsymbol{q}, \boldsymbol{\eta})$.

From the properties of D, we see that $j(\cdot)$ is a convex, positively homogeneous, nonnegative and l.s.c. functional.

Then the primal variational problem of elastoplasticity takes the following form.

Problem PRIM. Given $\ell \in H^1(0,T;Z^*)$ with $\ell(0) = 0$, find $\boldsymbol{w} = (\boldsymbol{u},\boldsymbol{p},\boldsymbol{\xi}) : [0,T] \to Z$ with $\boldsymbol{w}(0) = \mathbf{0}$, such that for almost all $t \in (0,T)$, $\dot{\boldsymbol{w}}(t) \in Z_p$ and

$$a(\boldsymbol{w}(t), \boldsymbol{z} - \dot{\boldsymbol{w}}(t)) + j(\boldsymbol{z}) - j(\dot{\boldsymbol{w}}(t)) \geq \langle \ell(t), \boldsymbol{z} - \dot{\boldsymbol{w}}(t)\rangle \quad \forall\, \boldsymbol{z} \in Z_p. \tag{2.17}$$

Later on, we will take as working examples the special cases of combined linear kinematic and isotropic hardening, or linear kinematic hardening only, together with the von Mises yield function. For the former case the unknown variables are the displacement $\boldsymbol{u}$, plastic strain $\boldsymbol{p}$, and isotropic hardening variable γ. We use the spaces V and Q_0 for displacements and plastic strains, and the space M of isotropic hardening variables is defined by $M = L^2(\Omega)$.

In this special context the subset Z_p of Z defined in (2.13)) is given by

$$Z_p = \{\boldsymbol{z} = (\boldsymbol{v}, \boldsymbol{q}, \mu) \in Z \ : \ |\boldsymbol{q}| \leq \mu \text{ a.e. in } \Omega\}.$$

The bilinear form $a : Z \times Z \to I\!R$ becomes

$$\begin{aligned} a(\boldsymbol{w},\boldsymbol{z}) &= \int_\Omega [\boldsymbol{C}(\boldsymbol{\epsilon}(\boldsymbol{u}) - \boldsymbol{p}) : (\boldsymbol{\epsilon}(\boldsymbol{v}) - \boldsymbol{q}) + k_1 \boldsymbol{p} : \boldsymbol{q} + k_2 \gamma\mu]\ dx \\ &= \int_\Omega [C_{ijkl}\ (\epsilon_{ij}(\boldsymbol{u}) - p_{ij})(\epsilon_{kl}(\boldsymbol{v}) - q_{kl}) + k_1 p_{ij} q_{ij} + k_2 \gamma\mu]\ dx, \end{aligned} \tag{2.18}$$

while the functional j is defined by (2.16), with the dissipation function D being given by

$$D(\boldsymbol{q},\mu) = \begin{cases} c_0|\boldsymbol{q}|, & \text{if } |\boldsymbol{q}| \leq \mu, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.19}$$

Here, $c_0 > 0$ is the constant in the von Mises yield condition. The linear functional $\ell(t)$ is as in (2.15). With these modifications, the primal variational problem of elastoplasticity with combined linear kinematic and isotropic hardening, and with the von Mises yield criterion, is

Problem KIN-ISO. Given $\ell \in H^1(0,T;Z^*)$, $\ell(0) = 0$, find $\boldsymbol{w} = (\boldsymbol{u},\boldsymbol{p},\gamma) : [0,T] \to Z$ with $\boldsymbol{w}(0) = \mathbf{0}$, such that for almost all $t \in (0,T)$, $\dot{\boldsymbol{w}}(t) \in Z_p$ and

$$a(\boldsymbol{w}(t), \boldsymbol{z} - \dot{\boldsymbol{w}}(t)) + j(\boldsymbol{z}) - j(\dot{\boldsymbol{w}}(t)) \geq \langle \ell(t), \boldsymbol{z} - \dot{\boldsymbol{w}}(t)\rangle \quad \forall\, \boldsymbol{z} = (\boldsymbol{v},\boldsymbol{q},\mu) \in Z_p. \tag{2.20}$$

The problem for the case of linear kinematic hardening with the von Mises yield crierion can be viewed as a degenerate case of Problem KIN-ISO, with $k_2 = 0$. The unknown variables for this case are the displacement $\boldsymbol{u}$ and the plastic strain $\boldsymbol{p}$. We still use V and Q_0 as previously defined, and the solution space is now $Z = V \times Q_0$, with the inner product

$$(\boldsymbol{w},\boldsymbol{z})_Z = (\boldsymbol{u},\boldsymbol{v})_V + (\boldsymbol{p},\boldsymbol{q})_Q$$

and the norm $\|\boldsymbol{z}\|_Z = (\boldsymbol{z},\boldsymbol{z})_Z^{1/2}$, where $\boldsymbol{w} = (\boldsymbol{u},\boldsymbol{p})$ and $\boldsymbol{z} = (\boldsymbol{v},\boldsymbol{q})$. This time, instead of (2.19), the dissipation function takes the simple form

$$D(\boldsymbol{q}) = c_0|\boldsymbol{q}| \quad \forall\, \boldsymbol{q} \in Q_0, \tag{2.21}$$

so that the functional

$$j(\boldsymbol{z}) = \int_\Omega D(\boldsymbol{q})\,dx \quad \text{for } \boldsymbol{z} = (\boldsymbol{v},\boldsymbol{q}) \in Z \tag{2.22}$$

is finite on the whole space Z. The bilinear form $a : Z \times Z \to \mathbb{R}$ is now

$$a(\boldsymbol{w},\boldsymbol{z}) = \int_\Omega \left[\boldsymbol{C}(\boldsymbol{\epsilon}(\boldsymbol{u}) - \boldsymbol{p}) : (\boldsymbol{\epsilon}(\boldsymbol{v}) - \boldsymbol{q}) + k_1\boldsymbol{p} : \boldsymbol{q}\right]\,dx \tag{2.23}$$

while the linear functional $\ell(t)$ is unchanged from (2.15). We can now define the primal variational problem corresponding to linear kinematic hardening with the von Mises yield function.

Problem Kin. Given $\ell \in H^1(0,T;Z^*)$, $\ell(0) = 0$, find $\boldsymbol{w} = (\boldsymbol{u},\boldsymbol{p}) : [0,T] \to Z$ with $\boldsymbol{w}(0) = \boldsymbol{0}$, such that for almost all $t \in (0,T)$,

$$a(\boldsymbol{w}(t),\boldsymbol{z}-\dot{\boldsymbol{w}}(t))+j(\boldsymbol{z})-j(\dot{\boldsymbol{w}}(t)) \geq \langle \ell(t),\boldsymbol{z}-\dot{\boldsymbol{w}}(t)\rangle \quad \forall\, \boldsymbol{z} = (\boldsymbol{v},\boldsymbol{q}) \in Z. \tag{2.24}$$

2.3 Properties of material parameters

The elasticity tensor $\boldsymbol{C}$ has the symmetry properties

$$C_{ijkl} = C_{jikl} = C_{klij}, \tag{2.25}$$

and we assume that

$$C_{ijkl} \in L^\infty(\Omega) \tag{2.26}$$

and that $\boldsymbol{C}$ is pointwise stable: that is, there exists a constant $C_0 > 0$ such that

$$C_{ijkl}(\boldsymbol{x})\zeta_{ij}\zeta_{kl} \geq C_0|\boldsymbol{\zeta}|^2 \quad \forall\, \boldsymbol{\zeta} \in \mathbb{R}^{d\times d}, \boldsymbol{\zeta}^T = \boldsymbol{\zeta}, \quad \text{a.e. in } \Omega. \tag{2.27}$$

The hardening modulus $\boldsymbol{H}$, viewed as a linear operator from $\mathbb{R}^m$ into itself, is assumed to possess the symmetry property

$$\boldsymbol{\xi} \cdot \boldsymbol{H}\boldsymbol{\lambda} = \boldsymbol{\lambda} \cdot \boldsymbol{H}\boldsymbol{\xi} \tag{2.28}$$

and it is further assumed that

$$H_{ij} \in L^\infty(\Omega) \tag{2.29}$$

and that a constant $H_0 > 0$ exists such that

$$\boldsymbol{\xi} \cdot \boldsymbol{H}\boldsymbol{\xi} \geq H_0|\boldsymbol{\xi}|^2 \;\; \forall\, \boldsymbol{\xi} \in \mathbb{R}^m, \text{ a.e. in } \Omega. \tag{2.30}$$

3 Time-discrete approximations of an abstract problem

The primal formulation is a particular case of the following abstract problem.

Problem ABS. Find $w : [0,T] \to H$, $w(0) = 0$, such that for almost all $t \in (0,T)$, $\dot{w}(t) \in K$ and

$$a(w(t), z - \dot{w}(t)) + j(z) - j(\dot{w}(t)) \geq \langle \ell(t), z - \dot{w}(t) \rangle \quad \forall\, z \in K. \tag{3.1}$$

Under the assumptions that

- H is a Hilbert space
- $K \subset H$ is a non-empty, closed, convex cone
- $a : H \times H \to I\!R$ is a bilinear form on H, symmetric, bounded and H-elliptic
- $\ell \in H^1(0,T;H^*)$, $\ell(0) = 0$
- $j : K \to I\!R$ is non-negative, convex, positively homogeneous and Lipschitz continuous

we have the existence of a unique solution $w \in H^1(0,T;H)$ of the problem ABS (cf. [3]).

In this section we will prove the strong convergence of time-discrete solutions of the problem ABS under the basic regularity assumption $w \in H^1(0,T;H)$.

The following elementary result will be used repeatedly:

$$a, b, x \geq 0 \ \text{ and } \ x^2 \leq a\, x + b \Longrightarrow x^2 \leq a^2 + 2\, b. \tag{3.2}$$

We analyze a family of semi-discrete schemes which are obtained by discretizing the time interval. For notational simplicity, we use a uniform partition of the time interval $[0,T]$ with node points $t_n = nk$, $0 \leq n \leq N$, where $k = T/N$ is the step-size. We will use $I_n = [t_{n-1}, t_n]$, $n = 1, \cdots, N$, to denote the time sub-intervals. We remark that the following analysis can be easily modified for the case of a non-uniform partition; in this case the step-size k in the error bounds is replaced by the maximum step-size of the partition. For the given linear functional $\ell \in H^1(0,T;H^*)$ and the solution $w \in H^1(0,T;H)$, we use the notation $\ell_n = \ell(t_n)$ and $w_n = w(t_n)$, which are well-defined because of the continuous embeddings $H^1(0,T;H^*) \subset C([0,T];H^*)$ and $H^1(0,T;H) \subset C([0,T];H)$. The symbol Δw_n is used to denote the backward difference $w_n - w_{n-1}$, and $\delta w_n = \Delta w_n / k$ denotes the backward divided difference.

Let $\theta \in [\frac{1}{2}, 1]$ be a parameter. A family of generalized mid-point time-discrete approximations of the problem ABS is

Problem ABS^k. Find $w^k = \{w_n^k\}_{n=0}^N \subset H$, $w_0^k = 0$, such that for $n = 1, \ldots, N$, $\delta w_n^k \in K$ and

$$\begin{aligned} &a(\theta\, w_n^k + (1-\theta)\, w_{n-1}^k, z - \delta w_n^k) + j(z) \\ &\quad -j(\delta w_n^k) \geq \langle \ell_{n-1+\theta}, z - \delta w_n^k \rangle \quad \forall\, z \in K. \end{aligned} \tag{3.3}$$

Here, $\ell_{n-1+\theta} = \ell(t_{n-1+\theta})$, and $t_{n-1+\theta} = (n-1+\theta)\,k = \theta\, t_n + (1-\theta)\,t_{n-1}$. It is shown in [3] that the use of $\theta \notin [\frac{1}{2}, 1]$ leads to a divergent scheme. For simplicity in writing, the dependence of the solution w^k on θ will not be indicated explicitly.

Set

$$E_{n,\theta}(w) = \theta\, w_n + (1-\theta)\, w_{n-1} - w_{n-1+\theta}; \tag{3.4}$$

then the following inequality is the basis for order error estimates (cf. [3]):

$$\begin{aligned}\max_{1\le n\le N} \|w_n - w_n^k\|_H \\ &\le c\left(\|E_{N,\theta}(w)\|_H + \sum_{n=1}^{N-1} \|E_{n,\theta}(w) - E_{n+1,\theta}(w)\|_H\right) \\ &+c\left\{k\sum_{n=1}^{N} \|\delta w_n - \dot{w}_{n-1+\theta}\|_H\right\}^{1/2}.\end{aligned} \tag{3.5}$$

Nevertheless, we cannot use (3.5) for a convergence analysis under the basic regularity condition $w \in H^1(0,T;H)$, because then the pointwise values $\dot{w}_{n-1+\theta}$ occurring in (3.5) are not well-defined. Thus it is necessary to derive a result similar to (3.5), in which the terms $\dot{w}_{n-1+\theta}$ are not present. For this purpose we will need the following density result (for a proof, see, e.g. [15]).

Theorem 3.1. *The space $C^\infty([0,T];H)$ is dense in $H^1(0,T;H)$; that is, given $w \in H^1(0,T;H)$, for any $\varepsilon > 0$ there is a function $\overline{w} \in C^\infty([0,T];H)$ such that*

$$\|w - \overline{w}\|_{H^1(0,T;H)} \le \varepsilon. \tag{3.6}$$

The inequality (3.6) can be equivalently written in the form

$$\int_0^T \|w(t) - \overline{w}(t)\|_H^2 dt + \int_0^T \|\dot{w}(t) - \dot{\overline{w}}(t)\|_H^2 dt \le \varepsilon^2.$$

In addition,

$$\begin{aligned}\|w - \overline{w}\|_{L^\infty(0,T;H)} &\equiv \operatorname{ess\,sup}_{0\le t\le T} \|w(t) - \overline{w}(t)\|_H \\ &\le c\,\|w - \overline{w}\|_{H^1(0,T;H)} \le c\,\varepsilon.\end{aligned} \tag{3.7}$$

We will also need the following results, proved in [3].

Lemma 3.2. *Under the assumption $\ddot{w} \in L^1(0,T;H)$,*

$$\|E_{n,\theta}(w)\|_H \le 2\,\theta\,(1-\theta)\,k\,\|\ddot{w}\|_{L^1(t_{n-1},t_n;H)}.$$

If it is assumed further that $\ddot{w} \in L^\infty(0,T;H)$, then

$$\|E_{n,\theta}(w)\|_H \le \frac{\theta\,(1-\theta)}{2}\,k^2 \|\ddot{w}\|_{L^\infty(0,T;H)}.$$

Lemma 3.3. *Under the assumption $\ddot{w} \in L^1(0,T;H)$,*

$$\|E_{n,\theta}(w) - E_{n+1,\theta}(w)\|_H \le c\, k\, \|\ddot{w}\|_{L^1(t_{n-1},t_{n+1};H)}.$$

If it is assumed further that $w^{(3)} \in L^1(0,T;H)$, then

$$\|E_{n,\theta}(w) - E_{n+1,\theta}(w)\|_H \le c\, k^2 \|w^{(3)}\|_{L^1(t_{n-1},t_{n+1};H)}.$$

3.1 A bound on $\boldsymbol{w}_n - \boldsymbol{w}_n^k$

Set $e_n = w_n - w_n^k$, $0 \le n \le N$, with $e_0 = 0$. From the assumptions on the bilinear form $a(\cdot,\cdot)$, we see that the term $\|w\|_a = a(w,w)^{1/2}$ defines a norm on H, which is equivalent to $\|w\|_H$. Consider the quantity

$$A_n = a(\theta\, e_n + (1-\theta)\, e_{n-1}, \delta e_n).$$

Since $\theta \in [\frac{1}{2}, 1]$, we have a lower bound

$$A_n \ge \frac{1}{2k} \left(\|e_n\|_a^2 - \|e_{n-1}\|_a^2\right). \tag{3.8}$$

Next, we derive an upper bound for A_n. We have

$$\begin{aligned} A_n = {} & a(\theta\, w_n + (1-\theta)\, w_{n-1}, \delta w_n - \delta w_n^k) \\ & - a(\theta\, w_n^k + (1-\theta)\, w_{n-1}^k, \delta w_n - \delta w_n^k). \end{aligned}$$

We use (3.3) with $z = \delta w_n$ for the second term on the right hand side of the above inequality to obtain

$$\begin{aligned} A_n \le {} & a(\theta\, w_n + (1-\theta)\, w_{n-1}, \delta w_n - \delta w_n^k) \\ & + j(\delta w_n) - j(\delta w_n^k) - \langle \ell_{n-1+\theta}, \delta w_n - \delta w_n^k \rangle. \end{aligned}$$

Combining the lower and upper bound, we have the inequality

$$\begin{aligned} & \frac{1}{2k} \left(\|e_n\|_a^2 - \|e_{n-1}\|_a^2\right) \\ & \quad \le a(\theta\, w_n + (1-\theta)\, w_{n-1}, \delta w_n - \delta w_n^k) + j(\delta w_n) \\ & \qquad - j(\delta w_n^k) - \langle \ell_{n-1+\theta}, \delta w_n - \delta w_n^k \rangle. \end{aligned} \tag{3.9}$$

Now take $z = \delta w_n^k$ in (3.1); then

$$a(w(t), \delta w_n^k - \dot{w}(t)) + j(\delta w_n^k) - j(\dot{w}(t)) \ge \langle \ell(t), \delta w_n^k - \dot{w}(t) \rangle.$$

We then integrate the relation over I_n to obtain

$$\begin{aligned} 0 \le {} & \frac{1}{k} \int_{I_n} a(w(t), \delta w_n^k - \dot{w}(t))\, dt + \frac{1}{k} \int_{I_n} j(\delta w_n^k) dt \\ & - \frac{1}{k} \int_{I_n} j(\dot{w}(t))\, dt - \frac{1}{k} \int_{I_n} \langle \ell(t), \delta w_n^k - \dot{w}(t) \rangle\, dt. \end{aligned} \tag{3.10}$$

Adding the inequalities (3.9) and (3.10) we find that

$$\frac{1}{2k}\left(\|e_n\|_a^2 - \|e_{n-1}\|_a^2\right) \le Q_1 + Q_2 + Q_3, \tag{3.11}$$

where

$$\begin{aligned}
Q_1 &= a(\theta\, w_n + (1-\theta)\, w_{n-1}, \delta w_n - \delta w_n^k) \\
&\quad + \frac{1}{k}\int_{I_n} a(w(t), \delta w_n^k - \dot w(t))\, dt, \\
Q_2 &= \frac{1}{k}\int_{I_n} \left[j(\delta w_n) - j(\dot w(t)) \right] dt, \\
Q_3 &= -\langle \ell_{n-1+\theta}, \delta w_n - \delta w_n^k \rangle - \frac{1}{k}\int_{I_n} \langle \ell(t), \delta w_n^k - \dot w(t) \rangle\, dt.
\end{aligned}$$

We now estimate each of these three terms. Define the local average of $w(t)$ by

$$w_n^a = \frac{1}{k}\int_{I_n} w(t)\, dt \in H, \quad n = 1, \cdots, N, \tag{3.12}$$

and by analogy with (3.4) introduce the quantities

$$E^a_{n,\theta}(w) = \theta\, w_n + (1-\theta)\, w_{n-1} - w_n^a. \tag{3.13}$$

Then

$$\begin{aligned}
Q_1 &= a(\theta\, w_n + (1-\theta)\, w_{n-1} - w_n^a, \delta w_n - \delta w_n^k) \\
&\quad + \frac{1}{k}\int_{I_n} a(w(t), \delta w_n)\, dt - \frac{1}{k}\int_{I_n} a(w(t), \dot w(t))\, dt \\
&= a(E^a_{n,\theta}(w), \delta w_n - \delta w_n^k) + a(w_n^a, \delta w_n) \\
&\quad - \frac{1}{2\,k}\left[a(w_n, w_n) - a(w_{n-1}, w_{n-1}) \right] \\
&= \frac{1}{k}\, a(E^a_{n,\theta}(w), e_n - e_{n-1}) \\
&\quad + \frac{1}{2\,k}\left[2\, a(w_n^a, w_n - w_{n-1}) - a(w_n, w_n) + a(w_{n-1}, w_{n-1}) \right].
\end{aligned}$$

Since

$$\begin{aligned}
&2\, a(w_n^a, w_n - w_{n-1}) - a(w_n, w_n) + a(w_{n-1}, w_{n-1}) \\
&\quad = a(2\, w_n^a - w_n - w_{n-1}, w_n - w_{n-1})
\end{aligned}$$

we see that

$$\begin{aligned}
Q_1 &= \frac{1}{k}\, a(E^a_{n,\theta}(w), e_n - e_{n-1}) \\
&\quad + \frac{1}{k}\, a(w_n^a - \tfrac{1}{2}\,(w_n + w_{n-1}), w_n - w_{n-1}).
\end{aligned} \tag{3.14}$$

For the second term Q_2, we use the Lipschitz continuity of $j(\cdot)$ on K to obtain

$$
\begin{aligned}
|Q_2| &\le \frac{c}{k}\int_{I_n} \|\delta w_n - \dot w(t)\|_H dt \\
&= \frac{c}{k}\int_{I_n} \left\| \frac{1}{k}\int_{I_n} (\dot w(s) - \dot w(t))\, ds \right\|_H dt \\
&\le \frac{c}{k^2}\int_{I_n}\int_{I_n} \|\dot w(s) - \dot w(t)\|_H ds\, dt \\
&\le \frac{c}{k^2}\int_{I_n}\int_{I_n} \big[\|\dot w(s) - \dot{\overline{w}}(s)\|_H + \|\dot w(t) - \dot{\overline{w}}(t)\|_H \\
&\qquad + \|\dot{\overline{w}}(s) - \dot{\overline{w}}(t)\|_H\big]\ ds\, dt \\
&= \frac{c}{k}\int_{I_n} \|\dot w(t) - \dot{\overline{w}}(t)\|_H\ dt + \frac{c}{k^2}\int_{I_n}\int_{I_n} \left\| \int_s^t \ddot{\overline{w}}(\tau)\, d\tau \right\|_H ds\, dt \\
&\le \frac{c}{k}\int_{I_n} \|\dot w(t) - \dot{\overline{w}}(t)\|_H\ dt + c\int_{I_n} \|\ddot{\overline{w}}(t)\|_H\ dt. \qquad (3.15)
\end{aligned}
$$

Analogously to (3.12), we define the local average of ℓ by

$$
\ell_n^a = \frac{1}{k}\int_{I_n} \ell(t)\, dt \in H^*, \quad n = 1, \cdots, N.
$$

Then

$$
Q_3 = \langle \ell_n^a - \ell_{n-1+\theta}, \delta w_n - \delta w_n^k\rangle + \frac{1}{k}\int_{I_n} \langle \ell(t), \dot w(t) - \delta w_n\rangle\, dt.
$$

Now

$$
\int_{I_n} \langle \ell(t), \delta w_n\rangle\, dt = \langle \frac{1}{k}\int_{I_n} \ell(t)\, dt, \int_{I_n} \dot w(s)\, ds\rangle = \int_{I_n} \langle \ell_n^a, \dot w(t)\rangle\, dt,
$$

so that

$$
Q_3 = \frac{1}{k}\langle \ell_n^a - \ell_{n-1+\theta}, e_n - e_{n-1}\rangle + \frac{1}{k}\int_{I_n} \langle \ell(t) - \ell_n^a, \dot w(t)\rangle\, dt. \qquad (3.16)
$$

Combining (3.11), (3.14), (3.15) and (3.16) we obtain

$$
\begin{aligned}
&\|e_n\|_a^2 - \|e_{n-1}\|_a^2 \le 2\, a(E_{n,\theta}^a(w), e_n - e_{n-1}) \\
&+ 2\, a(w_n^a - \tfrac{1}{2}(w_n + w_{n-1}), w_n - w_{n-1}) \\
&+ c\int_{I_n} \|\dot w(t) - \dot{\overline{w}}(t)\|_H dt + c\, k\int_{I_n} \|\ddot{\overline{w}}(t)\|_H dt \\
&+ 2\,\langle \ell_n^a - \ell_{n-1+\theta}, e_n - e_{n-1}\rangle + 2\int_{I_n} \langle \ell(t) - \ell_n^a, \dot w(t)\rangle\, dt.
\end{aligned}
$$

Adding the above inequalities from $n = 1$ to n, and observing that $e_0 = 0$, we have

$$
\begin{aligned}
\|e_n\|_a^2 &\le 2 \sum_{j=1}^{n} a(E^a_{j,\theta}(w), e_j - e_{j-1}) \\
&\quad + 2 \sum_{j=1}^{n} a(w^a_j - \tfrac{1}{2}(w_j + w_{j-1}), w_j - w_{j-1}) \\
&\quad + c \int_0^{t_n} \|\dot{w}(t) - \dot{\overline{w}}(t)\|_H dt + c\, k \int_0^{t_n} \|\ddot{\overline{w}}(t)\|_H dt \\
&\quad + 2 \sum_{j=1}^{n} \langle \ell^a_j - \ell_{j-1+\theta}, e_j - e_{j-1}\rangle + 2 \sum_{j=1}^{n} \int_{I_j} \langle \ell(t) - \ell^a_j, \dot{w}(t)\rangle\, dt \\
&= 2\, a(E^a_{n,\theta}(w), e_n) + 2 \sum_{j=1}^{n-1} a(E^a_{j,\theta}(w) - E^a_{j+1,\theta}(w), e_j) \\
&\quad + 2 \sum_{j=1}^{n} a(w^a_j - \tfrac{1}{2}(w_j + w_{j-1}), w_j - w_{j-1}) \\
&\quad + 2\, \langle \ell^a_n - \ell_{n-1+\theta}, e_n\rangle + 2 \sum_{j=1}^{n-1} \langle (\ell^a_j - \ell_{j-1+\theta}) - (\ell^a_{j+1} - \ell_{j+\theta}), e_j\rangle \\
&\quad + 2 \sum_{j=1}^{n} \int_{I_j} \langle \ell(t) - \ell^a_j, \dot{w}(t)\rangle\, dt \\
&\quad + c \int_0^{t_n} \|\dot{w}(t) - \dot{\overline{w}}(t)\|_H dt + c\, k \int_0^{t_n} \|\ddot{\overline{w}}(t)\|_H dt.
\end{aligned}
$$

Set $M = \max_{1 \le n \le N} \|e_n\|_a$. Then from the above inequality we get

$$
\begin{aligned}
M^2 \le c\, M \Bigg\{ & \|E^a_{N,\theta}(w)\|_H + \sum_{n=1}^{N-1} \|E^a_{n,\theta}(w) - E^a_{n+1,\theta}(w)\|_H \\
& + \|\ell^a_N - \ell_{N-1+\theta}\|_{H^*} + \sum_{n=1}^{N-1} \|(\ell^a_n - \ell_{n-1+\theta}) \\
& - (\ell^a_{n+1} - \ell_{n+\theta})\|_{H^*} \Bigg\} \\
& + c \sum_{n=1}^{N} \|w^a_n - \tfrac{1}{2}(w_n + w_{n-1})\|_H \|w_n - w_{n-1}\|_H
\end{aligned}
$$

$$
\begin{aligned}
&+ c \sum_{n=1}^{N} \int_{I_n} \|\ell(t) - \ell_n^a\|_{H^*} \|\dot{w}(t)\|_H dt \\
&+ c \int_0^T \|\dot{w}(t) - \dot{\overline{w}}(t)\|_H dt + c\, k \int_0^T \|\ddot{\overline{w}}(t)\|_H dt.
\end{aligned}
$$

Applying the result (3.2), we then have

$$
\begin{aligned}
&\max_{1\le n\le N} \|e_n\|_a \\
&\quad \le c \Bigg\{ \|E^a_{N,\theta}(w)\|_H + \sum_{n=1}^{N-1} \|E^a_{n,\theta}(w) - E^a_{n+1,\theta}(w)\|_H \\
&\qquad + \|\ell^a_N - \ell_{N-1+\theta}\|_{H^*} + \sum_{n=1}^{N-1} \|(\ell^a_n - \ell_{n-1+\theta}) - (\ell^a_{n+1} - \ell_{n+\theta})\|_{H^*} \Bigg\} \\
&\qquad + c \Bigg\{ \sum_{n=1}^{N} \|w^a_n - \tfrac{1}{2}(w_n + w_{n-1})\|_H \|w_n - w_{n-1}\|_H \\
&\qquad + \int_0^T \|\dot{w}(t) - \dot{\overline{w}}(t)\|_H dt \\
&\qquad + k \int_0^T \|\ddot{\overline{w}}(t)\|_H dt + \sum_{n=1}^{N} \int_{I_n} \|\ell(t) - \ell^a_n\|_{H^*} \|\dot{w}(t)\|_H dt \Bigg\}^{1/2}.
\end{aligned}
\tag{3.17}
$$

We now analyze each term on the right hand side of (3.17). First, for the terms involving $E^a_{n,\theta}(w)$, we have

$$
\begin{aligned}
&\|E^a_{n,\theta}(w) - E^a_{n+1,\theta}(w)\|_H \le \|E^a_{n,\theta}(\overline{w}) - E^a_{n+1,\theta}(\overline{w})\|_H \\
&\quad + \|E^a_{n,\theta}(w - \overline{w})\|_H + \|E^a_{n+1,\theta}(w - \overline{w})\|_H.
\end{aligned}
\tag{3.18}
$$

Since $w(t) - \overline{w}(t)$ is continuous in t, it follows that

$$
w^a_n - \overline{w}^a_n = \frac{1}{k} \int_{I_n} [w(t) - \overline{w}(t)]\, dt = w(\tau_n) - \overline{w}(\tau_n), \quad \text{for some } \tau_n \in I_n.
$$

Hence

$$
\begin{aligned}
E^a_{n,\theta}(w - \overline{w}) &= \theta\,(w_n - \overline{w}_n) + (1-\theta)\,(w_{n-1} - \overline{w}_{n-1}) \\
&\qquad - (w(\tau_n) - \overline{w}(\tau_n)) \\
&= \theta \int_{\tau_n}^{t_n} [\dot{w}(t) - \dot{\overline{w}}(t)]\, dt + (1-\theta) \int_{\tau_n}^{t_n} [\dot{w}(t) - \dot{\overline{w}}(t)]\, dt,
\end{aligned}
$$

and so

$$\|E^a_{n,\theta}(w-\overline{w})\|_H \le c\int_{t_{n-1}}^{t_n}\|\dot{w}(t)-\dot{\overline{w}}(t)\|_H dt.$$

Similarly, it can be shown that

$$\|E^a_{n+1,\theta}(w-\overline{w})\|_H \le c\int_{t_n}^{t_{n+1}}\|\dot{w}(t)-\dot{\overline{w}}(t)\|_H dt.$$

Now observing that

$$E^a_{n,\theta}(\overline{w}) = E_{n,\theta}(\overline{w}) + \overline{w}_{n-1+\theta} - \frac{1}{k}\int_{I_n}\overline{w}(t)\,dt,$$

we have

$$\begin{aligned}&E^a_{n,\theta}(\overline{w}) - E^a_{n+1,\theta}(\overline{w}) = (E_{n,\theta}(\overline{w}) - E_{n+1,\theta}(\overline{w}))\\ &+\Big(\overline{w}_{n-1+\theta} - \frac{1}{k}\int_{I_n}\overline{w}(t)\,dt\Big) - \Big(\overline{w}_{n+\theta} - \frac{1}{k}\int_{I_{n+1}}\overline{w}(t)\,dt\Big).\end{aligned}$$

By Lemma 3.3,

$$\|E_{n,\theta}(\overline{w}) - E_{n+1,\theta}(\overline{w})\|_H \le c\,k\,\|\ddot{\overline{w}}\|_{L^1(t_{n-1},t_{n+1};H)}.$$

We use the Taylor expansion

$$\overline{w}(t) = \overline{w}_{n-1+\theta} + \dot{\overline{w}}_{n-1+\theta}(t-t_{n-1+\theta}) + \int_{t_{n-1+\theta}}^{t}(t-s)\,\ddot{\overline{w}}(s)\,ds$$

to get

$$\begin{aligned}&\overline{w}_{n-1+\theta} - \frac{1}{k}\int_{I_n}\overline{w}(t)\,dt = -\frac{1-2\,\theta}{2}\,k\,\dot{\overline{w}}_{n-1+\theta}\\ &-\frac{1}{k}\int_{I_n}\int_{t_{n-1+\theta}}^{t}(t-s)\,\ddot{\overline{w}}(s)\,ds\,dt.\end{aligned}\tag{3.19}$$

So

$$\begin{aligned}&\Big(\overline{w}_{n-1+\theta} - \frac{1}{k}\int_{I_n}\overline{w}(t)\,dt\Big) - \Big(\overline{w}_{n+\theta} - \frac{1}{k}\int_{I_{n+1}}\overline{w}(t)\,dt\Big)\\ &= \frac{1-2\,\theta}{2}\,k\,\left[\dot{\overline{w}}_{n+\theta} - \dot{\overline{w}}_{n-1+\theta}\right]\\ &\quad -\frac{1}{k}\int_{I_n}\int_{t_{n-1+\theta}}^{t}(t-s)\,\ddot{\overline{w}}(s)\,ds\,dt + \frac{1}{k}\int_{I_{n+1}}\int_{t_{n+\theta}}^{t}(t-s)\,\ddot{\overline{w}}(s)\,ds\,dt\\ &= \frac{1-2\,\theta}{2}\,k\int_{t_{n-1+\theta}}^{t_{n+\theta}}\ddot{\overline{w}}(t)\,dt\\ &\quad -\frac{1}{k}\int_{I_n}\int_{t_{n-1+\theta}}^{t}(t-s)\,\ddot{\overline{w}}(s)\,ds\,dt + \frac{1}{k}\int_{I_{n+1}}\int_{t_{n+\theta}}^{t}(t-s)\,\ddot{\overline{w}}(s)\,ds\,dt,\end{aligned}$$

and hence

$$\|E^a_{n,\theta}(\overline{w}) - E^a_{n+1,\theta}(\overline{w})\|_H \le \|E_{n,\theta}(\overline{w}) - E_{n+1,\theta}(\overline{w})\|_H$$
$$+ \left\| \left(\overline{w}_{n-1+\theta} - \frac{1}{k}\int_{I_n} \overline{w}(t)\,dt\right) - \left(\overline{w}_{n+\theta} - \frac{1}{k}\int_{I_{n+1}} \overline{w}(t)\,dt\right)\right\|_H,$$

which implies that

$$\|E^a_{n,\theta}(\overline{w}) - E^a_{n+1,\theta}(\overline{w})\|_H \le c\,k\,\|\ddot{\overline{w}}\|_{L^1(t_{n-1},t_{n+1};H)}.$$

Therefore from (3.18),

$$\|E^a_{n,\theta}(w) - E^a_{n+1,\theta}(w)\|_H \le c\,k\,\|\ddot{\overline{w}}\|_{L^1(t_{n-1},t_{n+1};H)}$$
$$+c\int_{t_{n-1}}^{t_{n+1}} \|\dot{w}(t) - \dot{\overline{w}}(t)\|_H dt,$$

and thus

$$\sum_{n=1}^{N-1} \|E^a_{n,\theta}(w) - E^a_{n+1,\theta}(w)\|_H \le c\,k\,\|\ddot{\overline{w}}\|_{L^1(0,T;H)}$$
$$+c\,\|\dot{w} - \dot{\overline{w}}\|_{L^1(0,T;H)} \le c\,k\,\|\ddot{\overline{w}}\|_{L^1(0,T;H)} + c\,\varepsilon. \tag{3.20}$$

The formula (3.19) with $n = N$ implies that

$$\left\|\overline{w}_{N-1+\theta} - \frac{1}{k}\int_{I_N} \overline{w}(t)\,dt\right\|_H$$
$$\le c\,k\,\left[\|\dot{\overline{w}}\|_{L^\infty(t_{N-1},t_N;H)} + \|\ddot{\overline{w}}\|_{L^1(t_{N-1},t_N;H)}\right].$$

By Lemma 3.2,

$$\|E_{N,\theta}(\overline{w})\|_H \le c\,k\,\|\ddot{\overline{w}}\|_{L^1(t_{N-1},t_N;H)}.$$

It is not difficult to see that

$$\|E^a_{N,\theta}(w) - E^a_{N,\theta}(\overline{w})\|_H \le c \sup_{t_{N-1}\le t\le t_N} \|w(t) - \overline{w}(t)\|_H.$$

Applying (3.7), we then have

$$\|E^a_{N,\theta}(w) - E^a_{N,\theta}(\overline{w})\|_H \le c\,\varepsilon.$$

Using the last several bounds in the inequality

$$\|E^a_{N,\theta}(w)\|_H \le \|E^a_{N,\theta}(w) - E^a_{N,\theta}(\overline{w})\|_H$$
$$+\|E_{N,\theta}(\overline{w})\|_H + \left\|\overline{w}_{N-1+\theta} - \frac{1}{k}\int_{I_N} \overline{w}(t)\,dt\right\|_H$$

we obtain

$$\|E^a_{N,\theta}(w)\|_H \le c\,\varepsilon + c\,k\,\left[\|\dot{\overline{w}}\|_{L^\infty(t_{N-1},t_N;H)} + \|\ddot{\overline{w}}\|_{L^1(t_{N-1},t_N;H)}\right]. \tag{3.21}$$

We will carry out similar manipulations on the terms involving approximations of ℓ. First another application of Theorem 3.1 yields the existence of an $\overline{\ell} \in C^\infty([0,T];H^*)$ such that

$$\|\ell - \overline{\ell}\|_{H^1(0,T;H^*)} \le \varepsilon. \tag{3.22}$$

We also have

$$\|\ell - \overline{\ell}\|_{L^\infty(0,T;H^*)} \le c\,\|\ell - \overline{\ell}\|_{H^1(0,T;H^*)} \le c\,\varepsilon. \tag{3.23}$$

Thus

$$\begin{aligned}
\|\ell^a_N - \ell_{N-1+\theta}\|_{H^*} &\le \|\overline{\ell}^a_N - \overline{\ell}_{N-1+\theta}\|_{H^*} \\
&\qquad + \|\ell^a_N - \overline{\ell}^a_N\|_{H^*} + \|\ell_{N-1+\theta} - \overline{\ell}_{N-1+\theta}\|_{H^*} \\
&\le \|\overline{\ell}^a_N - \overline{\ell}_{N-1+\theta}\|_{H^*} + c\,\|\ell - \overline{\ell}\|_{L^\infty(0,T;H^*)} \\
&\le \|\overline{\ell}^a_N - \overline{\ell}_{N-1+\theta}\|_{H^*} + c\,\varepsilon.
\end{aligned}$$

Now applying the formula (3.19) to $\overline{\ell}$ we see that

$$\|\overline{\ell}^a_N - \overline{\ell}_{N-1+\theta}\|_{H^*} \le c\,k\,\left[\|\dot{\overline{\ell}}\|_{L^\infty(t_{N-1},t_N;H^*)} + \|\ddot{\overline{\ell}}\|_{L^1(t_{N-1},t_N;H^*)}\right].$$

Hence

$$\|\ell^a_N - \ell_{N-1+\theta}\|_{H^*} \le c\,\varepsilon + c\,k\,\left[\|\dot{\overline{\ell}}\|_{L^\infty(0,T;H^*)} + \|\ddot{\overline{\ell}}\|_{L^1(0,T;H^*)}\right]. \tag{3.24}$$

Similarly,

$$\begin{aligned}
&\|(\ell^a_n - \ell_{n-1+\theta}) - (\ell^a_{n+1} - \ell_{n+\theta})\|_{H^*} \\
&\quad \le \|(\overline{\ell}^a_n - \overline{\ell}_{n-1+\theta}) - (\overline{\ell}^a_{n+1} - \overline{\ell}_{n+\theta})\|_{H^*} \\
&\qquad + \|(\ell^a_n - \ell^a_{n+1}) - (\overline{\ell}^a_n - \overline{\ell}^a_{n+1})\|_{H^*} \\
&\qquad + \|(\ell_{n-1+\theta} - \ell_{n+\theta}) - (\overline{\ell}_{n-1+\theta} - \overline{\ell}_{n+\theta})\|_{H^*} \\
&\quad \le \|(\overline{\ell}^a_n - \overline{\ell}_{n-1+\theta}) - (\overline{\ell}^a_{n+1} - \overline{\ell}_{n+\theta})\|_{H^*} + c\,\|\dot{\ell} - \dot{\overline{\ell}}\|_{L^1(t_{n-1},t_{n+1};H^*)}.
\end{aligned}$$

Applying (3.19) to $\overline{\ell}$ once more, we have

$$\|(\overline{\ell}^a_n - \overline{\ell}_{n-1+\theta}) - (\overline{\ell}^a_{n+1} - \overline{\ell}_{n+\theta})\|_{H^*} \le c\,k\,\|\ddot{\overline{\ell}}\|_{L^1(t_{n-1},t_{n+1};H^*)},$$

and therefore

$$\sum_{n=1}^{N-1} \|(\ell_n^a - \ell_{n-1+\theta}) - (\ell_{n+1}^a - \ell_{n+\theta})\|_{H^*}$$
$$\leq c\,k\,\|\ddot{\overline{\ell}}\|_{L^1(0,T;H^*)} + c\,\|\dot{\ell} - \dot{\overline{\ell}}\|_{L^1(0,T;H^*)}.$$

Hence

$$\sum_{n=1}^{N-1} \|(\ell_n^a - \ell_{n-1+\theta}) - (\ell_{n+1}^a - \ell_{n+\theta})\|_{H^*} \leq c\,k\,\|\ddot{\overline{\ell}}\|_{L^1(0,T;H^*)} + c\,\varepsilon. \tag{3.25}$$

For the last sum in (3.17), we first use the Cauchy-Schwarz inequality in the form

$$\sum_{n=1}^{N} \int_{I_n} \|\ell(t) - \ell_n^a\|_{H^*} \|\dot{w}(t)\|_H dt$$
$$\leq \|\dot{w}\|_{L^2(0,T;H)} \left[\sum_{n=1}^{N} \int_{I_n} \|\ell(t) - \ell_n^a\|_{H^*}^2 dt \right]^{1/2},$$

then use the inequality

$$\|\ell(t) - \ell_n^a\|_{H^*}^2 \leq c \left[\|\overline{\ell}(t) - \overline{\ell}_n^a\|_{H^*}^2 + \|\ell_n^a - \overline{\ell}_n^a\|_{H^*}^2 + \|\ell(t) - \overline{\ell}(t)\|_{H^*}^2 \right],$$

to find that

$$\sum_{n=1}^{N} \int_{I_n} \|\ell(t) - \ell_n^a\|_{H^*}^2 dt \leq c \sum_{n=1}^{N} \int_{I_n} \|\overline{\ell}(t) - \overline{\ell}_n^a\|_{H^*}^2\, dt$$
$$+ c\,k \sum_{n=1}^{N} \|\ell_n^a - \overline{\ell}_n^a\|_{H^*}^2 + c\,\|\ell - \overline{\ell}\|_{L^2(0,T;H^*)}^2.$$

Now

$$\begin{aligned}
\sum_{n=1}^{N} \int_{I_n} \|\overline{\ell}(t) - \overline{\ell}_n^a\|_{H^*}^2\, dt &= \sum_{n=1}^{N} \int_{I_n} \left\| \frac{1}{k} \int_{I_n} [\overline{\ell}(t) - \overline{\ell}(s)]\, ds \right\|_{H^*}^2 dt \\
&= \sum_{n=1}^{N} \int_{I_n} \frac{1}{k^2} \Big\| \int_{I_n} \int_s^t \dot{\overline{\ell}}(\tau)\, d\tau\, ds \|_{H^*}^2 dt \\
&\leq c\,k^2 \sum_{n=1}^{N} \int_{I_n} \|\dot{\overline{\ell}}(t)\|_{H^*}^2 dt \\
&= c\,k^2 \|\dot{\overline{\ell}}\|_{L^2(0,T;H^*)}^2,
\end{aligned}$$

and

$$\sum_{n=1}^{N} \|\ell_n^a - \overline{\ell}_n^a\|_{H^*}^2 = \sum_{n=1}^{N} \left\| \frac{1}{k} \int_{I_n} [\ell(t) - \overline{\ell}(t)]\, dt \right\|_{H^*}^2$$
$$\leq \frac{1}{k} \sum_{n=1}^{N} \int_{I_n} \|\ell(t) - \overline{\ell}(t)\|_{H^*}^2\, dt$$
$$\leq \frac{1}{k} \|\ell - \overline{\ell}\|_{L^2(0,T;H^*)}^2.$$

Therefore,

$$\sum_{n=1}^{N} \int_{I_n} \|\ell(t) - \ell_n^a\|_{H^*}^2 dt \leq c\, k^2 \|\dot{\overline{\ell}}\|_{L^2(0,T;H^*)}^2 + c\, \|\ell - \overline{\ell}\|_{L^2(0,T;H^*)}^2,$$

and so

$$\sum_{n=1}^{N} \int_{I_n} \|\ell(t) - \ell_n^a\|_{H^*} \|\dot{w}(t)\|_H dt$$
$$\leq c\, \|\dot{w}\|_{L^2(0,T;H)} \left[k\, \|\dot{\overline{\ell}}\|_{L^2(0,T;H^*)} + \varepsilon\right]. \tag{3.26}$$

Finally, we estimate the terms $\|w_n^a - \frac{1}{2}(w_n + w_{n-1})\|_H$ and $\|w_n - w_{n-1}\|_H$. We have

$$w_n^a - \tfrac{1}{2}(w_n + w_{n-1}) = \frac{1}{k} \int_{I_n} \left[w(t) - \tfrac{1}{2}(w_n + w_{n-1})\right] dt$$
$$= -\frac{1}{2\,k} \int_{I_n} \left[\int_t^{t_n} \dot{w}(s)\, ds + \int_t^{t_{n-1}} \dot{w}(s)\, ds \right] dt,$$

and so

$$\|w_n^a - \tfrac{1}{2}(w_n + w_{n-1})\|_H \leq c \int_{I_n} \|\dot{w}(t)\|_H dt.$$

Also,

$$\|w_n - w_{n-1}\|_H = \left\| \int_{I_n} \dot{w}(t)\, dt \right\|_H \leq \int_{I_n} \|\dot{w}(t)\|_H\, dt.$$

Thus

$$\sum_{n=1}^{N} \|w_n^a - \tfrac{1}{2}(w_n + w_{n-1})\|_H \|w_n - w_{n-1}\|_H$$
$$\leq c \sum_{n=1}^{N} \left(\int_{I_n} \|\dot{w}(t)\|_H\, dt \right)^2 \leq c\, k\, \|\dot{w}\|_{L^2(0,T;H)}. \tag{3.27}$$

Summarizing then, by using (3.17), (3.20), (3.21), (3.24)—(3.27) and the inequality

$$\|\dot{w} - \dot{\overline{w}}\|_{L^1(0,T;H)} \le c\,\|\dot{w} - \dot{\overline{w}}\|_{L^2(0,T;H)} \le c\,\varepsilon,$$

we obtain the error estimate

$$\begin{aligned}\max_{1\le n\le N} \|w_n^k - w_n\|_H \le c\,\Big\{\varepsilon + k\,\Big(&\|\dot{\overline{w}}\|_{L^\infty(0,T;H)} + \|\ddot{\overline{w}}\|_{L^1(0,T;H)} \\ &+\|\dot{\overline{\ell}}\|_{L^\infty(0,T;H^*)} + \|\ddot{\overline{\ell}}\|_{L^1(0,T;H^*)}\Big)\Big\} \\ &+c\,\Big\{\varepsilon + k\,\Big(\|\dot{w}\|_{L^2(0,T;H)} + \|\ddot{\overline{w}}\|_{L^1(0,T;H)}\Big) \\ &+\|\dot{w}\|_{L^2(0,T;H)}\,\Big(\varepsilon + k\,\|\dot{\overline{\ell}}\|_{L^2(0,T;H^*)}\Big)\Big\}^{1/2}. \qquad (3.28)\end{aligned}$$

The following result is therefore valid.

Theorem 3.4. *For the abstract problem* ABS, *assume that H is a Hilbert space, $K \subset H$ a non-empty, closed, convex cone, $a : H \times H \to I\!R$ a symmetric, bounded and H-elliptic bilinear form on H, $\ell \in H^1(0,T;H^*)$, $\ell(0) = 0$ and $j : K \to I\!R$ non-negative, convex, positively homogeneous and Lipschitz continuous. Then the time-discrete solution w^k converges to w in the sense that*

$$\max_{1\le n\le N} \|w_n^k - w_n\|_H \to 0 \quad \text{as } k \to 0. \qquad (3.29)$$

4 Spatially and fully discrete approximations of the abstract problem

In this section, we prove the convergence of spatially and fully discrete solutions for the abstract Problem ABS.

In both the spatially and fully discrete approximations, the space H is replaced by a family of finite-dimensional subspaces $\{H^h\}$, and correspondingly, the set K is replaced by a family of finite-dimensional subsets $\{K^h\}$, defined by $K^h = H^h \cap K$. The subspaces $\{H^h\}$ are intended to be finite element spaces, though much of the analysis applies to more general situations. We use $h \in (0,1]$ for the mesh parameter of a triangulation of the domain Ω, and make the following additional assumptions about the function space and the finite element space.

Assumption **(H$_1$)**. There exists a subspace $H_0 \subset H$ with the property that $H^1(0,T;H_0) \cap H^1(0,T;K)$ is dense in $H^1(0,T;K)$ in the norm of $H^1(0,T;H)$.

Assumption **(H$_2$)**. For some constants c and $\alpha > 0$, the estimate

$$\inf_{z^h\in K^h} \|z - z^h\|_H \le c\,\|z\|_{H_0}\, h^\alpha \quad \forall\, z \in H_0 \cap K \qquad (4.1)$$

holds.

These two hypotheses will be verified later for the particular case of the primal problem in elastoplasticity.

Spatially discrete approximations

The spatially discrete approximation of Problem ABS is the following.

Problem ABS^h. Find $w^h : [0,T] \to H^h$, $w^h(0) = 0$, such that for almost all $t \in (0,T)$, $\dot{w}^h(t) \in K^h$ and

$$\begin{aligned} &a(w^h(t), z^h - \dot{w}^h(t)) + j(z^h) - j(\dot{w}^h(t)) \\ &\quad \geq \langle \ell(t), z^h - \dot{w}^h(t)\rangle \quad \forall\, z^h \in K^h. \end{aligned} \tag{4.2}$$

We note that for any given h, K^h is a non-empty, closed, convex cone in H^h, and the problem has a unique solution $w^h \in H^1(0,T;H)$. In [3], it is proved that

$$\|w - w^h\|_{L^\infty(0,T;H)} \leq c \inf_{z^h \in L^2(0,T;K^h)} \|\dot{w} - z^h\|^{1/2}_{L^2(0,T;H)}. \tag{4.3}$$

The inequality (4.3) is the basis for various asymptotic error estimates. Here we will use the inequality to prove the convergence of the approximate solution under the basic regularity condition $w \in H^1(0,T;H)$.

By Assumption (**H**$_1$), for any $\varepsilon > 0$ there exists $\tilde{w} \in H^1(0,T;H_0) \cap H^1(0,T;K)$ such that

$$\|w - \tilde{w}\|_{H^1(0,T;H)} \leq \varepsilon. \tag{4.4}$$

Since $\tilde{w} \in H^1(0,T;H_0) \cap H^1(0,T;K)$, by using Assumption (**H**$_2$) we have

$$\inf_{z^h \in L^2(0,T;K^h)} \|\dot{\tilde{w}} - z^h\|_{L^2(0,T;H)} \leq c\, h^\alpha \|\dot{\tilde{w}}\|_{L^2(0,T;H_0)}.$$

Therefore

$$\begin{aligned} \inf_{z^h \in L^2(0,T;K^h)} \|\dot{w} - z^h\|_{L^2(0,T;H)} &\leq \|\dot{w} - \dot{\tilde{w}}\|_{L^2(0,T;H)} \\ &\quad + \inf_{z^h \in L^2(0,T;K^h)} \|\dot{\tilde{w}} - z^h\|_{L^2(0,T;H)} \leq \varepsilon + c\, h^\alpha \|\dot{\tilde{w}}\|_{L^2(0,T;H_0)}. \end{aligned}$$

Now the inequality (4.3) yields

$$\|w - w^h\|_{L^\infty(0,T;H)} \leq c\sqrt{\varepsilon} + c\, h^{\alpha/2} \|\dot{\tilde{w}}\|^{1/2}_{L^2(0,T;H_0)},$$

and so we have the following result.

Theorem 4.1. *Let* $w \in H^1(0,T;H)$ *be the solution to Problem* $\textrm{ABS}$*, and* $w^h \in H^1(0,T;H^h)$ *the solution to the spatially discrete Problem* $\textrm{ABS}^h$*. Then under the set of assumptions following* (3.1)*, together with those on* H^h*, and Assumptions* $\mathbf{H}_1$ *and* $\mathbf{H}_2$*,* w^h *converges to* w *in the sense that*

$$\|w - w^h\|_{L^\infty(0,T;H)} \to 0 \quad \text{as } h \to 0. \tag{4.5}$$

Fully discrete approximations

We partition the time interval $I = [0,T]$ as in Sect. 3,and consider the following family of fully discrete schemes.

Problem ABS^{hk}. Find $w^{hk} = \{w_n^{hk}\}_{n=0}^N$ where $w_n^{hk} \in H^h$, $0 \le n \le N$, with $w_0^{hk} = 0$, such that for $n = 1, 2, \ldots, N$, $\delta w_n^{hk} \in K^h$ and

$$\begin{aligned} &a\left(\theta\, w_n^{hk} + (1-\theta)\, w_{n-1}^{hk}, z^h - \delta w_n^{hk}\right) + j(z^h) \\ &-j\left(\delta w_n^{hk}\right) \ge \left\langle \ell_{n-1+\theta}, z^h - \delta w_n^{hk}\right\rangle \quad \forall\, z^h \in K^h. \end{aligned} \tag{4.6}$$

It has been shown in [3] that Problem $\textrm{ABS}^{hk}$ has a unique solution, and that the estimate

$$\begin{aligned} \max_n \|w_n - w_n^{hk}\|_H \le c \Bigg(&\|E_{N,\theta}(w)\|_H + \sum_{j=1}^{N-1} \|E_{j,\theta}(w) \\ &-E_{j+1,\theta}(w)\|_H + k \sum_{j=1}^{N} \|\delta w_j - z_j^h\|_H \Bigg) \\ &+c \left\{ k \max_n \|E_{n,\theta}(w)\|_H \sum_{j=1}^{N} \|\delta w_j - z_j^h\|_H \right\}^{1/2} \\ &+c \left\{ k \sum_{j=1}^{N} \|\dot w_{j-1+\theta} - z_j^h\|_H \right\}^{1/2} \end{aligned} \tag{4.7}$$

holds. This estimate in turn leads to optimal order error estimates under suitable assumptions about the regularity of the solution. Here, as before, we are interested in proving the convergence of the numerical solution under the basic regularity condition $w \in H^1(0,T;H)$. For this purpose, the inequality (4.7) is no longer useful since it involves the pointwise values $\dot w_{j-1+\theta}$ which are not defined. We will derive an estimate similar to (4.7) without the occurrence of pointwise values of $\dot w$.

First note that, by Assumption ($\mathbf{H}_1$), (4.4) still holds.

We set $e_n = w_n - w_n^{hk}$, $0 \le n \le N$, with $e_0 = 0$, and consider the quantities

$$A_n = a(\theta\, e_n + (1-\theta)\, e_{n-1}, \delta e_n), \quad n = 1, \dots, N.$$

Since $\theta \in [\frac{1}{2}, 1]$, a lower bound of A_n is

$$A_n \ge \frac{1}{2k} \left(\|e_n\|_a^2 - \|e_{n-1}\|_a^2 \right). \tag{4.8}$$

To obtain an upper bound we begin with

$$\begin{aligned} A_n &= a(\theta\, w_n + (1-\theta)\, w_{n-1}, \delta w_n - \delta w_n^{hk}) \\ &\qquad - a(\theta\, w_n^{hk} + (1-\theta)\, w_{n-1}^{hk}, \delta w_n - \delta w_n^{hk}) \\ &= a(\theta\, w_n + (1-\theta)\, w_{n-1}, \delta w_n - \delta w_n^{hk}) \\ &\qquad - a(\theta\, w_n^{hk} + (1-\theta)\, w_{n-1}^{hk}, \delta w_n - z_n^h) \\ &\qquad - a(\theta\, w_n^{hk} + (1-\theta)\, w_{n-1}^{hk}, z_n^h - \delta w_n^{hk}), \end{aligned}$$

where $z_n^h \in K^h$ is arbitrary. Using (4.6) in the last term, we obtain

$$\begin{aligned} A_n &\le a(\theta\, w_n + (1-\theta)\, w_{n-1}, \delta w_n - \delta w_n^{hk}) \\ &\quad - a(\theta\, w_n^{hk} + (1-\theta)\, w_{n-1}^{hk}, \delta w_n - z_n^h) \\ &\quad + j(z_n^h) - j(\delta w_n^{hk}) - \langle \ell_{n-1+\theta}, z_n^h - \delta w_n^{hk} \rangle. \end{aligned} \tag{4.9}$$

Now integrate (3.1) with $z = \delta w_n^{hk} \in K$ from $t = t_{n-1}$ to $t = t_n$, to obtain

$$\begin{aligned} 0 &\le \frac{1}{k} \int_{I_n} a(w(t), \delta w_n^{hk} - \dot{w}(t))\, dt + \frac{1}{k} \int_{I_n} j(\delta w_n^{hk}) dt \\ &\quad - \frac{1}{k} \int_{I_n} j(\dot{w}(t))\, dt - \frac{1}{k} \int_{I_n} \langle \ell(t), \delta w_n^{hk} - \dot{w}(t) \rangle\, dt. \end{aligned} \tag{4.10}$$

We then add (4.10) to (4.9) to obtain

$$A_n \le R_1 + R_2 + R_3, \tag{4.11}$$

where

$$\begin{aligned} R_1 &= a(\theta\, w_n + (1-\theta)\, w_{n-1}, \delta e_n) \\ &\quad - a\left(\theta\, w_n^{hk} + (1-\theta)\, w_{n-1}^{hk}, \delta w_n - z_n^h\right) \\ &\quad + \frac{1}{k} \int_{I_n} a\left(w(t), \delta w_n^{hk} - \dot{w}(t)\right)\, dt, \\ R_2 &= \frac{1}{k} \int_{I_n} [j(z_n^h) - j(\dot{w}(t))]\, dt, \\ R_3 &= -\left\langle \ell_{n-1+\theta}, z_n^h - \delta w_n^{hk} \right\rangle - \frac{1}{k} \int_{I_n} \langle \ell(t), \delta w_n^{hk} - \dot{w}(t) \rangle\, dt. \end{aligned}$$

We now use the definitions (3.12) and (3.13) to obtain

$$
\begin{aligned}
R_1 = {}& a\left(E^a_{n,\theta}(w), \delta e_n\right) + \frac{1}{k}\int_{I_n} a(w(t), \delta w_n - \dot{w}(t))\, dt \\
&+ a\left(\theta\, e_n + (1-\theta)\, e_{n-1}, \delta w_n - z^h_n\right) \\
&- a\left(\theta\, w_n + (1-\theta)\; w_{n-1}, \delta w_n - z^h_n\right).
\end{aligned} \tag{4.12}
$$

Since

$$
R_2 = \frac{1}{k}\int_{I_n}\left[j(z^h_n) - j(\dot{w}(t))\right]\, dt,
$$

using the Lipschitz continuity of $j(\cdot)$ on K, we have

$$
|R_2| \le \frac{c}{k}\int_{I_n} \|z^h_n - \dot{w}(t)\|_H dt \le \frac{c}{k}\int_{I_n} \|z^h_n - \dot{w}(t)\|_H dt. \tag{4.13}
$$

Finally, R_3 can be rewritten as

$$
\begin{aligned}
Q_3 = {}& \langle \ell^a_n - \ell_{n-1+\theta}, \delta e_n\rangle \\
&+ \frac{1}{k}\int_{I_n}\langle \ell(t), \dot{w}(t) - \delta w_n\rangle\, dt - \langle \ell_{n-1+\theta}, z^h_n - \delta w_n\rangle.
\end{aligned} \tag{4.14}
$$

By combining (4.8) with (4.11)–(4.14), we find that

$$
\begin{aligned}
\|e_n\|^2_a - \|e_{n-1}\|^2_a \le {}& 2\, a(E^a_{n,\theta}(w), e_n - e_{n-1}) \\
&+ 2\int_{I_n} a(w(t), \delta w_n - \dot{w}(t))\, dt \\
&+ 2\, k\, a(\theta\, e_n + (1-\theta)\, e_{n-1}, \delta w_n - z^h_n) \\
&- 2\, k\, a(\theta\, w_n + (1-\theta)\, w_{n-1}, \delta w_n - z^h_n) \\
&+ c\int_{I_n} \|z^h_n - \dot{w}(t)\|_H dt + 2\,\langle \ell^a_n - \ell_{n-1+\theta}, e_n - e_{n-1}\rangle \\
&+ 2\int_{I_n}\langle \ell(t), \dot{w}(t) - \delta w_n\rangle\, dt - 2\, k\,\langle \ell_{n-1+\theta}, z^h_n - \delta w_n\rangle.
\end{aligned}
$$

Again we set $M = \max_{1\le n\le N} \|e_n\|_a$, and carry out a series of manipulations very similar to those leading to (3.17), to obtain eventually

$$
\begin{aligned}
M \le c\Bigg\{ & \|E^a_{N,\theta}(w)\|_H + \sum_{n=1}^{N-1} \|E^a_{n,\theta}(w) - E^a_{n+1,\theta}(w)\|_H \\
& + k\sum_{n=1}^{N} \|\delta w_n - z^h_n\|_H \\
& + \|\ell^a_N - \ell_{N-1+\theta}\|_{H^*} + \sum_{n=1}^{N-1} \|(\ell^a_n - \ell_{n-1+\theta}) \\
& - (\ell^a_{n+1} - \ell_{n+\theta})\|_{H^*}\Bigg\} \\
& + c\Bigg\{ k\left(\|w\|_{L^\infty(0,T;H)} + \|\ell\|_{L^\infty(0,T;H^*)}\right) \sum_{n=1}^{N} \|\delta w_n - z^h_n\|_H \\
& + \left(\|w\|_{L^\infty(0,T;H)} + \|\ell\|_{L^\infty(0,T;H^*)}\right) \sum_{n=1}^{N} \int_{I_n} \|\delta w_n - \dot w(t)\|_H\, dt \\
& + \sum_{n=1}^{N} \int_{I_n} \|\dot w(t) - z^h_n\|_H\, dt \Bigg\}^{1/2}.
\end{aligned}
\tag{4.15}
$$

We now estimate the term

$$\sum_{n=1}^{N} \int_{I_n} \|\delta w_n - \dot w(t)\|_H\, dt.$$

We have

$$
\begin{aligned}
&\int_{I_n} \|\delta w_n - \dot w(t)\|_H\, dt \\
&\qquad = \int_{I_n} \left\| \frac{1}{k} \int_{I_n} [\dot w(s) - \dot w(t)]\, ds \right\|_H dt \\
&\qquad \le \frac{1}{k} \int_{I_n}\int_{I_n} \big[\|\dot w(s) - \dot{\overline{w}}(s)\|_H + \|\dot w(t) \\
&\qquad\quad - \dot{\overline{w}}(t)\|_H + \|\dot{\overline{w}}(s) - \dot{\overline{w}}(t)\|_H\big]\, ds\, dt \\
&\qquad = c\int_{I_n} \|\dot w(t) - \dot{\overline{w}}(t)\|_H\, dt + \frac{1}{k}\int_{I_n}\int_{I_n} \left\| \int_t^s \ddot{\overline{w}}(\tau)\, d\tau \right\|_H ds\, dt \\
&\qquad \le c\int_{I_n} \|\dot w(t) - \dot{\overline{w}}(t)\|_H\, dt + c\, k \int_{I_n} \|\ddot{\overline{w}}(t)\|_H\, dt.
\end{aligned}
$$

Therefore,

$$\sum_{n=1}^{N} \int_{I_n} \|\delta w_n - \dot{w}(t)\|_H \, dt \le c \, \|\dot{w} - \dot{\tilde{w}}\|_{L^1(0,T;H)}$$

$$+c \, k \, \|\ddot{\tilde{w}}\|_{L^1(0,T;H)} \le c \, \varepsilon + c \, k \, \|\ddot{\tilde{w}}\|_{L^1(0,T;H)}. \tag{4.16}$$

Next, from

$$\|\dot{w}(t) - z_n^h\|_H \le \|\delta w_n - \dot{w}(t)\|_H + \|\delta w_n - z_n^h\|_H$$

we see that

$$\sum_{n=1}^{N} \int_{I_n} \|\dot{w}(t) - z_n^h\|_H \, dt \le \sum_{n=1}^{N} \int_{I_n} \|\delta w_n - \dot{w}(t)\|_H \, dt$$

$$+k \sum_{n=1}^{N} \|\delta w_n - z_n^h\|_H. \tag{4.17}$$

It remains to estimate the term

$$k \sum_{n=1}^{N} \|\delta w_n - z_n^h\|_H.$$

We have

$$\delta w_n - z_n^h = \frac{1}{k} \int_{I_n} \dot{w}(t) \, dt - z_n^h = \frac{1}{k} \int_{I_n} \left[\dot{w}(t) - \dot{\tilde{w}}(t)\right] \, dt + \delta \tilde{w}_n - z_n^h,$$

and so

$$\|\delta w_n - z_n^h\|_H \le \frac{1}{k} \int_{I_n} \|\dot{w}(t) - \dot{\tilde{w}}(t)\|_H \, dt + \|\delta \tilde{w}_n - z_n^h\|_H,$$

$$k \sum_{n=1}^{N} \|\delta w_n - z_n^h\|_H \le \|\dot{w} - \dot{\tilde{w}}\|_{L^1(0,T;H)}$$

$$+k \sum_{n=1}^{N} \|\delta \tilde{w}_n - z_n^h\|_H \le c \, \varepsilon + k \sum_{n=1}^{N} \|\delta \tilde{w}_n - z_n^h\|_H. \tag{4.18}$$

Using the bounds (3.20), (3.21), (3.24), (3.25), and (4.16)–(4.18) in the inequality (4.15) and observing the arbitrariness of $z_n^h \in K^h$, we get the estimate

$$\max_{1 \le n \le N} \|w_n^{hk} - w_n\|_H \le c \Big\{ \varepsilon + k \, \left(\|\dot{\tilde{w}}\|_{L^\infty(0,T;H)} + \|\ddot{\tilde{w}}\|_{L^1(0,T;H)} \right.$$

$$\left. + \|\dot{\tilde{\ell}}\|_{L^\infty(0,T;H^*)} + \|\ddot{\tilde{\ell}}\|_{L^1(0,T;H^*)} \right) + D_{hk}(\tilde{w}) \Big\} \tag{4.19}$$

$$+c \Big\{ \left(\varepsilon + k \, \|\ddot{\tilde{w}}\|_{L^1(0,T;H)} + D_{hk}(\tilde{w})\right) \left(\|\dot{w}\|_{L^\infty(0,T;H)}\right.$$

$$\left. + \|\dot{\ell}\|_{L^\infty(0,T;H^*)} + 1\right) \Big\}^{1/2}, \tag{4.20}$$

where

$$D_{hk}(\tilde{w}) = k \sum_{n=1}^{N} \inf_{z_n^h \in K^h} \|\delta \tilde{w}_n - z_n^h\|_H. \tag{4.21}$$

By Assumption ($\mathbf{H}_2$), we see that

$$\inf_{z_n^h \in K^h} \|\delta \tilde{w}_n - z_n^h\|_H \le c\, h^\alpha \|\delta \tilde{w}_n\|_{H_0} \le \frac{c\, h^\alpha}{k} \int_{I_n} \|\dot{\tilde{w}}(t)\|_{H_0} dt,$$

and thus

$$D_{hk}(\tilde{w}) = k \sum_{n=1}^{N} \inf_{z_n^h \in K^h} \|\delta \tilde{w}_n - z_n^h\|_H \le c\, h^\alpha \|\dot{\tilde{w}}\|_{L^1(0,T;H_0)}.$$

Using this inequality in the estimate (4.20), we obtain

$$\max_{1\le n\le N} \|w_n^{hk} - w_n\|_H \le c\Big\{\varepsilon + k\, \Big(\|\dot{\bar{w}}\|_{L^\infty(0,T;H)} + \|\ddot{\bar{w}}\|_{L^1(0,T;H)}$$

$$+\|\dot{\bar{\ell}}\|_{L^\infty(0,T;H^*)} + \|\ddot{\bar{\ell}}\|_{L^1(0,T;H^*)}\Big) + h^\alpha \|\dot{\tilde{w}}\|_{L^1(0,T;H_0)}\Big\} \tag{4.22}$$

$$+c\Big\{ \Big(\varepsilon + k\, \|\ddot{\bar{w}}\|_{L^1(0,T;H)} + h^\alpha \|\dot{\tilde{w}}\|_{L^1(0,T;H_0)}\Big) \Big(\|\dot{w}\|_{L^\infty(0,T;H)}$$

$$+\|\dot{\ell}\|_{L^\infty(0,T;H^*)} + 1\Big) \Big\}^{1/2}, \tag{4.23}$$

which in turn gives the following result.

Theorem 4.2. *Let $w \in H^1(0,T;H)$ be the solution to Problem* ABS, *and $w_n^{hk} \in H^h$ the solution to the fully discrete Problem* ABS^{hk}. *Then under the set of assumptions following* (3.1), *together with those on H^h, and Assumptions $\mathbf{H}_1$ and $\mathbf{H}_2$, w^{hk} converges to w in the sense that*

$$\max_{1\le n\le N} \|w_n^{hk} - w_n\|_H \to 0 \quad \text{as } h,\ k \to 0. \tag{4.24}$$

5 Convergence of approximations of the primal variational problem

5.1 Time-discrete approximations

We apply Theorem 3.4 to time-discrete solutions of Problem PRIM. First we need to verify the assumptions stated in the theorem. As noted in [3], in general there is no guarantee that the bilinear form (2.14) will be Z-elliptic, and thus the convergence result contained in Theorem 3.4 cannot be applied to the most general primal problem. On the other hand, all the assumptions of Theorem 3.4 are satisfied in the practically important special cases of Problems KIN-ISO and KIN, with the von Mises yield function.

Thus the convergence result is readily applicable to time-discrete solutions of these two problems, and without any assumptions on the regularity of the solution other than those implied by the existence result, we have, for the time-discrete solution $\{\boldsymbol{u}_n^k, \boldsymbol{p}_n^k, \gamma_n^k\}$ of Problem Kin-Iso,

$$\max_{1\le n\le N} \Big\{ \|\boldsymbol{u}_n^k - \boldsymbol{u}(t_n)\|_V + \|\boldsymbol{p}_n^k - \boldsymbol{p}(t_n)\|_Q \\ + \|\gamma_n^k - \gamma(t_n)\|_{L^2(\Omega)} \Big\} \to 0 \quad \text{as } k \to 0,$$

and for the time-discrete solution $\{\boldsymbol{u}_n^k, \boldsymbol{p}_n^k\}$ of Problem Kin,

$$\max_{1\le n\le N} \Big\{ \|\boldsymbol{u}_n^k - \boldsymbol{u}(t_n)\|_V + \|\boldsymbol{p}_n^k - \boldsymbol{p}(t_n)\|_Q \Big\} \to 0 \quad \text{as } k \to 0.$$

5.2 Spatially and fully discrete approximations

Here we want to apply Theorems 4.1 and 4.2 to Problems KIN-ISO and KIN. The main issue here is verification of the hypotheses **(H**$_1$**)** and **(H**$_2$**)** made in Sect. 4.It is easier to verify these hypotheses for Problem KIN than for KIN-ISO, so for brevity we will verify the hypotheses only for the latter problem.

In the context of Problem KIN-ISO,

$$H = Z = (H_0^1(\Omega))^d \times Q_0 \times L^2(\Omega),$$

$$K = Z_p = \{\boldsymbol{z} = (\boldsymbol{v}, \boldsymbol{q}, \mu) \in Z : |\boldsymbol{q}| \le \mu \text{ a.e. in } \Omega\}.$$

We will show that we can take

$$H_0 = (H_0^1(\Omega) \cap C^\infty(\overline{\Omega}))^d \times (Q_0 \cap C^\infty(\overline{\Omega})) \\ \times (L^2(\Omega) \cap C^\infty(\overline{\Omega})) \tag{5.1}$$

in **(H**$_1$**)** and **(H**$_2$**)**. For this purpose, we need to make some preparations.

The following result is found in [15] (Proposition 23.2).

Proposition 5.1. *Assume that X is a Banach space, $1 \le q < \infty$. Then the space $C([0,T];X)$ is dense in $L^q(0,T;X)$.*

Using this proposition, we can prove the next result.

Proposition 5.2. *Assume that X is a Banach space, $1 \le q < \infty$, and l a non-negative integer. Then the space $C^l([0,T];X)$ is dense in $W^{l,q}(0,T;X)$.*

Proof. We prove the result for $l = 1$. A similar argument applies for other values of l.

Let $u \in W^{1,q}(0,T;X)$. Then $u' \in L^q(0,T;X)$. By Proposition 5.1, we can find a sequence $\{v_n\} \subset C([0,T];X)$ such that

$$v_n \to u' \quad \text{in } L^q(0,T;X).$$

Define u_n by

$$u_n(t) = u(0) + \int_0^t v_n(t)\,dt.$$

Then $\{u_n\} \subset C^1([0,T];X)$ and converges to u in $W^{1,q}(0,T;X)$. $\square$

Define

$$P(0,T;X) = \{p : p(t) = \sum_{i=0}^{m} a_i t^i,\ a_i \in X,\ 0 \le i \le m,\ m = 0,1,\cdots\}.$$

Obviously, $P(0,T;X) \subset C^\infty([0,T];X)$. The following result is found in [15] (page 442).

Proposition 5.3. *Assume that X is a Banach space, $X_0 \subset X$ is dense in X, $1 \le q < \infty$, and l a non-negative integer. Then $P(0,T;X_0)$ is dense in $C^l([0,T];X)$.*

Combining Proposition 5.2 and Proposition 5.3, we have the following.

Proposition 5.4. *Assume that X is a Banach space, $X_0 \subset X$ is dense in X, $1 \le q < \infty$, and l is a non-negative integer. Then $P(0,T;X_0)$ is dense in $W^{l,q}(0,T;X)$.*

Now we recall the following two smooth density results. Let $k \ge 0$, $1 \le p < \infty$; then

(5.2) $$C_0^\infty(\Omega) \text{ is dense in } W_0^{k,p}(\Omega).$$

If the boundary $\partial\Omega$ is Lipschitz continuous, then

(5.3) $$C^\infty(\overline{\Omega}) \text{ is dense in } W^{k,p}(\Omega).$$

From Proposition 5.4, (5.2) and (5.3), we see that $H^1(0,T;C_0^\infty(\Omega))$ is dense in $H^1(0,T;H_0^1(\Omega))$, and $H^1(0,T;C^\infty(\overline{\Omega}))$ is dense in $H^1(0,T;L^2(\Omega))$. Thus given $\boldsymbol{w} = (\boldsymbol{u},\boldsymbol{p},\gamma) \in H^1(0,T;K)$, we can find a sequence $\boldsymbol{w}_n = (\boldsymbol{u}_n,\boldsymbol{p}_n,\gamma_n) \in H^1(0,T;(C_0^\infty(\Omega))^d \times (C^\infty(\overline{\Omega}))^{d\times d} \times C^\infty(\overline{\Omega}))$ converging to $\boldsymbol{w}$ in $H^1(0,T;Z)$. In order for the space H_0 defined in (5.1) to have the property

(5.4) $$H^1(0,T;H_0 \cap K) \text{ is dense in } H^1(0,T;K),$$

we require that

(5.5) $$\boldsymbol{p}_n \in Q_0, \quad |\boldsymbol{p}_n| \le \gamma_n \text{ in } \Omega.$$

To verify (5.5), let us briefly review a typical proof of the density result (5.3) (cf. [2]).

We first introduce some notation. For $\boldsymbol{x}_0 \in \mathbb{R}^d$ and $r > 0$, we denote by

$$B(\boldsymbol{x}_0,r) = \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}-\boldsymbol{x}_0\| < r\}$$

and

$$\overline{B}(\boldsymbol{x}_0,r) = \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}-\boldsymbol{x}_0\| \le r\}$$

the open and closed balls centered at $\boldsymbol{x}_0$ with radius r. Here and below, the vector norm in $I\!R^d$ is the Euclidean norm. Define

$$J(\boldsymbol{x}) = \begin{cases} c_0 e^{1/(\|\boldsymbol{x}\|^2-1)}, & \|\boldsymbol{x}\| < 1, \\ 0, & \|\boldsymbol{x}\| \geq 1, \end{cases}$$

where $c_0 > 0$ is chosen so that

$$\int_{I\!R^d} J(\boldsymbol{x})\, dx = 1.$$

The function $J(\cdot)$ is infinitely smooth. Then we define the standard mollifier

$$J_\epsilon(\boldsymbol{x}) = \frac{1}{\epsilon^d}\, J(\boldsymbol{x}/\epsilon)$$

which has the properties that

$$J_\epsilon \in C^\infty(I\!R^d), \quad \int_{I\!R^d} J_\epsilon(\boldsymbol{x})\, dx = 1, \quad J_\epsilon(\boldsymbol{x}) = 0 \text{ for } \|\boldsymbol{x}\| \geq \epsilon.$$

The main steps of the proof of (5.3) are as follows.

Step 1. Localization. Since $\partial\Omega$ is compact and is Lipschitz continuous, there exists a finite number of points $\boldsymbol{x}_m \in \partial\Omega$, positive numbers $r_m > 0$, and Lipschitz continuous functions $\gamma_m : I\!R^{d-1} \to I\!R$, $1 \leq m \leq M$, such that

$$\partial\Omega \subset \cup_{m=1}^M B(\boldsymbol{x}_m, r_m/2)$$

and for each m, after relabelling the coordinate axes if necessary,

$$\Omega \cap \overline{B}(\boldsymbol{x}_m, r_m) = \{\boldsymbol{x} \in \overline{B}(\boldsymbol{x}_m, r_m) : x_d > \gamma_m(x_1, \cdots, x_{d-1})\}.$$

Define

$$\Omega_m = \Omega \cap \overline{B}(\boldsymbol{x}_m, r_m/2), \quad 1 \leq m \leq M$$

and choose an open set $\Omega_0 \subset\subset \Omega$ with the property

$$\Omega \subset \cup_{m=0}^M \Omega_m.$$

Let $\{\zeta_m\}_{m=0}^M$ be a smooth partition of unity subordinate to $\{\Omega_m\}_{m=0}^M$; i.e., for each m, $\zeta_m \in C^\infty$, $\operatorname{supp}(\zeta_m) \subset \Omega_m$, and

$$\sum_{m=0}^M \zeta_m(\boldsymbol{x}) \equiv 1, \quad \boldsymbol{x} \in \Omega.$$

Then we have

$$u = u \sum_{m=0}^M \zeta_m = \sum_{m=0}^M u_m,$$

where

$$u_m = u\,\zeta_m.$$

Step 2. Smooth approximation of u_0. Take $\epsilon_0 \in (0, \mathrm{dist}\,(\partial\Omega_0, \partial\Omega))$, and consider only those ϵ with $\epsilon \le \epsilon_0/2$. Define $\Omega_0' = \{\boldsymbol{x} \in \Omega : \mathrm{dist}\,(\boldsymbol{x}, \partial\Omega) > \epsilon_0/2\}$, and define the mollification of u_0 with respect to the spatial variable by

$$u_0^\epsilon(\boldsymbol{x}) = (J_\epsilon * u_0)(\boldsymbol{x}) = \int_{B(\mathbf{0},\epsilon)} J_\epsilon(\boldsymbol{y})\, u_0(\boldsymbol{x}-\boldsymbol{y})\, dy.$$

We then have, for ϵ sufficiently small,

$$u_0^\epsilon \in C_0^\infty(\overline{\Omega_0'})$$

and

$$\lim_{\epsilon\to 0} \|u_0^\epsilon - u_0\|_{W^{k,p}(\Omega)} = 0.$$

Step 3. Smooth approximation of u_m, $1 \le m \le M$. Fix $m = 1, \cdots, M$ and consider the smooth approximation of u_m. Recall that there exist $r_m > 0$ and a Lipschitz continuous function γ_m such that, upon relabelling the coordinate axes if necessary,

$$\Omega \cap \overline{B}(\boldsymbol{x}_m, r_m) = \{\boldsymbol{x} \in \overline{B}(\boldsymbol{x}_m, r_m) : x_d > \gamma_m(x_1, \cdots, x_{d-1})\}.$$

Let $\Omega_m = \Omega \cap B(\boldsymbol{x}_m, r_m/2)$. For any $\boldsymbol{x} \in \Omega_m$, we define $\boldsymbol{x}^\epsilon = \boldsymbol{x} + \alpha\,\epsilon\,\boldsymbol{e}_n$, where we choose $\alpha = \sqrt{2}\,\max\{\mathrm{Lip}\,(\gamma_m), 1\}$. It can be verified that if ϵ is small enough, then

$$\overline{B}(\boldsymbol{x}^\epsilon, \epsilon) \subset \Omega \cap \overline{B}(\boldsymbol{x}, r_m).$$

Now we let

$$v_m^\epsilon(\boldsymbol{x}) = u_m(\boldsymbol{x}^\epsilon), \quad \boldsymbol{x} \in \Omega_m$$

and define

$$u_m^\epsilon(\boldsymbol{x}) = (J_\epsilon * v_m^\epsilon)(\boldsymbol{x}) = \int_{B(\mathbf{0},\epsilon)} J_\epsilon(\boldsymbol{y})\, v_m^\epsilon(\boldsymbol{x}-\boldsymbol{y})\, dy.$$

We have

$$u_m^\epsilon \in C^\infty(\overline{\Omega}_m)$$

and

$$\lim_{\epsilon\to 0} \|u_m^\epsilon - u_m\|_{W^{k,p}(\Omega)} = 0.$$

Step 4. Global smooth approximation with respect to $\boldsymbol{x}$. Define

$$u^\epsilon = \sum_{m=0}^{M} u_m^\epsilon.$$

Then

$$u^\epsilon \in C^\infty(\overline{\Omega})$$

and

$$u^\epsilon \to u \quad \text{in } W^{k,p}(\Omega) \text{ as } \epsilon \to 0.$$

It is evident from the proof that

$$\boldsymbol{p} \in Q_0 \Longrightarrow \boldsymbol{p}^\epsilon \in Q_0$$

and

$$|\boldsymbol{p}| \leq \gamma \text{ a.e. in } \Omega \Longrightarrow |\boldsymbol{p}^\epsilon| \leq \gamma^\epsilon \text{ in } \Omega.$$

Thus (5.4), and therefore **(H$_1$)**, is satisfied.

As for **(H$_2$)**, we assume that the finite element space V^h approximating $V = (H_0^1(\Omega))^d$ contains piecewise linear functions, and the finite element spaces $Q_0^h \subset Q_0$ and $M^h \subset M$ contain piecewise constants. Then for any $\boldsymbol{z} = (\boldsymbol{v}, \boldsymbol{q}, \mu) \in H_0 \cap K$, we define $\boldsymbol{z} = (\Pi^h \boldsymbol{v}, \Pi^h \boldsymbol{q}, \Pi^h \mu)$, where $\Pi^h \boldsymbol{v}$ is the piecewise linear interpolant of $\boldsymbol{v}$, $\Pi^h \boldsymbol{q}$ and $\Pi^h \mu$ are element-wise averages of $\boldsymbol{q}$ and μ. It is not difficult to see that $\boldsymbol{z}^h \in K^h$. By the standard theory of finite element interpolation (see, for example, [1]),

$$\|\boldsymbol{z} - \boldsymbol{z}^h\|_Z \leq c\, h\, (\|\boldsymbol{v}\|_{H^2(\Omega)} + \|\boldsymbol{q}\|_{H^1(\Omega)} + \|\mu\|_{H^1(\Omega)}).$$

Hence **(H$_2$)** is satisfied.

We now have the following theorem on convergence.

Theorem 5.5. *For the problem* Kin-Iso, *assume that the finite element space V^h contains piecewise linear functions, and that Q_0^h and M^h contain piecewise constants. Then for the spatially discrete solution $\boldsymbol{w}^h = (\boldsymbol{u}^h, \boldsymbol{p}^h, \gamma^h)$,*

$$\begin{aligned} \operatorname{ess\,sup}_{0\leq t\leq T} \Big\{ &\|\boldsymbol{u}(t) - \boldsymbol{u}^h(t)\|_V + \|\boldsymbol{p}(t) - \boldsymbol{p}^h(t)\|_Q \\ &+ \|\gamma(t) - \gamma^h(t)\|_M \Big\} \to 0 \quad \text{as } h \to 0, \end{aligned}$$

and for the fully discrete solution $\boldsymbol{w}^{hk} = (\boldsymbol{u}^{hk}, \boldsymbol{p}^{hk}, \gamma^{hk})$,

$$\begin{aligned} \max_{1\leq n\leq N} \Big\{ &\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V + \|\boldsymbol{p}_n - \boldsymbol{p}_n^{hk}\|_Q \\ &+ \|\gamma_n - \gamma_n^h\|_M \Big\} \to 0 \quad \text{as } h, k \to 0. \end{aligned}$$

For Problem Kin, *assume that the finite element space V^h contains piecewise linears, and that Q_0^h contains piecewise constants. Then for the spatially discrete solution $\boldsymbol{w}^h = (\boldsymbol{u}^h, \boldsymbol{p}^h)$,*

$$\operatorname{ess\,sup}_{0\leq t\leq T} \Big\{ \|\boldsymbol{u}(t) - \boldsymbol{u}^h(t)\|_V + \|\boldsymbol{p}(t) - \boldsymbol{p}^h(t)\|_Q \Big\} \to 0 \quad \text{as } h \to 0,$$

and for the fully discrete solution $\boldsymbol{w}^{hk} = (\boldsymbol{u}^{hk}, \boldsymbol{p}^{hk})$,

$$\max_{1\le n\le N} \left\{ \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V + \|\boldsymbol{p}_n - \boldsymbol{p}_n^{hk}\|_Q \right\} \to 0 \quad \text{as } h, k \to 0.$$

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