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On convergence of numerical methods for variational–hemivariational inequalities under minimal solution regularity ☆

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ABSTRACT

Hemivariational inequalities have been successfully employed for mathematical and numerical studies of application problems involving nonsmooth, nonmonotone and multivalued relations. In recent years, error estimates have been derived for numerical solutions of hemivariational inequalities under additional solution regularity assumptions. Since the solution regularity properties have not been rigorously proved for hemivariational inequalities, it is important to explore the convergence of numerical solutions of hemivariational inequalities without assuming additional solution regularity. In this paper, we present a general convergence result enhancing existing results in the literature.

The notion of hemivariational inequalities was introduced by Panagiotopoulos in early 1980s [1] for the study of engineering problems involving non-smooth, non-monotone and possibly multi-valued relations for deformable bodies. Substantial advances have been achieved on modeling, analysis, numerical approximation and computer simulations of hemivariational inequalities. Comprehensive references in the area include [2–4] in earlier years and [5–7] more recently. Recently, optimal order error estimates are derived for numerical solutions of hemivariational inequalities under certain solution regularity conditions (cf. [8–12]). Since the required solution regularity properties have not been proved, it is important to investigate the convergence issue for numerical solutions of hemivariational inequalities under the available minimal solution regularity. In [13], such a convergence analysis is carried out for numerical methods of both internal and external ones of general variational–hemivariational inequalities; see also [14]. In this paper, we present an alternative proof of a more general convergence result under weaker assumptions.

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For a normed space X, we denote X^* its dual space and by $\langle \cdot, \cdot \rangle$ the duality pairing between X^* and X. Strong convergence is indicated by the symbol $\to$, whereas weak convergence is indicated by the symbol $\rightharpoonup$. The general variational–hemivariational inequality to be studied is of the following type, the same as in [13,14].

Problem 1. Find an element $u \in K$ such that

$$\langle Au, v-u\rangle + \varphi(\gamma_\varphi u, \gamma_\varphi v) - \varphi(\gamma_\varphi u, \gamma_\varphi u) + j^0(\gamma_j u; \gamma_j v - \gamma_j u) \geq \langle f, v-u\rangle \quad \forall\, v \in K. \tag{1}$$

We begin with a list of the hypotheses on the data of Problem 1.

(H_K) X is a reflexive Banach space, K is a non-empty, closed and convex subset of X.

(H_A) $A\colon X \to X^*$ is bounded, continuous and strongly monotone.

We will denote the monotonicity constant of A by $m_A > 0$, i.e.

$$\langle Av_1 - Av_2, v_1 - v_2\rangle \geq m_A \|v_1 - v_2\|_X^2 \quad \forall\, v_1, v_2 \in X. \tag{2}$$

(H_φ) X_φ is a Banach space, $\gamma_\varphi \in \mathcal{L}(X, X_\varphi)$, $\varphi\colon X_\varphi \times X_\varphi \to \mathbb{R}$ is convex and continuous w.r.t. to its second argument, and there exists a constant $\alpha_\varphi > 0$ such that

$$\varphi(z_1, z_4) - \varphi(z_1, z_3) + \varphi(z_2, z_3) - \varphi(z_2, z_4) \leq \alpha_\varphi \|z_1 - z_2\|_{X_\varphi} \|z_3 - z_4\|_{X_\varphi} \quad \forall\, z_1, z_2, z_3, z_4 \in X_\varphi, \tag{3}$$

where $\mathcal{L}(X, X_\varphi)$ stands for the space of linear and continuous operators from X to X_φ.

(H_j) X_j is a Banach space with its dual X_j^*, $\gamma_j \in \mathcal{L}(X, X_j)$, $j\colon X_j \to \mathbb{R}$ is locally Lipschitz and there exist constants $c_0, c_1 \geq 0$ and $\alpha_j > 0$ such that

$$\|\partial j(z)\|_{X_j^*} \leq c_0 + c_1 \|z\|_{X_j} \quad \forall\, z \in X_j, \tag{4}$$

$$j^0(z_1; z_2 - z_1) + j^0(z_2; z_1 - z_2) \leq \alpha_j \|z_1 - z_2\|_{X_j}^2 \quad \forall\, z_1, z_2 \in X_j. \tag{5}$$

(H_f) $f \in X^*$.

We will denote by $c_\varphi, c_j \geq 0$ upper bounds of the operator norms of γ_φ and γ_j:

$$\|\gamma_\varphi v\|_{X_\varphi} \leq c_\varphi \|v\|_X, \quad \|\gamma_j v\|_{X_j} \leq c_j \|v\|_X \quad \forall\, v \in X. \tag{6}$$

In the formulation of Problem 1 and in (H_j), we use the notions of the generalized directional derivative and generalized subdifferential in the sense of Clarke (cf. [15,16]). Assume $\psi\colon X \to \mathbb{R}$ is locally Lipschitz continuous. The generalized directional derivative of ψ at $x \in X$ in the direction $v \in X$ is

$$\psi^0(x; v) := \limsup_{y \to x,\, \lambda \downarrow 0} \frac{\psi(y + \lambda v) - \psi(y)}{\lambda}.$$

The generalized subdifferential of ψ at x is a subset of the dual space X^* given by

$$\partial \psi(x) := \{\, \zeta \in X^* \mid \psi^0(x; v) \geq \langle \zeta, v\rangle_{X^* \times X} \ \forall\, v \in X \,\}.$$

We will make use of the following properties:

$$\psi^0(x; v_1 + v_2) \leq \psi^0(x; v_1) + \psi^0(x; v_2) \quad \forall\, x, v_1, v_2 \in X, \tag{7}$$

$$\psi^0(x; v) = \max\{\langle \zeta, v\rangle_{X^* \times X} \mid \zeta \in \partial\psi(x)\}, \tag{8}$$

$$x_n \to x \text{ and } v_n \to v \text{ in } X \quad \Longrightarrow \quad \limsup_{n\to\infty} \psi^0(x_n; v_n) \leq \psi^0(x; v). \tag{9}$$

The following unique solvability is adapted from a corresponding result from [17], cf. [13,14].

Theorem 2. *Assume* (H_K), (H_A), (H_φ), (H_j), (H_f), *and* $\alpha_\varphi c_\varphi^2 + \alpha_j c_j^2 < m_A$. *Then, Problem 1 has a unique solution* $u \in K$.

Henceforth, we will assume the conditions stated in Theorem 2 hold. Turning to numerical approximation, we let $X^h \subset X$ be a finite dimensional subspace characterized by a discretization parameter $h > 0$, with the expectation of convergence when $h \to 0$. Let K^h be a non-empty, closed and convex subset of X^h such that $\{K^h\}_h$ approximates K in the following sense (cf. [18]):

$$v^h \in K^h \text{ and } v^h \rightharpoonup v \text{ in } X \text{ imply } v \in K; \tag{10}$$

$$\forall\, v \in K,\ \exists\, v^h \in K^h \text{ such that } v^h \to v \text{ in } X \text{ as } h \to 0. \tag{11}$$

Then a Galerkin approximation of Problem 1 is the following.

Problem 3. Find an element $u^h \in K^h$ such that

$$\langle Au^h, v^h - u^h\rangle + \varphi(\gamma_\varphi u^h, \gamma_\varphi v^h) - \varphi(\gamma_\varphi u^h, \gamma_\varphi u^h) + j^0(\gamma_j u^h; \gamma_j v^h - \gamma_j u^h) \ge \langle f, v^h - u^h\rangle \quad \forall\, v^h \in K^h. \tag{12}$$

Problem 3 has a unique solution u^h and it can be proved (cf. [13,14]) that the numerical solutions defined by Problem 3 are uniformly bounded with respect to the parameter h, i.e., there exists a constant $M > 0$ such that $\|u^h\|_X \le M$ for all $h > 0$.

Theorem 4. *Keep the assumptions in Theorem 2. Assume further that (10)–(11) hold. Then, we have the convergence of the numerical method:* $u^h \to u$ *in* X *as* $h \to 0$.

Proof. The proof consists of three steps. The first two steps are on the weak and strong convergence of the numerical solutions. In the third step, we show that the limit is the solution u.

Since $\{u^h\}$ is bounded in X and X is reflexive, and since $\gamma_\varphi \in \mathcal{L}(X, X_\varphi)$ and $\gamma_j \in \mathcal{L}(X, X_j)$, there exist a subsequence $\{u^{h'}\} \subset \{u^h\}$ and an element $w \in X$ such that

$$u^{h'} \rightharpoonup w \text{ in } X, \quad \gamma_\varphi u^{h'} \rightharpoonup \gamma_\varphi w \text{ in } X_\varphi, \quad \gamma_j u^{h'} \rightharpoonup \gamma_j w \text{ in } X_j. \tag{13}$$

By the assumption (10), we know that $w \in K$.

Let us prove the strong convergence,

$$u^{h'} \to w \text{ in } X. \tag{14}$$

By (11), there exists a sequence $\{w^{h'}\} \subset X$ with $w^{h'} \in K^{h'}$, such that

$$w^{h'} \to w \text{ in } X, \quad \gamma_\varphi w^{h'} \to \gamma_\varphi w \text{ in } X_\varphi, \quad \gamma_j w^{h'} \to \gamma_j w \text{ in } X_j. \tag{15}$$

By the strong monotonicity of A, we have

$$m_A \|w - u^{h'}\|_X^2 \le \langle Aw - Au^{h'}, w - u^{h'}\rangle,$$

which is rewritten as

$$m_A \|w - u^{h'}\|_X^2 \le \langle Aw, w - u^{h'}\rangle - \langle Au^{h'}, w^{h'} - u^{h'}\rangle - \langle Au^{h'}, w - w^{h'}\rangle. \tag{16}$$

In (12) with $h = h'$, take $v^{h'} = w^{h'}$ to obtain

$$\begin{aligned} -\langle Au^{h'}, w^{h'} - u^{h'}\rangle &\le \varphi(\gamma_\varphi u^{h'}, \gamma_\varphi w^{h'}) - \varphi(\gamma_\varphi u^{h'}, \gamma_\varphi u^{h'}) \\ &\quad + j^0(\gamma_j u^{h'}; \gamma_j w^{h'} - \gamma_j u^{h'}) - \langle f, w^{h'} - u^{h'}\rangle. \end{aligned} \tag{17}$$

Apply (3) with $z_1 = z_3 = \gamma_\varphi u^{h'}$, $z_2 = \gamma_\varphi w$ and $z_4 = \gamma_\varphi w^{h'}$ to obtain

$$\begin{aligned}\varphi(\gamma_\varphi u^{h'}, \gamma_\varphi w^{h'}) - \varphi(\gamma_\varphi u^{h'}, \gamma_\varphi u^{h'}) &\le \varphi(\gamma_\varphi w, \gamma_\varphi w^{h'}) - \varphi(\gamma_\varphi w, \gamma_\varphi u^{h'}) \\ &\quad + \alpha_\varphi \|\gamma_\varphi(u^{h'} - w)\|_{X_\varphi} \|\gamma_\varphi(u^{h'} - w^{h'})\|_{X_\varphi}. \end{aligned} \tag{18}$$

Write $\|\gamma_\varphi(u^{h'} - w^{h'})\|_{X_\varphi} \le \|\gamma_\varphi(u^{h'} - w)\|_{X_\varphi} + \|\gamma_\varphi(w - w^{h'})\|_{X_\varphi}$ and use the inequality $a\,b \le \varepsilon\, a^2 + C\, b^2$ for any reals a and b, any $\varepsilon > 0$, with $C = 1/(4\,\varepsilon)$. Then for the last term of (18), for a constant $C(\epsilon)$ depending on ϵ, we have

$$\alpha_\varphi \|\gamma_\varphi(u^{h'} - w)\|_{X_\varphi} \|\gamma_\varphi(u^{h'} - w^{h'})\|_{X_\varphi} \le \left(\alpha_\varphi c_\varphi^2 + \epsilon\right) \|w - u^{h'}\|_X^2 + C(\epsilon)\, \|\gamma_\varphi(w - w^{h'})\|_{X_\varphi}^2. \tag{19}$$

Using the sub-additivity property (7), we have

$$\begin{aligned} j^0(\gamma_j u^{h'}; \gamma_j w^{h'} - \gamma_j u^{h'}) &\le j^0(\gamma_j u^{h'}; \gamma_j w^{h'} - \gamma_j w) + j^0(\gamma_j u^{h'}; \gamma_j w - \gamma_j u^{h'}) \\ &= j^0(\gamma_j u^{h'}; \gamma_j w - \gamma_j u^{h'}) + j^0(\gamma_j w; \gamma_j u^{h'} - \gamma_j w) \\ &\quad + j^0(\gamma_j u^{h'}; \gamma_j w^{h'} - \gamma_j w) - j^0(\gamma_j w; \gamma_j u^{h'} - \gamma_j w). \end{aligned} \tag{20}$$

By (5), it has

$$j^0(\gamma_j u^{h'}; \gamma_j w - \gamma_j u^{h'}) + j^0(\gamma_j w; \gamma_j u^{h'} - \gamma_j w) \le \alpha_j c_j^2 \|w - u^{h'}\|_X^2. \tag{21}$$

Use the relations (17)–(21) in (16):

$$\begin{aligned} \left(m_A - \alpha_\varphi c_\varphi^2 - \alpha_j c_j^2 - \epsilon\right) \|w - u^{h'}\|_X^2 &\le \langle Aw, w - u^{h'}\rangle - \langle Au^{h'}, w - w^{h'}\rangle - \langle f, w^{h'} - u^{h'}\rangle \\ &\quad + \varphi(\gamma_\varphi w, \gamma_\varphi w^{h'}) - \varphi(\gamma_\varphi w, \gamma_\varphi u^{h'}) + C(\epsilon)\, \|\gamma_\varphi(w - w^{h'})\|_{X_\varphi}^2 \\ &\quad + j^0(\gamma_j u^{h'}; \gamma_j w^{h'} - \gamma_j w) - j^0(\gamma_j w; \gamma_j u^{h'} - \gamma_j w). \end{aligned} \tag{22}$$

Consider the limits of the terms on the right side of (22) as $h' \to 0$. From the boundedness of A, and the convergence relation (13),

$$\langle Aw, w - u^{h'}\rangle \to 0.$$

From the boundedness of $\{u^{h'}\}$ and A, (15) ensures us to find

$$\langle Au^{h'}, w - w^{h'}\rangle \to 0.$$

From (13) and (15), one has

$$\langle f, w^{h'} - u^{h'}\rangle = \langle f, w^{h'} - w\rangle + \langle f, w - u^{h'}\rangle \to 0.$$

By the continuity of φ with respect to its second argument and (15),

$$\varphi(\gamma_\varphi w, \gamma_\varphi w^{h'}) \to \varphi(\gamma_\varphi w, \gamma_\varphi w).$$

As a consequence of the well-known Mazur Lemma, the convexity and continuity of φ with respect to its second argument imply that φ is weakly sequentially lower semicontinuous with respect to its second argument (cf. [19, p. 136]). Hence, by (13),

$$\limsup_{h'\to 0} \left[-\varphi(\gamma_\varphi w, \gamma_\varphi u^{h'})\right] = -\liminf_{h'\to 0} \left[\varphi(\gamma_\varphi w, \gamma_\varphi u^{h'})\right] \le -\varphi(\gamma_\varphi w, \gamma_\varphi w).$$

By (15),

$$\|\gamma_\varphi(w - w^{h'})\|_{X_\varphi} \to 0.$$

By (4), the boundedness of $\gamma_j u^{h'}$ in X_j, and (15),

$$\limsup_{h'\to 0} j^0(\gamma_j u^{h'}; \gamma_j w^{h'} - \gamma_j w) \le 0.$$

For any $\xi_w \in \partial j(\gamma_j w)$, by (8),

$$-j^0(\gamma_j w; \gamma_j u^{h'} - \gamma_j w) \le -\langle \xi_w, \gamma_j u^{h'} - \gamma_j w\rangle \to 0 \quad \text{as } h' \to 0.$$

Thus,

$$\limsup_{h'\to 0} \left[-j^0(\gamma_j w; \gamma_j u^{h'} - \gamma_j w)\right] \le 0.$$

Now in (22), we choose $\epsilon = (m_A - \alpha_\varphi c_\varphi^2 - \alpha_j c_j^2)/2$ and then take the upper limit of both sides as $h' \to 0$ to conclude that

$$\limsup_{h'\to 0} \|w - u^{h'}\|_X^2 \le 0,$$

i.e., strong convergence (14) holds.

Finally, let us show that the strong limit w is the unique solution of Problem 1. For any $v \in K$, we have a sequence $\{v^{h'}\} \subset X$ with $v^{h'} \in K^{h'}$, such that $v^{h'} \to v$ in X. Then, $\gamma_\varphi v^{h'} \to \gamma_\varphi v$ in X_φ, $\gamma_j v^{h'} \to \gamma_j v$ in X_j. By (12) with $h = h'$,

$$\langle A u^{h'}, v^{h'} - u^{h'}\rangle + \varphi(\gamma_\varphi u^{h'}, \gamma_\varphi v^{h'}) - \varphi(\gamma_\varphi u^{h'}, \gamma_\varphi u^{h'}) + j^0(\gamma_j u^{h'}; \gamma_j v^{h'} - \gamma_j u^{h'}) \ge \langle f, v^{h'} - u^{h'}\rangle. \tag{23}$$

Obviously,

$$\langle A u^{h'}, v^{h'} - u^{h'}\rangle \to \langle Aw, v - w\rangle, \quad \langle f, v^{h'} - u^{h'}\rangle \to \langle f, v - w\rangle \quad \text{as } h' \to 0. \tag{24}$$

An analogue of (18) is

$$\begin{aligned}\varphi(\gamma_\varphi u^{h'}, \gamma_\varphi v^{h'}) - \varphi(\gamma_\varphi u^{h'}, \gamma_\varphi u^{h'}) &\le \varphi(\gamma_\varphi w, \gamma_\varphi v^{h'}) - \varphi(\gamma_\varphi w, \gamma_\varphi u^{h'}) \\ &\quad + \alpha_\varphi \|\gamma_\varphi(u^{h'} - w)\|_{X_\varphi} \|\gamma_\varphi(u^{h'} - v^{h'})\|_{X_\varphi},\end{aligned} \tag{25}$$

in which,

$$\|\gamma_\varphi(u^{h'} - w)\|_{X_\varphi} \|\gamma_\varphi(u^{h'} - v^{h'})\|_{X_\varphi} \to 0 \tag{26}$$

since $\|\gamma_\varphi(u^{h'} - w)\|_{X_\varphi} \to 0$ and $\|\gamma_\varphi(u^{h'} - v^{h'})\|_{X_\varphi}$ is bounded. Note that $\gamma_j u^{h'} \to \gamma_j w$ and $\gamma_j v^{h'} \to \gamma_j v$ in X_j. So by the property (9),

$$j^0(\gamma_j w; \gamma_j v - \gamma_j w) \ge \limsup_{h'\to 0} j^0(\gamma_j u^{h'}; \gamma_j v^{h'} - \gamma_j u^{h'}). \tag{27}$$

We now take the upper limit $h' \to 0$ in (23) and make use of the relations (24)–(27) to obtain

$$\langle Aw, v - w\rangle + \varphi(\gamma_\varphi w, \gamma_\varphi v) - \varphi(\gamma_\varphi w, \gamma_\varphi w) + j^0(\gamma_j w; \gamma_j v - \gamma_j w) \ge \langle f, v - u\rangle,$$

which holds for any $v \in K$. Thus, w is a solution of Problem 1. Since a solution of Problem 1 is unique, so, $w = u$. Moreover, since the limit u does not depend on the subsequence, the entire family of the numerical solutions converge: $\|u - u^h\|_X \to 0$ as $h \to 0$. ■

We remark that in [13] and [14], convergence of the numerical method is proved under additional assumptions that $\gamma_\varphi \in \mathcal{L}(X, X_\varphi)$ and $\gamma_j \in \mathcal{L}(X, X_j)$ are compact.

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