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QUALITATIVE AND NUMERICAL ANALYSIS OF QUASI-STATIC PROBLEMS IN ELASTOPLASTICITY*

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Dedicated to Professor Ivo Babuška on the occasion of his 70th birthday.

Abstract. The quasi-static problem of elastoplasticity with combined kinematic-isotropic hardening is formulated as a time-dependent variational inequality (VI) of the mixed kind; that is, it is an inequality involving a nondifferentiable functional and is imposed on a subset of a space. This VI differs from the standard parabolic VI in that time derivatives of the unknown variable occur in all of its terms. The problem is shown to possess a unique solution.

We consider two types of approximations to the VI corresponding to the quasi-static problem of elastoplasticity: semidiscrete approximations, in which only the spatial domain is discretized, by finite elements; and fully discrete approximations, in which the spatial domain is again discretized by finite elements, and the temporal domain is discretized and the time-derivative appearing in the VI is approximated in an appropriate way.

Estimates of the errors inherent in the above two types of approximations, in suitable Sobolev norms, are obtained for the quasi-static problem of elastoplasticity; in particular, these estimates express rates of convergence of successive finite element approximations to the solution of the variational inequality in terms of element size h and, where appropriate, of the time step size k.

A major difficulty in solving the problems is caused by the presence of the nondifferentiable terms. We consider some regularization techniques for overcoming the difficulty. Besides the usual convergence estimates, we also provide a posteriori error estimates which enable us to estimate the error by using only the solution of a regularized problem.

Key words. elastoplastic problems with kinematic and/or isotropic hardening, variational inequality of mixed kind, semidiscrete approximations, fully discrete approximations, finite element method, backward Euler scheme, Crank–Nicolson scheme, convergence, error estimates, regularization method, a posteriori error estimates

AMS subject classifications. 65N30, 65M06, 65N15, 65M15, 73V05, 73E99

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1. Introduction.

The aim of this work is to provide a qualitative and numerical analysis of a problem arising in the description of quasi-static behavior of elastoplastic bodies. The quasi-static (as opposed to simply static or steady) nature of the problem is due to the fact that plastic behavior can only be correctly described in terms of rates of change of certain variables (such as plastic strain); thus these contribute to the presence of rate quantities, and the problem is not therefore merely a boundary-value problem. On the other hand, processes are assumed to occur sufficiently slowly so that inertial effects may be ignored. Thus acceleration does not appear in the problem. The quasi-static problem, while an approximation, is an important special case both mathematically and from a practical point of view, as is confirmed by the large number of papers on both of these aspects.

In abstract form, the problem is formulated as a time-dependent variational inequality (VI) of the mixed kind (see section 3). It is nonstandard, and differs from

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the standard parabolic VI (see (1.1) below) in that rate quantities occur in all of its terms. A similar VI arises in the study of elastic bodies which are subject to frictional contact (see [1, 6, 24, 25], for example).

The literature on VIs and their numerical approximation includes investigations of VIs arising in plasticity (see, for example, the works by Glowinski, Lions, and Trémolières [9] and by Hlaváček et al. [14]).The first systematic mathematical study of the elastoplastic problem is due to Duvaut and Lions [6], who considered the problem for a perfectly plastic material. Johnson [17] subsequently extended the analysis in [6] by approaching the problem in two stages; in the first stage the velocity is eliminated and the problem becomes a VI posed on a time-dependent convex set. The second stage involves the solution for the velocity. The contributions of Duvaut–Lions and Johnson also predated, and were therefore not in a position to draw on, the important works of Matthies [26], Matthies, Strang, and Christiansen [27], and Temam [33] on existence for the displacement problem in perfect plasticity. This work gave rise to the definition and study of the space $BD(\Omega)$ of functions of bounded deformation, which are central to a proper study of the existence problem for perfectly plastic materials.

Analyses of finite element approximations of the elastoplastic problem have enjoyed limited but steady attention, in contrast to the voluminous literature devoted to computational and algorithmic aspects of this problem. Havner and Patel [12] and Jiang [16] analyzed approximations of the so-called rate problem; this is an elliptic VI in which the primary unknowns are the velocity, rather than the displacement, and the plastic multiplier. Johnson [18] has considered a formulation of the elastoplasticity problem in which stress is the primary variable and has derived error estimates for the fully discrete (that is, discrete in both time and space) problem (see also related work by Hlaváček [13] and a summary account in [14]). In a later work, Johnson [19] considered fully discrete finite element approximations in the context of plasticity, while Brezzi, Johnson, and Mercier [3] have treated finite element approximations of the time-independent Hencky problem for elastoplastic plates.

All of the above studies differ from that undertaken here in that, first, the model problem investigated here is a VI *of the mixed kind*; it is an inequality both because of the presence of a nondifferentiable functional *and* because the problem is posed on a closed convex cone in a Hilbert space. Secondly, unlike the standard parabolic VI which is of the form: find $u : [0,T] \to V$ such that

$$(1.1) \qquad (\dot{u}, v-u) + a(u, v-u) + j(v) - j(u) \geq l(v-u) \quad \text{for all } v \in V,$$

rate quantities occur in *all of the terms* of the VI (see (4.1)).

We make use of a formulation which has been extensively treated both theoretically and computationally over the last decade by Reddy [29] and Reddy and Martin [31, 32]. The chief characteristic of this formulation is that, unlike conventional formulations in elastoplasticity (such as that presented, for example, in [6]), it is a logical extension of the standard displacement problem of linear elasticity in the sense that it reduces to this problem in the event that the body behaves elastically. We confine this study to one involving materials which undergo hardening; thus, solutions are sought in Sobolev spaces. The existence theory for this problem has been treated, for the case of kinematic hardening only, in [29]; here we extend this theory to accommodate the case of isotropic hardening.

The outline of the remainder of this work is as follows. In section 2 the model quasi-static problem is described, while the corresponding VI is formulated in section 3. This VI is considered in an abstract context in section 4, where conditions for its wellposedness are established. Section 5 is concerned with semidiscrete finite element approximations of the abstract VI, while section 6 is devoted to fully discrete approximations. In section 7 we apply the results of the previous sections to the quasi-static problem of elastoplasticity with combined kinematic-isotropic hardening as well as the special case of kinematic hardening only. In the latter case the VI reduces to one of the second kind, in which the nondifferentiable functional is present but the problem is posed on the entire space. We establish error estimates for the semidiscrete and fully discrete approximations. In the last section, we consider regularization techniques for handling the nondifferentiable terms and derive a posteriori error estimates.

2. Formulation of the problem. We consider the initial-boundary value problem for quasi-static behavior of an elastoplastic body which occupies a bounded domain Ω with Lipschitz boundary Γ. The plastic behavior of the material is assumed to be describable within the classical framework of a convex yield surface coupled with the normality law.

The material is assumed to undergo linear kinematic and isotropic hardening. The assumption of a hardening material, apart from the fact that it represents realistic material behavior, serves also to allow for a complete analysis within a Sobolev space framework, the case of perfect plasticity requiring special treatment (see, for example, [23]). The model incorporates also the classical assumption of no volume change accompanying plastic deformation.

Suppose that the system is initially at rest and that it is initially undeformed and unstressed. A time-dependent field of body force $\boldsymbol{f}(x,t)$ is given, with $\boldsymbol{f}(x,0)=\boldsymbol{0}$. We are required to find the displacement field $\boldsymbol{u}(x,t)$ and plastic strain field $\boldsymbol{p}(x,t)$ which satisfy for $0 \le t \le T$ the equilibrium equation

$$\operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u},\boldsymbol{p}) + \boldsymbol{f} = \boldsymbol{0}\ , \tag{2.1}$$

the elastic constitutive equation

$$\boldsymbol{\sigma}(\boldsymbol{u},\boldsymbol{p}) = \boldsymbol{C}(\boldsymbol{\epsilon}(\boldsymbol{u}) - \boldsymbol{p})\ , \tag{2.2}$$

the strain-displacement relation

$$\boldsymbol{\epsilon}(\boldsymbol{u}) = \tfrac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T), \tag{2.3}$$

and the condition of plastic incompressibility

$$\mathrm{tr}\boldsymbol{p} := \boldsymbol{I}\cdot\boldsymbol{p} = 0 \tag{2.4}$$

or $p_{kk}=0$. Here and henceforth summation is implied on repeated indices unless otherwise stated.

Equations (2.1)–(2.4) are required to hold in Ω; here $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{\epsilon}$ is the strain tensor, $\boldsymbol{u}$ is the displacement vector, and $\boldsymbol{p}$ is the plastic strain tensor. The quantity $\boldsymbol{C}$ is a fourth-order tensor of elastic coefficients.

In addition we have to specify the plastic flow law. For this purpose let γ represent the internal variable associated with isotropic hardening, and define the (thermodynamic) conjugate forces $\boldsymbol{\chi}$ and g by [32]

$$\boldsymbol{\chi} = \boldsymbol{\sigma} - k_1\boldsymbol{p}, \quad g = -k_2\gamma; \tag{2.5}$$

where k_1 and k_2 are nonnegative scalars. The region of admissible conjugate forces is then defined to be the set

$$\mathcal{K} = \{(\boldsymbol{\chi}, g):\ F(\boldsymbol{\chi}) + g \le c_0\}; \tag{2.6}$$

where c_0 is a positive constant and F is a convex function known as the yield function. The boundary of $\mathcal{K}$ is known as the yield surface, while its interior is known as the elastic region. The term $k_1\boldsymbol{p}$ in (2.5) is the back stress and defines the amount by which the yield surface is translated as a result of previous plastic behavior. The term g in the definition of $\mathcal{K}$ governs isotropic hardening, that is, the amount by which the admissible region expands as a result of previous plastic behavior.

For a quantity z, we will use $\dot{z}$ to denote the derivative of z with respect to time t. In its classical form the flow law is

$$(\dot{\boldsymbol{p}}, \dot{\gamma}) \in N_{\mathcal{K}}(\boldsymbol{\chi}, g), \tag{2.7}$$

where $N_{\mathcal{K}}(\boldsymbol{\chi}, g)$ denotes the normal cone to $\mathcal{K}$ at $(\boldsymbol{\chi}, g)$. If the yield function is smooth this may be rewritten in the form

$$\begin{pmatrix} \dot{\boldsymbol{p}} \\ \dot{\gamma} \end{pmatrix} = \lambda \begin{pmatrix} \nabla F(\boldsymbol{\chi}) \\ 1 \end{pmatrix}, \tag{2.8}$$

where λ is a nonnegative scalar. Thus λ may be identified with $\dot{\gamma}$.

We will find it advantageous to consider the flow law not in this form but in a form which is dual to the relation (2.7) in the sense of convex analysis. We introduce the support function D of $\mathcal{K}$, defined by

$$D : M^3 \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}, \qquad D(\boldsymbol{q}, \mu) = \sup\{\boldsymbol{\chi} \cdot \boldsymbol{q} + g\mu : (\boldsymbol{\chi}, g) \in \mathcal{K}\}. \tag{2.9}$$

Here M^3 is the set of all symmetric 3×3 matrices. The support function is denoted here by the symbol D since in plasticity it has the interpretation of the dissipation function. We will henceforth use this latter term to describe this function.

The dissipation function is a gauge; that is, it is proper, nonnegative, convex, positively homogeneous, and lower semicontinuous (l.s.c.):

$$D \not\equiv +\infty, \tag{2.10}$$

$$D(\boldsymbol{q}, \mu) \geq 0, \quad D(\boldsymbol{0}, 0) = 0, \tag{2.11}$$

$$\begin{aligned} &D(\theta\boldsymbol{p} + (1-\theta)\boldsymbol{q}, \theta\gamma + (1-\theta)\mu) \leq \theta D(\boldsymbol{p}, \gamma) + (1-\theta) D(\boldsymbol{q}, \mu), \\ &\qquad \forall\, \theta \in (0,1),\ \forall\, \boldsymbol{p}, \boldsymbol{q} \in M^3,\ \forall\, \gamma, \mu \in \mathbb{R} \end{aligned} \tag{2.12}$$

$$D(\alpha\boldsymbol{p}, \alpha\gamma) = \alpha D(\boldsymbol{p}, \gamma), \quad 0 < \alpha \in \mathbb{R} \tag{2.13}$$

$$\lim_{n\to\infty} D(\boldsymbol{q}_n, \mu_n) \geq D(\boldsymbol{q}, \gamma),\ \forall\, \{(\boldsymbol{q}_n, \mu_n)\},\ (\boldsymbol{q}_n, \mu_n) \to (\boldsymbol{q}, \gamma). \tag{2.14}$$

In analyzing the variational inequality for the elastoplasticity problem, it will be convenient to consider the dissipation function D on its effective domain $\operatorname{dom} D = \{(\boldsymbol{q}, \mu) \in M^3 \times \mathbb{R} :\ D(\boldsymbol{q}, \mu) < \infty\}$. It is easy to verify that $\operatorname{dom} D$ is a nonempty, convex, closed cone in $M^3 \times \mathbb{R}$. From Corollary 2.4 in [7], if D is bounded above over a nonempty open set, then D is locally Lipschitz continuous on $\operatorname{dom} D$. We will assume that

$$D \text{ is Lipschitz continuous on } \operatorname{dom} D. \tag{2.15}$$

As an example, we consider the popular von Mises yield condition. For this case we have

$$F(\boldsymbol{\chi}) = |\boldsymbol{\chi}^D| \equiv \sqrt{\chi_{ij}^D \chi_{ij}^D},$$

where $\boldsymbol{\chi}^D = \boldsymbol{\chi} - \frac{1}{3}(\mathrm{tr}\boldsymbol{\chi})\boldsymbol{I}$ is the deviatory part of $\boldsymbol{\chi}$. One can show that (see [32])

$$D(\dot{\boldsymbol{p}}, \dot{\gamma}) = \begin{cases} c_0|\dot{\boldsymbol{p}}| & \text{if} \quad |\dot{\boldsymbol{p}}| \le \dot{\gamma}, \\ +\infty & \text{if} \quad |\dot{\boldsymbol{p}}| > \dot{\gamma}. \end{cases} \tag{2.16}$$

Obviously, on $\mathrm{dom}\, D = \{(\boldsymbol{q}, \mu) : |\boldsymbol{q}| \le \mu \text{ a.e. in } \Omega\}$, $D(\boldsymbol{q}, \mu) = c_0|\boldsymbol{q}|$ is Lipschitz continuous.

By exploiting the fact that the dissipation function is the Legendre–Fenchel conjugate of the indicator function of $\mathcal{K}$ we can express the flow law (2.7) or (2.8) in the form

$$(\boldsymbol{\chi}, g) \in \partial D(\dot{\boldsymbol{p}}, \dot{\gamma}); \tag{2.17}$$

that is,

$$D(\boldsymbol{q}, \mu) \ge D(\dot{\boldsymbol{p}}, \dot{\gamma}) + \boldsymbol{\chi} \cdot (\boldsymbol{q} - \dot{\boldsymbol{p}}) + g\,(\mu - \dot{\gamma}) \quad \forall\, \boldsymbol{q} \in M^3,\ \mu \in \mathbb{R}, \tag{2.18}$$

where the inner product in M^3 is defined by $\boldsymbol{p} \cdot \boldsymbol{q} = p_{ij}q_{ij}$. The relation (2.18) is equivalent to $(\dot{\boldsymbol{p}}, \dot{\gamma}) \in \mathrm{dom} D$ and

$$D(\boldsymbol{q}, \mu) \ge D(\dot{\boldsymbol{p}}, \dot{\gamma}) + \boldsymbol{\chi} \cdot (\boldsymbol{q} - \dot{\boldsymbol{p}}) + g\,(\mu - \dot{\gamma}) \quad \forall\, (\boldsymbol{q}, \mu) \in \mathrm{dom}\, D. \tag{2.19}$$

We assume for the coefficient functions in (2.5) that k_1, $k_2 \in L^\infty(\Omega)$ and that there are constants $\bar{k}_i$ $(i = 1, 2)$ such that

$$k_i(x) \ge \bar{k}_i > 0 \quad \text{a.e. in } \Omega. \tag{2.20}$$

The elasticity tensor C has the symmetry properties

$$C_{ijkl} = C_{jikl} = C_{klij}, \tag{2.21}$$

and we assume that

$$C_{ijkl} \;\in\; L^\infty(\Omega) \tag{2.22}$$

and that C is pointwise stable: there exists a constant $c_0 > 0$ such that

$$C_{ijkl}(x)\zeta_{ij}\zeta_{kl} \ge c_0|\boldsymbol{\zeta}|^2 \quad \forall\, \boldsymbol{\zeta} \in M^3 \quad \text{a.e. in } \Omega. \tag{2.23}$$

Finally, we take the boundary condition to be

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma\ , \tag{2.24}$$

while the initial conditions are assumed to be

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{p}(\boldsymbol{x}, 0) = \boldsymbol{0}. \tag{2.25}$$

3. The variational problem. Function spaces. Before addressing the question of the variational formulation of the problem posed in section 2, we introduce the function spaces which will be required.

We use multi-index notation for derivatives of functions. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be an n-tuple of nonnegative integers and set $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Then $D^\alpha u$ denotes the αth derivative of a function u defined by

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

For integers $m \geq 1$, we denote by $H^m(\Omega)$ the space of (equivalence classes of) functions in $L^2(\Omega)$ whose distributional partial derivatives of order $|\alpha|$, $|\alpha| \leq m$, are in $L^2(\Omega)$. The space $H^m(\Omega)$ is a Hilbert space equipped with the inner product

$$(u, v)_{m,\Omega} = \int_\Omega \sum_{|\alpha|\leq m} D^\alpha u(\mathbf{x}) D^\alpha v(\mathbf{x})\, dx$$

so that

$$\|u\|_{m,\Omega} = \|u\|_{m,2,\Omega} = \left[(u,u)_{m,\Omega}\right]^{1/2}.$$

When the domain of a function is contextually apparent, we use the simpler notation

$$\|\cdot\|_{m,p} = \|\cdot\|_{m,p,\Omega}, \qquad \|\cdot\|_m = \|\cdot\|_{m,\Omega}.$$

We will also use the following seminorm defined on $H^m(\Omega)$:

$$|\, u\,|_{m,\Omega} = \left(\int_\Omega \sum_{|\alpha|=m} |\, D^\alpha u(\mathbf{x})\,|^2 \;\, dx \right)^{1/2}. \tag{3.1}$$

We denote by $H_0^1(\Omega)$ the subspace of $H^1(\Omega)$ comprising functions whose values vanish on the boundary Γ, in the sense of traces. The seminorm (3.1) with $m = 1$ is a norm on $H_0^1(\Omega)$, equivalent to the standard H^1-norm.

The spaces of displacements and plastic strains are defined, respectively, by

$$V = [H_0^1(\Omega)]^3, \quad Q = \{\boldsymbol{q} = (q_{ij}) : \; q_{ji} = q_{ij}, \; q_{ij} \in L^2(\Omega)\}.$$

We will also need the space Q_0 of traceless functions defined by $Q_0 = \{\boldsymbol{q} \in Q : \; \mathrm{tr}\, \boldsymbol{q} = 0$ a.e. in $\Omega\}$. Both V and Q are Hilbert spaces with inner products

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}\, dx \quad \text{and} \quad (\boldsymbol{p}, \boldsymbol{q})_Q = \int_\Omega \boldsymbol{p} \cdot \boldsymbol{q}\; dx = \int_\Omega p_{ij} q_{ij} dx$$

and norms $||\boldsymbol{v}||_V = (\boldsymbol{v}, \boldsymbol{v})_V^{1/2}$, $||\boldsymbol{q}||_Q = (\boldsymbol{q}, \boldsymbol{q})_Q^{1/2}$. Furthermore, Q_0 is a closed subspace of Q.

The space M of isotropic hardening variables is defined by $M = L^2(\Omega)$. We define the product space $Z = V \times Q_0 \times M$ which is a Hilbert space with the inner product

$$(\boldsymbol{w}, \boldsymbol{z})_Z \;:\; = (\boldsymbol{u}, \boldsymbol{v})_V + (\boldsymbol{p}, \boldsymbol{q})_Q + (\gamma, \mu)_M$$

and norm $||\boldsymbol{z}||_Z = (\boldsymbol{z}, \boldsymbol{z})_Z^{1/2}$, where $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{p}, \gamma)$ and $\boldsymbol{z} = (\boldsymbol{v}, \boldsymbol{q}, \mu)$. We also need a subset of Z, defined by

$$K = \{\boldsymbol{z} = (\boldsymbol{v}, \boldsymbol{q}, \mu) \in Z : |\boldsymbol{q}| \leq \mu \text{ a.e. in } \Omega\}.$$

Obviously, K is a nonempty, closed, convex cone in Z.

For any Banach space X, we denote by $C^m([0,T]; X)$ the space of continuous functions $u : [0,T] \to X$ that have continuous derivatives up to and including those of order m on $[0,T]$ with the norm

$$||u||_{C^m([0,T];X)} = \sum_{i=0}^m \max_{0\leq t\leq T} ||u^{(i)}(t)||_X \tag{3.2}$$

and by $L^p(0,T;X)$ for $1 \leq p < \infty$ the space of all measurable functions $u : (0,T) \to X$ for which

$$||u||_{L^p(0,T;X)} = \left(\int_0^T ||u(t)||_X^p \; dt \right)^{1/p} < \infty. \tag{3.3}$$

The space of measurable functions $u : (0,T) \to X$ which are essentially bounded is denoted by $L^\infty(0,T;X)$, and this space is endowed with the norm

$$||u||_{L^\infty(0,T;X)} = \text{ess sup}_{0 \leq t \leq T} ||u(t)||_X.$$

Some properties of these spaces are listed in the following theorem. For a proof, see Zeidler [35].

THEOREM 3.1. *Let m be a nonnegative integer and $1 \leq p \leq \infty$. Let X and Y be real Banach spaces. Then*

1. *$C^m([0,T];X)$ with the norm (3.2) is a Banach space;*
2. *$L^p(0,T;X)$ is a Banach space if we identify functions that are equal almost everywhere on $(0,T)$;*
3. *If X is a Hilbert space with inner product $(\cdot,\cdot)_X$, then $L^2(0,T;X)$ is also a Hilbert space with the inner product*

$$(u,v)_{L^2(0,T;X)} = \int_0^T (u(t),v(t))_X \; dt.$$

The topological dual of a Banach space X is denoted by X^*, and the operation of an element $u^* \in X^*$ on an element $u \in X$ is indicated by $\langle u^*, u \rangle$. If X is separable, then the space $L^1(0,T;X)$ is separable and

$$L^1(0,T;X)^* = L^\infty(0,T;X^*).$$

We define by $W^{1,2}(0,T;X)$ the space of functions $f \in L^2(0,T;X)$ such that $\dot{f} \in L^2(0,T;X)$, equipped with the norm

$$\|f\|^2_{W^{1,2}(0,T;X)} = \|f\|^2_{L^2(0,T;X)} \; + \; \|\dot{f}\|^2_{L^2(0,T;X)},$$

where $\dot{f}$ denotes the *generalized derivative* of f on $(0,T)$. We define $w = u^{(n)}$ to be the nth generalized derivative of the function u on $(0,T)$ iff

$$\int_0^T \phi^{(n)}(t)u(t)\,dt = (-1)^n \int_0^T \phi(t)w(t)\,dt \quad \forall \phi \in C_0^\infty(0,T)$$

is valid. Note that these integrals are defined whenever $u, w \in L^1(0,T;X)$ (see, for example, Zeidler [35, p. 418]). This generalized derivative is unique.

We record the fundamental inequality

$$\|f(t) - f(s)\|_X \leq \int_s^t \|\dot{f}(\tau)\|_X d\tau, \tag{3.4}$$

which holds for $s < t$ and $f \in W^{1,2}(0,T;X)$ (see, for example, Zeidler [35]). We have that $W^{1,2}(0,T;X) \subset C([0,T],X)$, with the embedding being continuous.

We introduce the bilinear form $a : Z \times Z \to \mathbb{R}$ defined by

$$a(\boldsymbol{w}, \boldsymbol{z}) = \int_\Omega [\boldsymbol{C}(\boldsymbol{\epsilon}(\boldsymbol{u}) - \boldsymbol{p}) \cdot (\boldsymbol{\epsilon}(\boldsymbol{v}) - \boldsymbol{q}) + k_1 \boldsymbol{p} \cdot \boldsymbol{q} + k_2 \gamma \mu] \ dx$$

$$= \int_\Omega [C_{ijkl} \ (\epsilon_{ij}(\boldsymbol{u}) - p_{ij})(\epsilon_{kl}(\boldsymbol{v}) - q_{kl}) + k_1 p_{ij} q_{ij} + k_2 \gamma \mu] \ dx, \tag{3.5}$$

the linear functional

$$\boldsymbol{l}(t) : Z \to \mathbb{R}, \quad \langle \boldsymbol{l}(t), \boldsymbol{z} \rangle = \int_\Omega \boldsymbol{f}(t) \cdot \boldsymbol{v} \ dx, \tag{3.6}$$

and the functional

$$j : Z \to \mathbb{R} \cup \{\infty\}, \quad j(\boldsymbol{z}) = \int_\Omega \ D(\boldsymbol{q}(x), \mu(x)) \, dx, \tag{3.7}$$

where as before $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{p}, \gamma)$ and $\boldsymbol{z} = (\boldsymbol{v}, \boldsymbol{q}, \mu)$.

The functional $\boldsymbol{l}(t)$ is easily shown to be bounded. From the properties of D, $j(\cdot)$ is a convex, positively homogeneous, nonnegative, l.s.c. functional on Z and is Lipschitz continuous on $\mathrm{dom} D = K$. Note, however, that j *is not differentiable.*

We are now ready to define the variational problem.

Problem EP. Given $\boldsymbol{l} \in W^{1,2}(0, T; Z^*)$ with $\boldsymbol{l}(0) = \boldsymbol{0}$, find $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{p}, \gamma) : [0, T] \to Z$ with $\boldsymbol{w}(0) = \boldsymbol{0}$, such that for almost all $t \in (0, T)$, $\dot{\boldsymbol{w}}(t) \in K$, and

$$a(\boldsymbol{w}(t), \boldsymbol{z} - \dot{\boldsymbol{w}}(t)) + j(\boldsymbol{z}) - j(\dot{\boldsymbol{w}}(t)) - \langle \boldsymbol{l}, \boldsymbol{z} - \dot{\boldsymbol{w}}(t) \rangle \geq 0 \quad \forall \boldsymbol{z} \in K. \tag{3.8}$$

The formal equivalence of problem EP to the classical problem defined by (2.1)–(2.4) is readily established (cf. [29, 31], for example). We take as fundamental the variational problem EP, though.

4. An abstract variational inequality.

4.1. The formulation. We study the elastoplastic problem EP in the framework of an abstract VI, of which problem EP is a specific example. This VI closely resembles a parabolic VI, with the important distinction that the rate quantity occurs in the arguments of all the functionals in the inequality. Apart from elastoplasticity, another application in which this VI may be found is that of elasticity with frictional contact (see [6] and [28]). In that case, though, the problem is posed on the entire space rather than on a convex subset. Furthermore, it will be seen that the methods adopted here are quite different from those used in the works cited, and we will present a wider range of results in the case of approximate problems.

We now take as fundamental the following abstract variational problem.

Problem P. Find $w : [0, T] \to H$, $w(0) = 0$, such that for almost all $t \in (0, T)$, $\dot{w}(t) \in K$ and

$$a(w(t), z - \dot{w}(t)) + j(z) - j(\dot{w}(t)) - \langle l(t), z - \dot{w}(t) \rangle \geq 0 \quad \forall z \in K. \tag{4.1}$$

Here H denotes a Hilbert space, K a nonempty, closed, convex cone in H. The bilinear form $a : H \times H \to \mathbb{R}$ is symmetric, bounded, and H-elliptic, $l \in W^{1,2}(0, T; H^*)$. The functional j: $K \to \mathbb{R}$ is nonnegative, convex, positively homogeneous, and Lipschitz continuous, which is *not* assumed to be differentiable. We call the problem P a variational inequality of the mixed kind, because it has features of VIs of both the first kind (the presence of the convex set K) and the second kind (the presence of the nondifferentiable term j). For a classification for the kinds of VI, see [8].

Questions of existence and uniqueness of solutions to this problem have been investigated in the context of elastoplasticity with kinematic hardening by Reddy [29].

We extend the functional j from K to the whole space H through

$$J(z) = \begin{cases} j(z), & z \in K, \\ +\infty, & z \notin K. \end{cases}$$

Since K is a nonempty, closed and convex cone and since j is nonnegative, convex, positively homogeneous, and Lipschitz continuous on K, the extended functional $J : H \to \mathbb{R} \cup \{\infty\}$ is proper, nonnegative, positively homogeneous, convex, and l.s.c. From now on, we will identify j with J; i.e., we will use the same notation $j(z)$ to denote the extension of $j(z)$ from K to H by ∞ for $z \notin K$. With this identification, we observe that (4.1) is equivalent to

$$a(w(t), z - \dot{w}(t)) + j(z) - j(\dot{w}(t)) - \langle l(t), z - \dot{w}(t)\rangle \geq 0 \quad \forall\, z \in H;$$

i.e., the inequality problem is not affected whether the test functions z are taken in H or only in K. We will have occasion to use this property later; in particular, we observe that problem P is equivalent to the problem of finding functions w: $[0,T] \to H$, $w(0) = 0$, and $w^*(t)$: $[0,T] \to H^*$ such that for almost all $t \in (0,T)$,

$$a(w(t), z) + \langle w^*(t), z\rangle = \langle l(t), z\rangle \quad \forall\, z \in H, \tag{4.2}$$

$$w^*(t) \in \partial j(\dot{w}(t)), \tag{4.3}$$

where $\partial j(\dot{w}(t))$ denotes the subdifferential of $j(\cdot)$ at $\dot{w}(t)$.

From the definition of the subdifferential, we observe that, because of the positive homogeneity of j, the relation $w^*(t) \in \partial j(\dot{w}(t))$ is equivalent to

$$\langle w^*(t), z\rangle \leq j(z) \;\; \forall z \in H \text{ and } \langle w^*(t), \dot{w}(t)\rangle = j(\dot{w}(t)). \tag{4.4}$$

A feature of the proof of the existence result presented below is that it employs a discretization method closely related to one which is used in practice for computational purposes (see, for example, Reddy and Martin [31], [32]). The method of proof has interesting parallels with the semidiscrete approximations of problem P, for which an estimate of the rate of convergence of the approximations is derived in section 5.

4.2. Existence, uniqueness, and stability.

Existence. The existence proof involves two stages: first, discretizing in time and establishing the existence of a family of solutions $\{w_n\}_{n=1}^N$ to the discrete problems. The second stage involves constructing a linear interpolate in time w_ϵ of the discrete solutions and showing that the limit, as the time step size ϵ approaches zero, of these interpolates is in fact a solution of problem P. The proof technique was employed in [29] for the problem with isotropic hardening only. Here, for convenience, we give a sketch of the proof for the existence of a solution to the problem under somewhat more general assumptions stated after (4.1).

Time discretization involves partitioning the time interval $[0,T]$ by $0 = t_0 < t_1 < \cdots < t_N = T$, where $t_n - t_{n-1} = \epsilon$. For given $l \in W^{1,2}(0,T;H^*)$, $l_n = l(t_n)$, which is well defined by the embedding $W^{1,2}(0,T;X) \hookrightarrow C([0,T];X)$ for any Banach space X

(see section 3). We define Δw_n to be the backward difference $w_n - w_{n-1}$ corresponding to a sequence $\{w_n\}_{n=0}^N$.

We start with the following.

LEMMA 4.1. *For any given* $\{l_n\}_{n=0}^N \subset H^*$, $l_0 = 0$, *there exists a unique sequence* $\{w_n\}_{n=0}^N \subset H$, *with* $w_0 = 0$, *such that for* $n = 1, 2, \ldots, N$, $\Delta w_n \in K$ *and*

$$a(w_n, z - \Delta w_n) + j(z) - j(\Delta w_n) - \langle l_n, z - \Delta w_n \rangle \geq 0 \quad \forall\, z \in H. \tag{4.5}$$

Furthermore, there exists a constant c, *independent of* ϵ, *such that*

$$\|\Delta w_n\|_H \leq c\, \|\Delta l_n\|_{H^*}, \quad n = 1, \ldots, N. \tag{4.6}$$

Proof. The inequality (4.5) may be rewritten as

$$\begin{aligned} &a(\Delta w_n, z - \Delta w_n) + j(z) - j(\Delta w_n) \\ &\geq \langle l_n, z - \Delta w_n \rangle - a(w_{n-1}, z - \Delta w_n) \quad \forall\, z \in H. \end{aligned} \tag{4.7}$$

We proceed inductively. For $n = 1$, since by the assumptions the bilinear form $a(\cdot,\cdot)$ is continuous and H-elliptic, the functional $j(\cdot)$ is proper, convex, and l.s.c., and the functional defined by the right-hand side of (4.7) is bounded and linear, the problem (4.7) has a unique solution $\Delta w_n = w_1$ (cf. [8]). Obviously, $j(\Delta w_n) < \infty$. Hence, $\Delta w_n \in K$. Assuming now that the solution w_{n-1} is known, we similarly show the existence of the solution $w_n = \Delta w_n + w_{n-1}$. To derive the estimate (4.6), set $z = 0$ in (4.7) to get

$$a(\Delta w_n, \Delta w_n) \leq \langle \Delta l_n, \Delta w_n \rangle - a(w_{n-1}, \Delta w_n) - j(\Delta w_n) + \langle l_{n-1}, \Delta w_n \rangle. \tag{4.8}$$

We now show that $-a(w_{n-1}, \Delta w_n) - j(\Delta w_n) + \langle l_{n-1}, \Delta w_n \rangle \leq 0$. By replacing n by $(n-1)$ and setting $z = \Delta w_{n-1} + \Delta w_n \in K$ in (4.5), we obtain

$$\begin{aligned} 0 &\leq a(w_{n-1}, \Delta w_n) - \langle l_{n-1}, \Delta w_n \rangle + j(\Delta w_{n-1} + \Delta w_n) - j(\Delta w_{n-1}) \\ &\leq a(w_{n-1}, \Delta w_n) - \langle l_{n-1}, \Delta w_n \rangle + j(\Delta w_n), \end{aligned}$$

where we have used the convexity and positive homogeneity of $j(\cdot)$. Hence from (4.8) we obtain the inequality

$$a(\Delta w_n, \Delta w_n) \leq \langle \Delta l_n, \Delta w_n \rangle,$$

from which estimate (4.6) follows by the H-ellipticity of $a(\cdot,\cdot)$. □

LEMMA 4.2. *Assume* $l \in W^{1,2}(0,T;H^*)$, $l(0) = 0$, *then the solution* $\{w_n\}_{n=0}^N$ *defined in Lemma* 4.1 *satisfies*

$$\max_{1 \leq n \leq N} \|w_n\|_H \leq c\, \|\dot{l}\|_{L^1(0,T;H^*)}, \tag{4.9}$$

$$\sum_{n=1}^N \|\Delta w_n\|_H^2 \leq c\,\epsilon\, \|\dot{l}\|_{L^2(0,T;H^*)}^2. \tag{4.10}$$

Proof. The estimates are consequences of (4.6) and (3.4) (see Reddy [29, Lemma 3]). □

We construct a piecewise linear interpolation w_ϵ of $\{w_n\}$ by setting

$$w_\epsilon(t) = w_{n-1} + \frac{\Delta w_n}{\epsilon}(t - t_{n-1})$$

for $t_{n-1} \le t \le t_n$. Clearly w_ϵ belongs to $L^\infty(0,T;H)$, while $\dot{w}_\epsilon \in L^2(0,T;H)$. For any sequence $\{z_n\}_{n=1}^N \subset H$, we define a step function $z(t)$ by

$$z(t) = z_n \quad \text{for } t_{n-1} \le t < t_n,\ n = 1, \dots, N-1,$$

$$z(t) = z_N \quad \text{for } t_{N-1} \le t \le t_N.$$

Let $z_{N+1} = 0$. We divide both sides of (4.5) by ϵ and use the positive homogeneity of j to obtain

$$a(w_n, z - \delta w_n) + j(z) - j(\delta w_n) - \langle l_n, z - \delta w_n\rangle \ge 0 \quad \forall\, z \in H,$$

where $\delta w_n = \Delta w_n/\epsilon$. Taking $z = (z_n + z_{n+1})/2$ in the above inequality, multiplying by ϵ, and summing over n, $n = 1, \dots, N$, we find that

$$\begin{aligned}(4.11)\qquad &\sum_{n=1}^N \epsilon\, a(w_n, (z_n + z_{n+1})/2 - \delta w_n) + \sum_{n=1}^N \epsilon\, j((z_n + z_{n+1})/2) \\ &- \sum_{n=1}^N \epsilon\, j(\delta w_n) - \sum_{n=1}^N \epsilon\, \langle l_n, (z_n + z_{n+1})/2 - \delta w_n\rangle \ge 0.\end{aligned}$$

We have

$$\sum_{n=1}^N \epsilon\, a(w_n, (z_n + z_{n+1})/2) = \int_0^T a(w_\epsilon(t), z(t))\, dt,$$

$$\sum_{n=1}^N \epsilon\, a(w_n, \delta w_n) \ge \int_0^T a(w_\epsilon(t), \dot{w}_\epsilon(t))\, dt,$$

$$\sum_{n=1}^N \epsilon\, j(\tfrac{1}{2}(z_n + z_{n+1})) \le \sum_{n=1}^N \epsilon\, \tfrac{1}{2}(j(z_n) + j(z_{n+1})) = \int_0^T j(z(t))\, dt - \tfrac{1}{2}\, \epsilon\, j(z_1)$$

(using the convexity of j),

$$\sum_{n=1}^N \epsilon\, j(\delta w_n) = \int_0^T j(\dot{w}_\epsilon(t))\, dt,$$

$$\sum_{n=1}^N \epsilon\, \langle l_n, \tfrac{1}{2}(z_n + z_{n+1})\rangle = \int_0^T \langle l_\epsilon(t), z(t)\rangle\, dt,$$

$$\begin{aligned}\sum_{n=1}^N \epsilon\, \langle l_n, \delta w_n\rangle &= \int_0^T \langle l_\epsilon(t), \dot{w}_\epsilon(t)\rangle\, dt + \sum_{n=1}^N \langle \Delta l_n, \Delta w_n\rangle \\ &\le \int_0^T \langle l_\epsilon(t), \dot{w}_\epsilon(t)\rangle\, dt + c\epsilon \int_0^T \|\dot{l}(t)\|_{H^*}^2\, dt,\end{aligned}$$

where, $l_\epsilon(t)$ represents the piecewise linear interpolation of $\{l_n\}_{n=0}^N$ and c is the constant appearing in (4.6).

Thus from (4.11) we see that w_ϵ satisfies the VI

$$0 \le J_\epsilon \equiv \int_0^T [a(w_\epsilon(t), z - \dot{w}_\epsilon(t)) + j(z) - j(\dot{w}_\epsilon(t)) - \langle l_\epsilon(t), z - \dot{w}_\epsilon(t)\rangle]\, dt$$

$$-\tfrac{1}{2}\epsilon j(z_1) + \tfrac{1}{2}c\epsilon \int_0^T \|\dot{l}(t)\|_{H^*}^2\, dt. \tag{4.12}$$

From (4.9), (4.10), and the definition of w_ϵ we see by direct evaluation that

$$\|w_\epsilon\|_{L^\infty(0,T;H)} \le C_1 \quad \text{and} \quad \|\dot{w}_\epsilon\|_{L^2(0,T;H)} \le C_2.$$

Now we fix a stepsize $\epsilon_0 > 0$ and consider the sequence of stepsizes $\epsilon_k = 2^{-k}\epsilon_0$, $k = 0, 1, \ldots$. It follows that there exists a subsequence $\{w_{\epsilon_{k_i}}\}$ of the sequence $\{w_{\epsilon_k}\}$ and a $w \in W^{1,2}(0,T;H)$ such that

$$w_{\epsilon_{k_i}} \overset{*}{\rightharpoonup} w \text{ in } L^\infty(0,T;H) \quad \text{and} \quad \dot{w}_{\epsilon_{k_i}} \rightharpoonup \dot{w} \text{ in } L^2(0,T;H) \text{ as } i \to \infty.$$

From the properties of j, it is easy to verify that the functional $\int_0^T j(v(t))\, dt$ is convex and l.s.c. on $L^1(0,T;K)$ and thus is weakly l.s.c. on $L^1(0,T;H)$. Since we also have

$$\dot{w}_{\epsilon_{k_i}} \rightharpoonup \dot{w} \text{ in } L^1(0,T;H) \text{ as } i \to \infty,$$

we obtain

$$\int_0^T j(\dot{w}(t))\, dt \le \liminf_{i\to\infty} \int_0^T j(\dot{w}_{\epsilon_{k_i}}(t))\, dt,$$

a relation needed in proving the next inequality below. In particular, the above relation implies that $\dot{w}(t) \in K$ for almost all $t \in [0,T]$.

It can then be proved that

$$0 \le \limsup_{i\to\infty} J_{\epsilon_{k_i}} \le \int_0^T \Big[a(w(t), z - \dot{w}(t)) + j(z) - j(\dot{w}(t)) - \langle l(t), z - \dot{w}(t)\rangle \Big]\, dt$$

for any step function z corresponding to a stepsize ϵ_{k_i}, $i = 1, 2, \ldots$. Approximating any $z \in L^2(0,T;K)$ by its piecewise averaging step functions $z_{\epsilon_{k_i}}$, it then follows that

$$\int_0^T \Big[a(w(t), z(t) - \dot{w}(t)) + j(z(t)) - j(\dot{w}(t)) - \langle l(t), z(t) - \dot{w}(t)\rangle \Big]\, dt \ge 0, \quad \forall\, z \in L^2(0,T;K)$$

Here we used the Lipschitz continuity of j on K and the fact that

$$z \in L^2(0,T;K) \Longrightarrow z_{\epsilon_{k_i}}(t) \in K \text{ a.e. } t.$$

By a standard procedure of passing to the pointwise inequality (see, for example, Duvaut and Lions [6]), we find from the above inequality that w satisfies the VI (4.1) a.e. on $[0,T]$. By the Sobolev embedding theorem, $W^{1,2}(0,T;H) \subset C([0,T];H)$, and we observe that $w \in L^\infty(0,T;H)$ and $\dot{w} \in L^2(0,T;H)$ is equivalent to $w \in W^{1,2}(0,T;H)$.

Uniqueness. Suppose that problem P has two solutions, w_1 and w_2. Denote by Δw the difference $w_1 - w_2$. From (4.1), on setting $w = w_1$, $z = \dot{w}_2 \in K$, and then $w = w_2$, $z = \dot{w}_1 \in K$, respectively, we have

$$a(w_1, \Delta\dot{w}) + j(\dot{w}_1) - j(\dot{w}_2) \leq \langle l, \Delta\dot{w}\rangle,$$
$$-a(w_2, \Delta\dot{w}) + j(\dot{w}_2) - j(\dot{w}_1) \leq -\langle l, \Delta\dot{w}\rangle.$$

Adding, we get

$$0 \geq a(\Delta w, \Delta\dot{w}) = \frac{1}{2}\frac{d}{dt}a(\Delta w, \Delta w).$$

Integration, the H-ellipticity of $a(\cdot,\cdot)$, and the initial conditions $w_1(0) = w_2(0) = 0$ together yield $w_2 = w_1$, as required.

We summarize the above analysis in the following theorem.

THEOREM 4.3 (Existence and uniqueness). *Let H be a Hilbert space; $K \subset H$ a nonempty, closed, convex cone; $a\colon H \times H \to \mathbb{R}$ a bilinear form which is symmetric, bounded, and H-elliptic; $l \in W^{1,2}(0,T;H^*)$ with $l(0) = 0$; and $j\colon K \to \mathbb{R}$ nonnegative, convex, positively homogeneous, and Lipschitz continuous. Then there exists a unique solution w of problem P satisfying $w \in W^{1,2}(0,T;H)$. Furthermore, $w\colon [0,T] \to H$ with $w(0) = 0$, is the solution of the problem P iff there is a function $w^*(t)\colon [0,T] \to H^*$ such that for almost all $t \in (0,T)$,*

$$a(w(t), z) + \langle w^*(t), z\rangle = \langle l(t), z\rangle \quad \forall\, z \in H, \tag{4.13}$$

$$w^*(t) \in \partial j(\dot{w}(t)). \tag{4.14}$$

We observe from (4.13) that w^* has the regularity property

$$w^* \in W^{1,2}(0,T;H^*). \tag{4.15}$$

If we assume $l \in W^{1,p}(0,T;H^*)$, $1 \leq p < \infty$, then (4.10) can be replaced by

$$\sum_{n=1}^{N} \|\Delta w_n\|_H^p \leq c\,\epsilon^{p-1}\|\dot{l}\|^p_{L^p(0,T;H^*)}.$$

As a result, $\{w_\epsilon\}$ is uniformly bounded in $W^{1,p}(0,T;H)$. Hence, from the existence proof above, the solution is $w \in W^{1,p}(0,T;H)$. Similarly, if $l \in W^{1,\infty}(0,T;H^*)$, then the solution $w \in W^{1,\infty}(0,T;H)$.

Stability. We discuss the stability of the solution w of (4.1) with respect to l. Let l_1, $l_2 \in W^{1,2}(0,T;H^*)$ be given, $l_1(0) = l_2(0) = 0$, and let w_1 and w_2 be the corresponding solutions whose existence is assured by Theorem 4.3. Thus, for almost all $t \in (0,T)$, $\dot{w}_1(t) \in K$, $\dot{w}_2(t) \in K$, and

$$a(w_1(t), z - \dot{w}_1(t)) + j(z) - j(\dot{w}_1(t)) - \langle l_1(t), z - \dot{w}_1(t)\rangle \geq 0 \quad \forall\, z \in K, \tag{4.16}$$

$$a(w_2(t), z - \dot{w}_2(t)) + j(z) - j(\dot{w}_2(t)) - \langle l_2(t), z - \dot{w}_2(t)\rangle \geq 0 \quad \forall\, z \in K. \tag{4.17}$$

Take $z = \dot{w}_2(t) \in K$ in (4.16) and $z = \dot{w}_1(t) \in K$ in (4.17) and add the two resultant inequalities to obtain

$$-\frac{1}{2}\frac{d}{dt}a\left(w_1(t) - w_2(t), w_1(t) - w_2(t)\right) + \langle l_1(t) - l_2(t), \dot{w}_1(t) - \dot{w}_2(t)\rangle \geq 0;$$

that is,

$$\frac{1}{2}\,\frac{d}{dt}a\left(w_1(t)-w_2(t),w_1(t)-w_2(t)\right)\le\langle l_1(t)-l_2(t),\dot w_1(t)-\dot w_2(t)\rangle.$$

Denote $e=w_1-w_2$. Observing that $e(0)=0$, we have

$$\begin{aligned}\tfrac{1}{2}\,a\left(e(t),e(t)\right)&\le\int_0^t\langle l_1(t)-l_2(t),\dot e(t)\rangle\,dt\\&=\langle l_1(t)-l_2(t),e(t)\rangle-\int_0^t\langle \dot l_1(t)-\dot l_2(t),e(t)\rangle\,dt.\end{aligned}$$

Since a is H-elliptic, we have

$$\|e(t)\|_H^2\le c\,\|l_1(t)-l_2(t)\|_{H^*}\|e(t)\|_H+c\int_0^t\|\dot l_1(t)-\dot l_2(t)\|_{H^*}\|e(t)\|_H dt.$$

Let $M=\sup_{0\le t\le T}\|e(t)\|_H$, then

$$\|e(t)\|_H^2\le c\,\|l_1(t)-l_2(t)\|_{H^*}M+c\int_0^t\|\dot l_1(t)-\dot l_2(t)\|_{H^*}M\,dt.$$

Hence,

$$M^2\le c\,M\,\|l_1-l_2\|_{L^\infty(0,T;H^*)}+c\,M\,\|\dot l_1-\dot l_2\|_{L^1(0,T;H^*)},$$

and

$$M\le c\left(\|l_1-l_2\|_{L^\infty(0,T;H^*)}+\|\dot l_1-\dot l_2\|_{L^1(0,T;H^*)}\right).$$

In conclusion, we have proved the following.

THEOREM 4.4 (Stability). *Under the assumptions of Theorem 4.3, the solution of the problem (4.1) depends continuously on l: for $l_1,l_2\in W^{1,2}(0,T;H^*)$ with $l_1(0)=l_2(0)=0$, the corresponding solutions w_1 and w_2 satisfy*

$$\|w_1-w_2\|_{L^\infty(0,T;H)}\le c\left(\|l_1-l_2\|_{L^\infty(0,T;H^*)}+\|\dot l_1-\dot l_2\|_{L^1(0,T;H^*)}\right).$$

5. Semidiscrete internal approximations. In this section we consider semidiscrete internal approximations of the model problem P. As in the last section, we assume that H is a Hilbert space; $K\subset H$ is a nonempty, closed, convex cone; $a:H\times H\to\mathbb{R}$ is bilinear, symmetric, bounded and H-elliptic; and $l\in W^{1,2}(0,T;H^*)$ with $l(0)=0$. The functional $j:K\to\mathbb{R}$ is nonnegative, convex, positively homogeneous, and Lipschitz continuous; i.e.,

$$|j(z_1)-j(z_2)|\le c\,\|z_1-z_2\|_H\quad\forall\,z_1,z_2\in K.$$

Let $h\in(0,1]$ be a mesh parameter and $\{H^h\}$ a family of finite-dimensional subspaces of H, with the property that

$$\lim_{h\to0}\|z-z^h\|_H=0\quad\forall\,z\in H.\tag{5.1}$$

Denote $K^h=H^h\cap K$. Then a semidiscrete internal approximation of the model problem P is as follows.

Problem $\mathbf{P}^h$. *Find $w^h : [0,T] \to H^h$, $w^h(0) = 0$, such that for almost all $t \in (0,T)$, $\dot{w}^h(t) \in K^h$ and*

$$a(w^h(t), z^h - \dot{w}^h(t)) + j(z^h) - j(\dot{w}^h(t)) - \langle l(t), z^h - \dot{w}^h(t)\rangle \geq 0 \quad \forall\, z^h \in K^h. \tag{5.2}$$

We note that for any given h, K^h is a nonempty, closed, convex cone in H^h. Thus, the existence of a unique solution w^h to problem P^h follows from Theorem 4.3 with H and K replaced by H^h and K^h. We also note from Theorem 4.3 that $w^h \in W^{1,2}(0,T;H)$. This regularity result implies that $w^h \in C([0,T];H)$; in particular, the value $w^h(0)$ is well defined. From Theorem 4.4, we have the stability estimate

$$\|w_1^h - w_2^h\|_{L^\infty(0,T;H)} \leq c \left(\|l_1 - l_2\|_{L^\infty(0,T;H^*)} + \|\dot{l}_1 - \dot{l}_2\|_{L^1(0,T;H^*)}\right)$$

for semidiscrete solutions w_1^h and w_2^h corresponding to l_1 and l_2.

The main purpose of the section is to give an estimate for the semidiscrete approximation error $w - w^h$. For convenience, we will use the notation

$$\|w\|_a^2 = a(w,w).$$

Note that $\|\cdot\|_a$ is a norm equivalent to $\|\cdot\|_H$. The strategy used in the following to derive the error estimate is inspired by ideas contained in [5], although the problems and analyses differ greatly.

Set $z = \dot{w}^h(t) \in K$ in (4.1) to obtain

$$a(w(t), \dot{w}^h(t) - \dot{w}(t)) + j(\dot{w}^h(t)) - j(\dot{w}(t)) \geq \langle l(t), \dot{w}^h(t) - \dot{w}(t)\rangle. \tag{5.3}$$

We now add (5.3) to (5.2) and obtain

$$\begin{aligned} &a(w(t), \dot{w}^h(t) - \dot{w}(t)) + a(w^h(t), z^h - \dot{w}^h(t)) + j(z^h) - j(\dot{w}(t)) \\ &\geq \langle l(t), z^h - \dot{w}(t)\rangle. \end{aligned} \tag{5.4}$$

Using (5.4), Theorem 4.3, and (4.4), we have for any $z^h \in K^h$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w(t) - w^h(t)\|_a^2 \\ &= a(w(t) - w^h(t), \dot{w}(t) - \dot{w}^h(t)) \\ &= a(w(t) - w^h(t), \dot{w}(t) - z^h) + a(w(t) - w^h(t), z^h - \dot{w}^h(t)) \\ &\leq a(w(t) - w^h(t), \dot{w}(t) - z^h) + a(w(t), z^h - \dot{w}^h(t)) \\ &\quad + a(w(t), \dot{w}^h(t) - \dot{w}(t)) + j(z^h) - j(\dot{w}(t)) - \langle l(t), z^h - \dot{w}(t)\rangle \\ &= a(w(t) - w^h(t), \dot{w}(t) - z^h) + j(z^h) - j(\dot{w}(t)) - \langle w^*(t), z^h - \dot{w}(t)\rangle \\ &\leq a(w(t) - w^h(t), \dot{w}(t) - z^h) + j(z^h) - j(\dot{w}(t)) + j(\dot{w}(t) - z^h), \end{aligned}$$

where in the last step, we used (4.4) which in turn is derived based on the positive homogeneity of $j(\cdot)$. On the other hand, we have the regularity estimate (4.15) directly from (4.13). Thus, we have $w^* \in C([0,T];H^*)$ and

$$-\langle w^*(t), z^h - \dot{w}(t)\rangle \leq c\,\|z^h - \dot{w}(t)\|_H$$

which can be used in deriving (5.5) below. Now, using the convexity, positive homogeneity, and Lipschitz continuity of $j(\cdot)$, we find that

$$\frac{1}{2}\frac{d}{dt}\|w(t)-w^h(t)\|_a^2 \le a(w(t)-w^h(t),\dot{w}(t)-z^h) + C\,\|z^h-\dot{w}(t)\|_H \quad \forall\, z^h\in K^h. \tag{5.5}$$

Since

$$\begin{aligned} & a(w(t)-w^h(t),\dot{w}(t)-z^h) \\ & \le a(w(t)-w^h(t),w(t)-w^h(t))^{1/2} a(\dot{w}(t)-z^h,\dot{w}(t)-z^h)^{1/2} \\ & \le c\left(\|w(t)-w^h(t)\|_a^2 + \|\dot{w}(t)-z^h\|_H^2\right), \end{aligned}$$

from (5.5), we find that for any $z^h = z^h(t)\in K^h$,

$$\begin{aligned} & \frac{d}{dt}\|w(t)-w^h(t)\|_a^2 \\ & \le c\left(\|w(t)-w^h(t)\|_a^2 + \|\dot{w}(t)-z^h(t)\|_H^2 + \|\dot{w}(t)-z^h(t)\|_H\right). \end{aligned} \tag{5.6}$$

We multiply the inequality (5.6) by e^{-ct} and integrate from 0 to t to obtain

$$\|w(t)-w^h(t)\|_a^2 \le c\,e^{ct}\int_0^t e^{-cs}\left(\|\dot{w}(s)-z^h(s)\|_H^2 + \|\dot{w}(s)-z^h(s)\|_H\right)\,ds.$$

Therefore, we have the Céa-type inequality

$$\|w(t)-w^h(t)\|_{L^\infty(0,T;H)} \le c \inf_{z^h\in L^2(0,T;K^h)} \|\dot{w}-z^h\|_{L^2(0,T;H)}^{1/2}. \tag{5.7}$$

The inequality (5.7) is the basis for various convergence order estimates (see section 7).

6. Fully discrete internal approximations. In this section, we consider the simultaneous discretizations of the temporal variable and the spatial variables. We keep the same assumptions on the data as in section 5. We divide the time interval $I = [0,T]$ into N equal parts. Denote $k = T/N$, the stepsize, $t_n = nk$, $n = 0,1,\dots,N$, the nodal points, and $I_n = [t_{n-1},t_n]$, $n = 1,2,\dots,N$, the subintervals. For a continuous function $v(t)$, with values in H or H^*, we use the notation $v_n = v(t_n)$, $v_{n-1/2} = v((t_n+t_{n-1})/2)$, $\Delta v_n = v_n - v_{n-1}$, $\Delta v_{n-1/2} = v_{n-1/2} - v_{n-3/2}$, and $\delta v_n = (v_n - v_{n-1})/k$. In this and later sections, no summation is implied over the repeated index n.

We will consider two different kinds of discretizations for the differentiation with respect to the temporal variable t.

6.1. Backward Euler scheme. We first consider a backward Euler scheme. The fully discrete approximation problem is as follows.

Problem $\mathbf{P}_1^{hk}$. *Find $w^{hk} = \{w_n^{hk}\}_{n=0}^N$, where $w_n^{hk}\in H^h$, $0\le n\le N$, and $w_0^{hk} = 0$, such that for $n = 1,2,\dots,N$, $\delta w_n^{hk}\in K^h$ and*

$$a(w_n^{hk}, z^h - \delta w_n^{hk}) + j(z^h) - j(\delta w_n^{hk}) - \langle l_n, z^h - \delta w_n^{hk}\rangle \ge 0 \quad \forall\, z^h\in K^h. \tag{6.1}$$

The existence and uniqueness of the solution w^{hk} to the problem $\mathbf{P}_1^{hk}$ follow when the argument in proving Lemma 4.1 is applied to the case of a finite-dimensional space.

It can also be similarly proved that

$$\max_{1\le n\le N} \|w_n^{hk}\|_H \le c_1,$$

$$\sum_{n=1}^{N} k\,\|\delta w_n^{hk}\|_H^2 \le c_2.$$

We also have a stability result for the fully discrete solution. Let $l_1, l_2 \in W^{1,2}(0,T;H^*)$, and let $w_{1,n}^{hk}$ and $w_{2,n}^{hk}$, $0 \le n \le N$, be the corresponding fully discrete solutions. Then for $n = 1, 2, \ldots, N$, we have $\delta w_{1,n}^{hk}, \delta w_{2,n}^{hk} \in K^h$, and

$$a(w_{1,n}^{hk}, z^h - \delta w_{1,n}^{hk}) + j(z^h) - j(\delta w_{1,n}^{hk}) - \langle l_{1,n}, z^h - \delta w_{1,n}^{hk}\rangle \ge 0 \quad \forall\, z^h \in K^h, \tag{6.2}$$

$$a(w_{2,n}^{hk}, z^h - \delta w_{2,n}^{hk}) + j(z^h) - j(\delta w_{2,n}^{hk}) - \langle l_{2,n}, z^h - \delta w_{2,n}^{hk}\rangle \ge 0 \quad \forall\, z^h \in K^h. \tag{6.3}$$

Denote $e_n = w_{1,n}^{hk} - w_{2,n}^{hk}$. We take $z^h = \delta w_{2,n}^{hk}$ in (6.2) and $z^h = \delta w_{1,n}^{hk}$ in (6.3) and add the two resultant inequalities to obtain

$$a(e_n, \delta e_n) \le \langle l_{1,n} - l_{2,n}, \delta e_n\rangle.$$

Since

$$a(e_n, \delta e_n) = \frac{1}{k}\,\left(a(e_n, e_n) - a(e_n, e_{n-1})\right) \ge \frac{1}{2k}\,\left(a(e_n, e_n) - a(e_{n-1}, e_{n-1})\right),$$

we have

$$a(e_n, e_n) - a(e_{n-1}, e_{n-1}) \le 2\,\langle l_{1,n} - l_{2,n}, e_n - e_{n-1}\rangle.$$

Hence, noticing that $e_0 = 0$,

$$\begin{aligned} a(e_n, e_n) &\le 2\sum_{j=1}^{n} \langle l_{1,j} - l_{2,j}, e_j - e_{j-1}\rangle \\ &= -2\sum_{j=1}^{n-1} \langle (l_{1,j+1} - l_{1,j}) - (l_{2,j+1} - l_{2,j}), e_j\rangle + 2\,\langle l_{1,n} - l_{2,n}, e_n\rangle \\ &\le c\sum_{j=1}^{n-1} \|(l_{1,j+1} - l_{1,j}) - (l_{2,j+1} - l_{2,j})\|_{H^*}\|e_j\|_H + c\,\|l_{1,n} - l_{2,n}\|_{H^*}\|e_n\|_H. \end{aligned}$$

Using (3.4) and the H-ellipticity of a, we obtain the inequality

$$\|e_n\|_H^2 \le c\sum_{j=1}^{n-1} \int_{t_j}^{t_{j+1}} \|\dot{l}_1(t) - \dot{l}_2(t)\|_{H^*} dt\,\|e_j\|_H + c\,\|l_{1,n} - l_{2,n}\|_{H^*}\|e_n\|, \quad 1 \le n \le N.$$

Let $M = \max_{0\le n\le N} \|e_n\|_H$. We find from the above inequality that

$$M^2 \le c\,M \int_0^T \|\dot{l}_1(t) - \dot{l}_2(t)\|_{H^*} dt + c\,M\,\|l_1 - l_2\|_{L^\infty(0,T;H^*)}.$$

Hence, we get the desired stability estimate

$$\max_{0\le n\le N} \|w_{1,n}^{hk} - w_{2,n}^{hk}\|_H \le c \left(\|\dot{l}_1 - \dot{l}_2\|_{L^1(0,T;H^*)} + \|l_1 - l_2\|_{L^\infty(0,T;H^*)} \right).$$

Our purpose now is to derive error estimates for $w_n - w_n^{hk}$.

THEOREM 6.1. *For the solution* w *of* $\mathbf{P}$ *and the solution* w^{hk} *of* $\mathbf{P}_1^{hk}$*, we have the inequalities*

$$\|w_n - w_n^{hk}\|_H^2 \le c\,k \sum_{j=1}^{n} \left(\inf_{z^h\in H^h} \|z^h - \dot{w}_j\|_H + \|\delta w_j - \dot{w}_j\|_H \right),\ \ n = 1, 2, \ldots, N$$

and

$$\max_{0\le n\le N} \|w_n - w_n^{hk}\|_H^2 \le c\,k \sum_{n=1}^{N} \left(\inf_{z^h\in H^h} \|z^h - \dot{w}_n\|_H + \|\delta w_n - \dot{w}_n\|_H \right).$$

Proof. Denote $e_n = w_n - w_n^{hk}$, $n = 0, 1, \ldots, N$. Consider the quantity $a(e_n, \delta e_n)$. First we have a lower bound:

$$a(e_n, \delta e_n) = \frac{1}{k}\ [a(e_n, e_n) - a(e_n, e_{n-1})] \ge \frac{1}{2k}\ [a(e_n, e_n) - a(e_{n-1}, e_{n-1})]\,.$$

For an upper bound we write for any $z^h \in K^h$,

$$\begin{aligned} &a(e_n, \delta e_n)\\ &= a(w_n, \delta w_n - \delta w_n^{hk}) - a(w_n^{hk}, \delta w_n - \delta w_n^{hk})\\ &= a(w_n, \dot{w}_n - \delta w_n^{hk}) + a(w_n, \delta w_n - \dot{w}_n) - a(w_n^{hk}, z^h - \delta w_n^{hk}) - a(w_n^{hk}, \delta w_n - z^h)\\ &= \underbrace{a(w_n, \dot{w}_n - \delta w_n^{hk})}_{I_1} + \underbrace{a(w_n, z^h - \dot{w}_n)}_{I_2} + \underbrace{a(w_n^{hk}, \delta w_n^{hk} - z^h)}_{I_3} + \underbrace{a(w_n - w_n^{hk}, \delta w_n - z^h)}_{I_4}. \end{aligned}$$

Since

$$I_1 = -a(w_n, \delta w_n^{hk} - \dot{w}_n) \le j(\delta w_n^{hk}) - j(\dot{w}_n) - \langle l_n, \delta w_n^{hk} - \dot{w}_n\rangle$$

and

$$I_3 = -a(w_n^{hk}, z^h - \delta w_n^{hk}) \le j(z^h) - j(\delta w_n^{hk}) - \langle l_n, z^h - \delta w_n^{hk}\rangle,$$

we have

$$I_1 + I_3 \le j(z^h) - j(\dot{w}_n) - \langle l_n, z^h - \dot{w}_n\rangle.$$

From Theorem 4.3 and (4.4),

$$I_2 = \langle l_n, z^h - \dot{w}_n\rangle - \langle w_n^*, z^h - \dot{w}_n\rangle \le \langle l_n, z^h - \dot{w}_n\rangle + j(z^h - \dot{w}_n).$$

Hence,

$$a(e_n, \delta e_n) \le j(z^h) - j(\dot{w}_n) + j(z^h - \dot{w}_n) + a(e_n, \delta w_n - z^h) \quad \forall\, z^h \in K^h.$$

Combining the upper and lower bounds for $a(e_n, \delta e_n)$, we find that

$$\begin{aligned} &\frac{1}{2k}\, [a(e_n, e_n) - a(e_{n-1}, e_{n-1})] \\ &\le a(e_n, \delta w_n - z^h) + j(z^h) - j(\dot{w}_n) + j(z^h - \dot{w}_n) \\ &\le \tfrac{1}{2}\, a(e_n, e_n) + c\, \|\delta w_n - z^h\|_H^2 + j(z^h) - j(\dot{w}_n) + j(z^h - \dot{w}_n). \end{aligned}$$

Using the properties of j, we get

$$\begin{aligned} &\|w_n - w_n^{hk}\|_a^2 \\ &\le \frac{1}{1-k}\, \|w_{n-1} - w_{n-1}^{hk}\|_a^2 + c\, k\, \left(\|z^h - \dot{w}_n\|_H + \|\delta w_n - z^h\|_H^2\right). \end{aligned} \tag{6.4}$$

Hence,

$$\begin{aligned} &\|w_n - w_n^{hk}\|_a^2 \\ &\le \frac{1}{1-k}\, \|w_{n-1} - w_{n-1}^{hk}\|_a^2 + c\, k\, \left(\inf_{z^h \in K^h} \|z^h - \dot{w}_n\|_H + \|\delta w_n - \dot{w}_n\|_H^2 \right). \end{aligned}$$

Applying the above inequality recursively, we find that

$$\|w_n - w_n^{hk}\|_a^2 \le c\, k\, \sum_{j=1}^{n} (1-k)^{j-n} \left(\inf_{z^h \in K^h} \|z^h - \dot{w}_j\|_H + \|\delta w_j - \dot{w}_j\|_H^2 \right).$$

Since $(1-k)^{-n} \le c$ for $n \le T/k$, we obtain the first inequality. The second inequality is an obvious consequence of the first. ∎

As a simple consequence of Theorem 6.1, we have the following.

COROLLARY 6.2. *Assume the solution to the problem* $\mathbf{P}$ *satisfies* $\ddot{w} \in L^2(0, T; H)$*; then*

$$\max_{0 \le n \le N} \|w_n - w_n^{hk}\|_H^2 \le c\, k\, \sum_{n=1}^{N} \inf_{z^h \in K^h} \|z^h - \dot{w}_n\|_H + c\, k^2 \|\ddot{w}_n\|_{L^2(0,T;H)}^2.$$

Proof. We rewrite the term $\delta w_n - \dot{w}_n$:

$$\begin{aligned} \delta w_n - \dot{w}_n &= \frac{1}{k} \int_{t_{n-1}}^{t_n} \dot{w}(t)\, dt - \dot{w}_n = \frac{1}{k} \int_{t_{n-1}}^{t_n} \left(\dot{w}(t) - \dot{w}_n\right)\, dt \\ &= \frac{1}{k} \int_{t_{n-1}}^{t_n} \int_{t_n}^{t} \ddot{w}(\tau)\, d\tau\, dt = -\frac{1}{k} \int_{t_{n-1}}^{t_n} \ddot{w}(\tau)\, d\tau \int_{t_{n-1}}^{\tau} dt \\ &= -\frac{1}{k} \int_{t_{n-1}}^{t_n} (\tau - t_{n-1})\, \ddot{w}(\tau)\, d\tau. \end{aligned}$$

Thus,

$$\|\delta w_n - \dot{w}_n\|_H^2 \le \frac{1}{k^2} \int_{t_{n-1}}^{t_n} (\tau - t_{n-1})^2 d\tau \int_{t_{n-1}}^{t_n} \|\ddot{w}(\tau)\|_H^2\, d\tau \le \frac{k}{3}\, \|\ddot{w}\|_{L^2(t_{n-1}, t_n; H)}^2.$$

Hence, Corollary 6.2 follows from Theorem 6.1. ∎

We have seen that the scheme $\mathbf{P}_1^{hk}$ is first-order accurate in time t, which is expected since we used the backward difference to approximate the derivative. When the solution w is smooth with respect to the time variable, it is natural to use a scheme with a higher order accuracy in time. We present and analyze a Crank–Nicolson-type scheme for solving the problem $\mathbf{P}$ in the next subsection.

6.2. Crank–Nicolson scheme. The discrete problem is as follows.

Problem $\mathbf{P}_2^{hk}$. Find $w^{hk} = \{w_n^{hk}\}_{n=0}^N$, where $w_n^{hk} \in H^h$, $0 \le n \le N$, $w_0^{hk} = 0$, such that for $n = 1, 2, \dots, N$, $\delta w_n^{hk} \in K^h$, and

$$a\left(\tfrac{1}{2}(w_n^{hk} + w_{n-1}^{hk}), z^h - \delta w_n^{hk}\right) + j(z^h) - j(\delta w_n^{hk}) - \langle l_{n-1/2}, z^h - \delta w_n^{hk}\rangle \ge 0 \quad \forall\, z^h \in K^h. \tag{6.5}$$

For the existence and uniqueness of the solution w^{hk} to the problem $\mathbf{P}_2^{hk}$, we have the following.

PROPOSITION 6.3. *The problem $\mathbf{P}_2^{hk}$ has a unique solution w^{hk}.*

Proof. Since j is positively homogeneous, (6.5) is equivalent to

$$a\left((w_n^{hk} + w_{n-1}^{hk})/2, z^h - \Delta w_n^{hk}\right) + j(z^h) - j(\Delta w_n^{hk}) \ge \langle l_{n-1/2}, z^h - \Delta w_n^{hk}\rangle \quad \forall\, z^h \in K^h \tag{6.6}$$

which can be written as

$$\begin{aligned} &\tfrac{1}{2}\, a\left(\Delta w_n^{hk}, z^h - \Delta w_n^{hk}\right) + j(z^h) - j(\Delta w_n^{hk}) \\ &\ge \langle l_{n-1/2}, z^h - \Delta w_n^{hk}\rangle - a\left(w_{n-1}^{hk}, z^h - \Delta w_n^{hk}\right) \quad \forall\, z^h \in K^h. \end{aligned} \tag{6.7}$$

We can then prove the existence of a unique solution by mathematical induction, as in the case of Lemma 4.1. ∎

For the Crank–Nicolson solution, we have the following stability result.

PROPOSITION 6.4. *Assume that $l_1, l_2 \in W^{1,2}(0,T;H^*)$. For the corresponding Crank–Nicolson solutions $w_{1,n}^{hk}$ and $w_{2,n}^{hk}$, $0 \le n \le N$, the inequality*

$$\max_{0\le n\le N} \|w_{1,n}^{hk} - w_{2,n}^{hk}\|_H \le c\left(\|\dot{l}_1 - \dot{l}_2\|_{L^1(0,T;H^*)} + \|l_1 - l_2\|_{L^\infty(0,T;H^*)}\right)$$

holds.

Proof. The fully discrete solutions $w_{1,n}^{hk}$ and $w_{2,n}^{hk}$ satisfy $\delta w_{1,n}^{hk}, \delta w_{2,n}^{hk} \in K^h$ and

$$a\left(\tfrac{1}{2}(w_{1,n}^{hk} + w_{1,n-1}^{hk}), z^h - \delta w_{1,n}^{hk}\right) + j(z^h) - j(\delta w_{1,n}^{hk}) \ge \langle l_{1,n-1/2}, z^h - \delta w_{1,n}^{hk}\rangle \quad \forall\, z^h \in K^h, \tag{6.8}$$

$$a\left(\tfrac{1}{2}(w_{2,n}^{hk} + w_{2,n-1}^{hk}), z^h - \delta w_{2,n}^{hk}\right) + j(z^h) - j(\delta w_{2,n}^{hk}) \ge \langle l_{2,n-1/2}, z^h - \delta w_{2,n}^{hk}\rangle \quad \forall\, z^h \in K^h. \tag{6.9}$$

Denote $e_n = w_{1,n}^{hk} - w_{2,n}^{hk}$. We take $z^h = \delta w_{2,n}^{hk}$ in (6.8) and $z^h = \delta w_{1,n}^{hk}$ in (6.9) and add the two inequalities to obtain

$$a\left((e_n + e_{n-1})/2, \delta e_n\right) \le \langle l_{1,n-1/2} - l_{2,n-1/2}, \delta e_n\rangle;$$

that is,

$$\|e_n\|_a^2 - \|e_{n-1}\|_a^2 \le 2\,\langle l_{1,n-1/2} - l_{2,n-1/2}, e_n - e_{n-1}\rangle.$$

Thus

$$\begin{aligned}
&\|e_n\|_a^2 \\
&\le 2 \sum_{j=1}^{n} \langle l_{1,j-1/2} - l_{2,j-1/2}, e_j - e_{j-1} \rangle \\
&= 2 \sum_{j=1}^{n-1} \langle (l_{1,j-1/2} - l_{1,j+1/2}) - (l_{2,j-1/2} - l_{2,j+1/2}), e_j \rangle + 2 \, \langle l_{1,n-1/2} - l_{2,n-1/2}, e_n \rangle.
\end{aligned}$$

We can then proceed as in the stability proof for the backward Euler solution to obtain the required inequality. □

In particular, if we take $l_2(t) \equiv 0$, then $w_{2,n}^{hk} = 0$, $0 \le n \le N$, and from Proposition 6.4, we get a bound on the discrete solution of the problem $\mathbf{P}_2^{hk}$.

COROLLARY 6.5. *For the solution w_n^{hk}, $0 \le n \le N$, of problem $\mathbf{P}_2^{hk}$, we have*

$$\max_{0 \le n \le N} \|w_n^{hk}\|_H \le c \left(\|\dot{l}\|_{L^1(0,T;H^*)} + \|l\|_{L^\infty(0,T;H^*)} \right).$$

Before proving an error estimate for the Crank–Nicolson solution, we need some preliminary results. We will use the notation

$$D w_n = \tfrac{1}{2} \, (w_n + w_{n-1}) - w_{n-1/2}, \quad n = 1, \ldots, N.$$

LEMMA 6.6. *Assume that $\ddot{w} \in L^\infty(0,T;H)$; then*

$$\|D w_n\|_H \le \frac{k^2}{8} \, \|\ddot{w}\|_{L^\infty(0,T;H)}.$$

Proof. We use Taylor expansions at $t_{n-1/2}$:

$$\begin{aligned}
w_n &= w_{n-1/2} + \frac{k}{2} \, \dot{w}_{n-1/2} + \int_{t_{n-1/2}}^{t_n} (t_n - t) \, \ddot{w}(t) \, dt, \\
w_{n-1} &= w_{n-1/2} - \frac{k}{2} \, \dot{w}_{n-1/2} + \int_{t_{n-1/2}}^{t_{n-1}} (t_{n-1} - t) \, \ddot{w}(t) \, dt.
\end{aligned}$$

Hence

$$\tfrac{1}{2} \, (w_n + w_{n-1}) - w_{n-1/2} = \tfrac{1}{2} \, \left[\int_{t_{n-1/2}}^{t_n} (t_n - t) \, \ddot{w}(t) \, dt + \int_{t_{n-1}}^{t_{n-1/2}} (t - t_{n-1}) \, \ddot{w}(t) \, dt \right].$$

Therefore

$$\|(w_n + w_{n-1})/2 - w_{n-1/2}\|_H \le \frac{k^2}{8} \, \|\ddot{w}\|_{L^\infty(0,T;H)}. \qquad \square$$

LEMMA 6.7. *Assume that $w^{(3)} \in L^1(0,T;H)$; then*

$$\|D w_n - D w_{n+1}\|_H \le c \, k^2 \|w^{(3)}\|_{L^1(t_{n-1},t_{n+1};H)}.$$

Proof. We again use Taylor expansions at $t_{n-1/2}$:

$$w_n = w_{n-1/2} + \frac{k}{2}\,\dot{w}_{n-1/2} + \frac{1}{2}\left(\frac{k}{2}\right)^2 \ddot{w}_{n-1/2} + \frac{1}{2}\int_{t_{n-1/2}}^{t_n} (t_n - t)^2\, w^{(3)}(t)\, dt \tag{6.10}$$

and

$$w_{n-1} = w_{n-1/2} - \frac{k}{2}\,\dot{w}_{n-1/2} + \frac{1}{2}\left(\frac{k}{2}\right)^2 \ddot{w}_{n-1/2} + \frac{1}{2}\int_{t_{n-1/2}}^{t_{n-1}} (t_{n-1} - t)^2\, w^{(3)}(t)\, dt. \tag{6.11}$$

Thus

$$Dw_n = \frac{k^2}{8}\,\ddot{w}_{n-1/2} + \frac{1}{4}\left[\int_{t_{n-1/2}}^{t_n} (t_n - t)^2\, w^{(3)}(t)\, dt - \int_{t_{n-1}}^{t_{n-1/2}} (t_{n-1} - t)^2\, w^{(3)}(t)\, dt\right]$$

and so

$$\begin{aligned} &Dw_n - Dw_{n+1} \\ &= \frac{k^2}{8}\,\left(\ddot{w}_{n-1/2} - \ddot{w}_{n+1/2}\right) \\ &\quad + \frac{1}{4}\left[\int_{t_{n-1/2}}^{t_n} (t_n - t)^2\, w^{(3)}(t)\, dt - \int_{t_{n-1}}^{t_{n-1/2}} (t_{n-1} - t)^2\, w^{(3)}(t)\, dt\right] \\ &\quad - \frac{1}{4}\left[\int_{t_{n+1/2}}^{t_{n+1}} (t_{n+1} - t)^2\, w^{(3)}(t)\, dt - \int_{t_n}^{t_{n+1/2}} (t_n - t)^2\, w^{(3)}(t)\, dt\right]. \end{aligned}$$

The estimate follows from the above inequality and (3.4). ∎

LEMMA 6.8. *Assume that* $w^{(3)} \in L^1(0,T;H)$; *then*

$$\|\delta w_n - \dot{w}_{n-1/2}\|_H \le \frac{k^2}{8}\,\|w^{(3)}\|_{L^1(t_{n-1},t_n;H)}.$$

Proof. Using (6.10) and (6.11), we have

$$\delta w_n - \dot{w}_{n-1/2} = \frac{1}{2k}\left[\int_{t_{n-1/2}}^{t_n} (t_n - t)^2\, w^{(3)}(t)\, dt + \int_{t_{n-1}}^{t_{n-1/2}} (t_{n-1} - t)^2\, w^{(3)}(t)\, dt\right].$$

Hence,

$$\|\delta w_n - \dot{w}_{n-1/2}\|_H \le \frac{k^2}{8}\,\|w^{(3)}\|_{L^1(t_{n-1},t_n;H)}. \qquad ∎$$

From the next theorem we see that the Crank–Nicolson-type scheme provides a more accurate approximation to the solution of problem **P** when the solution of the original problem has greater regularity.

THEOREM 6.9. *For the solution w of problem $\mathbf{P}$ and the solution w^{hk} of problem $\mathbf{P}_2^{hk}$, if $w \in W^{2,\infty}(0,T;H)$ and $w^{(3)} \in L^1(0,T;H)$, then the following inequality holds*

$$\max_{0\le n\le N} \|w_n - w_n^{hk}\|_H^2 \le c\,k^4 + c\,k \sum_{n=1}^{N} \inf_{z^h\in K^h} \|z^h - \dot w_{n-1/2}\|_H$$

for some constant c depending on $\|\dot l\|_{L^1(0,T;H^*)}$, $\|w\|_{W^{2,\infty}(0,T;H)}$, and $\|w^{(3)}\|_{L^1(0,T;H)}$ only.

Proof. Denote the error by $e_n = w_n - w_n^{hk}$, $n = 0,1,\ldots,N$. We will use c to denote a generic constant independent of n and k, but which may in general depend on $\|\dot l\|_{L^1(0,T;H^*)}$, $\|w\|_{W^{2,\infty}(0,T;H)}$, and $\|w^{(3)}\|_{L^1(0,T;H)}$. Consider the expression

$$D_n = \tfrac{1}{2} a\left((w_n + w_{n-1}) - (w_n^{hk} + w_{n-1}^{hk}), \delta w_n - \delta w_n^{hk}\right).$$

It is easy to verify that

$$D_n = \frac{1}{2k}\left(\|e_n\|_a^2 - \|e_{n-1}\|_a^2\right).$$

On the other hand, for any $z^h \in K^h$ we write

$$D_n = \tfrac{1}{2}a((e_n+e_{n-1}), \delta w_n - z^h) + \tfrac{1}{2}a((e_n+e_{n-1}), z^h - \delta w_n^{hk}).$$

Now

$$\begin{aligned}
&a(\tfrac{1}{2}(e_n+e_{n-1}), z^h - \delta w_n^{hk})\\
&= a(\tfrac{1}{2}(w_n+w_{n-1}), z^h - \delta w_n^{hk}) - a(\tfrac{1}{2}(w_n^{hk}+w_{n-1}^{hk}), z^h - \delta w_n^{hk})\\
&\le a(\tfrac{1}{2}(w_n+w_{n-1}), z^h - \delta w_n^{hk}) + j(z^h) - j(\delta w_n^{hk}) - \langle l_{n-1/2}, z^h - \delta w_n^{hk}\rangle.
\end{aligned}$$

From problem P at $t = t_{n-1/2}$ we have

$$a(w_{n-1/2}, \delta w_n^{hk} - \dot w_{n-1/2}) + j(\delta w_n^{hk}) - j(\dot w_{n-1/2}) - \langle l_{n-1/2}, \delta w_n^{hk} - \dot w_{n-1/2}\rangle \ge 0.$$

Hence

$$\begin{aligned}
&a((e_n+e_{n-1})/2, z^h - \delta w_n^{hk})\\
&\le a((w_n+w_{n-1})/2, z^h - \delta w_n^{hk}) + j(z^h) - j(\dot w_{n-1/2})\\
&\quad + a(w_{n-1/2}, \delta w_n^{hk} - \dot w_{n-1/2}) - \langle l_{n-1/2}, z^h - \dot w_{n-1/2}\rangle\\
&= a((w_n+w_{n-1})/2, z^h - \delta w_n^{hk}) + j(z^h) - j(\dot w_{n-1/2})\\
&\quad + a(w_{n-1/2}, \delta w_n^{hk} - \dot w_{n-1/2}) - a(w_{n-1/2}, z^h - \dot w_{n-1/2}) - \langle w_{n-1/2}^*, z^h - \dot w_{n-1/2}\rangle\\
&\le a(Dw_n, z^h - \delta w_n^{hk}) + j(z^h) - j(\dot w_{n-1/2}) + j(z^h - \dot w_{n-1/2})\\
&= a(Dw_n, \delta w_n - \delta w_n^{hk}) + a(Dw_n, z^h - \delta w_n) + j(z^h) - j(\dot w_{n-1/2}) + j(z^h - \dot w_{n-1/2}).
\end{aligned}$$

Thus, using the Lipschitz continuity of j and Lemma 6.6,

$$\begin{aligned}
&a((e_n+e_{n-1})/2, z^h - \delta w_n^{hk})\\
&\le a(Dw_n, \delta w_n - \delta w_n^{hk}) + c\,k^2\|z^h - \delta w_n\|_H + c\,\|z^h - \dot w_{n-1/2}\|_H.
\end{aligned} \tag{6.12}$$

So an upper bound for D_n is given by the expression

$$\begin{aligned}
&c\,(\|e_n\|_a + \|e_{n-1}\|_a)\,\|z^h - \delta w_n\|_H + a(Dw_n, \delta w_n - \delta w_n^{hk})\\
&+ c\,k^2\|z^h - \delta w_n\|_H + c\,\|z^h - \dot w_{n-1/2}\|_H.
\end{aligned}$$

To further simplify the presentation we introduce the notation

$$c_{n,h} = \inf_{z^h \in K^h} \|z^h - \dot{w}_{n-1/2}\|_H, \quad n = 1, 2, \ldots, N.$$

Now from Lemma 6.8,

$$\inf_{z^h \in K^h} \|z^h - \delta w_n\|_H \le c_{n,h} + \|\delta w_n - \dot{w}_{n-1/2}\|_H \le c_{n,h} + c\, k^2. \tag{6.13}$$

Combining the various estimates above, we have

$$\begin{aligned} D_n &= \frac{1}{2k} \left(\|e_n\|_a^2 - \|e_{n-1}\|_a^2\right) \\ &\le c\, (\|e_n\|_a + \|e_{n-1}\|_a)\, (c_{n,h} + c\, k^2) \\ &\qquad + \frac{1}{k}\, a(Dw_n, e_n - e_{n-1}) + c\, k^2 (c_{n,h} + k^2) + c\, c_{n,h}. \end{aligned}$$

Thus

$$\|e_n\|_a^2 - \|e_{n-1}\|_a^2 \le c\, k\, (M + k^2)\, (c_{n,h} + k^2) + c\, k\, c_{n,h} + a(Dw_n, e_n - e_{n-1}),$$

where

$$M = \max_{0 \le n \le N} \|e_n\|_a.$$

We then get

$$\begin{aligned} &\|e_n\|_a^2 \\ &\le c\, (M + k^2) \left(k \sum_{j=1}^n c_{j,h} + k^2 \right) + c\, k \sum_{j=1}^n c_{j,h} + \sum_{j=1}^n a(Dw_j, e_j - e_{j-1}) \\ &= c\, (M + k^2) \left(k \sum_{j=1}^n c_{j,h} + k^2 \right) + c\, k \sum_{j=1}^n c_{j,h} + \sum_{j=1}^{n-1} a(Dw_j - Dw_{j+1}, e_j) + a(Dw_n, e_n) \\ &\le c\, (M + k^2) \left(k \sum_{j=1}^n c_{j,h} + k^2 \right) + c\, k \sum_{j=1}^n c_{j,h} + c \sum_{j=1}^{n-1} \|Dw_j - Dw_{j+1}\|_H M \\ &\quad + c\, \|Dw_n\|_H M \\ &= c\, M \left(k \sum_{j=1}^n c_{j,h} + k^2 + \sum_{j=1}^{n-1} \|Dw_j - Dw_{j+1}\|_H + \|Dw_n\|_H \right) + c\, k^4 + c\, k \sum_{j=1}^n c_{j,h}. \end{aligned}$$

Hence, using Lemmas 6.6 and 6.7, we find that M satisfies the relation

$$M^2 \le c\, M \left(k \sum_{n=1}^N c_{n,h} + c\, k^2 \right) + c\, k^4 + c\, k \sum_{n=1}^N c_{n,h}. \tag{6.14}$$

It is easy to verify that if $a, b, x \ge 0$ and $x^2 \le a\, x + b$, then $x^2 \le a^2 + 2\, b$. Thus, the required error estimate follows from (6.14). ∎

7. Applications to the elastoplastic problem. The problem with combined kinematic-isotropic hardening. First we apply the results of the previous three sections to the quasi-static elastoplastic problem with combined linear kinematic-isotropic hardening considered in section 2. The variational problem is problem EP and is given by (3.8).

We apply Theorem 4.3 to obtain an existence and uniqueness result for problem EP. We identify H in Theorem 4.3 with Z and define $K = \{\boldsymbol{z} = (\boldsymbol{v}, \boldsymbol{q}, \mu) \in Z : |\boldsymbol{q}| \leq \mu \text{ a.e. in } \Omega\}$. We will show that the bilinear form $a(\cdot, \cdot)$ is Z-elliptic; the remaining assumptions of Theorem 4.3 are obviously true. In particular, j inherits the properties that Theorem 4.3 requires of it from the corresponding properties of the dissipation function D.

LEMMA 7.1. *The bilinear form $a : Z \times Z \to \mathbb{R}$ is Z-elliptic; that is, there exists $\alpha > 0$ such that*

$$a(\boldsymbol{z}, \boldsymbol{z}) \geq \alpha \, \|\boldsymbol{z}\|_Z^2 \quad \forall \, \boldsymbol{z} \in Z.$$

Proof. For any $\boldsymbol{z} = (\boldsymbol{v}, \boldsymbol{q}, \mu) \in Z$ we have, using the pointwise stability assumption on $\boldsymbol{C}$,

$$\begin{aligned} a(\boldsymbol{z}, \boldsymbol{z}) &\geq c_0 \int_\Omega |\boldsymbol{\epsilon}(\boldsymbol{v}) - \boldsymbol{q}|^2 dx + \bar{k}_1 \int_\Omega |\boldsymbol{q}|^2 dx + \bar{k}_2 \int_\Omega |\mu|^2 dx \\ &\geq c_0 \theta \int_\Omega |\boldsymbol{\epsilon}(\boldsymbol{v})|^2 dx + \left(\bar{k}_1 - \frac{1}{1 - \theta} \right) \int_\Omega |\boldsymbol{q}|^2 dx + \bar{k}_2 \int_\Omega |\mu|^2 dx \end{aligned}$$

for any $\theta \in (0, 1)$. The result then follows by choosing $\theta = \bar{k}_1/(2c_0 + \bar{k}_1)$ and using Korn's inequality (see, for example, [6]). ∎

Applying Theorems 4.3 and 4.4 to problem EP, we thus have the following.

THEOREM 7.2. *Under the assumptions made on the data in section 2, the quasi-static elastoplasticity problem EP has a unique solution $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{p}, \gamma) \in W^{1,2}(0, T; Z)$. Furthermore, if $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are the solutions corresponding to $\boldsymbol{l}_1, \boldsymbol{l}_2 \in W^{1,2}(0, T; Z^*)$ with $\boldsymbol{l}_1(0) = \boldsymbol{l}_2(0) = \boldsymbol{0}$, then*

$$\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{L^\infty(0,T;Z)} \leq c \left(\|\boldsymbol{l}_1 - \boldsymbol{l}_2\|_{L^\infty(0,T;Z^*)} + \|\dot{\boldsymbol{l}}_1 - \dot{\boldsymbol{l}}_2\|_{L^1(0,T;Z^*)} \right).$$

Let $Z^h = V^h \times Q_0^h \times M^h$ be a finite-dimensional subspace of Z. Let $K^h = Z^h \cap K = V^h \times K_0^h$, where, $K_0^h = \{(\boldsymbol{q}^h, \mu^h) \in Q_0^h \times M^h : |\boldsymbol{q}^h| \leq \mu^h \text{ in } \Omega\}$. Then in the semidiscrete internal approximation of the problem $\mathbf{EP}_1$, we find $\boldsymbol{w}^h = (\boldsymbol{u}^h, \boldsymbol{p}^h, \gamma^h) : [0, T] \to Z^h$, $\boldsymbol{w}^h(0) = \boldsymbol{0}$ such that for almost all $t \in (0, T)$, $\dot{\boldsymbol{w}}^h(t) \in K^h$, and

$$a(\boldsymbol{w}^h(t), \boldsymbol{z}^h - \dot{\boldsymbol{w}}^h(t)) + j(\boldsymbol{z}^h) - j(\dot{\boldsymbol{w}}^h(t)) \geq \langle \boldsymbol{l}_n, \boldsymbol{z}^h - \dot{\boldsymbol{w}}^h(t) \rangle \;\forall \, \boldsymbol{z}^h \in K^h. \tag{7.1}$$

From the discussion in section 5, we know that the discrete problem has a unique solution $\boldsymbol{w}^h$. Since $j(\boldsymbol{z})$ depends on $\boldsymbol{q}$ only, a careful examination of the argument in section 3 shows we may modify the error estimate (5.7) to read

$$\begin{aligned} \|\boldsymbol{w} - \boldsymbol{w}^h\|_{L^\infty(0,T;Z)} \leq c \Bigg[&\inf_{\boldsymbol{v}^h \in L^2(0,T;V^h)} \|\dot{\boldsymbol{u}} - \boldsymbol{v}^h\|_{L^2(0,T;V)} \\ &+ \inf_{(\boldsymbol{q}^h, \mu^h) \in L^2(0,T;K_0^h)} \left(\|\dot{\boldsymbol{p}} - \boldsymbol{q}^h\|_{L^2(0,T;Q)}^{1/2} + \|\dot{\gamma} - \mu^h\|_{L^2(0,T;M)} \right) \Bigg]. \end{aligned} \tag{7.2}$$

The inequality (7.2) is the basis for various error estimates. For example, suppose that we use linear elements for V^h and piecewise constants for both Q_0^h and M^h. Assume that $\dot{\boldsymbol{u}} \in L^2(0,T;(H^2(\Omega))^3)$, $\dot{\boldsymbol{p}} \in L^2(0,T;(H^1(\Omega))^{3\times 3})$, and $\dot{\gamma} \in L^2(0,T;H^1(\Omega))$. Then from standard interpolation error estimates for finite elements (cf. [2], [4]), we have

$$\inf_{\boldsymbol{v}^h \in L^2(0,T;V^h)} \|\dot{\boldsymbol{u}} - \boldsymbol{v}^h\|_{L^2(0,T;V)} \le c\,h.$$

Let $\boldsymbol{q}^h = \Pi_h \dot{\boldsymbol{p}}$ be the orthogonal projection of $\dot{\boldsymbol{p}}$ onto Q_0^h with respect to the inner product of Q. We observe that on each element, $\Pi_h \dot{\boldsymbol{p}}$ is the average value of $\dot{\boldsymbol{p}}$ on the element. Similarly we take $\mu^h = \Pi_h \dot{\gamma}$ to be the orthogonal projection of $\dot{\gamma}$ onto M^h with respect to the inner product of M. Since $\dot{\boldsymbol{w}} \in K$, we have $(\Pi_h \dot{\boldsymbol{p}}, \Pi_h \dot{\gamma}) \in K_0^h$. Thus, from (7.2) and the inequalities

$$\|\dot{\boldsymbol{p}} - \Pi_h \dot{\boldsymbol{p}}\|_{L^2(0,T;Q)} \le c\,h,$$
$$\|\dot{\gamma} - \Pi_h \dot{\gamma}\|_{L^2(0,T;M)} \le c\,h,$$

we get the error estimate

$$\|\boldsymbol{w} - \boldsymbol{w}^h\|_{L^\infty(0,T;Z)} \le c\,h^{1/2}. \tag{7.3}$$

If $\dot{\boldsymbol{p}} \in L^2(0,T;(H^2(\Omega))^{3\times 3})$ and $\dot{\gamma} \in L^2(0,T;H^2(\Omega))$, we can use either discontinuous or continuous piecewise linear functions for both Q_0^h and M^h. By choosing $\Pi_h \dot{\boldsymbol{p}}$ and $\Pi_h \dot{\gamma}$ to be the piecewise linear interpolations of $\dot{\boldsymbol{p}}$ and $\dot{\gamma}$, we have $(\Pi_h \dot{\boldsymbol{p}}, \Pi_h \dot{\gamma}) \in K_0^h$ and

$$\|\dot{\boldsymbol{p}} - \Pi_h \dot{\boldsymbol{p}}\|_{L^2(0,T;Q)} \le c\,h^2,$$
$$\|\dot{\gamma} - \Pi_h \dot{\gamma}\|_{L^2(0,T;M)} \le c\,h^2.$$

Then the error estimate for this case becomes

$$\|\boldsymbol{w} - \boldsymbol{w}^h\|_{L^\infty(0,T;Z)} \le c\,h. \tag{7.4}$$

Now let us consider fully discrete approximations. As in section 6, we divide the time interval $[0,T]$ by evenly spaced nodes $t_n = nk$, $n = 0,1,\dots,N$, with $k = T/N$ the stepsize.

In the backward Euler approximation of the problem EP, we compute $\boldsymbol{w}^h = (\boldsymbol{u}^h, \boldsymbol{p}^h, \gamma^h) : [0,T] \to Z^h$, $\boldsymbol{w}^h(0) = \boldsymbol{0}$ such that for $n = 1,2,\dots,N$, $\delta \boldsymbol{w}_n^{hk} \in K^h$ and

$$a(\boldsymbol{w}_n^{hk}, \boldsymbol{z}^h - \delta \boldsymbol{w}_n^{hk}) + j(\boldsymbol{z}^h) - j(\delta \boldsymbol{w}_n^{hk}) - \langle \boldsymbol{l}_n, \boldsymbol{z}^h - \delta \boldsymbol{w}_n^{hk} \rangle \ge 0 \quad \forall\, \boldsymbol{z}^h \in K^h. \tag{7.5}$$

We have a unique solution for the backward Euler scheme. By Corollary 6.2, again noticing that $j(\boldsymbol{z})$ depends only on $\boldsymbol{q}$, we find that if $\ddot{\boldsymbol{w}} \in L^2(0,T;Z)$, then

$$\begin{aligned} \max_{0\le n\le N} \|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_Z^2 \le c\,k \sum_{n=1}^{N} \Bigg[&\inf_{\boldsymbol{v}^h \in V^h} \|\dot{\boldsymbol{u}} - \boldsymbol{v}^h\|_V^2 \\ &+ \inf_{(\boldsymbol{q}^h,\mu^h)\in K_0^h} \left(\|\dot{\boldsymbol{p}} - \boldsymbol{q}^h\|_Q + \|\dot{\gamma} - \mu^h\|_M^2 \right) \Bigg] + c\,k^2. \end{aligned} \tag{7.6}$$

Assume $\dot{\boldsymbol{u}} \in L^2(0,T;(H^2(\Omega))^3)$, $\dot{\boldsymbol{p}} \in L^2(0,T;(H^1(\Omega))^{3\times 3})$, and $\dot{\gamma} \in L^2(0,T;H^1(\Omega))$. If we use linear elements for V^h, piecewise constants for both Q_0^h and M^h, then similarly as above, we have $(\Pi_h \dot{\boldsymbol{p}}, \Pi_h \dot{\gamma}) \in K_0^h$ and

$$\inf_{\boldsymbol{v}^h \in L^2(0,T;V^h)} \|\dot{\boldsymbol{u}} - \boldsymbol{v}^h\|_{L^2(0,T;V)} \le c\, h,$$
$$\|\dot{\boldsymbol{p}} - \Pi_h \dot{\boldsymbol{p}}\|_{L^2(0,T;Q)} \le c\, h,$$
$$\|\dot{\gamma} - \Pi_h \dot{\gamma}\|_{L^2(0,T;M)} \le c\, h.$$

Therefore we have the error estimate

$$\max_{0\le n\le N} \|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_Z \le c\,(h^{1/2} + k). \tag{7.7}$$

If $\dot{\boldsymbol{p}} \in L^2(0,T;(H^2(\Omega))^{3\times 3})$, $\dot{\gamma} \in L^2(0,T;H^2(\Omega))$, and we use either discontinuous or continuous piecewise linear functions for both Q_0^h and M^h, then the error estimate for this case becomes

$$\max_{0\le n\le N} \|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_Z \le c\,(h + k). \tag{7.8}$$

Similarly, the Crank–Nicolson scheme for the problem has a unique solution, and for the two different choices of the finite element spaces, under suitable smoothness assumptions on the solution of the original problems, we have the error estimates

$$\max_{0\le n\le N} \|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_Z \le c\,(h^{1/2} + k^2), \tag{7.9}$$

$$\max_{0\le n\le N} \|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_Z \le c\,(h + k^2), \tag{7.10}$$

to replace (7.7) and (7.8), respectively.

Finally, we make a remark on implementation of fully discrete schemes. As an example, we consider the backward Euler scheme (7.5), which is equivalent to

$$\begin{aligned} \delta\boldsymbol{w}_n^{hk} \in K^h, \quad & k\, a(\delta\boldsymbol{w}_n^{hk}, \boldsymbol{z}^h - \delta\boldsymbol{w}_n^{hk}) + j(\boldsymbol{z}^h) - j(\delta\boldsymbol{w}_n^{hk}) \\ & \ge \langle \boldsymbol{l}_n, \boldsymbol{z}^h - \delta\boldsymbol{w}_n^{hk}\rangle - a(\boldsymbol{w}_{n-1}^{hk}, \boldsymbol{z}^h - \delta\boldsymbol{w}_n^{hk}) \quad \forall\, \boldsymbol{z}^h \in K^h. \end{aligned} \tag{7.11}$$

Clearly $\delta\boldsymbol{w}_n^{hk} \in K^h$ is the minimimizer of the problem

$$\inf\{J(\boldsymbol{z}^h) : \boldsymbol{z}^h \in K^h\}, \tag{7.12}$$

where

$$J(\boldsymbol{z}^h) = \frac{k}{2}\, a(\boldsymbol{z}^h, \boldsymbol{z}^h) + j(\boldsymbol{z}^h) - \langle \boldsymbol{l}_n, \boldsymbol{z}^h\rangle + a(\boldsymbol{w}_{n-1}^{hk}, \boldsymbol{z}^h). \tag{7.13}$$

Once we have the solution of the problem (7.12), called $\delta\boldsymbol{w}_n^{hk}$, we compute $\boldsymbol{w}_n^{hk}$ through the formula

$$\boldsymbol{w}_n^{hk} = \boldsymbol{w}_{n-1}^{hk} + k\,\delta\boldsymbol{w}_n^{hk}.$$

We observe that $J(\boldsymbol{z})$ is a strictly convex function of $\boldsymbol{z}$. Algorithms based on direct minimization of J have been treated, for example, in [32]. An alternative approach is to observe that J is not differentiable, due to the nondifferentiability of the term j, and to use a regularization technique to overcome this difficulty (cf. [8], [9], [10], [20], [30]). More precisely, let j_ε be a sequence of differentiable functions approximating j when $\varepsilon \to 0$. For example, we may take

$$j_\varepsilon(\boldsymbol{z}) = \int_\Omega c_0 \sqrt{|\boldsymbol{q}(x)|^2 + \varepsilon^2}\, dx.$$

Then, instead of the problem (7.12), we solve a sequence of approximate differentiable optimization problems:

$$\inf\{J_\varepsilon(\boldsymbol{z}^h) : \boldsymbol{z}^h \in K^h\}, \tag{7.14}$$

where

$$J_\varepsilon(\boldsymbol{z}^h) = \frac{k}{2}\, a(\boldsymbol{z}^h, \boldsymbol{z}^h) + j_\varepsilon(\boldsymbol{z}^h) - \langle \boldsymbol{l}_n, \boldsymbol{z}^h \rangle + a(\boldsymbol{w}^{hk}_{n-1}, \boldsymbol{z}^h). \tag{7.15}$$

For the finite element subspaces discussed earlier, the requirement $\boldsymbol{z}^h \in K^h$ is equivalent to a set of linear inequalities. Thus, an element in K^h satisfying Kuhn–Tucker conditions is the solution of the problem (7.14) (cf. [22]). For details on the regularization technique, as well as on a posteriori error estimates for solutions of the regularized problems, see section 8.

The problem with kinematic hardening. The quasi-static problem of elastoplasticity with kinematic hardening is a special case of the more general problem with combined kinematic-isotropic hardening. Besides its importance in certain applications, the problem with kinematic hardening allows a simpler treatment. The simplification comes from the fact that in this case, the functional j is Lipschitz continuous on the whole space. Indeed, for the case in which plastic behavior is governed by the von Mises condition, the conjugate force is $\boldsymbol{\chi} = \boldsymbol{\sigma} - k_1 \boldsymbol{p}$, instead of $(\boldsymbol{\chi}, g)$ as defined in (2.5). The region of admissible conjugate forces is $\mathcal{K} = \{\boldsymbol{\chi} : F(\boldsymbol{\chi}) \le c_0\}$, and the dissipation function is

$$D(\boldsymbol{q}) = c_0 |\boldsymbol{q}|, \quad \boldsymbol{q} \in M^3.$$

We assume the remaining ingredients of the problem setting are the same as in section 2. Then the variational problem is, instead of (3.8), to find $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{p}) : [0, T] \to Z$ with $\boldsymbol{w}(0) = \boldsymbol{0}$, such that for almost all $t \in (0, T)$,

$$a(\boldsymbol{w}(t), \boldsymbol{z} - \dot{\boldsymbol{w}}(t)) + j(\boldsymbol{z}) - j(\dot{\boldsymbol{w}}(t)) - \langle \boldsymbol{l}, \boldsymbol{z} - \dot{\boldsymbol{w}}(t) \rangle \ge 0 \quad \forall\, \boldsymbol{z} = (\boldsymbol{v}, \boldsymbol{q}) \in Z,$$

where $Z = V \times Q_0$ and

$$a(\boldsymbol{w}, \boldsymbol{z}) = \int_\Omega [\boldsymbol{C}(\boldsymbol{\epsilon}(\boldsymbol{u}) - \boldsymbol{p}) \cdot (\boldsymbol{\epsilon}(\boldsymbol{v}) - \boldsymbol{q}) + k_1 \boldsymbol{p} \cdot \boldsymbol{q}]\; dx \tag{7.16}$$

and

$$j(\boldsymbol{z}) = \int_\Omega c_0\, |\boldsymbol{q}(x)|\, \mathrm{d}x. \tag{7.17}$$

From Lemma 7.1 (with $k_2 = 0$) it is seen that a is Z-elliptic. Thus we have the following.

THEOREM 7.3. *Under the assumptions made in section* 2*, the quasi-static elastoplasticity problem* EP *with kinematic hardening has a unique solution* $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{p}) \in W^{1,2}(0,T;Z)$*. Furthermore, if* $\boldsymbol{w}_1$ *and* $\boldsymbol{w}_2$ *are the solutions corresponding to* $\boldsymbol{l}_1, \boldsymbol{l}_2 \in W^{1,2}(0,T;Z^*)$*, then*

$$\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{L^\infty(0,T;Z)} \le c \left(\|\boldsymbol{l}_1 - \boldsymbol{l}_2\|_{L^\infty(0,T;Z^*)} + \|\dot{\boldsymbol{l}}_1 - \dot{\boldsymbol{l}}_2\|_{L^1(0,T;Z^*)} \right).$$

Now we consider discrete approximations to the solution $\boldsymbol{w}$ of this problem. Let V^h and Q_0^h be finite element subspaces of V and Q_0 and set $Z^h = V^h \times Q_0^h$. Then a semidiscrete approximation of the problem is to find $\boldsymbol{w}^h = (\boldsymbol{u}^h, \boldsymbol{p}^h) \in Z^h$, $\boldsymbol{w}^h(0) = \boldsymbol{0}$, such that

$$\begin{aligned} &a(\boldsymbol{w}^h(t), \boldsymbol{z}^h - \dot{\boldsymbol{w}}^h(t)) + j(\boldsymbol{z}^h) - j(\dot{\boldsymbol{w}}^h(t)) \ge \langle \boldsymbol{l}(t), \boldsymbol{z}^h - \dot{\boldsymbol{w}}^h(t)\rangle \\ &\qquad\qquad \forall\, \boldsymbol{z}^h = (\boldsymbol{v}^h, \boldsymbol{q}^h) \in Z^h. \end{aligned} \tag{7.18}$$

The semidiscrete approximation problem has a unique solution $\boldsymbol{w}^h(t)$, $t \in [0,T]$. Since the functional $j(\boldsymbol{z})$ depends on the second component $\boldsymbol{q}$ of $\boldsymbol{z}$ only, the term $c\,\|z^h - \dot{w}(t)\|_H$ on the right-hand side of (5.5) may be replaced by $c\,\|\boldsymbol{q}^h - \dot{\boldsymbol{p}}(t)\|_Q$. Thus, the error estimate (5.7) becomes, for this case,

$$\begin{aligned} &\sup_{0\le t\le T} \|\boldsymbol{w}(t) - \boldsymbol{w}^h(t)\|_Z^2 \\ &\le c \left\{ \inf_{\boldsymbol{v}^h \in L^2(0,T;V^h)} \|\dot{\boldsymbol{u}} - \boldsymbol{v}^h\|^2_{L^2(0,T;V)} + \inf_{\boldsymbol{q}^h \in L^2(0,T;Q_0^h)} \|\dot{\boldsymbol{p}} - \boldsymbol{q}^h\|_{L^1(0,T;Q)} \right\}. \end{aligned} \tag{7.19}$$

Now we consider fully discrete approximations of the problem. As in section 6, we divide $[0,T]$ into N equal parts and use $k = T/N$ for the stepsize. The backward Euler method amounts to finding $\boldsymbol{w}^{hk} = \{\boldsymbol{w}_n^{hk}\}_{n=0}^N$, where $\boldsymbol{w}_n^{hk} = (\boldsymbol{u}_n^{hk}, \boldsymbol{p}_n^{hk}) \in Z^h$, $0 \le n \le N$, $\boldsymbol{w}_0^{hk} = \boldsymbol{0}$, such that for $n = 1, 2, \ldots, N$,

$$\begin{aligned} &a(\boldsymbol{w}_n^{hk}, \boldsymbol{z}^h - \delta\boldsymbol{w}_n^{hk}) + j(\boldsymbol{z}^h) - j(\delta\boldsymbol{w}_n^{hk}) \ge \langle \boldsymbol{l}_n, \boldsymbol{z}^h - \delta\boldsymbol{w}_n^{hk}\rangle \\ &\qquad\qquad \forall\, \boldsymbol{z}^h = (\boldsymbol{v}^h, \boldsymbol{q}^h) \in Z^h. \end{aligned} \tag{7.20}$$

The discrete problem has a unique solution. Once again we observe that the term $\|z^h - \dot{w}_n\|_H$ on the right-hand side of the inequality (6.4) may be replaced by $\|\boldsymbol{q}^h - \dot{\boldsymbol{p}}_n\|_Q$. Therefore, the error estimate from Theorem 6.1 and Corollary 6.2 for the case of the problem (7.18) becomes

$$\begin{aligned} &\max_{0\le n\le N} \|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_Z^2 \\ &\le c\,k \sum_{n=1}^N \left(\inf_{\boldsymbol{q}^h \in Q_0^h} \|\boldsymbol{q}^h - \dot{\boldsymbol{p}}_n\|_Q + \inf_{\boldsymbol{v}^h \in V^h} \|\boldsymbol{v}^h - \dot{\boldsymbol{u}}_n\|_V^2 \right) + c\,k^2 \|\ddot{\boldsymbol{w}}\|^2_{L^2(0,T;Z)}. \end{aligned} \tag{7.21}$$

For the Crank–Nicolson scheme, we have to compute $\boldsymbol{w}^{hk} = \{\boldsymbol{w}_n^{hk}\}_{n=0}^N$, where $\boldsymbol{w}_n^{hk} = (\boldsymbol{u}_n^{hk}, \boldsymbol{p}_n^{hk}) \in Z^h$, $0 \le n \le N$, $\boldsymbol{w}_0^{hk} = \boldsymbol{0}$, such that for $n = 1, 2, \ldots, N$,

$$\begin{aligned} &a((\boldsymbol{w}_n^{hk} + \boldsymbol{w}_{n-1}^{hk})/2, \boldsymbol{z}^h - \delta\boldsymbol{w}_n^{hk}) + j(\boldsymbol{z}^h) - j(\delta\boldsymbol{w}_n^{hk}) \ge \langle \boldsymbol{l}_{n-1/2}, \boldsymbol{z}^h - \delta\boldsymbol{w}_n^{hk}\rangle \\ &\qquad\qquad \forall\, \boldsymbol{z}^h = (\boldsymbol{v}^h, \boldsymbol{q}^h) \in Z^h. \end{aligned} \tag{7.22}$$

The discrete problem has a unique solution. Assuming that $\boldsymbol{w} \in W^{2,\infty}(0,T;Z)$ and $\boldsymbol{w}^{(3)} \in L^1(0,T;Z)$, we have the error estimate

$$\begin{aligned} &\max_{0\le n\le N} \|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_Z^2 \\ &\le c\,k \sum_{n=1}^{N} \left(\inf_{\boldsymbol{q}^h\in Q_0^h} \|\boldsymbol{q}^h - \dot{\boldsymbol{p}}_{n-1/2}\|_Q + \inf_{\boldsymbol{v}^h\in V^h} \|\boldsymbol{v}^h - \dot{\boldsymbol{u}}_{n-1/2}\|_V^2 \right) + c\,k^4. \end{aligned} \tag{7.23}$$

The inequalities (7.19), (7.21), and (7.23) are the basis for various convergence order estimates, which can be obtained as in the combined isotropic-kinematic case.

8. Regularization technique and a posteriori error estimates. In this section, we take the backward Euler method (7.20) for the problem with kinematic hardening as an example. We will apply the regularization technique to solve (7.20). Besides an a priori error estimate showing the convergence of the regularization sequence, we will also develop an a posteriori error estimate giving a computable error bound once the solution of a regularized problem is computed. The discussion on the model problem (7.20) can be extended without difficulty to the regularization technique with other discrete schemes and the schemes for the problem with combined kinematic-isotropic hardening.

First we notice that, by the positive homogeneity of j, (7.20) can be equivalently written as

$$\begin{aligned} a(\boldsymbol{w}_n^{hk}, \boldsymbol{z}^h - \Delta\boldsymbol{w}_n^{hk}) + j(\boldsymbol{z}^h) - j(\Delta\boldsymbol{w}_n^{hk}) \ge \langle \boldsymbol{l}_n, \boldsymbol{z}^h - \Delta\boldsymbol{w}_n^{hk}\rangle \\ \forall\, \boldsymbol{z}^h = (\boldsymbol{v}^h, \boldsymbol{q}^h) \in Z^h, \end{aligned} \tag{8.1}$$

or

$$\begin{aligned} a(\boldsymbol{w}_n^{hk}, \boldsymbol{z}^h - \boldsymbol{w}_n^{hk}) + j(\boldsymbol{z}^h - \boldsymbol{w}_{n-1}^{hk}) - j(\boldsymbol{w}_n^{hk} - \boldsymbol{w}_{n-1}^{hk}) \ge \langle \boldsymbol{l}_n, \boldsymbol{z}^h - \boldsymbol{w}_n^{hk}\rangle \\ \forall\, \boldsymbol{z}^h \in Z^h. \end{aligned} \tag{8.2}$$

A difficulty in solving (8.2) is caused by the nondifferentiability of the functional j (cf. (7.17)).

The idea of the regularization technique, which has been widely used in applications (cf. [8, 9, 20, 30]), is to approximate j by a family of differentiable functionals j_ε, $\varepsilon \in (0,1)$ and to solve a sequence of approximation problems for (8.2) with j replaced by j_ε. Thus, let us introduce

$$j_\varepsilon(\boldsymbol{z}) = \int_\Omega \phi_\varepsilon(\boldsymbol{q}(x))\,dx, \tag{8.3}$$

where ϕ_ε are differentiable functions approximating ϕ. Specifically, the conditions satisfied by ϕ_ε are as follows:

$$\phi_\varepsilon(\boldsymbol{q}) \text{ is convex and continuously differentiable,} \quad |\phi_\varepsilon(\boldsymbol{q}) - \phi(\boldsymbol{q})| \le c\,\varepsilon \ \forall\, \boldsymbol{q}. \tag{8.4}$$

There are many regularization functions satisfying (8.4). For the von Mises condition we will make the popular choice

$$\phi_\varepsilon(\boldsymbol{q}) = c_0\sqrt{|\boldsymbol{q}|^2 + \varepsilon^2}. \tag{8.5}$$

The regularization technique corresponding to a given regularization function for (8.2) is to compute $\boldsymbol{w}_{\varepsilon,n}^{hk} \in Z^h$, such that

$$(8.6)\quad \begin{aligned} &a(\boldsymbol{w}_{\varepsilon,n}^{hk}, \boldsymbol{z}^h - \boldsymbol{w}_{\varepsilon,n}^{hk}) + j_\varepsilon(\boldsymbol{z}^h - \boldsymbol{w}_{n-1}^{hk}) - j_\varepsilon(\boldsymbol{w}_{\varepsilon,n}^{hk} - \boldsymbol{w}_{n-1}^{hk}) \geq \langle \boldsymbol{l}_n, \boldsymbol{z}^h - \boldsymbol{w}_{\varepsilon,n}^{hk} \rangle \\ &\qquad \forall\, \boldsymbol{z}^h \in Z^h. \end{aligned}$$

Since j_ε is differentiable, (8.6) is equivalent to

$$(8.7)\quad a(\boldsymbol{w}_{\varepsilon,n}^{hk}, \boldsymbol{z}^h) + \langle j_\varepsilon'(\boldsymbol{w}_{\varepsilon,n}^{hk} - \boldsymbol{w}_{n-1}^{hk}), \boldsymbol{z}^h \rangle = \langle \boldsymbol{l}_n, \boldsymbol{z}^h \rangle \quad \forall\, \boldsymbol{z}^h = (\boldsymbol{v}^h, \boldsymbol{q}^h) \in Z^h.$$

As usual, we have an a priori error estimate.

THEOREM 8.1. *The regularization method converges,* $\boldsymbol{w}_{\varepsilon,n}^{hk} \to \boldsymbol{w}_n^{hk}$ *in* Z *as* $\varepsilon \to 0$, *and*

$$\|\boldsymbol{w}_{\varepsilon,n}^{hk} - \boldsymbol{w}_n^{hk}\|_Z \leq c\,\sqrt{\varepsilon}.$$

Proof. We take $\boldsymbol{z}^h = \boldsymbol{w}_{\varepsilon,n}^{hk}$ in (8.2), $\boldsymbol{z}^h = \boldsymbol{w}_n^{hk}$ in (8.6), add the two inequalities, and use the Z-ellipticity of a and (8.4) to obtain

$$\begin{aligned} &\alpha\, \|\boldsymbol{w}_{\varepsilon,n}^{hk} - \boldsymbol{w}_n^{hk}\|_Z^2 \\ &\leq a(\boldsymbol{w}_{\varepsilon,n}^{hk} - \boldsymbol{w}_n^{hk}, \boldsymbol{w}_{\varepsilon,n}^{hk} - \boldsymbol{w}_n^{hk}) \\ &\leq j(\boldsymbol{w}_{\varepsilon,n}^{hk} - \boldsymbol{w}_{n-1}^{hk}) - j_\varepsilon(\boldsymbol{w}_{\varepsilon,n}^{hk} - \boldsymbol{w}_{n-1}^{hk}) + j_\varepsilon(\boldsymbol{w}_n^{hk} - \boldsymbol{w}_{n-1}^{hk}) - j(\boldsymbol{w}_n^{hk} - \boldsymbol{w}_{n-1}^{hk}) \\ &\leq c\,\varepsilon. \qquad \square \end{aligned}$$

The main part of the section is devoted to a posteriori error estimations for the regularization method. To do this, we need a result from convex analysis (cf. [7], [34]). A posteriori error estimates for the regularization technique for other application problems can be found in [10], [11], and [15].

Let Z, P be two normed spaces and Z^*, P^* their dual spaces. Assume there exists a linear continuous operator $\Lambda \in \mathcal{L}(Z, P)$, with transpose $\Lambda^* \in \mathcal{L}(Z^*, P^*)$. Let J be a function mapping $Z \times P$ into $\overline{R}$, the extended real line. Define the conjugate function of J by

$$J^*(z^*, p^*) = \sup_{z \in Z, p \in P} \left[\langle z, z^* \rangle + \langle p, p^* \rangle - J(z, p)\right].$$

THEOREM 8.2. *Assume that*

(1) *Z is a reflexive Banach space, P a normed space;*

(2) *$J : Z \times P \to \overline{R}$ is a proper, l.s.c., strictly convex function;*

(3) *$\exists\, z_0 \in Z$, such that $J(z_0, \Lambda z_0) < \infty$ and $p \mapsto J(z_0, p)$ is continuous at Λz_0;*

(4) *$J(z, \Lambda z) \to +\infty$, as $\|z\| \to \infty, z \in V$.*

Then the problem

$$(8.8)\qquad \inf_{z \in Z} J(z, \Lambda z)$$

has a unique solution $y \in Z$, and

$$(8.9)\qquad -J(y, \Lambda y) \leq J^*(\Lambda^* p^*, -p^*) \quad \forall\, p^* \in P^*.$$

For definiteness, assume we are using piecewise linear elements for V^h and piecewise constants for Q^h and $Q_0^h = \{\boldsymbol{q}^h \in Q^h : \operatorname{tr} \boldsymbol{q}^h = 0\}$. Let us apply Theorem 8.2 to the following problem setting:

$$
\begin{aligned}
&Z = Z^h, \text{ with the norm of } V \times Q,\\
&P = Q^h, \text{ with the norm of } Q,\\
&\Lambda \boldsymbol{z}^h = \boldsymbol{\epsilon}(\boldsymbol{v}^h),\\
&J(\boldsymbol{z}^h, \boldsymbol{s}) = \int_\Omega \left[\frac{1}{2} \boldsymbol{C}\,(\boldsymbol{s} - \boldsymbol{q}^h) \cdot (\boldsymbol{s} - \boldsymbol{q}^h) + \frac{k_1}{2}\,|\boldsymbol{q}^h|^2 + c_0 |\boldsymbol{q}^h - \boldsymbol{p}_{n-1}^{hk}| - \boldsymbol{f}_n \cdot \boldsymbol{v}^h \right] dx.
\end{aligned}
$$

We identify Q^{h*} with Q^h and use $\boldsymbol{s}^*$ to denote a generic element in Q^{h*}. It is easily seen that the discrete problem (8.2) is equivalent to the minimization problem (8.8) with the above identification. After a lengthy computation, from the definition of the conjugate function we find that

$$
J^*(\Lambda^* \boldsymbol{s}^*, -\boldsymbol{s}^*)
= \begin{cases} \displaystyle\int_\Omega \left[\frac{1}{2k_1} \left(|k_1 \boldsymbol{p}_{n-1}^{hk} + \boldsymbol{s}^{*D}| - c_0 \right)_+^2 - \frac{k_1}{2}\,|\boldsymbol{p}_{n-1}^{hk}|^2 - \boldsymbol{p}_{n-1}^{hk} \cdot \boldsymbol{s}^* + \frac{1}{2} \boldsymbol{C}^{-1} \boldsymbol{s}^* \cdot \boldsymbol{s}^* \right] dx \\ \qquad \text{if } \displaystyle\int_\Omega \left[\boldsymbol{\epsilon}(\boldsymbol{v}^h) \cdot \boldsymbol{s}^* + \boldsymbol{f}_n \cdot \boldsymbol{v}^h \right] dx = 0 \ \forall\, \boldsymbol{v}^h \in V^h, \\ +\infty \quad \text{otherwise,} \end{cases}
\tag{8.10}
$$

where, $\boldsymbol{s}^{*D} = \boldsymbol{s}^* - (1/3) \operatorname{tr}(\boldsymbol{s}^*)\, I$, and $x_+ = \max\{x, 0\}$.

Now let us consider the difference

$$
D = J(\boldsymbol{w}_{\varepsilon,n}^{hk}, \Lambda \boldsymbol{w}_{\varepsilon,n}^{hk}) - J(\boldsymbol{w}_n^{hk}, \Lambda \boldsymbol{w}_n^{hk}).
$$

First we derive a lower bound for D. From (8.2) with $\boldsymbol{z}^h = \boldsymbol{w}_{\varepsilon,n}^{hk}$, we get the inequality

$$
\begin{aligned}
&\int_\Omega \Big[\boldsymbol{C}\,\left(\boldsymbol{\epsilon}(\boldsymbol{u}_n^{hk}) - \boldsymbol{p}_n^{hk} \right) \cdot \left[\left(\boldsymbol{\epsilon}(\boldsymbol{u}_{\varepsilon,n}^{hk}) - \boldsymbol{p}_{\varepsilon,n}^{hk} \right) - \left(\boldsymbol{\epsilon}(\boldsymbol{u}_n^{hk}) - \boldsymbol{p}_n^{hk} \right) \right] \\
&\qquad + k_1\, \boldsymbol{p}_n^{hk} \cdot \left(\boldsymbol{p}_{\varepsilon,n}^{hk} - \boldsymbol{p}_n^{hk} \right) + c_0 |\boldsymbol{p}_{\varepsilon,n}^{hk} - \boldsymbol{p}_{n-1}^{hk}| - c_0 |\boldsymbol{p}_n^{hk} - \boldsymbol{p}_{n-1}^{hk}| \Big]\, dx \\
&\geq \int_\Omega \boldsymbol{f}_n \cdot (\boldsymbol{u}_{\varepsilon,n}^{hk} - \boldsymbol{u}_n^{hk})\, dx.
\end{aligned}
$$

Thus

$$
\begin{aligned}
D = &\int_\Omega \Big[\frac{1}{2} \boldsymbol{C}\,\left(\boldsymbol{\epsilon}(\boldsymbol{u}_{\varepsilon,n}^{hk}) - \boldsymbol{p}_{\varepsilon,n}^{hk} \right) \cdot \left(\boldsymbol{\epsilon}(\boldsymbol{u}_{\varepsilon,n}^{hk}) - \boldsymbol{p}_{\varepsilon,n}^{hk} \right) + \frac{k_1}{2}\,|\boldsymbol{p}_{\varepsilon,n}^{hk}|^2 + c_0 |\boldsymbol{p}_{\varepsilon,n}^{hk} - \boldsymbol{p}_{n-1}^{hk}| \\
&\quad - \boldsymbol{f}_n \cdot \boldsymbol{u}_{\varepsilon,n}^{hk} - \frac{1}{2} \boldsymbol{C}\,\left(\boldsymbol{\epsilon}(\boldsymbol{u}_n^{hk}) - \boldsymbol{p}_n^{hk} \right) \cdot \left(\boldsymbol{\epsilon}(\boldsymbol{u}_n^{hk}) - \boldsymbol{p}_n^{hk} \right) \\
&\quad - \frac{k_1}{2}\,|\boldsymbol{p}_n^{hk}|^2 - c_0 |\boldsymbol{p}_n^{hk} - \boldsymbol{p}_{n-1}^{hk}| + \boldsymbol{f}_n \cdot \boldsymbol{u}_n^{hk} \Big]\, dx \\
\geq &\int_\Omega \Big[\frac{1}{2} \boldsymbol{C} \quad \left(\left(\boldsymbol{\epsilon}(\boldsymbol{u}_{\varepsilon,n}^{hk}) - \boldsymbol{\epsilon}(\boldsymbol{u}_n^{hk}) \right) - \left(\boldsymbol{p}_{\varepsilon,n}^{hk} - \boldsymbol{p}_n^{hk} \right) \right) \cdot \left(\left(\boldsymbol{\epsilon}(\boldsymbol{u}_{\varepsilon,n}^{hk}) - \boldsymbol{\epsilon}(\boldsymbol{u}_n^{hk}) \right) - \left(\boldsymbol{p}_{\varepsilon,n}^{hk} - \boldsymbol{p}_n^{hk} \right) \right) \\
&\quad + \frac{k_1}{2}\,|\boldsymbol{p}_{\varepsilon,n}^{hk} - \boldsymbol{p}_n^{hk}|^2 \Big]\, dx \\
\geq &\ \alpha\, \|\boldsymbol{w}_{\varepsilon,n}^{hk} - \boldsymbol{w}_n^{hk}\|_Z^2.
\end{aligned}
$$

In the last step above, we used the Z-ellipticity of the bilinear form a. On the other hand, we have an upper bound for D from (8.9) and (8.10):

$$\begin{aligned} D \le \int_\Omega \Big[&\frac{1}{2}\, \boldsymbol{C}\big(\boldsymbol{\epsilon}(\boldsymbol{u}^{hk}_{\varepsilon,n}) - \boldsymbol{p}^{hk}_{\varepsilon,n}\big)\cdot\big(\boldsymbol{\epsilon}(\boldsymbol{u}^{hk}_{\varepsilon,n}) - \boldsymbol{p}^{hk}_{\varepsilon,n}\big) \\ &+ \frac{k_1}{2}\,|\boldsymbol{p}^{hk}_{\varepsilon,n}|^2 + c_0|\boldsymbol{p}^{hk}_{\varepsilon,n} - \boldsymbol{p}^{hk}_{n-1}| - \boldsymbol{f}_n\cdot\boldsymbol{u}^{hk}_{\varepsilon,n} \\ &+ \frac{1}{2k_1}\,\big(|k_1\boldsymbol{p}^{hk}_{n-1} + \boldsymbol{s}^{*D}| - c_0\big)_+^2 - \frac{k_1}{2}\,|\boldsymbol{p}^{hk}_{n-1}|^2 - \boldsymbol{p}^{hk}_{n-1}\cdot\boldsymbol{s}^* + \tfrac{1}{2}\,\boldsymbol{C}^{-1}\boldsymbol{s}^*\cdot\boldsymbol{s}^*\Big]\,dx \\ &\forall\, \boldsymbol{s}^* \in Q^{h*} \text{ such that } \int_\Omega \big[\boldsymbol{\epsilon}(\boldsymbol{v}^h)\cdot\boldsymbol{s}^* + \boldsymbol{f}_n\cdot\boldsymbol{v}^h\big]\,dx = 0 \quad \forall\, \boldsymbol{v}^h \in V^h. \end{aligned}$$

Now we choose $\boldsymbol{s}^*$ by using the solution $\boldsymbol{w}^{hk}_{\varepsilon,n}$ of the regularized problem (8.7). We have from (8.7) that $\forall\, \boldsymbol{z}^h \in Z^h$,

$$\begin{aligned} \int_\Omega \Bigg[&\boldsymbol{C}\,\big(\boldsymbol{\epsilon}(\boldsymbol{u}^{hk}_{\varepsilon,n}) - \boldsymbol{p}^{hk}_{\varepsilon,n}\big)\cdot\big(\boldsymbol{\epsilon}(\boldsymbol{v}^h) - \boldsymbol{q}^h\big) + k_1\,\boldsymbol{p}^{hk}_{\varepsilon,n}\cdot\boldsymbol{q}^h \\ &+ c_0\frac{\big(\boldsymbol{p}^{hk}_{\varepsilon,n} - \boldsymbol{p}^{hk}_{n-1}\big)\cdot\boldsymbol{q}^h}{\sqrt{|\boldsymbol{p}^{hk}_{\varepsilon,n} - \boldsymbol{p}^{hk}_{n-1}|^2 + \varepsilon^2}} - \boldsymbol{f}_n\cdot\boldsymbol{v}^h\Bigg]\,dx = 0; \end{aligned}$$

that is,

$$\int_\Omega \big[\boldsymbol{C}\,\big(\boldsymbol{\epsilon}(\boldsymbol{u}^{hk}_{\varepsilon,n}) - \boldsymbol{p}^{hk}_{\varepsilon,n}\big)\cdot\boldsymbol{\epsilon}(\boldsymbol{v}^h) - \boldsymbol{f}_n\cdot\boldsymbol{v}^h\big]\,dx = 0 \quad \forall\, \boldsymbol{v}^h \in V^h \tag{8.11}$$

and

$$\begin{aligned} \int_\Omega \Bigg[&-\boldsymbol{C}\,\big(\boldsymbol{\epsilon}(\boldsymbol{u}^{hk}_{\varepsilon,n}) - \boldsymbol{p}^{hk}_{\varepsilon,n}\big)\cdot\boldsymbol{q}^h + k_1\,\boldsymbol{p}^{hk}_{\varepsilon,n}\cdot\boldsymbol{q}^h \\ &+ c_0\frac{\big(\boldsymbol{p}^{hk}_{\varepsilon,n} - \boldsymbol{p}^{hk}_{n-1}\big)\cdot\boldsymbol{q}^h}{\sqrt{|\boldsymbol{p}^{hk}_{\varepsilon,n} - \boldsymbol{p}^{hk}_{n-1}|^2 + \varepsilon^2}}\Bigg]\,dx = 0 \quad \forall\, \boldsymbol{q}^h \in Q^h_0. \end{aligned} \tag{8.12}$$

Because of the relation (8.11), an admissible choice for $\boldsymbol{s}^*$ is

$$\boldsymbol{s}^* = -\boldsymbol{C}\,\big(\boldsymbol{\epsilon}(\boldsymbol{u}^{hk}_{\varepsilon,n}) - \boldsymbol{p}^{hk}_{\varepsilon,n}\big)\,. \tag{8.13}$$

From (8.12), we then find a useful relation

$$\boldsymbol{s}^{*D} + k_1\,\boldsymbol{p}^{hk}_{\varepsilon,n} + c_0\frac{\boldsymbol{p}^{hk}_{\varepsilon,n} - \boldsymbol{p}^{hk}_{n-1}}{\sqrt{|\boldsymbol{p}^{hk}_{\varepsilon,n} - \boldsymbol{p}^{hk}_{n-1}|^2 + \varepsilon^2}} = \boldsymbol{0}. \tag{8.14}$$

And we find

$$|k_1\boldsymbol{p}^{hk}_{n-1} + \boldsymbol{s}^{*D}| = \left(k_1 + \frac{c_0}{\sqrt{|\boldsymbol{p}^{hk}_{\varepsilon,n} - \boldsymbol{p}^{hk}_{n-1}|^2 + \varepsilon^2}}\right)\,|\boldsymbol{p}^{hk}_{\varepsilon,n} - \boldsymbol{p}^{hk}_{n-1}|.$$

After some simplifications, we obtain

$$D \le \int_\Omega \left\{ \frac{c_0 |\boldsymbol{p}^{hk}_{\varepsilon,n} - \boldsymbol{p}^{hk}_{n-1}|\, \varepsilon^2}{E(\boldsymbol{p}^{hk}_{\varepsilon,n}, \boldsymbol{p}^{hk}_{n-1}, \varepsilon)} - \frac{k_1}{2} |\boldsymbol{p}^{hk}_{\varepsilon,n} - \boldsymbol{p}^{hk}_{n-1}|^2 + \frac{1}{2k_1} \left[k_1 |\boldsymbol{p}^{hk}_{\varepsilon,n} - \boldsymbol{p}^{hk}_{n-1}| - \frac{c_0 \varepsilon^2}{E(\boldsymbol{p}^{hk}_{\varepsilon,n}, \boldsymbol{p}^{hk}_{n-1}, \varepsilon)} \right]_+^2 \right\} dx,$$

where

$$E(\boldsymbol{p}, \boldsymbol{q}, \varepsilon) = \sqrt{|\boldsymbol{p} - \boldsymbol{q}|^2 + \varepsilon^2} \left(\sqrt{|\boldsymbol{p} - \boldsymbol{q}|^2 + \varepsilon^2} + |\boldsymbol{p} - \boldsymbol{q}| \right). \tag{8.15}$$

Combining the lower bound and the upper bound for D, we get the a posteriori error estimate

$$\begin{aligned} &\alpha \|\boldsymbol{w}^{hk}_{\varepsilon,n} - \boldsymbol{w}^{hk}_n\|^2_Z \\ &\le \int_\Omega \left\{ \frac{c_0 |\boldsymbol{p}^{hk}_{\varepsilon,n} - \boldsymbol{p}^{hk}_{n-1}|\, \varepsilon^2}{E(\boldsymbol{p}^{hk}_{\varepsilon,n}, \boldsymbol{p}^{hk}_{n-1}, \varepsilon)} - \frac{k_1}{2} |\boldsymbol{p}^{hk}_{\varepsilon,n} - \boldsymbol{p}^{hk}_{n-1}|^2 \right. \\ &\qquad \left. + \frac{1}{2k_1} \left[k_1 |\boldsymbol{p}^{hk}_{\varepsilon,n} - \boldsymbol{p}^{hk}_{n-1}| - \frac{c_0 \varepsilon^2}{E(\boldsymbol{p}^{hk}_{\varepsilon,n}, \boldsymbol{p}^{hk}_{n-1}, \varepsilon)} \right]_+^2 \right\} dx, \end{aligned} \tag{8.16}$$

where $E(\boldsymbol{p}^{hk}_{\varepsilon,n}, \boldsymbol{p}^{hk}_{n-1}, \varepsilon)$ is defined by (8.15).

We observe that the summation of the last two terms of the integrand on the right-hand side is nonpositive, so that a simple consequence of (8.16) is

$$\alpha \|\boldsymbol{w}^{hk}_{\varepsilon,n} - \boldsymbol{w}^{hk}_n\|^2_Z \le \int_\Omega \frac{c_0 |\boldsymbol{p}^{hk}_{\varepsilon,n} - \boldsymbol{p}^{hk}_{n-1}|\, \varepsilon^2}{E(\boldsymbol{p}^{hk}_{\varepsilon,n}, \boldsymbol{p}^{hk}_{n-1}, \varepsilon)}\, dx,$$

which shows the efficiency of the a posteriori error estimate (8.16).

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