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EVOLUTIONARY VARIATIONAL INEQUALITIES ARISING IN VISCOELASTIC CONTACT PROBLEMS*

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Abstract. We consider a class of evolutionary variational inequalities arising in various frictional contact problems for viscoelastic materials. Under the smallness assumption of a certain coefficient, we prove an existence and uniqueness result using Banach's fixed point theorem. We then study two numerical approximation schemes of the problem: a semidiscrete scheme and a fully discrete scheme. For both schemes, we show the existence of a unique solution and derive error estimates. Finally, all these results are applied to the analysis and numerical approximations of a viscoelastic frictional contact problem, with the finite element method used to discretize the spatial domain.

Key words. evolutionary variational inequality, frictional contact problem, viscoelasticity, numerical schemes, finite element method, convergence analysis, error estimation

AMS subject classifications. 65M12, 65M15, 65M60, 65N12, 65N15, 65N60, 74D10, 74S05, 74S20

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1. Introduction.

This work concerns the study of a class of abstract evolutionary variational inequalities modeling the frictional contact between a viscoelastic body and a rigid foundation. Situations involving dynamic or quasi-static frictional contact abound in industry, especially in engines, motors, and transmissions. For this reason considerable engineering literature exists dealing with frictional contact problems. Invariably, the engineering papers deal with specific situations, geometries, or settings and the emphasis is on numerical approaches or experimental results. An early attempt to study frictional contact problems within the framework of variational inequalities was made in [7]. An excellent reference on analysis and numerical approximations of contact problems involving elastic materials with or without friction is [12]. The mathematical, mechanical, and numerical state of the art can be found in the proceedings [17].

Quasi-static contact problems arise when the forces applied to a system vary slowly in time so that acceleration is negligible. Rigorous mathematical treatment of quasi-static problems is recent. The reason lies in the considerable difficulties that the process of frictional contact presents in the modeling and the analysis because of the complicated surface phenomena involved. The variational analysis of some quasi-static contact problems can be found, for instance, in [1, 5, 13] within linearized elasticity and in [18, 19, 21] within nonlinear viscoelasticity.

Recently it is shown in [20] that a number of quasi-static frictional contact problems for viscoelastic materials leads to variational models of the following form: Find

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a displacement field $u : [0,T] \to V$ such that for $t \in [0,T]$,

$$(A\dot{u}(t), v - \dot{u}(t))_V + (Bu(t), v - \dot{u}(t))_V + j(\dot{u}(t), v) - j(\dot{u}(t), \dot{u}(t)) \geq (f(t), v - \dot{u}(t))_V \qquad \forall\, v \in V, \tag{1.1}$$

and

$$u(0) = u_0. \tag{1.2}$$

Here, V is a function space of admissible displacements, A and B are nonlinear operators related to the viscoelastic constitutive law, and the functional j is determined by contact boundary conditions. The data f is related to the given body forces and surface tractions, and u_0 represents the initial displacement. In this paper, $[0,T]$ is the time interval of interest, and the dot above a quantity denotes the derivative of the quantity with respect to the time variable t.

The aim of this paper is to provide variational and numerical analysis for the abstract Cauchy problem of the form (1.1)–(1.2) and to apply these results in the study of some viscoelastic frictional contact problems. The literature is abundant in numerical treatment of elliptic or evolution variational inequalities; see, for instance, [8, 10, 11]. Here we follow [10] (see also [9]) to analyze semidiscrete and fully discrete approximation schemes.

The rest of the paper is structured as follows. In section 2 we show an existence and uniqueness result to the problem (1.1)–(1.2). The result is proved based on standard arguments for elliptic variational inequalities followed by applying Banach's fixed point theorem twice. In sections 3 and 4 we analyze a semidiscrete scheme and a fully discrete scheme, respectively. For both schemes, we show the existence of a unique solution derive error estimates. Under suitable solution regularities, convergence order error estimates can be obtained. Since solution regularity results for the problem (1.1)–(1.2) are not available and since the solution does not likely enjoy high degree regularity, it is important to know if the numerical solution converges to the exact solution without any assumption on the solution regularity. In section 5, we show the convergence of the two schemes under the basic solution regularity proved in section 2. In the final section, we apply all the results proved in sections 2–5 to a concrete example of viscoelastic frictional contact problem modeled by an evolutionary variational inequality of the form (1.1)–(1.2). For numerical approximations of this example we use the finite element method to discretize the spatial domain.

2. An existence and uniqueness result.

In this section we list the assumptions on the data and present an existence and uniqueness result in the study of the Cauchy problem (1.1)–(1.2).

We suppose in what follows that V is a real Hilbert space endowed with the inner product $(\cdot,\cdot)_V$ and the associated norm $\|\cdot\|_V$. Let $T > 0$. We will use the space $C[0,T]$ of real-valued continuous functions on $[0,T]$ and denote by $C([0,T];V)$ ($C^1([0,T];V)$) the space of continuous (respectively, continuously differentiable) functions from $[0,T]$ to V, with norms

$$\|u\|_{C([0,T];V)} = \max_{t\in[0,T]} \|u(t)\|_V$$

and

$$\|u\|_{C^1([0,T];V)} = \max_{t\in[0,T]} \|u(t)\|_V + \max_{t\in[0,T]} \|\dot{u}(t)\|_V,$$

respectively. For $p \in [1,\infty]$, we will also use the Sobolev space $W^{1,p}(0,T;V)$ with the norm

$$\|u\|_{W^{1,p}(0,T;V)} = \left[\int_0^T (\|u(t)\|_V^p + \|\dot{u}(t)\|_V^p)\, dt \right]^{1/p} \quad \text{if } 1 \le p < \infty,$$

or

$$\|u\|_{W^{1,\infty}(0,T;V)} = \max\{\|u(t)\|_{L^\infty(0,T;V)}, \|\dot{u}(t)\|_{L^\infty(0,T;V)}\},$$

where a dot now represents the weak derivative with respect to the time variable. We assume that $A : V \to V$ is a strongly monotone, Lipschitz continuous operator, i.e.,

$$\text{(2.1)} \qquad \begin{cases} \text{(a) } \exists\, M > 0 \text{ such that} \\ \qquad (Au_1 - Au_2, u_1 - u_2)_V \ge M\|u_1 - u_2\|_V^2 \quad \forall\, u_1, u_2 \in V; \\ \text{(b) } \exists\, L_A > 0 \text{ such that} \\ \qquad \|Au_1 - Au_2\|_V \le L_A \|u_1 - u_2\|_V \quad \forall\, u_1, u_2 \in V. \end{cases}$$

The nonlinear operator $B : V \to V$ is Lipschitz continuous, i.e.,

$$\text{(2.2)} \qquad \exists\, L_B > 0 \text{ such that } \|B(u_1) - B(u_2)\|_V \le L_B\, \|u_1 - u_2\|_V \quad \forall\, u_1, u_2 \in V.$$

The functional $j : V \times V \to \mathbb{R}$ satisfies

$$\text{(2.3)} \qquad \begin{cases} \text{(a) } \forall g \in V,\ j(g,\cdot) \text{ is convex and lower semicontinuous on } V; \\ \text{(b) } \exists\, m > 0 \text{ such that} \\ \qquad j(g_1, v_2) - j(g_1, v_1) + j(g_2, v_1) - j(g_2, v_2) \\ \qquad\quad \le m\, \|g_1 - g_2\|_V\, \|v_1 - v_2\|_V \quad \forall\, g_1, g_2, v_1, v_2 \in V. \end{cases}$$

Finally, we assume that

$$\text{(2.4)} \qquad f \in C([0,T];V)$$

and

$$\text{(2.5)} \qquad u_0 \in V.$$

The main result of this section is the following.

THEOREM 2.1. *Let* (2.1)–(2.5) *hold. Then, if $M > m$, there exists a unique solution $u \in C^1([0,T];V)$ to the problem* (1.1)–(1.2).

The proof of Theorem 2.1 is based on fixed point arguments similar to those used in [18] and [21]. It will be established in several steps. We assume in what follows that (2.1)–(2.5) hold. To simplify the notation, sometimes we will not indicate explicitly the dependence of various functions on the time variable t.

In the first step let $\eta \in C([0,T];V)$ and $g \in C([0,T];V)$ be given and we consider the following variational inequality of finding $v_{\eta g} : [0,T] \to V$, such that for $t \in [0,T]$,

$$\text{(2.6)} \qquad \begin{aligned} &(Av_{\eta g}(t), v - v_{\eta g}(t))_V + (\eta(t), v - v_{\eta g}(t))_V + j(g(t), v) - j(g(t), v_{\eta g}(t)) \\ &\quad \ge (f(t), v - v_{\eta g}(t))_V \quad \forall\, v \in V. \end{aligned}$$

LEMMA 2.2. *There exists a unique solution $v_{\eta g} \in C([0,T];V)$ to the problem* (2.6).

Proof. It follows from classical results for elliptic variational inequalities (see, e.g., [2]) that there exists a unique element $v_{\eta g}(t) \in V$ that solves (2.6) for each $t \in [0,T]$.

Let us show that $v_{\eta g} : [0,T] \to V$ is continuous. Let $t_1, t_2 \in [0,T]$. For the sake of simplicity in writing we denote $v_{\eta g}(t_i) = v_i$, $\eta(t_i) = \eta_i$, $g(t_i) = g_i$ for $i = 1,2$. Using (2.6) we easily derive the relation

$$(Av_1 - Av_2, v_1 - v_2)_V \le (f_1 - f_2, v_1 - v_2)_V + (\eta_1 - \eta_2, v_2 - v_1)_V + j(g_1, v_2) - j(g_1, v_1) + j(g_2, v_1) - j(g_2, v_2).$$

Then we use the conditions (2.1)(a) and (2.3)(b) to obtain

$$M\|v_1 - v_2\|_V \le \|f_1 - f_2\|_V + \|\eta_1 - \eta_2\|_V + m\|g_1 - g_2\|_V. \tag{2.7}$$

Therefore, $v_{\eta g} : [0,T] \to V$ is a continuous function. ◻

For each $\eta \in C([0,T];V)$, we now consider the operator $\Lambda_\eta : C([0,T];V) \to C([0,T];V)$ defined by

$$\Lambda_\eta g = v_{\eta g} \quad \forall\, g \in C([0,T];V). \tag{2.8}$$

We have the following result.

Lemma 2.3. *Let $M > m$. Then, the operator Λ_η has a unique fixed point $g_\eta \in C([0,T];V)$.*

Proof. Let $g_1, g_2 \in C([0,T];V)$, $\eta \in C([0,T];V)$, and let v_i, $i = 1,2$, denote the solution of (2.6) for $g = g_i$, i.e., $v_i = v_{\eta g_i}$. From the definition (2.8) we have

$$\|\Lambda_\eta g_1 - \Lambda_\eta g_2\|_V = \|v_1 - v_2\|_V. \tag{2.9}$$

An argument similar to that in the proof of Lemma 2.2 shows

$$M\,\|v_1 - v_2\|_V \le m\,\|g_1 - g_2\|_V. \tag{2.10}$$

Thus if $M > m$, then the operator Λ_η is a contraction on the Banach space $C([0,T];V)$. The result of the lemma follows from the Banach's fixed point theorem. ◻

In what follows we suppose that $M > m$ and let $\eta \in C([0,T];V)$. We denote by g_η the fixed point given in Lemma 2.3 and let $v_\eta \in C([0,T];V)$ be the function defined by

$$v_\eta = v_{\eta g_\eta}. \tag{2.11}$$

We have $\Lambda_\eta g_\eta = g_\eta$ and from (2.8) and (2.11),

$$v_\eta = g_\eta. \tag{2.12}$$

Therefore, taking $g = g_\eta$ in (2.6) and using (2.11) and (2.12), we see that $v_\eta(t) \in V$ satisfies

$$(Av_\eta(t), v - v_\eta(t))_V + (\eta(t), v - v_\eta(t))_V + j(v_\eta(t), v) - j(v_\eta(t), v_\eta(t)) \ge (f(t), v - v_\eta(t))_V \quad \forall\, v \in V,\ t \in [0,T]. \tag{2.13}$$

We now denote by $u_\eta \in C^1([0,T];V)$ the function given by

$$u_\eta(t) = \int_0^t v_\eta(s)\, ds + u_0, \quad t \in [0,T] \tag{2.14}$$

and define the operator $\Lambda : C([0,T];V) \to C([0,T];V)$ by

$$\Lambda\eta = Bu_\eta \quad \forall\, \eta \in C([0,T];V). \tag{2.15}$$

We have the next result.

LEMMA 2.4. *Let $M > m$. Then the operator Λ has a unique fixed point $\eta^* \in C([0,T];V)$.*

Proof. For the proof of this lemma, we will use

$$\|v\|^*_{C([0,T];V)} = \max_{t\in[0,T]} e^{-\beta\, t}\|v(t)\|_V$$

with $\beta > L_B/(M-m)$ as the norm in the space $C([0,T];V)$. This norm is equivalent to the standard norm $\|v\|_{C([0,T];V)}$. Let $\eta_1, \eta_2 \in C([0,T];V)$ and let $v_i = v_{\eta_i}$, $u_i = u_{\eta_i}$ for $i = 1, 2$. Using (2.13) and the estimates in the proof of Lemma 2.2 (see (2.7)) we deduce that

$$M\,\|v_1 - v_2\|_V \le \|\eta_1 - \eta_2\|_V + m\,\|v_1 - v_2\|_V,$$

which implies

$$\|v_1 - v_2\|_V \le \frac{1}{M-m}\,\|\eta_1 - \eta_2\|_V. \tag{2.16}$$

Now using (2.15), (2.2), and (2.14) we obtain

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_V \le L_B \int_0^t \|v_1(s) - v_2(s)\|_V\, ds \quad \forall\, t \in [0,T]$$

and recalling (2.16) it follows that

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_V \le \frac{L_B}{M-m} \int_0^t \|\eta_1(s) - \eta_2(s)\|_V\, ds \quad \forall\, t \in [0,T].$$

We then have

$$\begin{aligned} e^{-\beta\, t}\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_V &\le \frac{L_B}{M-m}\, e^{-\beta\, t} \int_0^t e^{\beta\, s} e^{-\beta\, s}\|\eta_1(s) - \eta_2(s)\|_V\, ds \\ &\le \frac{L_B}{M-m}\, e^{-\beta\, t} \int_0^t e^{\beta\, s}\, ds\, \|\eta_1 - \eta_2\|^*_{C([0,T];V)} \\ &\le \frac{L_B}{M-m}\,\frac{1}{\beta}\, \|\eta_1 - \eta_2\|^*_{C([0,T];V)}, \end{aligned}$$

and so

$$\|\Lambda\eta_1 - \Lambda\eta_2\|^*_{C([0,T];V)} \le \frac{L_B}{M-m}\,\frac{1}{\beta}\, \|\eta_1 - \eta_2\|^*_{C([0,T];V)}.$$

Since $\beta > L_B/(M-m)$, the operator Λ is a contraction on the space $C([0,T];V)$ when the equivalent norm $\|\cdot\|^*_{C([0,T];V)}$ is used. By the Banach fixed point theorem, Λ has a unique fixed point $\eta^* \in C([0,T];V)$. ∎

We now have all the ingredients to prove Theorem 2.1.

Proof of Theorem 2.1. *Existence.* Let $\eta^* \in C([0,T];V)$ be the fixed point of Λ and let $u_{\eta^*} \in C^1([0,T];V)$ be the function given by (2.14) for $\eta = \eta^*$. We have $\dot{u}_{\eta^*} = v_{\eta^*}$ and using (2.13) for $\eta = \eta^*$ it follows that for any $t \in [0,T]$,

$$(A\dot{u}_{\eta^*}(t), v - \dot{u}_{\eta^*}(t))_V + (\eta^*(t), v - \dot{u}_{\eta^*}(t))_V + j(\dot{u}_{\eta^*}(t), v) - j(\dot{u}_{\eta^*}(t), \dot{u}_{\eta^*}(t)) \geq (f(t), v - \dot{u}_{\eta^*}(t))_V \quad \forall\, v \in V. \tag{2.17}$$

The inequality (1.1) now follows from (2.17) and (2.15) since $\eta^* = \Lambda\eta^* = Bu_{\eta^*}$ and (1.2) results from (2.14). We conclude that u_{η^*} is a solution of (1.1), (1.2).

Uniqueness. To prove the uniqueness of the solution let u_{η^*} be the solution of (1.1), (1.2) obtained above and let u be another solution such that $u \in C^1([0,T];V)$. We denote by $\eta \in C([0,T];V)$ the function given by

$$\eta = Bu \tag{2.18}$$

and let

$$\dot{u} = w. \tag{2.19}$$

Using (1.1) we obtain that w is a solution of the variational inequality (2.13) and since this problem has a unique solution $v_\eta \in C([0,T];V)$ (see, e.g., (2.16)) we conclude that

$$w = v_\eta. \tag{2.20}$$

Moreover, it follows from (2.19), (1.2), (2.20), and (2.14) that

$$u = u_\eta. \tag{2.21}$$

Now using (2.15), (2.18), and (2.21) we obtain that $\Lambda\eta = \eta$ and by the uniqueness of the fixed point of Λ we have

$$\eta = \eta^*. \tag{2.22}$$

The uniqueness of the solution is a consequence of (2.21) and (2.22). The proof of Theorem 2.1 is now complete. ∎

In what follows, c will represent a positive constant whose value may change from place to place.

COROLLARY 2.5. *Under the conditions stated in Theorem* 2.1*, if* $f \in W^{1,p}(0,T;V)$ *for some* $p \in [1,\infty]$*, then* $\dot{u} \in W^{1,p}(0,T;V)$ *and*

$$\|\dot{u}\|_{W^{1,p}(0,T;V)} \leq c\,(\|f\|_{W^{1,p}(0,T;V)} + \|u\|_{C^1([0,T];V)}).$$

Proof. For any $t_1, t_2 \in [0,T]$, we apply the inequality (2.7) to the inequality (1.1) to obtain

$$M\,\|\dot{u}(t_1) - \dot{u}(t_2)\|_V \leq \|f(t_1) - f(t_2)\|_V + \|Bu(t_1) - Bu(t_2)\|_V + m\,\|\dot{u}(t_1) - \dot{u}(t_2)\|_V.$$

By the assumptions $M > m$ and (2.2), we find that

$$\|\dot{u}(t_1) - \dot{u}(t_2)\|_V \leq c\,(\|f(t_1) - f(t_2)\|_V + \|u(t_1) - u(t_2)\|_V),$$

from which the result of the corollary follows. ∎

In the next three sections, we will assume all the conditions stated in Theorem 2.1 are satisfied.

3. Semidiscrete approximation. In this section we consider an approximation of the problem (1.1)–(1.2) by discretizing only the space V. Let $V^h \subset V$ be a finite-dimensional space which, for example, can be constructed by the finite element method. Then a semidiscrete scheme can be formed as in the following problem.

PROBLEM $\mathbf{P}^h$. Find $u^h : [0,T] \to V^h$ such that for $t \in [0,T]$,

$$\begin{aligned}(A\dot{u}^h(t), v^h - \dot{u}^h(t))_V + (Bu^h(t), v^h - \dot{u}^h(t))_V + j(\dot{u}^h(t), v^h)\\ - j(\dot{u}^h(t), \dot{u}^h(t)) \geq (f(t), v^h - \dot{u}^h(t))_V \quad \forall\, v^h \in V^h,\end{aligned} \tag{3.1}$$

and

$$u^h(0) = u_0^h. \tag{3.2}$$

Here, $u_0^h \in V^h$ is an appropriate approximation of u_0.

Using the arguments presented in the previous section, we see that under the conditions stated in Theorem 2.1, Problem $\mathbf{P}^h$ has a unique solution $u^h \in C^1([0,T]; V^h)$. Our main purpose here is to derive an estimate for the error $u - u^h$.

To simplify the writing, we introduce the velocity variable

$$w(t) = \dot{u}(t). \tag{3.3}$$

Then by using the initial value condition (1.2), we have the relation

$$u(t) = \int_0^t w(s)\, ds + u_0. \tag{3.4}$$

Similarly, we introduce the discrete velocity variable

$$w^h(t) = \dot{u}^h(t). \tag{3.5}$$

With the initial value condition (3.2), we have

$$u^h(t) = \int_0^t w^h(s)\, ds + u_0^h. \tag{3.6}$$

Now the variational inequalities (1.1) and (3.1) can be rewritten as

$$\begin{aligned}(Aw(t), v - w(t))_V + (Bu(t), v - w(t))_V + j(w(t), v) - j(w(t), w(t))\\ \geq (f(t), v - w(t))_V \qquad \forall\, v \in V,\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}(Aw^h(t), v^h - w^h(t))_V + (Bu^h(t), v^h - w^h(t))_V + j(w^h(t), v^h)\\ - j(w^h(t), w^h(t)) \geq (f(t), v^h - w^h(t))_V \qquad \forall\, v^h \in V^h.\end{aligned} \tag{3.8}$$

We take $v = w^h(t)$ in (3.7) and add the inequality to (3.8) with $v^h = v^h(t) \in V^h$. After some manipulations, we have

$$\begin{aligned}&(Aw(t) - Aw^h(t), w(t) - w^h(t))_V\\ &\quad \leq (Aw^h(t), v^h(t) - w(t))_V + (Bu(t), w^h(t) - w(t))_V\\ &\qquad + (Bu^h(t), v^h(t) - w^h(t))_V + (f(t), w(t) - v^h(t))_V\\ &\qquad + j(w(t), w^h(t)) - j(w(t), w(t)) + j(w^h(t), v^h(t)) - j(w^h(t), w^h(t))\\ &\quad \leq (Aw^h(t) - Aw(t), v^h(t) - w(t))_V + (Bu(t) - Bu^h(t), w^h(t) - v^h(t))_V\\ &\qquad + j(w(t), w^h(t)) - j(w(t), v^h(t)) + j(w^h(t), v^h(t)) - j(w^h(t), w^h(t))\\ &\qquad + R(t; v^h(t), w(t)),\end{aligned}$$

where

$$R(t;v^h(t),w(t)) = (Aw(t),v^h(t)-w(t))_V + (Bu(t),v^h(t)-w(t))_V + j(w(t),v^h(t)) - j(w(t),w(t)) - (f(t),v^h(t)-w(t))_V \tag{3.9}$$

represents a residual quantity. Using the assumptions (2.1), (2.2), and (2.3), we have

$$\begin{aligned} M\,\|w(t)-w^h(t)\|_V^2 &\le L_A\|w(t)-w^h(t)\|_V\,\|w(t)-v^h(t)\|_V + |R(t;v^h(t),w(t))| \\ &\quad + (L_B\|u(t)-u^h(t)\|_V + m\,\|w(t)-w^h(t)\|_V)\,\|w^h(t)-v^h(t)\|_V \\ &\le L_A\|w(t)-w^h(t)\|_V\,\|w(t)-v^h(t)\|_V + |R(t;v^h(t),w(t))| \\ &\quad + (L_B\|u(t)-u^h(t)\|_V + m\,\|w(t)-w^h(t)\|_V) \\ &\quad \cdot (\|w(t)-w^h(t)\|_V + \|w(t)-v^h(t)\|_V). \end{aligned}$$

Thus, under the assumption $M > m$, we have the inequality

$$\|w(t)-w^h(t)\|_V^2 \le c\,\big(\|w(t)-v^h(t)\|_V^2 + \|u(t)-u^h(t)\|_V^2 + |R(t;v^h(t),w(t))|\big). \tag{3.10}$$

By (3.4) and (3.6), we have

$$u(t)-u^h(t) = \int_0^t (w(s)-w^h(s))\,ds + u_0 - u_0^h,$$

and so

$$\|u(t)-u^h(t)\|_V^2 \le c\left(\int_0^t \|w(s)-w^h(s)\|_V^2 ds + \|u_0-u_0^h\|_V^2\right). \tag{3.11}$$

Then the inequality (3.10) can be rewritten as

$$\|w(t)-w^h(t)\|_V^2 \le c\left(\|w(t)-v^h(t)\|_V^2 + \int_0^t \|w(s)-w^h(s)\|_V^2\,ds + \|u_0-u_0^h\|_V^2 + |R(t;v^h(t),w(t))|\right).$$

Applying the Gronwall inequality, we have

$$\begin{aligned} &\|w-w^h\|_{C([0,T];V)} \\ &\quad \le c \inf_{v^h\in C([0,T];V^h)} \left(\|w-v^h\|_{C([0,T];V)} + \|R(\cdot;v^h(\cdot),w(\cdot))\|_{C([0,T])}^{1/2}\right) \\ &\qquad + c\,\|u_0-u_0^h\|_V. \end{aligned} \tag{3.12}$$

Summarizing, with (3.11) and (3.12), we have proved the following result.

THEOREM 3.1. *Assume the conditions* (2.1)–(2.5) *and* $M > m$. *Then for the error of the spatially semidiscrete solution of* (3.1)–(3.2), *we have the estimate*

$$\begin{aligned} &\|u-u^h\|_{C^1([0,T];V)} \\ &\quad \le c \inf_{v^h\in C([0,T];V^h)} \left(\|\dot u-v^h\|_{C([0,T];V)} + \|R(\cdot;v^h(\cdot),\dot u(\cdot))\|_{C([0,T])}^{1/2}\right) \\ &\qquad + c\,\|u_0-u_0^h\|_V, \end{aligned} \tag{3.13}$$

where $R(\cdot;v^h(\cdot),\dot u(\cdot))$ *is defined in* (3.9).

The inequality (3.11) is the basis for a convergence analysis (see section 5) and for error estimates, as is shown in section 6 in the context of a frictional contact problem.

4. Fully discrete approximation. In this section we consider a fully discrete approximation of the problem (1.1)–(1.2). In addition to the finite dimensional space V^h introduced in the last section, we need a partition of the time interval $[0, T]$: $0 = t_0 < t_1 < \cdots < t_N = T$. We denote the time step-size $k_n = t_n - t_{n-1}$ for $n = 1, \ldots, N$. We allow nonuniform partition of the time interval, and let $k = \max_n k_n$ be the maximal step-size. For a continuous function $w(t)$, we use the notation $w_n = w(t_n)$. For a sequence $\{w_n\}_{n=0}^N$, we denote $\Delta w_n = w_n - w_{n-1}$ for the difference, and $\delta w_n = \Delta w_n / k_n$ the corresponding divided difference. No summation is implied over the repeated index n.

The fully discrete approximation method we will analyze is the following.

PROBLEM $\mathbf{P}^{hk}$. Find $\{u_n^{hk}\}_{n=0}^N \subset V^h$ such that for $n = 1, \ldots, N$,

$$(A\delta u_n^{hk}, v^h - \delta u_n^{hk})_V + (Bu_{n-1}^{hk}, v^h - \delta u_n^{hk})_V + j(\delta u_n^{hk}, v^h) - j(\delta u_n^{hk}, \delta u_n^{hk}) \geq (f_n, v^h - \delta u_n^{hk})_V \quad \forall\, v^h \in V^h, \tag{4.1}$$

and

$$u_0^{hk} = u_0^h. \tag{4.2}$$

Here, $u_0^h \in V^h$ is an appropriate approximation of u_0.

To again simplify the notation, we introduce the discrete velocity

$$w_n^{hk} = \delta u_n^{hk}, \quad n = 1, \ldots, N. \tag{4.3}$$

Then using the initial value condition (4.2), we have the relation

$$u_n^{hk} = \sum_{j=1}^{n} w_j^{hk} k_j + u_0^h. \tag{4.4}$$

We can rewrite (4.1) in the form

$$(Aw_n^{hk}, v^h - w_n^{hk})_V + (Bu_{n-1}^{hk}, v^h - w_n^{hk})_V + j(w_n^{hk}, v^h) - j(w_n^{hk}, w_n^{hk}) \geq (f_n, v^h - w_n^{hk})_V \quad \forall\, v^h \in V^h. \tag{4.5}$$

By a discrete analogue of Lemma 2.3, we see that given $u_{n-1}^{hk} \in V^h$, the inequality (4.5) has a unique solution $w_n^{hk} \in V^h$. Note that $u_0^{hk} = u_0^h$ is given and we have the relation (4.4) between $\{u_n^{hk}\}_{n=1}^N$ and $\{w_n^{hk}\}_{n=1}^N$. A mathematical induction argument yields the existence and uniqueness of a solution of the problem $\mathbf{P}^{hk}$. Our main objective of the section is to derive an error estimate for the fully discrete solution.

Take $v = w_n^{hk}$ in (3.7) at $t = t_n$,

$$(Aw_n, w_n^{hk} - w_n)_V + (Bu_n, w_n^{hk} - w_n) + j(w_n, w_n^{hk}) - j(w_n, w_n) \geq (f_n, w_n^{hk} - w_n)_V. \tag{4.6}$$

We now add (4.5) with $v^h = v_n^h \in V^h$ and (4.6) to obtain an error relation

$$\begin{aligned}
&(Aw_n - Aw_n^{hk}, w_n - w_n^{hk})_V \\
&\quad \leq (Aw_n^{hk}, v_n^h - w_n)_V + (Bu_{n-1}^{hk}, v_n^h - w_n^{hk})_V \\
&\qquad + (Bu_n, w_n^{hk} - w_n)_V - (f_n, v_n^h - w_n)_V \\
&\qquad + j(w_n^{hk}, v_n^h) - j(w_n^{hk}, w_n^{hk}) + j(w_n, w_n^{hk}) - j(w_n, w_n) \\
&\quad = (Aw_n^{hk} - Aw_n, v_n^h - w_n)_V + (Bu_{n-1}^{hk} - Bu_n, v_n^h - w_n^{hk})_V + R_n(v_n^h, w_n) \\
&\qquad + j(w_n^{hk}, v_n^h) - j(w_n, v_n^h) + j(w_n, w_n^{hk}) - j(w_n^{hk}, w_n^{hk}),
\end{aligned}$$

where

$$R_n(v_n^h, w_n) = (Aw_n, v_n^h - w_n)_V + (Bu_n, v_n^h - w_n)_V + j(w_n, v_n^h) - j(w_n, w_n) - (f_n, v_n^h - w_n)_V. \tag{4.7}$$

By the assumptions (2.1), (2.2), and (2.3), we have

$$M\,\|w_n - w_n^{hk}\|_V^2 \le L_A\|w_n - w_n^{hk}\|_V\,\|w_n - v_n^h\|_V + L_B\|u_{n-1}^{hk} - u_n\|_V\,\|v_n^h - w_n^{hk}\|_V + |R_n(v_n^h, w_n)| + m\,\|w_n - w_n^{hk}\|_V\,\|v_n^h - w_n^{hk}\|_V.$$

Here $\|v_n^h - w_n^{hk}\|_V$ will be bounded as follows:

$$\|v_n^h - w_n^{hk}\|_V \le \|v_n^h - w_n\|_V + \|w_n - w_n^{hk}\|_V.$$

Since $M > m$, we get the relation

$$\|w_n - w_n^{hk}\|_V^2 \le c\left\{\|v_n^h - w_n\|_V^2 + \|u_{n-1}^{hk} - u_n\|_V^2 + |R_n(v_n^h, w_n)|\right\},$$

or

$$\|w_n - w_n^{hk}\|_V \le c\left\{\|v_n^h - w_n\|_V + |R_n(v_n^h, w_n)|^{1/2} + \|u_{n-1}^{hk} - u_n\|_V\right\}. \tag{4.8}$$

Let us bound the term $\|u_{n-1}^{hk} - u_n\|_V$. We have

$$\begin{aligned} u_{n-1}^{hk} - u_n &= \sum_{j=1}^{n-1} w_j^{hk} k_j + u_0^h - \int_0^{t_n} w(s)\,ds - u_0 \\ &= \sum_{j=1}^{n-1} (w_j^{hk} - w_j)\,k_j + u_0^h - u_0 \\ &\quad + \sum_{j=1}^{n-1}\left(w_j k_j - \int_{t_{j-1}}^{t_j} w(s)\,ds\right) - \int_{t_{n-1}}^{t_n} w(s)\,ds. \end{aligned}$$

Now

$$\begin{aligned} \left\|\sum_{j=1}^{n-1}\left(w_j k_j - \int_{t_{j-1}}^{t_j} w(s)\,ds\right)\right\|_V &= \left\|\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j} (w_j - w(s))\,ds\right\|_V \\ &\le \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j} \|w_j - w(s)\|_V ds, \end{aligned}$$

and also

$$\left\|\int_{t_{n-1}}^{t_n} w(s)\,ds\right\|_V \le \int_{t_{n-1}}^{t_n} \|w(s)\|_V\,ds \le k\,\|w\|_{C([0,T];V)}.$$

Hence,

$$\|u_{n-1}^{hk} - u_n\|_V \le \sum_{j=1}^{n-1} \|w_j^{hk} - w_j\|_V k_j + \|u_0^h - u_0\|_V + I_k(w), \tag{4.9}$$

where

$$I_k(w) = \sum_{j=1}^{N}\int_{t_{j-1}}^{t_j} \|w_j - w(s)\|_V ds. \tag{4.10}$$

Therefore, from (4.8), we have

$$\|w_n - w_n^{hk}\|_V \le c\Big\{\|v_n^h - w_n\|_V + |R_n(v_n^h, w_n)|^{1/2} + \|u_0^h - u_0\|_V + k\,\|u\|_{C^1([0,T];V)} + I_k(w) + \sum_{j=1}^{n-1}\|w_j^{hk} - w_j\|_V k_j\Big\}. \tag{4.11}$$

To proceed further, we need the following result.

LEMMA 4.1. *Assume $\{g_n\}_{n=1}^N$ and $\{e_n\}_{n=1}^N$ are two sequences of nonnegative numbers satisfying*

$$e_n \le c\,g_n + c\sum_{j=1}^{n-1} k_j e_j.$$

Then

$$e_n \le c\left(g_n + \sum_{j=1}^{n-1} k_j g_j\right), \quad n = 1, \dots, N. \tag{4.12}$$

Therefore,

$$\max_{1\le n\le N} e_n \le c \max_{1\le n\le N} g_n. \tag{4.13}$$

Proof. Denote

$$E_n = \sum_{j=1}^{n} k_j e_j, \quad 1 \le n \le N,$$

and $E_0 = 0$. Then, from the given condition,

$$e_n \le c\,g_n + c\,E_{n-1}, \quad n = 1, \dots, N. \tag{4.14}$$

Now

$$E_n - E_{n-1} = k_n e_n \le c\,k_n g_n + c\,k_n E_{n-1},$$

which implies

$$E_n - (1 + c\,k_n)\,E_{n-1} \le c\,k_n g_n. \tag{4.15}$$

We introduce a sequence of numbers $\{z_n\}_{n=0}^N$ by $z_0 = 1$ and

$$z_n = \prod_{j=1}^{n}(1 + c\,k_j), \quad 1 \le n \le N.$$

Using the inequalities

$$1 \le 1 + c\,k_j \le e^{c\,k_j}, \quad j = 1, \dots, N,$$

we have the following bounds:

$$1 \le \prod_{i=j+1}^{N}(1 + ck_i) \le e^{c(T-t_j)}, \quad j = 1, \dots, N. \tag{4.16}$$

With the use of the sequence $\{z_n\}_{n=0}^N$, the inequality (4.15) can be rewritten as

$$\frac{E_n}{z_n} - \frac{E_{n-1}}{z_{n-1}} \le \frac{c\,k_n g_n}{z_n}.$$

A simple induction argument shows

$$E_n \le c\, z_n \sum_{j=1}^{n} k_j \frac{g_j}{z_j} = c \sum_{j=1}^{n} k_j \prod_{i=j+1}^{n} (1 + ck_i) g_j,$$

which can be combined with (4.14) and (4.16) to yield (4.12). The inequality (4.13) follows easily from (4.12). □

Applying Lemma 4.1 to the inequality (4.11), we obtain the following estimate:

$$\max_n \|w_n - w_n^{hk}\|_V \le c \max_n \left\{ \|v_n^h - w_n\|_V + |R_n(v_n^h, w_n)|^{1/2} \right\} + c\, \|u_0^h - u_0\|_V \\ + c\,(I_k(w) + k\, \|u\|_{C^1([0,T];V)}).$$

Similar to (4.9), we have

$$\|u_n^{hk} - u_n\|_V \le \sum_{j=1}^{n-1} \|w_j^{hk} - w_j\|_V k_j + \|u_0^h - u_0\|_V + c\, k\, \|u\|_{C^1([0,T];V)} + I_k(w).$$

Therefore, we have proved the following result.

THEOREM 4.2. *Assume the conditions* (2.1)–(2.5) *and* $M > m$. *Then for the error of the fully discrete solution of* (4.1)–(4.2)*, we have the estimate*

$$\begin{aligned} (4.17)\quad & \max_{1\le n\le N} (\|u_n - u_n^{hk}\|_V + \|\dot u_n - \delta u_n^{hk}\|_V) \\ &\le c \max_{1\le n\le N} \inf_{v_n^h\in V^h} \left\{ \|v_n^h - \dot u_n\|_V + |R_n(v_n^h, \dot u_n)|^{1/2} \right\} + c\, \|u_0^h - u_0\|_V \\ &\quad + c\,(I_k(\dot u) + k\, \|u\|_{C^1([0,T];V)}), \end{aligned}$$

where $R_n(v_n^h, \dot u_n)$ *is defined by* (4.7)*,* $I_k(\dot u) = I_k(w)$ *is defined in* (4.10)*.*

When the data f is smoother, we can derive a more convenient error bound from (4.17). For this purpose, we assume additionally $f \in W^{1,1}(0,T;V)$. Then by Corollary 2.5, we have $w \in W^{1,1}(0,T;V)$ and

$$\|w\|_{W^{1,1}(0,T;V)} \le c\,(\|f\|_{W^{1,1}(0,T;V)} + \|u\|_{C^1([0,T];V)}).$$

Recall that $W^{1,1}(0,T;V) \subset C([0,T];V)$ and

$$\|w\|_{C([0,T];V)} \le c\, \|w\|_{W^{1,1}(0,T;V)}.$$

By writing

$$w_j - w(s) = \int_s^{t_j} \dot w(\tau)\, d\tau$$

we easily find the following bound for $I_k(w)$ defined in (4.10):

$$I_k(w) \le c\, k\, \|\dot w\|_{L^1(0,T;V)} \le c\, k\, \|f\|_{W^{1,1}(0,T;V)}.$$

Therefore, under the additional assumption $f \in W^{1,1}(0,T;V)$, the estimate (4.17) can be replaced by

$$\begin{aligned} (4.18)\quad & \max_{1\le n\le N} (\|u_n - u_n^{hk}\|_V + \|\dot u_n - \delta u_n^{hk}\|_V) \\ &\le c \max_{1\le n\le N} \inf_{v_n^h\in V^h} \left\{ \|v_n^h - \dot u_n\|_V + |R_n(v_n^h, \dot u_n)|^{1/2} \right\} + c\, \|u_0^h - u_0\|_V \\ &\quad + c\, k\,(\|f\|_{W^{1,1}(0,T;V)} + \|u\|_{C^1([0,T];V)}). \end{aligned}$$

5. Convergence under the basic solution regularity. The inequalities (3.13) and (4.17) (or (4.18)) lead to order error estimates under additional solution regularity assumptions. Since solution regularity results for the problem (1.1)–(1.2) are not available, and since the solution does not likely enjoy high degree regularity, it is important to know if the numerical solution converges to the exact solution without any assumption on the solution regularity. In this section, we provide a convergence analysis for the two schemes studied in the previous two sections under the basic solution regularity $u \in C^1([0,T];V)$. For this purpose, we introduce two hypotheses.

HYPOTHESIS $\mathbf{H}_1$. *There exists a dense subset $V_0 \subset V$ and a function $\alpha(h) > 0$ with the property $\lim_{h\to 0+} \alpha(h) = 0$ such that*

$$\|v - \mathcal{P}^h v\|_V \le \alpha(h)\, \|v\|_{V_0} \quad \forall\, v \in V_0.$$

Here $\mathcal{P}^h : V \to V^h$ is a projection operator defined by

$$\|v - \mathcal{P}^h v\|_V = \inf_{v^h \in V^h} \|v - v^h\|_V, \quad v \in V.$$

Since V is a Hilbert space and V^h is finite-dimensional, the operator $\mathcal{P}^h$ is well-defined and is linear and nonexpansive:

$$\|\mathcal{P}^h v_1 - \mathcal{P}^h v_2\|_V \le \|v_1 - v_2\|_V \quad \forall\, v_1, v_2 \in V.$$

Thus for $v \in C([0,T];V)$, we have $\mathcal{P}^h v \in C([0,T];V)$.

HYPOTHESIS $\mathbf{H}_2$. *For any bounded set $B \subset V$, the functional $j(\cdot,\cdot)$ is uniformly continuous with respect to its second argument in $B \times V$, i.e., $\forall\, \varepsilon > 0$, $\exists\, \delta = \delta(B) > 0$ such that*

$$|j(g, v_1) - j(g, v_2)| < \varepsilon \quad \forall\, g \in B\ \forall\, v_1, v_2 \in V \text{ with } \|v_1 - v_2\|_V < \delta.$$

As preparation for convergence analysis, we first prove some lemmas.

LEMMA 5.1. *If V_0 is dense in V, then $C([0,T];V_0)$ is dense in $C([0,T];V)$.*

Proof. Let $v \in C([0,T];V)$. Then $v(t)$ is a uniformly continuous function. Thus, for any $\varepsilon > 0$, we can find an integer $n > 0$ such that

$$\|v(t) - v(s)\|_V < \frac{\varepsilon}{2} \quad \text{if } |t - s| \le \frac{1}{n}.$$

Denote $t_j = j\,T/n$, $0 \le j \le n$. For any j, we choose $z_j \in V_0$ satisfying

$$\|z_j - v(t_j)\|_V < \frac{\varepsilon}{2}.$$

Then we define a function $z : [0,T] \to V_0$ by the formula

$$z(t) = n\,(t_j - t)\, z_{j-1} + n\,(t - t_{j-1})\, z_j, \quad t_{j-1} \le t \le t_j,\ 1 \le j \le n.$$

Obviously, $z \in C([0,T];V_0)$. For $t \in [t_{j-1}, t_j]$, we have

$$z(t) - v(t) = n\,(t_j - t)\,(z_{j-1} - v(t)) + n\,(t - t_{j-1})\,(z_j - v(t)).$$

Thus

$$\begin{aligned} \|z(t) - v(t)\|_V &\le n\,(t_j - t)\,(\|z_{j-1} - v_{j-1}\|_V + \|v_{j-1} - v(t)\|_V) \\ &\quad + n\,(t - t_{j-1})\,(\|z_j - v_j\|_V + \|v_j - v(t)\|_V) \\ &< \varepsilon. \end{aligned}$$

Therefore,

$$\|z - v\|_{C([0,T];V)} < \varepsilon,$$

and $C([0,T];V_0)$ is dense in $C([0,T];V)$. ∎

LEMMA 5.2. *Assume Hypothesis* $\mathbf{H}_1$ *holds. Then for any* $v \in C([0,T];V)$*, we have the convergence*

$$\|v - \mathcal{P}^h v\|_{C([0,T];V)} \to 0 \quad \text{as } h \to 0.$$

Proof. For any $\varepsilon > 0$, using Lemma 5.1, we can find a $z \in C([0,T];V_0)$ such that

$$\|z - v\|_{C([0,T];V)} < \frac{\varepsilon}{4}.$$

By Hypothesis $\mathbf{H}_1$,

$$\|z(t) - \mathcal{P}^h z(t)\|_V \le \alpha(h)\,\|z(t)\|_{V_0}.$$

Therefore

$$\|z - \mathcal{P}^h z\|_{C([0,T];V)} \le \alpha(h)\,\|z\|_{C([0,T];V_0)}.$$

Since $\alpha(h) \to 0$ as $h \to 0+$, for sufficiently small h, we have

$$\|z - \mathcal{P}^h z\|_{C([0,T];V)} < \frac{\varepsilon}{2}.$$

Then, by the properties of $\mathcal{P}^h$, we have

$$\begin{aligned}\|v - \mathcal{P}^h v\|_{C([0,T];V)} &= \|(v-z) - \mathcal{P}^h(v-z) + (z - \mathcal{P}^h z)\|_{C([0,T];V)}\\ &\le 2\,\|v - z\|_{C([0,T];V)} + \|z - \mathcal{P}^h z\|_{C([0,T];V)}\\ &< \varepsilon,\end{aligned}$$

i.e., the result is true. ∎

Concerning the convergence of the semidiscrete solution we have the following result.

THEOREM 5.3. *Assume the conditions* (2.1)–(2.5), $\mathbf{H}_1$, $\mathbf{H}_2$ *and* $M > m$. *Then if*

$$\|u_0 - u_0^h\|_V \to 0 \quad \text{as } h \to 0,$$

the semidiscrete solution of the problem $\mathbf{P}^h$ *converges:*

$$\|u - u^h\|_{C^1([0,T];V)} \to 0 \quad \text{as } h \to 0.$$

Proof. Let us apply Theorem 3.1. We first bound the term $R(\cdot; v^h(\cdot), \dot u(\cdot))$. From the definition (3.9) and the properties of A, B, f and the solution u, we obtain

$$|R(t; v^h(t), \dot u(t))| \le c\,\|v^h(t) - \dot u(t)\|_V + |j(\dot u(t), v^h(t)) - j(\dot u(t), \dot u(t))|,$$

where the constant c depends on L_A, L_B, $\|f\|_{C([0,T];V)}$ and $\|u\|_{C^1([0,T];V)}$. Taking $v^h = \mathcal{P}^h \dot u \in C([0,T];V^h)$ in (3.13), we then have

$$\begin{aligned}\|u - u^h\|_{C^1([0,T];V)} \le c\big(&\|\dot u - \mathcal{P}^h \dot u\|_{C([0,T];V)} + \|\dot u - \mathcal{P}^h \dot u\|_{C([0,T];V)}^{1/2}\\ &+ \|j(\dot u, \mathcal{P}^h \dot u) - j(\dot u, \dot u)\|_{C[0,T]}^{1/2} + \|u_0 - u_0^h\|_V\big).\end{aligned}$$

By Lemma 5.2,

$$\|\dot{u} - \mathcal{P}^h \dot{u}\|_{C([0,T];V)} \to 0 \quad \text{as } h \to 0.$$

By Hypothesis $\mathbf{H}_2$ and Lemma 5.2,

$$\|j(\dot{u}, \mathcal{P}^h \dot{u}) - j(\dot{u}, \dot{u})\|_{C[0,T]} \to 0 \quad \text{as } h \to 0.$$

Therefore, $\|u - u^h\|_{C^1([0,T];V)} \to 0$ as $h \to 0$. ∎

Now we consider the convergence of the fully discrete solution. We need one more lemma.

LEMMA 5.4. *The quantity* $I_k(w)$ *defined in* (4.10) *converges to zero as* $k \to 0$.

Proof. Since $w \in C([0,T];V)$, $t \mapsto w(t)$ is uniformly continuous on $[0,T]$. Thus for any $\varepsilon > 0$, there exists a $k_0 > 0$ such that if $k < k_0$, we have

$$\|w(t) - w(s)\|_V < \frac{\varepsilon}{T} \quad \forall\, s, t \in [0,T],\ |t - s| < k.$$

Then, by the definition (4.10), we have

$$I_k(w) \le \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \frac{\varepsilon}{T}\, ds = \varepsilon.$$

Hence, $I_k(w) \to 0$ as $k \to 0$. ∎

THEOREM 5.5. *Assume the conditions* (2.1)–(2.5), $\mathbf{H}_1$, $\mathbf{H}_2$, *and* $M > m$. *Then if*

$$\|u_0 - u_0^h\|_V \to 0 \quad \text{as } h \to 0,$$

the fully discrete solution of the problem $\mathbf{P}^{hk}$ *converges:*

$$\max_{1 \le n \le N} (\|u_n - u_n^{hk}\|_V + \|\dot{u}_n - \delta u_n^{hk}\|_V) \to 0 \quad \text{as } h, k \to 0.$$

Proof. We take $v_n^h = \mathcal{P}^h \dot{u}_n$ in (4.17). Then the convergence result follows from Lemma 5.4 together with an argument similar to the proof of Theorem 5.3. ∎

6. Applications in frictional contact problems for viscoelastic materials.

In this section we apply the abstract results of sections 2–5 in the study of a frictional contact problem for viscoelastic materials.

6.1. The contact problem.

The physical setting is as follows. A viscoelastic body occupies an open, bounded, connected set $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3. The boundary $\Gamma = \partial\Omega$ is assumed to be Lipschitz continuous and has the decomposition $\Gamma = \cup_{i=1}^3 \overline{\Gamma}_i$ with mutually disjoint, relatively open sets Γ_1, Γ_2, and Γ_3, with Lipschitz relative boundaries if $d = 3$. We assume meas $(\Gamma_1) > 0$. We are interested in the evolution process of the mechanical state of the body in the time interval $[0,T]$ with $T > 0$. The body is clamped on $\Gamma_1 \times (0,T)$ and so the displacement field vanishes there. Surface tractions of density $\boldsymbol{f}_2$ act on $\Gamma_2 \times (0,T)$ and volume forces of density $\boldsymbol{f}_0$ act in $\Omega \times (0,T)$. We assume that the forces and tractions change slowly in time so that the acceleration of the system is negligible. Moreover, the body is in frictional contact with a rigid foundation on $\Gamma_3 \times (0,T)$. The constitutive law and the contact conditions on the contact surface are assumed as in [20] and will be discussed below.

Under these conditions, the classical formulation of the mechanical problem of frictional contact of the viscoelastic body is the following: Find a displacement $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$ such that

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega \times (0,T), \tag{6.1}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \mathbf{0} \quad \text{in } \Omega \times (0,T), \tag{6.2}$$

$$\boldsymbol{u} = \mathbf{0} \quad \text{on } \Gamma_1 \times (0,T), \tag{6.3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{6.4}$$

$$\left.\begin{array}{l} -\sigma_\nu = p_\nu(\dot{u}_\nu), \ |\boldsymbol{\sigma}_\tau| \le p_\tau(\dot{u}_\nu) \\ |\boldsymbol{\sigma}_\tau| < p_\tau(\dot{u}_\nu) \Rightarrow \dot{\boldsymbol{u}}_\tau = 0 \\ |\boldsymbol{\sigma}_\tau| = p_\tau(\dot{u}_\nu) \Rightarrow \boldsymbol{\sigma}_\tau = -\lambda\, \dot{\boldsymbol{u}}_\tau, \ \lambda \ge 0 \end{array}\right\} \quad \text{on } \Gamma_3 \times (0,T), \tag{6.5}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega. \tag{6.6}$$

Here $\mathbb{S}^d$ represents the space of second order symmetric tensors on $\mathbb{R}^d$. The relation (6.1) is the viscoelastic constitutive law in which $\mathcal{A}$ and $\mathcal{B}$ are given nonlinear operators, called the viscosity operator and elasticity operator, respectively. As usual, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the infinitesimal strain tensor. Relation (6.2) represents the equilibrium equation, (6.3) and (6.4) are the displacement-traction boundary conditions in which $\boldsymbol{\nu}$ represents the unit outward normal vector to Γ. The function $\boldsymbol{u}_0$ in (6.6) denotes the initial displacement.

We make some comments on the contact condition (6.5). Here σ_ν denotes the normal stress, $\boldsymbol{\sigma}_\tau$ represents the tangential traction, u_ν and $\boldsymbol{u}_\tau$ are the normal and tangential components of the displacement, respectively. Properties of the functions p_ν and p_τ will be listed below. In [19] the following form of the function p_ν is employed:

$$p_\nu(r) = \beta\, r_+ + p_0 \tag{6.7}$$

in order to model setting when the foundation is covered with a thin lubricant layer, say oil. Here β is the damping resistance coefficient, assumed positive, $r_+ = \max\{0, r\}$ and p_0 is the oil pressure, which is given and nonnegative. In this case the lubricant layer presents resistance, or damping, only when the surface moves towards the foundation, but does nothing when it recedes. Another choice of p_ν is

$$p_\nu(r) = S, \tag{6.8}$$

where S is a given positive function. This type of contact conditions in which the normal stress is prescribed arises in the study of some mechanisms and was considered by a number of authors (see, e.g., [7, 15]).

The conditions (6.5) represent an appropriate version of Coulomb's law of friction. They state that the tangential shear stress cannot exceed the maximal frictional resistance p_τ. When the strict inequality holds the surface adheres to the foundation and is in the so-called *stick* state; and when the equality holds then there is relative sliding between the surface and the foundation; this is the so-called *slip* state. Taking in (6.5)

$$p_\tau = \mu\, p_\nu \tag{6.9}$$

with $\mu \geq 0$, we obtain the classical Coulomb's law of friction (see, e.g., [7, 12]). We also remark that a modified version of Coulomb's law of friction has been recently derived in [22, 23], and it is in the form

$$p_\tau = \mu p_\nu (1 - \delta p_\nu)_+, \tag{6.10}$$

where δ is a small positive material constant related to the wear and penetration hardness of the surface, and μ is the coefficient of friction, assumed positive. This change in Coulomb's law means that when the magnitude $|\sigma_\nu|$ of the normal stress exceeds $1/\delta$ the surface disintegrates and offers no resistance to the motion.

We denote in what follows by "$\cdot$" and $|\cdot|$ the inner product and the Euclidean norm on the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$ and we introduce the spaces

$$\begin{aligned} V &= \{\boldsymbol{v} = (v_i) \in (H^1(\Omega))^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\},\\ Q &= \{\boldsymbol{\tau} = (\tau_{ij}) \in (L^2(\Omega))^{d\times d} : \tau_{ij} = \tau_{ji},\ 1 \leq i,j \leq d\},\\ Q_1 &= \{\boldsymbol{\tau} \in Q : \operatorname{Div} \boldsymbol{\tau} \in (L^2(\Omega))^d\}. \end{aligned}$$

These are real Hilbert spaces with their canonical inner products. Since meas$\,(\Gamma_1) > 0$, Korn's inequality holds:

$$\|\boldsymbol{v}\|_{(H^1(\Omega))^d} \leq c_K\, \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_Q \quad \forall\, \boldsymbol{v} \in V. \tag{6.11}$$

Here $c_K > 0$ is a constant depending only on Ω and Γ_1 and $\boldsymbol{\varepsilon} : H^1(\Omega)^d \to Q$ is the deformation operator. A proof of Korn's inequality can be found in, for instance, [14, p. 79].

Over the space V, we use the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V. \tag{6.12}$$

It follows from (6.11) that $\|\cdot\|_{H^1(\Omega)^d}$ and $\|\cdot\|_V$ are equivalent norms on V and therefore $(V, \|\cdot\|_V)$ is a real Hilbert space.

Finally, $\forall \boldsymbol{v} \in V$ we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ given by

$$v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}, \quad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}.$$

In the study of the mechanical problem (6.1)–(6.6) we assume that the *viscosity operator* $\mathcal{A}$ and the *elasticity operator* $\mathcal{B}$ satisfy

$$\left\{ \begin{array}{l} \text{(a) } \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d.\\ \text{(b) There exists } L_A > 0 \text{ such that}\\ \qquad |\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)| \leq L_A |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| \quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_1 \in \mathbb{S}^d,\\ \qquad \text{almost everywhere (a.e.) } \boldsymbol{x} \in \Omega.\\ \text{(c) There exists } M > 0 \text{ such that}\\ \qquad (\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq M\, |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2 \quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_1 \in \mathbb{S}^d,\\ \qquad \text{a.e. } \boldsymbol{x} \in \Omega.\\ \text{(d) For any } \boldsymbol{\varepsilon} \in \mathbb{S}^d,\ \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega.\\ \text{(e) The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{0}) \in Q. \end{array} \right. \tag{6.13}$$

$$(6.14)\qquad \begin{cases} \text{(a) } \mathcal{B} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) There exists an } L_B > 0 \text{ such that} \\ \qquad |\mathcal{B}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{B}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)| \le L_B \, |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| \quad \forall \, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d \text{ a.e. in } \Omega. \\ \text{(c) For any } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \, \boldsymbol{x} \mapsto \mathcal{B}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is measurable.} \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{B}(\boldsymbol{x}, \boldsymbol{0}) \in Q. \end{cases}$$

The contact functions p_r $(r = \nu, \tau)$ satisfy

$$(6.15)\qquad \begin{cases} \text{(a) } p_r : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) There exists an } L_r > 0 \text{ such that} \\ \qquad |p_r(\boldsymbol{x}, u_1) - p_r(\boldsymbol{x}, u_2)| \le L_r \, |u_1 - u_2| \quad \forall \, u_1, u_2 \in \mathbb{R} \text{ a.e. in } \Omega. \\ \text{(c) For any } u \in \mathbb{R}, \, \boldsymbol{x} \mapsto p_r(\boldsymbol{x}, u) \text{ is measurable.} \\ \text{(d) The mapping } \boldsymbol{x} \mapsto p_r(\boldsymbol{x}, 0) \in L^2(\Gamma_3). \end{cases}$$

We observe that the assumptions (6.15) on the functions p_ν and p_τ are pretty general. The only severe restriction comes from the condition (b), which, roughly speaking, requires the functions to grow at most linearly. Certainly the functions defined in (6.7) and (6.8) satisfy the condition (6.15)(b). We also observe that if the functions p_ν and p_τ are related by (6.9) or (6.10) and p_ν satisfies the condition (6.15)(b), then p_τ also satisfies the condition (6.15)(b) with $L_\tau = \mu \, L_\nu$.

We also assume that the forces and tractions satisfy

$$(6.16)\qquad \boldsymbol{f}_0 \in C([0,T]; (L^2(\Omega))^d), \quad \boldsymbol{f}_2 \in C([0,T]; (L^2(\Gamma_2))^d),$$

and finally

$$(6.17)\qquad \boldsymbol{u}_0 \in V.$$

Next we denote by $\boldsymbol{f}(t)$ the element of V given by

$$(6.18)\qquad (\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v} \, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v} \, da$$

$\forall \, \boldsymbol{v} \in V$ and $t \in [0,T]$, and we note that conditions (6.16) imply

$$(6.19)\qquad \boldsymbol{f} \in C([0,T]; V).$$

Let $j : V \times V \to \mathbb{R}$ be the functional

$$(6.20)\qquad j(\boldsymbol{v}, \boldsymbol{w}) = \int_{\Gamma_3} p_\nu(v_\nu) \, w_\nu \, da + \int_{\Gamma_3} p_\tau(v_\nu) \, |\boldsymbol{w}_\tau| \, da \quad \forall \, \boldsymbol{v}, \boldsymbol{w} \in V.$$

With these notations, it follows from [20] that if $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ are sufficiently regular functions satisfying (6.1)–(6.6), then $\boldsymbol{u}(t) \in V$ and $\forall \, t \in [0,T]$,

$$(6.21)\qquad \begin{aligned} &(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_V + (\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_V \\ &\quad + j(\dot{\boldsymbol{u}}(t), \boldsymbol{v}) - j(\dot{\boldsymbol{u}}(t), \dot{\boldsymbol{u}}(t)) \ge (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \quad \forall \, \boldsymbol{v} \in V. \end{aligned}$$

Thus we obtain the following variational formulation of problem (6.1)–(6.6) in terms of displacements.

Problem $\mathbf{P}_0$. *Find a displacement* $\boldsymbol{u} : \Omega \times [0,T] \to V$ *which satisfies* (6.21) *and* (6.6).

The well-posedness of the problem $\mathbf{P}_0$ follows from an application of Theorem 2.1.

THEOREM 6.1. *Assume that* (6.13)–(6.17) *hold. Then there exists* $L_0 > 0$ *which depends only on* Ω, Γ_1, Γ_3 *and* $\mathcal{A}$ *such that the problem* $\mathbf{P}_0$ *has a unique solution if* $L_\nu + L_\tau < L_0$. *Moreover, the solution satisfies* $\boldsymbol{u} \in C^1([0,T];V)$.

Proof. Let $A : V \to V$ and $B : V \to V$ be the operators defined by

$$(Av, w)_V = (\mathcal{A}\varepsilon(v), \varepsilon(w))_Q, \quad (Bv, w)_V = (\mathcal{B}\varepsilon(v), \varepsilon(w))_Q \tag{6.22}$$

$\forall\, \boldsymbol{v}, \boldsymbol{w} \in V$. Using (6.13) and (6.14) it follows that A and B are Lipschitz continuous operators. Using again (6.13) and (6.12) we deduce that A is a strongly monotone operator on V:

$$(A\boldsymbol{v}_1 - A\boldsymbol{v}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2)_V \geq M\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V^2 \quad \forall\, \boldsymbol{v}_1\, \boldsymbol{v}_2 \in V. \tag{6.23}$$

Moreover, from (6.15), (6.11), and (6.12) it follows that the function j defined by (6.20) satisfies (2.3) and

$$\begin{aligned} &j(\boldsymbol{g}_1, \boldsymbol{v}_2) - j(\boldsymbol{g}_1, \boldsymbol{v}_1) + j(\boldsymbol{g}_2, \boldsymbol{v}_1) - j(\boldsymbol{g}_2, \boldsymbol{v}_2) \\ &\quad \leq c_0(L_\nu + L_\tau)\, \|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_V\, \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V \quad \forall\, \boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{v}_1, \boldsymbol{v}_2 \in V, \end{aligned} \tag{6.24}$$

where $c_0 > 0$ depends only on Ω, Γ_1 and Γ_3. Applying Theorem 2.1, we conclude that if

$$c_0(L_\nu + L_\tau) < M$$

then the problem $\mathbf{P}_0$ has a unique solution $\boldsymbol{u} \in C^1([0,T];V)$, and we may take $L_0 = M/c_0$. ◻

Now let $\boldsymbol{u} \in C^1([0,T];V)$ be the solution of the problem $\mathbf{P}_0$ and let $\boldsymbol{\sigma}$ be the stress field given by (6.1). Using (6.21) and (6.16) it can be shown that $\operatorname{Div}\boldsymbol{\sigma} \in C([0,T];L^2(\Omega)^d)$ and therefore $\boldsymbol{\sigma} \in C([0,T];Q_1)$. A pair of functions $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ which satisfies (6.1), (6.6), and (6.21) is called a *weak solution* of the problem (6.1)–(6.6). We conclude that the problem (6.1)–(6.6) has a unique weak solution provided $L_\nu + L_\tau$ is sufficiently small, which represents a result already obtained in [20]. The critical value L_0 depends only on the viscosity operator and on the geometry of the problem but does not depend on the elasticity operator, nor on the external forces, nor on the initial displacement.

We end this section with some mechanical interpretation of the condition $L_\nu + L_\tau < L_0$ which guarantees the unique solvability of the problem $\mathbf{P}_0$. The verification of this condition as well as its interpretation depends on the specific mechanical problem. For example, consider the mechanical problem (6.1)–(6.6) in which the function p_ν is given by (6.7) and the function p_τ is given by (6.9) or by (6.10). It follows that assumption (6.15)(b) is satisfied with $L_\nu = \beta$ and $L_\tau = \mu\beta$ and therefore the condition $L_\nu + L_\tau < L_0$ holds if $\beta \leq L_0/(\mu + 1)$ which may be interpreted as a smallness assumption on the damping resistance coefficient. We conclude that the corresponding mechanical problem has a unique weak solution if the damping resistance coefficient of the oil layer is small enough. Consider now the mechanical problem (6.1)–(6.6) in the case when the function p_ν is given by (6.8) with $S \in L^\infty(\Gamma_3)$ and the function p_τ is given by (6.9) or by (6.10). In this case the assumption (6.15)(b) is satisfied with $L_\nu = L_\tau = 0$ and therefore the condition $L_\nu + L_\tau < L_0$ trivially holds. We conclude that the corresponding mechanical problem has a unique weak solution without any supplementary restriction on the coefficients μ and δ.

6.2. Numerical approximations. Now we state some sample results on error estimates for numerical approximations of the problem $\mathbf{P}_0$.

We first briefly describe how to construct the finite dimensional space V^h via the finite element method. Details can be found in [3]. For simplicity, we assume Ω is a polygon or polyhedron. We have $\overline{\Gamma}_3 = \cup_{i=1}^{I} \overline{\Gamma}_{3,i}$ with each piece $\overline{\Gamma}_{3,i}$ represented by an affine function. Let $\mathcal{T}^h$ be a regular finite element partition of Ω in such a way that if a side of an element lies on the boundary, the side belongs entirely to one of the subsets $\overline{\Gamma}_1$, $\overline{\Gamma}_2$ and $\overline{\Gamma}_3$. Let h be the maximal diameter of the elements. Let $V^h \subset V$ be the finite element space consisting of piecewise linear functions, corresponding to the partition $\mathcal{T}^h$. If the solution $\boldsymbol{u}$ has higher regularity, we may use higher order elements, and the error analysis presented below can be easily extended to such a situation.

For convergence analysis under the basic solution regularity, we need to verify Hypotheses $\mathbf{H}_1$ and $\mathbf{H}_2$. The following density result is proved in [6].

PROPOSITION 6.2. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be an open, bounded, Lipschitz domain, and let $\Gamma_1 \subset \partial\Omega$ be a relatively open set with a Lipschitz relative boundary. Then the space* $\{v \in C^\infty(\overline{\Omega}) : v = 0 \text{ in a neighborhood of } \Gamma_1\}$ *is dense in* $\{v \in H^1(\Omega) : v = 0 \text{ a.e. on } \Gamma_1\}$.

From this proposition, we see immediately that the space

$$V_0 = \{\boldsymbol{v} \in [C^\infty(\overline{\Omega})]^d : \boldsymbol{v} = \mathbf{0} \text{ in a neighborhood of } \Gamma_1\}$$

is dense in V.

Let $\Pi^h : V \to V^h$ be the piecewise linear interpolation operator. Then

$$\|\boldsymbol{v} - \mathcal{P}^h \boldsymbol{v}\|_V \leq \|\boldsymbol{v} - \Pi^h \boldsymbol{v}\|_V \leq c\, h\, \|\boldsymbol{v}\|_{H^2(\Omega)} \quad \forall\, \boldsymbol{v} \in V_0.$$

Thus, Hypothesis $\mathbf{H}_1$ is valid.

The functional $j(\boldsymbol{g}, \boldsymbol{v})$ defined in (6.20) is Lipschitz continuous with respect to $\boldsymbol{v}$. Then it follows from (6.15) that Hypothesis $\mathbf{H}_2$ is valid.

We first consider a spatially semidiscrete approximation of the problem $\mathbf{P}_0$.

PROBLEM $\mathbf{P}_0^h$. Find the displacement field $\boldsymbol{u}^h : [0,T] \to V^h$, such that for $t \in [0,T]$,

$$\begin{aligned}(6.25)\quad & (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^h(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^h(t)))_Q + (\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}^h(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^h(t)))_Q \\ & + j(\dot{\boldsymbol{u}}^h(t), \boldsymbol{v}^h) - j(\dot{\boldsymbol{u}}^h(t), \dot{\boldsymbol{u}}^h(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v}^h - \dot{\boldsymbol{u}}^h(t))_V \quad \forall\, \boldsymbol{v}^h \in V^h,\end{aligned}$$

$$(6.26)\qquad \boldsymbol{u}^h(0) = \boldsymbol{u}_0^h,$$

where $\boldsymbol{u}_0^h \in V^h$ is a suitable approximation of $\boldsymbol{u}_0$.

From the discussions in section 3, the problem $\mathbf{P}_0^h$ has a unique solution. By Theorem 5.3, if we choose $\boldsymbol{u}_0^h$ such that $\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V \to 0$, then we have the convergence

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_{C^1([0,T];V)} \to 0 \quad \text{as } h \to 0.$$

For convergence order error estimates, we have the following estimate from (3.13):

$$\begin{aligned}(6.27)\quad & \|\boldsymbol{u} - \boldsymbol{u}^h\|_{C^1([0,T];V)} \\ & \leq c \inf_{\boldsymbol{v}^h \in C([0,T];V^h)} \left(\|\dot{\boldsymbol{u}} - \boldsymbol{v}^h\|_{C([0,T];V)} + \|R(\cdot; \boldsymbol{v}^h(\cdot), \dot{\boldsymbol{u}}(\cdot))\|_{C([0,T])}^{1/2} \right) \\ & \quad + c\, \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V.\end{aligned}$$

Let us present some sample error estimates under additional solution smoothness assumptions. Assume

$$\boldsymbol{\sigma\nu} \in C([0,T];(L^2(\Gamma))^d), \tag{6.28}$$

then the following equalities hold for $t \in [0,T]$:

$$\mathrm{Div}\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{a.e. in } \Omega, \tag{6.29}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{a.e. on } \Gamma_2, \tag{6.30}$$

$$-\sigma_\nu(t) = p_\nu(\dot{u}_\nu(t)) \quad \text{a.e. on } \Gamma_3. \tag{6.31}$$

Using assumptions (6.15)–(6.16), the proof of (6.29)–(6.31) follows from standard arguments (see, e.g., [12]).

By the definition (3.9) and (6.29)–(6.31), we then have

$$\begin{aligned} R(t;\boldsymbol{v}^h(t),\dot{\boldsymbol{u}}(t)) &= (\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{v}^h(t)) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q + j(\dot{\boldsymbol{u}}(t),\boldsymbol{v}^h(t)) - j(\dot{\boldsymbol{u}}(t),\dot{\boldsymbol{u}}(t)) \\ &\quad - (\boldsymbol{f}(t),\boldsymbol{v}^h(t) - \dot{\boldsymbol{u}}(t))_V \\ &= \int_{\Gamma_3} \Big(\boldsymbol{\sigma}_\tau(t)\cdot(\boldsymbol{v}^h_\tau(t) - \dot{\boldsymbol{u}}_\tau(t)) + p_\tau(\dot{u}_\nu(t))\,(|\boldsymbol{v}^h_\tau(t)| - |\dot{\boldsymbol{u}}_\tau(t)|)\Big)\, da. \end{aligned}$$

Since $p_\tau(\dot{u}_\nu) \in C([0,T];L^2(\Gamma_3))$ from the regularity of $\boldsymbol{u}$ and the properties of p_τ, we have

$$|R(t;\boldsymbol{v}^h(t),\dot{\boldsymbol{u}}(t))| \le c\,\|\boldsymbol{v}^h_\tau(t) - \dot{\boldsymbol{u}}_\tau(t)\|_{(L^2(\Gamma_3))^d}. \tag{6.32}$$

Therefore, the estimate (6.27) in this case reduces to

$$\begin{aligned} &\|\boldsymbol{u} - \boldsymbol{u}^h\|_{C^1([0,T];V)} \\ &\quad \le c \inf_{\boldsymbol{v}^h\in C([0,T];V^h)} \Big(\|\dot{\boldsymbol{u}} - \boldsymbol{v}^h\|_{C([0,T];V)} + \|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^h_\tau\|^{1/2}_{C([0,T];L^2(\Gamma_3)^d)} \Big) \\ &\qquad + c\,\|\boldsymbol{u}_0 - \boldsymbol{u}^h_0\|_V. \end{aligned}$$

Assume

$$\dot{\boldsymbol{u}} \in C([0,T];H^2(\Omega)^d). \tag{6.33}$$

Then

$$\dot{\boldsymbol{u}}_\tau|_{\Gamma_{3,i}} \in C([0,T];H^{3/2}(\Gamma_{3,i})^d),\ 1 \le i \le I.$$

We use $\Pi^h\dot{\boldsymbol{u}}(t)$ to denote the piecewise Lagrange interpolant of $\dot{\boldsymbol{u}}(t)$ (cf. [3]), and use the same symbol Π^h for the interpolation on Γ_3. Then we have the interpolation error estimates for $t \in [0,T]$,

$$\|\dot{\boldsymbol{u}}(t) - \Pi^h\dot{\boldsymbol{u}}(t)\|_V \le c\,h\,\|\dot{\boldsymbol{u}}(t)\|_{H^2(\Omega)^d}, \tag{6.34}$$

$$\|\dot{\boldsymbol{u}}_\tau(t) - \Pi^h\dot{\boldsymbol{u}}_\tau(t)\|_{L^2(\Gamma_3)^d} \le c\,h^{3/2}\sum_{i=1}^{I}\|\dot{\boldsymbol{u}}_\tau(t)\|_{H^{3/2}(\Gamma_{3,i})^d}. \tag{6.35}$$

Assume the initial value satisfies

$$\boldsymbol{u}_0 \in H^2(\Omega)^d. \tag{6.36}$$

Then

$$\|\boldsymbol{u}_0 - \Pi^h \boldsymbol{u}_0\|_V \le c\, h\, |\boldsymbol{u}_0|_{H^2(\Omega)^d}. \tag{6.37}$$

Summarizing under the additional assumptions (6.28), (6.33), and (6.36), if we take $\boldsymbol{u}_0^h = \Pi^h \boldsymbol{u}_0$, then using the estimates (6.27), (6.32), (6.34), (6.35), and (6.37), we have the error estimate

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_{C^1([0,T];V)} \le O(h^{3/4}). \tag{6.38}$$

If we further assume

$$\dot{\boldsymbol{u}}_\tau|_{\Gamma_{3,i}} \in C([0,T]; H^2(\Gamma_{3,i})^d),\ 1 \le i \le I, \tag{6.39}$$

then the estimate (6.35) can be replaced by

$$\|\dot{\boldsymbol{u}}_\tau(t) - \Pi^h \dot{\boldsymbol{u}}_\tau(t)\|_{L^2(\Gamma_3)^d} \le c\, h^2 \sum_{i=1}^{I} \|\dot{\boldsymbol{u}}_\tau(t)\|_{H^2(\Gamma_{3,i})^d},$$

and we have the optimal order error estimate

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_{C^1([0,T];V)} \le O(h). \tag{6.40}$$

For fully discrete approximations, we need the partition of the time interval introduced in section 4. Then a fully discrete approximation for the problem $\mathbf{P}_0$ is what follows.

PROBLEM $\mathbf{P}_0^{hk}$. Find the displacement field $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset V^h$ such that for $n = 1, \ldots, N$,

$$\begin{aligned}(\mathcal{A}\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\delta \boldsymbol{u}_n^{hk}))_Q + (\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\delta \boldsymbol{u}_n^{hk}))_Q \\ + j(\delta \boldsymbol{u}_n^{hk}, \boldsymbol{v}^h) - j(\delta \boldsymbol{u}_n^{hk}, \delta \boldsymbol{u}_n^{hk}) \ge (\boldsymbol{f}(t), \boldsymbol{v}^h - \delta \boldsymbol{u}_n^{hk})_V \qquad \forall\, \boldsymbol{v}^h \in V^h,\end{aligned} \tag{6.41}$$

$$\boldsymbol{u}_0^{hk} = \boldsymbol{u}_0^h, \tag{6.42}$$

where again $\boldsymbol{u}_0^h \in V^h$ is a suitable approximation of $\boldsymbol{u}_0$.

From the discussions in section 4, the problem $\mathbf{P}_0^{hk}$ has a unique solution. By Theorem 5.5, if we choose $\boldsymbol{u}_0^h$ such that $\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V \to 0$, then we have the convergence

$$\max_{1 \le n \le N} (\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V + \|\dot{\boldsymbol{u}}_n - \delta \boldsymbol{u}_n^{hk}\|_V) \to 0 \quad \text{as } h, k \to 0.$$

For order error estimate, assuming $\boldsymbol{f} \in W^{1,1}(0,T;V)$, we have the following estimate from (4.17):

$$\begin{aligned}&\max_{1 \le n \le N} (\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V + \|\dot{\boldsymbol{u}}_n - \delta \boldsymbol{u}_n^{hk}\|_V) \\ &\quad \le c \max_{1 \le n \le N} \inf_{\boldsymbol{v}_n^h \in V^h} \left\{ \|\boldsymbol{v}_n^h - \dot{\boldsymbol{u}}_n\|_V + |R_n(\boldsymbol{v}_n^h, \dot{\boldsymbol{u}}_n)|^{1/2} \right\} + c\, \|\boldsymbol{u}_0^h - \boldsymbol{u}_0\|_V \\ &\qquad + c\, k\, (\|\boldsymbol{f}\|_{W^{1,1}(0,T;V)} + \|\boldsymbol{u}\|_{C^1([0,T];V)}).\end{aligned} \tag{6.43}$$

The term $R_n(\boldsymbol{v}_n^h, \dot{\boldsymbol{u}}_n)$ is defined in (4.7). Similar to (6.32), we have

$$|R_n(\boldsymbol{v}_n^h, \dot{\boldsymbol{u}}_n)| \le c\, \|(\boldsymbol{v}_n^h)_\tau - (\dot{\boldsymbol{u}}_n)_\tau\|_{(L^2(\Gamma_3))^d}.$$

Then under the additional regularity conditions (6.33) and (6.36), we have the following error estimate for the fully discrete solution:

$$\max_{1\le n\le N}(\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V+\|\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n^{hk}\|_V)\le O(h^{3/4}+k). \tag{6.44}$$

If we further assume (6.39), then we have the optimal order error estimate

$$\max_{1\le n\le N}(\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V+\|\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n^{hk}\|_V)\le O(h+k). \tag{6.45}$$

We emphasize that the error estimates (6.38), (6.40), (6.44), and (6.45) are only sample results under the stated regularity conditions. If the regularity conditions are different, the error estimates need to be changed accordingly, but that follows easily from (6.27) and (6.43). In particular when $\dot{\boldsymbol{u}}(t)\notin C(\overline{\Omega})$, we should use Clément's interpolation operator (cf. [4]) or projection operator (cf. [16]) to replace the piecewise Lagrange interpolation operator in the error estimations.

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