SIAM J. NUMER. ANAL.
Vol. 40, No. 5, pp. 1683–1697

NONCONFORMING FINITE ELEMENT ANALYSIS FOR A PLATE CONTACT PROBLEM*

WEIMIN HAN† AND LIE-HENG WANG‡

Abstract. In this paper we analyze nonconforming finite element methods for solving a fourth order elliptic variational inequality of the second kind arising in a plate frictional contact problem. The variational inequality involves a nondifferentiable term due to the frictional contact. Optimal order error estimates are derived for both continuous and discontinuous nonconforming finite elements.

Key words. nonconforming finite element method, elliptic variational inequality of fourth order, plate frictional contact problem, optimal order error estimate

AMS subject classifications. 65N30, 74S05

PII. S0036142901390731

1. Introduction.

Variational inequalities form an important family of nonlinear boundary value or initial-boundary value problems. Interest in variational inequalities originates in applications from mechanics and physics. A partial list of the applications that lead to variational inequalities include the following: contact mechanics, non-Newtonian fluid flows such as Bingham fluids, obstacle problems, optimal control, plasticity, Stefan problems, unilateral problems, and so on. An early comprehensive reference on the topic is [8], where many problems in mechanics and physics are formulated and studied in the framework of variational inequalities. More recent references on the mathematical analysis of variational inequalities include [1, 10, 20, 22, 23]. Comprehensive references concerning the numerical analysis of variational inequalities, especially those arising in mechanical problems, include [12, 13, 14, 15, 16, 17, 19]. These references focus on numerical analysis for variational inequalities involving second order differential operators. Numerical study of fourth order variational inequalities is also available from some of these references; e.g., a duality approach based on conforming finite elements is analyzed in [16].

Nonconforming finite element methods are a natural choice in employing finite element methods for solving fourth order boundary value problems since the smoothness requirement on finite element functions is weakened. An early reference on the mathematical analysis of nonconforming finite element methods for the plate bending problem is [21]. Application of nonconforming finite element methods is not limited to fourth order problems; they offer more efficient solution algorithms for numerous other problems (cf. [3, p. 208]). Convergence and error estimation of nonconforming finite element methods are more involved compared to that of conforming finite element methods. A patch test was proposed and is widely used by engineers for convergence analysis of nonconforming finite element methods (cf. [2, 18]). However, it

*Received by the editors June 12, 2001; accepted for publication (in revised form) April 16, 2002; published electronically November 14, 2002.

http://www.siam.org/journals/sinum/40-5/39073.html

†Department of Mathematics, University of Iowa, Iowa City, IA 52242 (whan@math.uiowa.edu). The work of this author was supported by K. C. Wong Education Foundation, and the National Science Foundation under grants DMS-9874015 and DMS-0106781.

‡State Key Laboratory of Scientific and Engineering Computing, ICMSEC, Chinese Academy of Sciences, Beijing 100080, People's Republic of China (wlh@lsec.cc.ac.cn). The work of this author was supported by the National Natural Science Foundation of China.

is shown in [26] that the patch test is neither a necessary nor a sufficient condition for convergence. One finds in [26] a rigorous necessary and sufficient condition for convergence of nonconforming finite element solutions to variational equations of some boundary value problems. Some further developments along this line can be found in [25, 30], where convergence conditions are studied which are easier to examine. A summary account of nonconforming finite element methods can be found in [5] or, more recently, in [6]. In particular, in these latter references, one can find some discussions of the four nonconforming finite elements mentioned later in this paper: the continuous nonconforming elements of the Zienkiewicz triangle and Adini's rectangle, and the discontinuous nonconforming elements of Morley's triangle and the Fraeijs De Veubeke triangle.

In this paper, we derive error estimates for continuous and discontinuous nonconforming finite elements in solving a fourth order elliptic variational inequality of the second kind. A variational inequality of the second kind is featured by the presence of nondifferentiable terms in the formulation. Variational inequalities of the second kind are commonly seen in frictional contact problems. In this paper we adopt a plate frictional contact problem as our model fourth order variational inequality of the second kind for error analysis of nonconforming finite element methods; the ideas and results reported here can be extended to nonconforming finite element methods for other fourth order elliptic variational inequalities of the second kind. Literature on nonconforming finite element methods for fourth order variational inequalities is rather small at the moment. The only papers on this topic we know of are [27, 28, 29]. Note that in these papers the variational inequalities being approximated are of the first kind; i.e., they are imposed over convex sets, and no nondifferentiable terms are involved. To analyze nonconforming finite elements for fourth order variational inequalities of the second kind, we need to employ new techniques.

The paper is organized as follows. In section 2, we introduce the plate contact problem and show some properties for the solution of the problem. In section 3, we present an abstract result for nonconforming methods that will be used in deriving error estimates later in the paper. Sections 4 and 5 are devoted to error estimation of continuous and discontinuous nonconforming finite element methods for the plate contact problem, respectively.

2. The plate contact problem. Consider a thin flat plate $\Omega \times (-d/2, d/2)$, where $\Omega \subset \mathbb{R}^2$, $d > 0$ is the thickness of the plate and is assumed to be small. Assume the three-dimensional material is isotropic, linearly elastic with Poisson's ratio $\nu \in (0, 1/2)$ and Young's modulus $E > 0$. The plate is subject to a normal force of density $D_0 f(\boldsymbol{x})$ with the stiffness coefficient of the plate

$$D_0 = \frac{E\, d^3}{12\,(1-\nu^2)}.$$

Denote by $u = u(\boldsymbol{x})$, $\boldsymbol{x} \in \overline{\Omega}$, the vertical deflection of the plate. Let the boundary $\Gamma = \partial\Omega$ of the plate be decomposed into three mutually disjoint parts: $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$ such that Γ_1, Γ_2, and Γ_3 are relatively open, $\overline{\Gamma}_1 \cap \overline{\Gamma}_3 = \emptyset$, and $\operatorname{meas}(\Gamma_1) > 0$. The boundary is assumed to be Lipschitz continuous, and the unit outward normal vector is denoted by $\boldsymbol{n} = (n_1, n_2)^T$. The tangential vector is $\boldsymbol{\tau} = (\tau_1, \tau_2)^T$ with $\tau_1 = -n_2$, $\tau_2 = n_1$. Both $\boldsymbol{n}$ and $\boldsymbol{\tau}$ exist a.e. on Γ. Assume the plate is clamped on Γ_1:

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_1,$$

is free on Γ_2:

$$M(u) = N(u) = 0 \quad \text{on } \Gamma_2,$$

and is in frictional contact with a rigid foundation on Γ_3. Here,

$$M(u) = -\Delta u + (1-\nu)\,\partial_{\tau\tau} u,$$
$$N(u) = \partial_n \Delta u + (1-\nu)\,\partial_\tau(\partial_{n\tau} u).$$

Notice that for a smooth function u, $M(u)$ and $N(u)$ are defined a.e. on Γ. The quantity $M(u)$ can be interpreted as the tangential moment, while $-N(u)$ represents a force. Here and throughout the paper, we use the following notations:

$$\partial_{11} u = \frac{\partial^2 u}{\partial x_1^2}, \quad \partial_{12} u = \frac{\partial^2 u}{\partial x_1 \partial x_2}, \quad \partial_n u = \frac{\partial u}{\partial n}, \quad \partial_\tau u = \frac{\partial u}{\partial \tau}, \quad \cdots .$$

Introduce the function space

$$V = \{\, v \in H^2(\Omega) \mid v = \partial_n v = 0 \text{ on } \Gamma_1 \,\}. \tag{2.1}$$

Over the space V, we define a bilinear form

$$a(u,v) = \int_\Omega \left[\Delta u\, \Delta v + (1-\nu)\,(2\,\partial_{12} u\, \partial_{12} v - \partial_{11} u\, \partial_{22} v - \partial_{22} u\, \partial_{11} v)\right] dx, \tag{2.2}$$

and a functional

$$j(v) = \int_{\Gamma_3} g\,|v|\,ds, \tag{2.3}$$

where g is given. For the data of the problem, we assume

$$f \in L^2(\Omega), \quad g \in L^2(\Gamma_3),\ g > 0 \text{ a.e. on } \Gamma_3. \tag{2.4}$$

We will use the notation

$$(f,v) = \int_\Omega f\, v\, dx.$$

The plate frictional contact problem is defined through a minimal energy principle:

$$u \in V,\ J(u) = \inf_{v \in V} J(v),$$

where

$$J(v) = \frac{1}{2}\, a(v,v) + j(v) - (f,v).$$

The quantity $D_0 J(v)$ is the total energy, and $j(v)$ is the contribution from the frictional contact. It is easy to show that the minimization problem is equivalent to the following variational inequality.

Problem 2.1. *Find $u \in V$ such that*

$$a(u, v-u) + j(v) - j(u) \ge (f, v-u) \quad \forall\, v \in V. \tag{2.5}$$

Wellposedness of Problem 2.1 follows from a standard argument.

THEOREM 2.2. *Problem* 2.1 *has a unique solution.*

Proof. Since meas $(\Gamma_1) > 0$, the bilinear form is coercive on V:

$$a(v,v) \ge \alpha \, \|v\|_V^2 \quad \forall \, v \in V.$$

We also observe that over the space V, $a(\cdot,\cdot)$ is continuous, $j(\cdot)$ is continuous and convex, and f defines a linear continuous functional. Thus Problem 2.1 is an elliptic variational inequality of the second kind and has a unique solution (cf. [12]). ∎

To obtain the corresponding strong form of the boundary value problem, we assume the solution u is smooth (say, $u \in C^4(\overline{\Omega})$). For any $v \in H^2(\Omega)$, we have

$$\int_\Omega \Delta^2 u \, v \, dx = \int_\Omega \Delta u \, \Delta v \, dx + \int_\Gamma \partial_n \Delta u \, v \, ds - \int_\Gamma \Delta u \, \partial_n v \, ds. \tag{2.6}$$

It is easy to verify the equality

$$\int_\Omega (2\,\partial_{12} u \, \partial_{12} v - \partial_{11} u \, \partial_{22} v - \partial_{22} u \, \partial_{11} v) \, dx = \int_\Gamma \left(-\partial_{\tau\tau} u \, \partial_n v + \partial_{n\tau} u \, \partial_\tau v \right) \, ds.$$

If the boundary Γ is smooth, we further have

$$\int_\Omega (2\,\partial_{12} u \, \partial_{12} v - \partial_{11} u \, \partial_{22} v - \partial_{22} u \, \partial_{11} v) \, dx = -\int_\Gamma \left(\partial_{\tau\tau} u \, \partial_n v + \partial_\tau (\partial_{n\tau} u) \, v \right) \, ds. \tag{2.7}$$

Then from (2.6) we have

$$a(u,v) = \int_\Omega \Delta^2 u \, v \, dx - \int_\Gamma N(u) \, v \, ds - \int_\Gamma M(u) \, \partial_n v \, ds \quad \forall \, v \in H^2(\Omega). \tag{2.8}$$

Using this relation in the variational inequality (2.5), we can follow a standard argument (cf., e.g., [8, 19]) to conclude that u satisfies the relations

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{2.9}$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_1, \tag{2.10}$$

$$M(u) = N(u) = 0 \quad \text{on } \Gamma_2, \tag{2.11}$$

$$\left. \begin{array}{l} M(u) = 0, \quad |N(u)| \le g, \\ |N(u)| < g \Longrightarrow u = 0, \\ |N(u)| = g \Longrightarrow u = \lambda \, N(u) \text{ for some } \lambda \ge 0 \end{array} \right\} \text{on } \Gamma_3. \tag{2.12}$$

This is the strong form of the plate contact problem studied in [8]. We comment that $g > 0$ can be interpreted as the frictional bound.

When the boundary Γ is only piecewise smooth, the right-hand side of the relation (2.7) needs to be replaced by

$$\sum_P \big(\partial_{n\tau} u(P_-) - \partial_{n\tau} u(P_+) \big) \, v(P) - \int_\Gamma \left(\partial_{\tau\tau} u \, \partial_n v + \partial_\tau (\partial_{n\tau} u) \, v \right) \, ds,$$

where P is any corner point on the boundary and $\partial_{n\tau} u(P_-)$ and $\partial_{n\tau} u(P_+)$ are the left and right limiting values of $\partial_{n\tau} u$ at P along Γ directed counterclockwise. Then the relations (2.9)–(2.12) are to be supplemented with continuity conditions of the form

$$\partial_{n\tau} u(P_-) = \partial_{n\tau} u(P_+).$$

Such conditions can be interpreted as "corner force conditions" (cf. [7]).

The main purpose of the paper is to analyze nonconforming finite element methods for Problem 2.1. For this, we need a characterization of the solution of Problem 2.1, following an idea found in [13]. Let

$$\Lambda = \{ \mu \in L^\infty(\Gamma_3) \mid |\mu| \le 1 \text{ a.e. on } \Gamma_3 \}.$$

THEOREM 2.3. *A function u is a solution of Problem* 2.1 *if and only if there exists $\lambda \in \Lambda$ such that*

$$a(u,v) + \int_{\Gamma_3} g\,\lambda\,v\,ds = (f,v) \quad \forall\, v \in V, \tag{2.13}$$

$$\lambda\,u = |u| \quad \text{a.e. on } \Gamma_3. \tag{2.14}$$

Proof. Suppose u is a solution of Problem 2.1. By taking $v = 0$ and $2\,u$ in (2.5), we obtain

$$a(u,u) + j(u) = (f,u). \tag{2.15}$$

Then, from (2.5),

$$a(u,v) + j(v) \ge (f,v) \quad \forall\, v \in V. \tag{2.16}$$

It is easy to see that (2.5) is equivalent to (2.15) and (2.16). From (2.16), we get

$$(f,v) - a(u,v) \le j(v) \quad \forall\, v \in V.$$

Replacing v by $-v$ in this inequality, we obtain

$$(f,v) - a(u,v) \ge -j(v) \quad \forall\, v \in V.$$

Therefore,

$$|(f,v) - a(u,v)| \le j(v) \quad \forall\, v \in V. \tag{2.17}$$

Let γ be the trace operator defined on V and denote $H_{\Gamma_3} = \gamma(V)|_{\Gamma_3}$ with the norm

$$\|v\|_{H_{\Gamma_3}} = \inf\{ \|w\|_V \mid w \in V,\, w|_{\Gamma_3} = v \}.$$

Then, from (2.17), we see that $H_{\Gamma_3} \ni v \mapsto (f,v) - a(u,v)$ defines a linear mapping on H_{Γ_3}; here, we use the same symbol v for a function from V with the trace v on Γ_3. Thus $\ell(v) = (f,v) - a(u,v)$ is a linear mapping on H_{Γ_3} and, from (2.17),

$$|\ell(v)| \le \int_{\Gamma_3} g\,|v|\,ds \quad \forall\, v \in H_{\Gamma_3}.$$

Obviously, $H_{\Gamma_3} \subset L^1(\Gamma_3)$. By the Hahn–Banach theorem, the linear functional ℓ can be the extended to the space $L^1(\Gamma_3)$, and we have the existence of $\lambda \in \Lambda$ such that

$$\ell(v) = \int_{\Gamma_3} g\,\lambda\,v\,ds \quad \forall\, v \in L^1(\Gamma_3).$$

Therefore, (2.13) holds. Using (2.15), we then have

$$\int_{\Gamma_3} g\,(\lambda\,u - |u|)\,ds = 0.$$

Since $|\lambda| \le 1$ a.e. on Γ_3, we have the relation (2.14).

Proof of the converse statement is easy and is hence omitted. ∎

By comparing (2.8) with (2.13) and using the equality boundary conditions of the solution u, we find that

$$g\,\lambda = N(u) \quad \text{on } \Gamma_3.$$

Thus, the "Lagrange multiplier" λ can be interpreted as a scaled shearing force.

Note that the relations (2.9)–(2.12) are valid only if the solution $u \in V$ of Problem 2.1 is smooth, e.g., $u \in C^4(\overline{\Omega})$. Consequently, these relations cannot be used in error analysis. In finite element error analysis, we need a reasonable solution regularity stronger than $u \in V$. Such a regularity result for Problem 2.1 does not seem to be available in the current literature. In this paper, we will assume

$$u \in H^3(\Omega). \tag{2.18}$$

Now let us derive some relations for the solution $u \in V$ of Problem 2.1. In (2.13), we let $v \in C_0^\infty(\Omega)$ to obtain

$$\Delta^2 u = f \quad \text{in the sense of distributions.}$$

Since $f \in L^2(\Omega)$, we actually have

$$\Delta^2 u = f \quad \text{in } L^2(\Omega), \tag{2.19}$$

and then also

$$\Delta^2 u = f \quad \text{a.e. in } \Omega. \tag{2.20}$$

Since $\Delta^2 u \in L^2(\Omega)$ and $u \in H^3(\Omega)$, we can define $\partial_n \Delta u \in H^{-1/2}(\Gamma)$ by the relation (cf., e.g., [11])

$$\langle \partial_n \Delta u, v\rangle_{1/2,\Gamma} = \int_\Omega \left[\Delta^2 u\, v + \nabla(\Delta u)\cdot\nabla v\right] dx \quad \forall\, v \in H^1(\Omega). \tag{2.21}$$

For the bilinear form (2.2), we then have

$$\begin{aligned} a(u,v) &= \int_\Omega \Delta^2 u\, v\, dx - \langle \partial_n \Delta u, v\rangle_{1/2,\Gamma} + \int_\Gamma \Delta u\, \partial_n v\, ds \\ &\quad + (1-\nu)\int_\Gamma (-\partial_{\tau\tau} u\, \partial_n v + \partial_{n\tau} u\, \partial_\tau v)\, ds \\ &= (f,v) - \int_\Gamma M(u)\, \partial_n v\, ds - \langle \partial_n \Delta u, v\rangle_{1/2,\Gamma} + (1-\nu)\int_\Gamma \partial_{n\tau} u\, \partial_\tau v\, ds. \end{aligned}$$

Thus by (2.13) we have

$$-\int_\Gamma M(u)\, \partial_n v\, ds - \langle \partial_n \Delta u, v\rangle_{1/2,\Gamma} + (1-\nu)\int_\Gamma \partial_{n\tau} u\, \partial_\tau v\, ds + \int_{\Gamma_3} g\, \lambda\, v\, ds = 0 \quad \forall\, v \in V. \tag{2.22}$$

By a standard procedure (cf. [8]), it can then be established that

$$M(u) = 0 \quad \text{a.e. on } \Gamma_2 \cup \Gamma_3. \tag{2.23}$$

Then from (2.22) we obtain

$$-\langle \partial_n \Delta u, v\rangle_{1/2,\Gamma} + (1-\nu)\int_\Gamma \partial_{n\tau} u\, \partial_\tau v\, ds + \int_{\Gamma_3} g\,\lambda\, v\, ds = 0 \quad \forall\, v \in V. \tag{2.24}$$

Now the closure of V in $H^1(\Omega)$ is

$$H^1_{\Gamma_1}(\Omega) = \{\, v \in H^1(\Omega) \mid v = 0 \text{ a.e. on } \Gamma_1 \,\}.$$

Denote

$$\tilde{H}^1_{\Gamma_1}(\Omega) = \{\, v \in H^1_{\Gamma_1}(\Omega) \mid \partial_\tau v \in L^2(\Gamma) \,\}.$$

Then from (2.24) we conclude that

$$-\langle \partial_n \Delta u, v\rangle_{1/2,\Gamma} + (1-\nu)\int_\Gamma \partial_{n\tau} u\, \partial_\tau v\, ds + \int_{\Gamma_3} g\,\lambda\, v\, ds = 0 \quad \forall\, v \in \tilde{H}^1_{\Gamma_1}(\Omega). \tag{2.25}$$

3. An abstract error estimate. Let $\{\mathcal{T}_h\}_h$ be a family of finite element partitions of the domain $\overline{\Omega}$. Here $h \to 0+$ is a discretization parameter. A typical element in $\mathcal{T}_h$ is denoted by T. Let $\{V_h\}_h$ be a family of corresponding finite element spaces used to approximate the space V. We consider the case of nonconforming approximation. Thus, in general, $V_h \not\subset V$. Then the discrete approximation problem is the following.

PROBLEM 3.1. *Find $u \in V_h$ such that*

$$a_h(u_h, v_h - u_h) + j(v_h) - j(u_h) \ge (f, v_h - u_h) \quad \forall\, v_h \in V_h, \tag{3.1}$$

where the discrete bilinear form is

$$a_h(u,v) = \sum_{T\in\mathcal{T}_h} \int_T \left[\Delta u\, \Delta v + (1-\nu)\,(2\,\partial_{12} u\, \partial_{12} v - \partial_{11} u\, \partial_{22} v - \partial_{22} u\, \partial_{11} v)\right] dx. \tag{3.2}$$

Assume

$$\|v_h\|_h = \left\{ \sum_{T\in\mathcal{T}_h} |v_h|^2_{2,T} \right\}^{1/2}, \quad v_h \in V_h,$$

is a norm on V_h. Then the bilinear form (3.2) is coercive on V_h. Obviously, $a_h(\cdot,\cdot)$ is continuous:

$$|a_h(u,v)| \le M\, \|u_h\|_h \|v_h\|_h \quad \forall\, u_h, v_h \in V + V_h.$$

We also observe that $j(\cdot)$ is continuous and convex on V_h, and f defines a linear continuous functional on V_h. Therefore, Problem 3.1 has a unique solution.

The following abstract error estimate is inspired by Falk's work [9] and the work of Brezzi, Hager, and Raviart [4]. It plays an important role in error analysis for the approximation of the variational inequality and can be viewed as an extension of the Strang lemma (cf. [5]) for variational equations to variational inequalities.

THEOREM 3.2. *For the solutions of the Problems* 2.1 *and* 3.1, *we have the inequality*

$$\|u - u_h\|_h^2 \le c \inf_{v_h \in V_h} \left\{ \|u - v_h\|_h^2 + R_h(v_h, u_h) \right\}, \tag{3.3}$$

where

$$R_h(v_h, u_h) = a_h(u, v_h - u_h) + j(v_h) - j(u_h) - (f, v_h - u_h) \tag{3.4}$$

is a discrete residual.

Proof. For any $v_h \in V_h$, we have

$$\begin{aligned}\alpha \|u_h - v_h\|_h^2 &\le a_h(u_h - v_h, u_h - v_h)\\ &= a_h(u - v_h, u_h - v_h) + a_h(u_h - u, u_h - v_h)\\ &\le M \|u - v_h\|_h \|u_h - v_h\|_h + a_h(u_h, u_h - v_h) - a_h(u, u_h - v_h)\\ &\le M \|u - v_h\|_h \|u_h - v_h\|_h + R_h(v_h, u_h),\end{aligned}$$

where in the last step we used the defining inequality (3.1). Using the inequality

$$M \|u - v_h\|_h \|u_h - v_h\|_h \le \frac{\alpha}{2} \|u_h - v_h\|_h^2 + \frac{M^2}{2\,\alpha} \|u - v_h\|_h^2$$

we obtain

$$\|u_h - v_h\|_h^2 \le c \left\{\|u - v_h\|_h^2 + R_h(v_h, u_h)\right\}.$$

Now the relation (3.3) follows from

$$\|u - u_h\|_h \le \|u - v_h\|_h + \|u - v_h\|_h$$

and the arbitrariness of $v_h \in V_h$. ◻

4. Continuous nonconforming finite element approximation. We consider some continuous nonconforming plate elements in this section. Let $\{\mathcal{T}_h\}_h$ be a family of regular triangulation of $\overline{\Omega}$, and let $\{V_h\}_h \subset C^0(\overline{\Omega})$ be a corresponding family of nonconforming finite element subspaces of V. We assume

$$\Big| \sum_T \int_{\partial T} w\, \partial_n v_h\, ds \Big| \le c\, h \|w\|_{1,\Omega} \|v_h\|_h \quad \forall\, v_h \in V_h, \tag{4.1}$$

and the finite element interpolation error estimate

$$\|w - \Pi_h w\|_h \le c\, h \|w\|_{3,\Omega} \quad \forall\, w \in V \cap H^3(\Omega). \tag{4.2}$$

Here $\Pi_h w \in V_h$ denotes the finite element interpolant of w.

THEOREM 4.1. *Assume* (2.18), (4.1), *and* (4.2). *Then we have the error estimate*

$$\|u - u_h\|_h \le c\, h\, (\|u\|_{3,\Omega} + h^{1/4} \|g\|_{0,\Gamma_3}). \tag{4.3}$$

Proof. Let us first estimate the terms involved in the residual $R_h(v_h, u_h)$. We have

(4.4)

$$\begin{aligned}a_h(u, u_h - v_h) &= \sum_T \int_T \big\{\Delta u\, \Delta(u_h - v_h) + (1-\nu)\, (2\, \partial_{12} u\, \partial_{12}(u_h - v_h)\\ &\qquad - \partial_{11} u\, \partial_{22}(u_h - v_h) - \partial_{22} u\, \partial_{11}(u_h - v_h))\big\}\, ds\\ &= -\sum_T \int_T \nabla(\Delta u) \cdot \nabla(u_h - v_h)\, dx + \sum_T \int_{\partial T} \Delta u\, \partial_n(u_h - v_h)\, ds\\ &\quad + (1-\nu) \sum_T \int_{\partial T} \big\{ - \partial_{\tau\tau} u\, \partial_n(u_h - v_h) + \partial_{n\tau} u\, \partial_\tau(u_h - v_h)\big\}\, ds.\end{aligned}$$

Since $V_h \subset C(\overline{\Omega})$, we have $u_h, v_h \in H^1(\Omega)$ and

(4.5)
$$-\sum_T \int_T \nabla(\Delta u)\cdot\nabla(u_h - v_h)\,dx = -\int_\Omega \nabla(\Delta u)\cdot\nabla(u_h - v_h)\,dx$$
$$= \int_\Omega \Delta^2 u\,(u_h - v_h)\,dx - \langle \partial_n \Delta u, u_h - v_h\rangle_{1/2,\Gamma}.$$

Then

$$\begin{aligned}(f, u_h - v_h) - a_h(u, u_h - v_h)\\ &= -\sum_T \int_{\partial T}\left[\Delta u - (1-\nu)\,\partial_{\tau\tau} u\right]\partial_n(u_h - v_h)\,ds + \langle \partial_n \Delta u, u_h - v_h\rangle_{1/2,\Gamma}\\ &\quad - (1-\nu)\sum_T \int_{\partial T} \partial_{n\tau} u\,\partial_\tau(u_h - v_h)\,ds\\ &= -\sum_T \int_{\partial T} M(u)\,\partial_n(u_h - v_h)\,ds - (1-\nu)\sum_T \sum_{\substack{\gamma\subset\partial T\\ \gamma\not\subset\Gamma}} \int_\gamma \partial_{n\tau} u\,\partial_\tau(u_h - v_h)\,ds\\ &\quad - (1-\nu)\sum_T \sum_{\substack{\gamma\subset\partial T\\ \gamma\subset\Gamma}} \int_\gamma \partial_{n\tau} u\,\partial_\tau(u_h - v_h)\,ds + \langle \partial_n \Delta u, u_h - v_h\rangle_{1/2,\Gamma}.\end{aligned}$$

Since $u_h, v_h \in C(\overline{\Omega})$, we have

$$\sum_T \sum_{\substack{\gamma\subset\partial T\\ \gamma\not\subset\Gamma}} \int_\gamma \partial_{n\tau} u\,\partial_\tau(u_h - v_h)\,ds = 0.$$

Thus,

$$\begin{aligned}(f, u_h - v_h) - a_h(u, u_h - v_h) &= -\sum_T \int_{\partial T} M(u)\,\partial_n(u_h - v_h)\,ds + \langle \partial_n \Delta u, u_h - v_h\rangle_{1/2,\Gamma}\\ &\quad - (1-\nu)\int_\Gamma \partial_{n\tau} u\,\partial_\tau(u_h - v_h)\,ds.\end{aligned}$$

Using the relation (2.25), we obtain

(4.6)
$$(f, u_h - v_h) - a_h(u, u_h - v_h) = -\sum_T \int_{\partial T} M(u)\,\partial_n(u_h - v_h)\,ds - \int_{\Gamma_3} g\,\lambda\,(u_h - v_h)\,ds.$$

Then

(4.7)
$$R_h(v_h, u_h) = \int_{\Gamma_3} g\,(|v_h| - \lambda\,v_h - |u_h| + \lambda\,u_h)\,ds - \sum_T \int_{\partial T} M(u)\,\partial_n(u_h - v_h)\,ds.$$

The last term on the right-hand side of (4.7) is estimated by (4.1):

$$\left|\sum_T \int_{\partial T} M(u)\,\partial_n(v_h - u_h)\,ds\right| \le c\,h\,\|u\|_{3,\Omega}\|v_h - u_h\|_h.$$

We now estimate the first term on the right-hand side of (4.7). Since $|\lambda| \le 1$ a.e. on Γ_3,

$$\int_{\Gamma_3} g\,(|v_h| - \lambda\, v_h - |u_h| + \lambda\, u_h)\, ds \le \int_{\Gamma_3} g\,(|v_h| - \lambda\, v_h)\, ds.$$

In the following, we choose $v_h = \Pi_h u$. We have

$$\begin{aligned}\int_{\Gamma_3} g\,(|\Pi_h u| - \lambda\, \Pi_h u)\, ds &= \int_{\Gamma_3} g\,\left(|\Pi_h u| - |u| - \lambda\,(\Pi_h u - u)\right)\, ds \\ &\le 2 \int_{\Gamma_3} g\,|u - \Pi_h u|\, ds \\ &\le 2\, \|g\|_{0,\Gamma_3} \|u - \Pi_h u\|_{0,\Gamma_3}.\end{aligned}$$

From [26], for any element side $\gamma \subset \Gamma_3$, denoting T the element that has the side γ, we have

$$\begin{aligned}\|u - \Pi_h u\|_{0,\gamma} &\le c\left(h^{-1}\|u - \Pi_h u\|_{0,T}^2 + h\,|u - \Pi_h u|_{1,T}^2\right)^{1/2} \\ &\le c\left(h^{-1} h^6 |u|_{3,T}^2 + h\, h^4 |u|_{3,T}^2\right)^{1/2} \\ &\le c\, h^{5/2} |u|_{3,T}.\end{aligned}$$

Thus,

$$\begin{aligned}\|u - \Pi_h u\|_{0,\Gamma_3} &= \left(\sum_{\gamma \subset \Gamma_3} \|u - \Pi_h u\|_{0,\gamma}^2\right)^{1/2} \\ &\le c\, h^{5/2} \left(\sum_{T:\partial T \cap \Gamma_3 \ne \emptyset} |u|_{3,T}^2\right)^{1/2} \\ &\le c\, h^{5/2} |u|_{3,\Omega}.\end{aligned}$$

Summarizing, we have the bound

$$R_h(\Pi_h u, u_h) \le c\, h\, \|u\|_{3,\Omega} \|\Pi_h u - u_h\|_h + c\, h^{5/2} \|g\|_{0,\Gamma_3} \|u\|_{3,\Omega}.$$

So, from (3.3),

$$\|u - u_h\|_h^2 \le c\,\{\|u - \Pi_h u\|_h^2 + h\, \|u\|_{3,\Omega} \|\Pi_h u - u_h\|_h + h^{5/2} \|g\|_{0,\Gamma_3} \|u\|_{3,\Omega}\}.$$

The term $\|\Pi_h u - u_h\|_h$ is bounded by $\|\Pi_h u - u\|_h + \|u - u_h\|_h$. Using the interpolation error estimate (4.2), we then obtain the error estimate (4.3). ∎

One example of a continuous nonconforming finite element is the Zienkiewicz triangle. Assume $\overline{\Omega}$ is such that it is possible to split it into triangles with all sides parallel to three fixed directions. This property is valid if Ω is the union of rectangles with sides parallel to two fixed directions and right triangles with two sides parallel to the two fixed directions. Let $\{\mathcal{T}_h\}$ be a regular family of partitions of $\overline{\Omega}$ into such triangles. Then the Zienkiewicz triangle consists of piecewise incomplete polynomials of degree less than or equal to 3. On each triangle, the polynomial is determined by its values and the values of its two first order derivatives at the three vertices; for details, cf. [5]. For this element, we have (4.1) and (4.2) (cf. [24]).

Another example is Adini's rectangle. Assume $\overline{\Omega} \subset \mathbb{R}^2$ can be partitioned into rectangles (e.g., if Ω is the union of rectangles with sides parallel to two fixed directions). Let $\{\mathcal{T}_h\}_h$ be a regular family of partitions of $\overline{\Omega}$ into rectangles with sides parallel to the coordinate axes. Then Adini's rectangle is defined as a piecewise polynomial corresponding to the partition $\mathcal{T}_h$ such that, on each element, it is a polynomial from the space $\mathcal{P}_3(\mathbb{R}^2) + [x_1^3 x_2, x_1 x_2^3]$, with the values of function and of the two first partial derivatives with respect to x_1 and x_2 at the four vertices of the element as the finite element parameters. For the vertices on $\overline{\Gamma}_1$, the parameters are taken to be zero for V_h. Then, from [26], we have (4.1) and (4.2).

We conclude that for both the Zienkiewicz triangle and Adini's rectangle, the optimal order error estimate (4.3) holds.

5. Discontinuous nonconforming finite element approximation. In this section, we consider discontinuous nonconforming finite element approximations of the plate contact problem. Let $\{V_h\}_h \not\subset C^0(\overline{\Omega})$ be a family of nonconforming finite element subspaces of V corresponding to a regular family $\{\mathcal{T}_h\}_h$ of triangulations of $\overline{\Omega}$ such that the finite element functions are continuous at the vertices of the corresponding triangulation. We still assume (4.1) and (4.2).

THEOREM 5.1. *Assume* (2.18)*,* (4.1)*, and* (4.2)*. Also assume the finite element functions are continuous at the vertices of the corresponding triangulation. Then we have the error estimate*

$$\|u - u_h\|_h \le c\, h \left\{ \|u\|_{3,\Omega} + h^{1/4} \|g\|_{0,\Gamma_3} + h\, \|f\|_{0,\Omega} \right\}. \tag{5.1}$$

Proof. Since $V_h \not\subset C(\overline{\Omega})$ implies $V_h \not\subset H^1(\Omega)$, we must modify the expression (4.5) as follows. Denote $w_h = u_h - v_h$ and let w_h^I be the continuous piecewise linear interpolant of w_h. Since $w_h^I \in C(\overline{\Omega})$, $w_h^I \in H^1(\Omega)$. First we write

$$\begin{aligned}
-\sum_T \int_T \nabla(\Delta u) \cdot \nabla(u_h - v_h)\, dx &= -\sum_T \int_T \nabla(\Delta u) \cdot \nabla w_h\, dx \\
&= -\sum_T \int_T \nabla(\Delta u) \cdot \nabla w_h^I\, dx \\
&\quad - \sum_T \int_T \nabla(\Delta u) \cdot \nabla(w_h - w_h^I)\, dx \\
&= \int_\Omega \Delta^2 u\, w_h^I\, dx - \langle \partial_n \Delta u, w_h^I \rangle_{1/2,\Gamma} \\
&\quad - \sum_T \int_T \nabla(\Delta u) \cdot \nabla(w_h - w_h^I)\, dx.
\end{aligned}$$

Then

$$\begin{aligned}
a_h(u, u_h - v_h) = &\int_\Omega \Delta^2 u\, w_h^I\, dx - \langle \partial_n \Delta u, w_h^I \rangle_{1/2,\Gamma} - \sum_T \int_T \nabla(\Delta u) \cdot \nabla(w_h - w_h^I)\, dx \\
&+ \sum_T \int_{\partial T} \Delta u\, \partial_n w_h ds + (1-\nu) \sum_T \int_{\partial T} (-\partial_{\tau\tau} u\, \partial_n w_h + \partial_{n\tau} u\, \partial_\tau w_h)\, ds.
\end{aligned}$$

Use the relation (2.20),

$$a_h(u, u_h - v_h) = (f, w_h^I) - \langle \partial_n \Delta u, w_h^I \rangle_{1/2,\Gamma} + \sum_T \int_{\partial T} \left\{ \Delta u - (1-\nu)\,\partial_{\tau\tau} u \right\} \partial_n w_h \, ds$$
$$+ (1-\nu) \sum_T \int_{\partial T} \partial_{n\tau} u \partial_\tau w_h \, ds - \sum_T \int_T \nabla(\Delta u) \cdot \nabla(w_h - w_h^I)\, dx.$$

Hence,

$$\begin{aligned}
&(f, u_h - v_h) - a_h(u, u_h - v_h)\\
&\quad = (f, w_h - w_h^I) + \sum_T \int_T \nabla(\Delta u) \cdot \nabla(w_h - w_h^I)\, dx + \langle \partial_n \Delta u, w_h^I \rangle_{1/2,\Gamma}\\
&\qquad - \sum_T \int_{\partial T} M(u)\, \partial_n w_h \, ds - (1-\nu) \sum_T \int_{\partial T} \partial_{n\tau} u \, \partial_\tau w_h \, ds.
\end{aligned}$$

Now

$$\sum_T \int_{\partial T} \partial_{n\tau} u \, \partial_\tau w_h \, ds = \sum_T \int_{\partial T} \partial_{n\tau} u \, \partial_\tau w_h^I \, ds + \sum_T \int_{\partial T} \partial_{n\tau} u \, \partial_\tau (w_h - w_h^I)\, ds.$$

For each side γ of the elements, define a piecewise constant projection operator $P_0^\gamma : L^1(\gamma) \to \mathbb{R}$ by

$$P_0^\gamma(v) = \frac{1}{|\gamma|} \int_\gamma v \, ds.$$

Since

$$\int_\gamma \partial_\tau (w_h - w_h^I)\, ds = 0,$$

we have

$$\begin{aligned}
\sum_T \int_{\partial T} \partial_{n\tau} u \, \partial_\tau (w_h - w_h^I)\, ds &= \sum_T \sum_{\gamma \subset \partial T} \int_\gamma \partial_{n\tau} u \, \partial_\tau (w_h - w_h^I)\, ds\\
&= \sum_T \sum_{\gamma \subset \partial T} \int_\gamma (\partial_{n\tau} u - P_0^\gamma(\partial_{n\tau} u))\, \partial_\tau (w_h - w_h^I)\, ds\\
&\le c\, h\, |u|_{3,\Omega} \|w_h\|_h.
\end{aligned}$$

Using the fact $w_h^I \in C(\overline{\Omega})$ we have

$$\sum_T \int_{\partial T} \partial_{n\tau} u \, \partial_\tau w_h^I \, ds = \sum_{\gamma \subset \Gamma} \int_\gamma \partial_{n\tau} u \, \partial_\tau w_h^I \, ds.$$

Also,

$$(f, w_h - w_h^I) \le \|f\|_{0,\Omega} \|w_h - w_h^I\|_{0,\Omega} \le c\, h^2 \|f\|_{0,\Omega} \|w_h\|_h$$

and

$$\sum_T \int_T \nabla(\Delta u) \cdot \nabla(w_h - w_h^I) \le \sum_T |u|_{3,T} |w_h - w_h^I|_{1,T} \le c\, h\, |u|_{3,\Omega} \|w_h\|_h.$$

Using the estimate (4.1), we have

$$(f, u_h - v_h) - a_h(u, u_h - v_h) \le c\, h\, (|u|_{3,\Omega} + h\, \|f\|_{0,\Omega})\, \|w_h\|_h + \langle \partial_n \Delta u, w_h^I \rangle_{1/2,\Gamma} - (1-\nu) \int_\Gamma \partial_{n\tau} u\, \partial_\tau w_h^I\, ds.$$

By (2.25), we then obtain

$$(f, u_h - v_h) - a_h(u, u_h - v_h) \le c\, h\, (|u|_{3,\Omega} + h\, \|f\|_{0,\Omega})\, \|w_h\|_h + \int_{\Gamma_3} g\, \lambda\, w_h^I\, ds.$$

Thus, for the residual term defined in (3.4), we have

$$R_h(v_h, u_h) \le \int_{\Gamma_3} g\, (|v_h| - |u_h| + \lambda\, w_h^I)\, ds + c\, h\, (|u|_{3,\Omega} + h\, \|f\|_{0,\Omega})\, \|w_h\|_h \tag{5.2}$$
$$= \int_{\Gamma_3} g\, (|v_h| - |u_h| + \lambda\, w_h)\, ds + \int_{\Gamma_3} g\, \lambda\, (w_h^I - w_h)\, ds + c\, h\, (|u|_{3,\Omega} + h\, \|f\|_{0,\Omega})\, \|w_h\|_h.$$

The second term on the right is bounded as follows:

$$\begin{aligned}
\int_{\Gamma_3} g\, \lambda\, (w_h^I - w_h)\, ds &\le \sum_{\gamma \subset \Gamma_3} \int_\gamma g\, |w_h - w_h^I|\, ds \\
&\le \|g\|_{0,\Gamma_3} \left(\sum_{\gamma \subset \Gamma_3} \|w_h - w_h^I\|_{0,\gamma}^2 \right)^{1/2} \\
&\le c\, \|g\|_{0,\Gamma_3} \left(\sum_{\partial T \cap \Gamma_3 \neq \emptyset} \left[h^{-1} \|w_h - w_h^I\|_{0,T}^2 + h\, |w_h - w_h^I|_{1,T}^2 \right] \right)^{1/2} \\
&\le c\, \|g\|_{0,\Gamma_3} h^{3/2} \left(\sum_{T : \partial T \cap \Gamma_3 \neq \emptyset} |w_h|_{2,T}^2 \right)^{1/2} \\
&\le c\, h^{3/2} \|g\|_{0,\Gamma_3} \|w_h\|_h.
\end{aligned}$$

The first term $\int_{\Gamma_3} g\, (|v_h| - |u_h| + \lambda\, w_h)\, ds$ can be handled similarly as in the proof of Theorem 4.1. So, with $v_h = \Pi_h u$ in (5.2), we have the bound

$$R_h(\Pi_h u, u_h) \le c\, (h^{3/2} \|g\|_{0,\Gamma_3} + h\, |u|_{3,\Omega} + h^2 \|f\|_{0,\Omega})\, \|\Pi_h u - u_h\|_h + c\, h^{5/2} \|g\|_{0,\Gamma_3} \|u\|_{3,\Omega}. \tag{5.3}$$

Now, by (3.3), we have

$$\|u - u_h\|_h^2 \le c\, (\|u - \Pi_h u\|_h^2 + (h^{3/2} \|g\|_{0,\Gamma_3} + h\, |u|_{3,\Omega} + h^2 \|f\|_{0,\Omega})\, \|\Pi_h u - u_h\|_h + \ h^{5/2} \|g\|_{0,\Gamma_3} \|u\|_{3,\Omega}),$$

from which we can derive the error estimate (5.1) as in the proof of Theorem 4.1. $\square$

As examples of discontinuous nonconforming finite element spaces for the plate contact problem, we mention Morley's triangle and the Fraeijs De Veubeke triangle.

Assume Ω is a polygonal domain and let $\{\mathcal{T}_h\}$ be a regular family of partitions of $\overline{\Omega}$ into triangles. For Morley's triangle, on each element, the finite element function is quadratic and is uniquely determined by the function values at the three vertices and the normal derivative at the three midside nodes. For the Fraeijs De Veubeke triangle, on each element, the finite element function is cubic and is uniquely determined by the function values at the three vertices and at the center and the normal derivative at the Gaussian points of second order on each side. From their constructions, we see that for both Morley's triangle and the Fraeijs De Veubeke triangle, the finite element functions are continuous at the vertices of the corresponding triangulation. For both elements, (4.1) and (4.2) are valid (cf. [26]).

We conclude that for both Morley's triangle and the Fraeijs De Veubeke triangle, the optimal order error estimate (5.1) holds.

Acknowledgment. We thank the two referees whose suggestions led to an improvement of this paper.

REFERENCES

[1] C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities: Applications to Free-Boundary Problems*, John Wiley, New York, 1984.

[2] G.P. Bazeley, Y.K. Cheung, B.M. Irons, and O.C. Zienkiewicz, *Triangular elements in bending—conforming and non-conforming solutions*, in Proceedings of the Conference on Matrix Methods in Structural Mechanics, Air Force Institute of Technology, Wright-Patterson Air Force Base, OH, 1965.

[3] S.C. Brenner and L.R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, New York, 1994.

[4] F. Brezzi, W.W. Hager, and P.A. Raviart, *Error estimates for the finite element solution of variational inequalities, Part* I. *Primal theory*, Numer. Math., 28 (1977), pp. 431–443.

[5] P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

[6] P.G. Ciarlet, *Basic error estimates for elliptic problems*, in Handbook of Numerical Analysis, Vol. II, P.G. Ciarlet and J.-L. Lions, eds., North-Holland, Amsterdam, 1991, pp. 17–351.

[7] I. Doghri, *Mechanics of Deformable Solids*, Springer-Verlag, Berlin, 2000.

[8] G. Duvaut and J.-L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1976.

[9] R. Falk, *Error estimates for the approximation of a class of variational inequalities*, Math. Comput., 28 (1974), pp. 963–971.

[10] A. Friedman, *Variational Principles and Free-boundary Problems*, John Wiley, New York, 1982.

[11] V. Girault and P.A. Raviart, *Finite Element Methods for Navier-Stokes Equations, Theory and Algorithms*, Springer-Verlag, New York, 1986.

[12] R. Glowinski, *Numerical Methods for Nonlinear Variational Problems*, Springer-Verlag, New York, 1984.

[13] R. Glowinski, J.-L. Lions, and R. Trémolières, *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, 1981.

[14] W. Han and B.D. Reddy, *Plasticity: Mathematical Theory and Numerical Analysis*, Springer-Verlag, New York, 1999.

[15] W. Han and M. Sofonea, *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*, AMS and International Press, 2002.

[16] J. Haslinger, I. Hlaváček, and J. Nečas, *Numerical methods for unilateral problems in solid mechanics*, in Handbook of Numerical Analysis, Vol. IV, P.G. Ciarlet and J.-L. Lions, eds., North-Holland, Amsterdam, 1996, pp. 313–485.

[17] I. Hlaváček, J. Haslinger, J. Nečas, and J. Lovíšek, *Solution of Variational Inequalities in Mechanics*, Springer-Verlag, New York, 1988.

[18] B.M. Irons and A. Razzaque, *Experience with the patch test for convergence of finite element methods*, in Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, A.R. Aziz, ed., Academic Press, New York, 1972, pp. 557–587.

[19] N. Kikuchi and J.T. Oden, *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*, SIAM Stud. Appl. Math. 8, SIAM, Philadelphia, 1988.
[20] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.
[21] P. Lascaux and P. Lesaint, *Some nonconforming finite elements for the plate bending problem*, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge Anal. Numér., 9 (1975), pp. 9–53.
[22] P.D. Panagiotopoulos, *Inequality Problems in Mechanics and Applications*, Birkhäuser, Boston, 1985.
[23] J.F. Rodrigues, *Obstacle Problems in Mathematical Physics*, North-Holland, Amsterdam, 1987.
[24] Z.C. Shi, *The generalized patch test for Zienkiewicz's triangles*, J. Comput. Math., 2 (1984), pp. 279–286.
[25] Z.C. Shi, *The F-E-M-Test for nonconforming finite elements*, Math. Comp., 49 (1987), pp. 391–405.
[26] F. Stummel, *The generalized patch test*, SIAM J. Numer. Anal., 16 (1979), pp. 449–471.
[27] L.H. Wang, *Morley's element approximation to a fourth order variational inequality with curvature obstacle*, Math. Numer. Sinica, 12 (1990), pp. 279–284.
[28] L.H. Wang, *Some nonconforming finite element approximations of a fourth order variational inequality with displacement obstacle*, Math. Numer. Sinica, 12 (1990), pp. 352–356.
[29] L.H. Wang, *Some strongly discontinuous nonconforming finite element approximations for a fourth order variational inequality with displacement obstacle*, Math. Numer. Sinica, 14 (1992), pp. 98–101.
[30] M. Wang, *On the necessity and sufficiency of the patch test for convergence of nonconforming finite elements*, SIAM J. Numer. Anal., 39 (2001), pp. 363–384.