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SINGULAR PERTURBATIONS OF VARIATIONAL-HEMIVARIATIONAL INEQUALITIES*

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Dedicated to Professor Kendall Atkinson on the occasion of his 80th birthday

Abstract. This paper is devoted to an analysis of singular perturbations of inequality problems. For a general variational-hemivariational inequality, it is shown rigorously that under appropriate conditions, as the singular perturbation parameter approaches zero, the solution of the singularly perturbed problem converges to the solution of the limiting problem. As corollaries of this general result, we have similar convergence results for singularly perturbed problems of "pure" hemivariational inequalities and of variational inequalities. The results are illustrated in the study of an obstacle plate bending problem.

Key words. singular perturbation, hemivariational inequality, variational-hemivariational inequality, variational inequality, convergence

AMS subject classifications. 47J20, 47J22

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1. Introduction.

Variational inequalities and hemivariational inequalities are important nonlinear mathematical models for applications in complicated problems of science and engineering. Rigorous mathematical analysis of variational inequalities started in the 1960s. Modeling, mathematical theory, numerical analysis, and applications of variational inequalities have been well documented in the literature (cf., e.g., monographs [6, 8, 7, 27, 16, 19, 13, 12] and the references therein). In comparison, research on hemivariational inequalities is more recent. It was started in the early 1980s [24] due to the need in engineering applications for problems involving nonsmooth, nonmonotone, and possibly multivalued relations for deformable bodies. Since then, hemivariational inequalities have attracted steady attention in the engineering and applied mathematics communities. Some comprehensive references in the area include [26, 23, 15, 2, 21, 29]. In addition to mathematical theories of hemivariational inequalities, recent years have witnessed substantial progress on numerical analysis of hemivariational inequalities (cf. [14]).

Singular perturbations of variational inequalities were studied as early as the late 1960s and early 1970s (e.g., [17, 18, 20, 27]). A representative result can be found in Corollary 4.3 as a consequence of the results proved in section 3. The aim of the paper is to study singular perturbations of more general inequality problems. Following the relevant references (e.g., [11, 22, 29]), we use the term variational-hemivariational inequality to mean an inequality problem over a function space or a closed and convex set of the space such that both nonsmooth convex functionals and nonsmooth nonconvex functionals are present; for examples of a variational-hemivariational inequality, see (2.1), (3.16), or (3.17). When the nonsmooth convex functionals are dropped, the variational-hemivariational inequality is reduced to a "pure" hemivariational inequality; for examples of a hemivariational inequality, see

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(4.1) or (4.2). When the nonsmooth nonconvex functionals are dropped, or when both nonsmooth convex and nonsmooth nonconvex functionals are dropped and the problem is posed over a closed and convex set, we have a variational inequality; for examples of a variational inequality, see (4.3), (4.4), (4.5), or (4.6).

The rest of the paper is organized as follows. In section 2, we introduce a general variational-hemivariational inequality and recall a solution existence and uniqueness result. In addition, we present a Minty lemma for the variational-hemivariational inequality, which will be needed for the singular perturbation analysis. In section 3, we study the singular perturbation problem of the variational-hemivariational inequality and show the convergence of its solution to the solution of the limiting variational-hemivariational inequality as the perturbation parameter approaches zero. In section 4, we specialize the results proved in section 3 for singular perturbations of "pure" hemivariational inequalities and of variational inequalities. Finally, in section 5, we illustrate the theoretical results in the study of a singular perturbation problem for the plate bending with an obstacle.

2. A general variational-hemivariational inequality. First, we introduce a general variational-hemivariational inequality. Let V be a normed space, and let K be a set in V. Denote by V^* its dual space and by $\langle\cdot,\cdot\rangle$ the duality pairing between V^* and V. We use the symbol $\to$ for strong convergence (convergence in norm) and use the symbol $\rightharpoonup$ for weak convergence. The variational-hemivariational inequality of concern is as follows.

PROBLEM 2.1. *Find an element $u \in K$ such that*

$$\langle Au, v-u\rangle + \Phi(u,v) - \Phi(u,u) + J^0(u; v-u) \geq \langle f, v-u\rangle \quad \forall\, v \in K. \tag{2.1}$$

Let us list the hypotheses on the data. We note that these hypotheses are commonly used in the study of Problem 2.1 (cf. [22, 29, 14]).

(H_K) V is a reflexive Banach space, and K is a nonempty, closed, and convex subset of V.

(H_A) $A\colon V \to V^*$ is bounded, continuous, and strongly monotone with a monotonicity constant $m_A > 0$,

$$\langle Av_1 - Av_2, v_1 - v_2\rangle \geq m_A \|v_1 - v_2\|_V^2 \quad \forall\, v_1, v_2 \in V. \tag{2.2}$$

(H_Φ) $\Phi\colon V\times V \to \mathbb{R}$ is convex and continuous with respect to its second argument, and there exists a constant $m_\Phi > 0$ such that

$$\Phi(v_1,v_4) - \Phi(v_1,v_3) + \Phi(v_2,v_3) - \Phi(v_2,v_4) \leq m_\Phi \|v_1 - v_2\|_V \|v_3 - v_4\|_V \quad \forall\, v_1, v_2, v_3, v_4 \in V. \tag{2.3}$$

(H_J) $J\colon V \to \mathbb{R}$ is locally Lipschitz and there exist constants $c_0, c_1 \geq 0$ and $m_J > 0$ such that

$$\|\partial J(v)\|_{V^*} \leq c_0 + c_1\|v\|_V \quad \forall\, v \in V, \tag{2.4}$$

$$J^0(v_1; v_2 - v_1) + J^0(v_2; v_1 - v_2) \leq m_J \|v_1 - v_2\|_V^2 \quad \forall\, v_1, v_2 \in V. \tag{2.5}$$

(H_f) $f \in V^*$.

In the formulation of Problem 2.1 and in (H_J), we use the notions of the generalized directional derivative and generalized subdifferential in the sense of Clarke (cf. [4, 5]), which are recalled here. Assume $\psi\colon V \to \mathbb{R}$ is locally Lipschitz continuous.

The generalized directional derivative of ψ at $u \in V$ in the direction $v \in V$ is defined to be

$$\psi^0(u;v) := \limsup_{w\to u,\, \lambda\downarrow 0} \frac{\psi(w+\lambda v)-\psi(w)}{\lambda}\,.$$

The generalized subdifferential of ψ at u is a subset of the dual space V^* given by

$$\partial\psi(u) := \left\{\xi \in V^* \mid \psi^0(u;v) \geq \langle \xi, v\rangle \ \forall\, v \in V\right\}.$$

We note the following properties:

$$\psi^0(u;t\,v) = t\,\psi^0(u;v) \quad \forall\, u,v \in V,\ t \geq 0, \tag{2.6}$$

$$\psi^0(u;v_1+v_2) \leq \psi^0(u;v_1)+\psi^0(u;v_2) \quad \forall\, u,v_1,v_2 \in V, \tag{2.7}$$

$$\psi^0(u;v) = \max\{\langle\zeta,v\rangle \mid \zeta \in \partial\psi(u)\} \quad \forall\, u,v \in V, \tag{2.8}$$

$$u_n \to u \text{ and } v_n \to v \text{ in } V \quad \Longrightarrow \quad \limsup_{n\to\infty} \psi^0(u_n;v_n) \leq \psi^0(u;v). \tag{2.9}$$

Note that $\partial J(v)$ is a set and (2.4) means

$$\|\xi\|_{V^*} \leq c_0 + c_1\|v\|_V \quad \forall\, v \in V,\ \xi \in \partial J(v).$$

LEMMA 2.2. *Let $J\colon V \to \mathbb{R}$ be locally Lipschitz continuous. Then*

$$J^0(u;v) \geq -J^0(u;-v) \quad \forall\, u,v \in V. \tag{2.10}$$

Proof. We start with the identity

$$0 = \frac{1}{\lambda}\left[J(w+\lambda\, v) - J(w)\right] + \frac{1}{\lambda}\left[J(w) - J(w+\lambda\, v)\right]$$

for any $\lambda > 0$ and $w \in V$. Take the upper limit of both sides,

$$0 \leq \limsup_{w\to u,\, \lambda\downarrow 0} \frac{1}{\lambda}\left[J(w+\lambda\, v) - J(w)\right] + \limsup_{w\to u,\, \lambda\downarrow 0} \frac{1}{\lambda}\left[J(w) - J(w+\lambda\, v)\right].$$

With the substitution $w' = w + \lambda\, v$, we rewrite the second term on the right side as

$$\limsup_{w\to u,\, \lambda\downarrow 0} \frac{1}{\lambda}\left[J(w) - J(w+\lambda\, v)\right] = \limsup_{w'\to u,\, \lambda\downarrow 0} \frac{1}{\lambda}\left[J(w'+\lambda\,(-v)) - J(w')\right] = J^0(u;-v).$$

So

$$0 \leq J^0(u;v) + J^0(u;-v),$$

i.e., (2.10) holds. ∎

Regarding the solvability of Problem 2.1, a representative result is the following.

THEOREM 2.3. *Assume (H_K), (H_A), (H_Φ), (H_J), (H_f), and*

$$m_\Phi + m_J < m_A.$$

Then, Problem 2.1 *has a unique solution $u \in K$.*

This solution existence and uniqueness result (in a slightly different form) was first proved in [22]. Its variant is provided in [9, 14]. The form presented in Theorem 2.3 is adapted from [14, Theorem 4.2]. We comment that the assumption on A

can be replaced by a more general one: $A\colon V \to V^*$ is pseudomonotone and strongly monotone. However, in applications, it is sufficient to assume (H_A), which is more accessible to researchers from the applied communities. The condition (2.3) is standard in the study of the corresponding variational inequality:

$$u \in K, \quad \langle Au, v-u\rangle + \Phi(u,v) - \Phi(u,u) \geq \langle f, v-u\rangle \quad \forall\, v \in K.$$

The condition (2.4) limits the allowable growth rate of the generalized subdifferential of the functional J. The condition (2.5) is known as a relaxed monotonicity condition; when J is convex, (2.5) holds with $m_J = 0$.

The assumption $m_\Phi + m_J < m_A$ is called a smallness condition. Such a condition is usually adopted in the literature in the well-posedness analysis of hemivariational inequalities.

The next result provides a characterization of the solution of Problem 2.1, which can be called a Minty lemma.

THEOREM 2.4. *Assume (H_K), (H_A), (H_f), and assume $\Phi\colon V\times V \to \mathbb{R}$ is convex with respect to its second argument, $J\colon V \to \mathbb{R}$ is locally Lipschitz, and* (2.5) *holds with $m_J \leq m_A$. Then $u \in K$ is a solution of Problem* 2.1 *if and only if it satisfies*

$$\langle Av, v-u\rangle + \Phi(u,v) - \Phi(u,u) + J^0(v; v-u) \geq \langle f, v-u\rangle \quad \forall\, v \in K. \tag{2.11}$$

Proof. Let $u \in K$ be the solution of Problem 2.1. Fix an arbitrary $v \in K$. By (2.10) and (2.5),

$$J^0(v; v-u) - J^0(u; v-u) \geq -\left[J^0(v; u-v) + J^0(u; v-u)\right] \geq -m_J \|v-u\|_V^2.$$

Then by (2.2) and the assumption $m_J \leq m_A$,

$$\langle Av - Au, v-u\rangle + J^0(v; v-u) - J^0(u; v-u) \geq m_A\|v-u\|_V^2 - m_J\|v-u\|_V^2 \geq 0,$$

i.e.,

$$\langle Av, v-u\rangle + J^0(v; v-u) \geq \langle Au, v-u\rangle + J^0(u; v-u).$$

Then it is obvious that u satisfies the inequality (2.11).

Conversely, assume $u \in K$ satisfies (2.11). Since K is convex, for any $v \in K$ and any $t \in [0,1]$, $u + t\,(v-u) \in K$. We replace v by $u + t\,(v-u)$ in (2.11),

$$\begin{aligned} t\,\langle A\,(u + t\,(v-u)), v-u\rangle + \Phi(u, u + t\,(v-u)) - \Phi(u,u) \\ + t\,J^0(u + t\,(v-u); v-u) \geq t\langle f, v-u\rangle. \end{aligned} \tag{2.12}$$

Note that $\Phi(u,\cdot)$ is convex,

$$\Phi(u, u + t\,(v-u)) \leq t\,\Phi(u,v) + (1-t)\,\Phi(u,u).$$

We deduce from (2.12) that for $t \in (0,1)$,

$$\langle A\,(u + t\,(v-u)), v-u\rangle + \Phi(u,v) - \Phi(u,u) + J^0(u + t\,(v-u); v-u) \geq \langle f, v-u\rangle.$$

Recall that $A\colon V \to V^*$ is continuous, and the generalized directional derivative has the upper semicontinuity property (2.9). We take the upper limit with $t \to 0+$ in the left side of the above inequality to recover the inequality (2.1). ∎

3. Singular perturbation of a variational-hemivariational inequality. In this section, we consider singular perturbation for a general variational-hemivariational inequality. For this purpose, we first list assumptions on the data. We need two normed spaces V_1 and V_0 and their subsets K_1 and K_0.

(H_{K_1}) V_1 is a reflexive Banach space, and K_1 is a nonempty, closed, and convex subset of V_1.

(H_{K_0}) V_0 is a reflexive Banach space, and K_0 is a nonempty, closed, and convex subset of V_0.

These spaces and subsets are related as follows.

(H_{K_1,K_0}) V_1 is continuously and densely embedded in V_0, and K_0 is the closure of K_1 in V_0.

Note that (H_{K_1}) and (H_{K_1,K_0}) together imply that K_0 is a nonempty, closed, and convex subset of V_0.

For $i = 0, 1$, we denote by $\|\cdot\|_i$ the norm of the space V_i, by V_i^* the dual space of V_i, by $\|\cdot\|_{i*}$ the norm of the space V_i^*, and by $\langle\cdot,\cdot\rangle_i$ the duality pairing between V_i^* and V_i. Denote by $\alpha > 0$ an embedding constant from V_1 to V_0:

$$\|v\|_0 \le \alpha\,\|v\|_1 \quad \forall\, v \in V_1. \tag{3.1}$$

Corresponding to the two spaces, we introduce two operators A_1 and A_0.

(H_{A_1}) $A_1\colon V_1 \to V_1^*$, and for some constants $L_1 > 0$, $m_1 > 0$ and $m_2 \ge 0$,

$$\|A_1 u - A_1 v\|_{1*} \le L_1 \|u - v\|_1 \quad \forall\, u, v \in V_1, \tag{3.2}$$

$$\langle A_1 u - A_1 v, u - v\rangle_1 \ge m_1 \|u - v\|_1^2 - m_2 \|u - v\|_0^2 \quad \forall\, u, v \in V_1. \tag{3.3}$$

(H_{A_0}) $A_0\colon V_0 \to V_0^*$, and for some constants $L_0 > 0$ and $m_0 > 0$,

$$\|A_0 u - A_0 v\|_{0*} \le L_0 \|u - v\|_0 \quad \forall\, u, v \in V_0, \tag{3.4}$$

$$\langle A_0 u - A_0 v, u - v\rangle_0 \ge m_0 \|u - v\|_0^2 \quad \forall\, u, v \in V_0. \tag{3.5}$$

The assumption on Φ is similar to (H_Φ) except that it is a condition over V_0.

(H_{Φ,V_0}) $\Phi\colon V_0 \times V_0 \to \mathbb{R}$ is convex and continuous with respect to its second argument, and for some constant $m_\Phi > 0$,

$$\Phi(v_1, v_4) - \Phi(v_1, v_3) + \Phi(v_2, v_3) - \Phi(v_2, v_4) \le m_\Phi \|v_1 - v_2\|_0 \|v_3 - v_4\|_0 \quad \forall\, v_1, v_2, v_3, v_4 \in V_0. \tag{3.6}$$

Similarly, the conditions on J and f are over V_0.

(H_{J,V_0}) $J\colon V_0 \to \mathbb{R}$ is locally Lipschitz and there exist constants $c_0, c_1 \ge 0$, and $m_J > 0$ such that

$$\|\partial J(v)\|_{0*} \le c_0 + c_1 \|v\|_0 \quad \forall\, v \in V_0, \tag{3.7}$$

$$J^0(v_1; v_2 - v_1) + J^0(v_2; v_1 - v_2) \le m_J \|v_1 - v_2\|_0^2 \quad \forall\, v_1, v_2 \in V_0. \tag{3.8}$$

(H_{f,V_0}) $f \in V_0^*$.

Finally, the smallness assumption takes the following form:

(H_s)

$$m_\Phi + m_J < m_0. \tag{3.9}$$

In the study of the singular perturbation problem, we will use $\tilde{J}$ to stand for either $J\colon V_0 \to \mathbb{R}$ or its restriction $J|_{V_1}\colon V_1 \to \mathbb{R}$. Note that $J|_{V_1}$ is locally Lipschitz continuous on V_1. The symbol $\tilde{J}^0$ can mean the generalized directional derivative of J either in V_1 or V_0:

$$\tilde{J}^0(u;v) = \limsup_{w\to u \text{ in } V_1,\, \lambda\downarrow 0} \frac{J(w+\lambda v)-J(w)}{\lambda} \quad \text{for } \tilde{J} = J|_{V_1},\ u,v\in V_1 \tag{3.10}$$

or

$$\tilde{J}^0(u;v) = \limsup_{w\to u \text{ in } V_0,\, \lambda\downarrow 0} \frac{J(w+\lambda v)-J(w)}{\lambda} \quad \text{for } \tilde{J} = J,\ u,v\in V_0. \tag{3.11}$$

Note that in the case $\tilde{J} = J$,

$$\tilde{J}^0(u;v) = J^0(u;v) \quad \forall\, u,v\in V_1;$$

and in the case $\tilde{J} = J|_{V_1}$,

$$\tilde{J}^0(u;v) \le J^0(u;v) \quad \forall\, u,v\in V_1.$$

So in both cases,

$$\tilde{J}^0(u;v) \le J^0(u;v) \quad \forall\, u,v\in V_1. \tag{3.12}$$

In terms of the generalized subdifferential, (3.12) is equivalent to

$$\partial\tilde{J}(u) \subset \partial J(u) \quad \forall\, u\in V_1. \tag{3.13}$$

Consider the case where $\tilde{J} = J|_{V_1}$. The assumption (3.7) implies

$$\|\partial\tilde{J}(v)\|_{1*} \le \alpha\, c_0 + \alpha\, c_1\|v\|_0 \quad \forall\, v\in V_1. \tag{3.14}$$

This is proved as follows. Let $\xi\in\partial\tilde{J}(v)$. By definition,

$$\|\xi\|_{1*} = \sup_{\|w\|_1=1} \langle \xi, w\rangle_1.$$

By (2.8) and then by (3.12),

$$\|\xi\|_{1*} \le \sup_{\|w\|_1=1} \tilde{J}^0(v;w) \le \sup_{\|w\|_1=1} J^0(v;w).$$

From (2.8) and (3.7), for $w\in V_1$ with $\|w\|_1 = 1$,

$$J^0(v;w) \le (c_0 + c_1\|v\|_0)\,\|w\|_0 \le (c_0 + c_1\|v\|_0)\,\alpha\,\|w\|_1 = \alpha\,(c_0 + c_1\|v\|_0)\,.$$

Thus, (3.14) holds. Corresponding to (3.8), thanks to (3.12), we have

$$\tilde{J}^0(v_1;v_2-v_1) + \tilde{J}^0(v_2;v_1-v_2) \le m_J\|v_1-v_2\|_0^2 \quad \forall\, v_1,v_2\in V_1. \tag{3.15}$$

Let $\varepsilon > 0$ be a small singular perturbation parameter. We consider the singularly perturbed variational-hemivariational inequality

$$\begin{aligned} u_\varepsilon\in K_1,\quad \varepsilon\,\langle A_1u_\varepsilon, v-u_\varepsilon\rangle_1 + \langle A_0u_\varepsilon, v-u_\varepsilon\rangle_0 + \Phi(u_\varepsilon,v) - \Phi(u_\varepsilon,u_\varepsilon)\\ + \tilde{J}^0(u_\varepsilon; v-u_\varepsilon) \ge \langle f, v-u_\varepsilon\rangle_0 \quad \forall\, v\in K_1 \end{aligned} \tag{3.16}$$

and the limiting problem

$$\begin{aligned} u\in K_0,\quad \langle A_0u, v-u\rangle_0 + \Phi(u,v) - \Phi(u,u) + J^0(u;v-u)\\ \ge \langle f, v-u\rangle_0 \quad \forall\, v\in K_0. \end{aligned} \tag{3.17}$$

We have the following unique solvability result on the problem (3.16).

THEOREM 3.1. *Assume* (H_{K_1}), (H_{K_0}), (H_{K_1,K_0}), (H_{A_1}), (H_{A_0}), (H_{Φ,V_0}), (H_{J,V_0}), (H_{f,V_0}), *and* (H_s). *If* $\varepsilon > 0$ *is so small that* $\varepsilon\, m_2 \le m_0 - m_J$, *then the problem* (3.16) *has a unique solution.*

Proof. First consider the case $\tilde{J} = J|_{V_1}$. Define $A\colon V_1 \to V_1^*$ by $A = \varepsilon\, A_1 + A_0$, i.e.,

$$\langle Au, v\rangle_1 = \varepsilon\, \langle A_1 u, v\rangle_1 + \langle A_0 u, v\rangle_0, \quad u, v \in V_1,$$

and view Φ, J, and f as being defined over V_1. Obviously, $A\colon V_1 \to V_1^*$ is bounded and continuous. We can then follow the proof of Theorem 2.3 in [22]; here, we only point out how to verify the strong monotonicity of the operator $T_1\colon V_1 \to 2^{V_1^*}$ defined by $T_1 v = Av + \partial\tilde{J}(v)$, since the rest of the proof is identical. For any $u, v \in V_1$,

$$\langle T_1 u - T_1 v, u - v\rangle_1 = \varepsilon\, \langle A_1 u - A_1 v, u - v\rangle_1 + \langle A_0 u - A_0 v, u - v\rangle_0 \\ + \langle \partial\tilde{J}(u) - \partial\tilde{J}(v), u - v\rangle_1.$$

From (3.15),

$$\langle \partial\tilde{J}(u) - \partial\tilde{J}(v), u - v\rangle_1 \ge -m_J \|u - v\|_0^2.$$

Use this inequality and (3.3), (3.5) to obtain

$$\langle T_1 u - T_1 v, u - v\rangle_1 \ge \varepsilon\, m_1 \|u - v\|_1^2 + (m_0 - \varepsilon\, m_2)\, \|u - v\|_0^2 - m_J \|u - v\|_0^2 \\ = \varepsilon\, m_1 \|u - v\|_1^2 + (m_0 - m_J - \varepsilon\, m_2)\, \|u - v\|_0^2.$$

Thus, if $\varepsilon > 0$ is so small that $\varepsilon\, m_2 \le m_0 - m_J$, then

$$\langle T_1 u - T_1 v, u - v\rangle_1 \ge \varepsilon\, m_1 \|u - v\|_1^2, \quad u, v \in V_1,$$

and so $T_1\colon V_1 \to 2^{V_1^*}$ is strongly monotone.

Next, consider the case $\tilde{J} = J$. By (3.12), the solution $u_\varepsilon \in K_1$ of the problem (3.16) with $\tilde{J} = J|_{V_1}$ satisfies the relation

$$\varepsilon\, \langle A_1 u_\varepsilon, v - u_\varepsilon\rangle_1 + \langle A_0 u_\varepsilon, v - u_\varepsilon\rangle_0 + \Phi(u_\varepsilon, v) - \Phi(u_\varepsilon, u_\varepsilon) + J^0(u_\varepsilon; v - u_\varepsilon) \\ \ge \langle f, v - u_\varepsilon\rangle_0 \quad \forall\, v \in K_1.$$

So the problem (3.16) with $\tilde{J} = J$ has a solution. The uniqueness of the solution of the problem (3.16) follows from a standard argument. ∎

Note that if $m_2 = 0$ for the assumption (3.3), then there is no restriction on ε in the statement of Theorem 3.1.

THEOREM 3.2. *Under the assumptions* (H_{K_0}), (H_{A_0}), (H_{Φ,V_0}), (H_{J,V_0}), (H_{f,V_0}), *and* (H_s), *the problem* (3.17) *has a unique solution.*

Proof. We apply Theorem 2.3 by letting $V = V_0$, $K = K_0$, $A = A_0$. Then it is straightforward to conclude that the problem (3.17) has a unique solution. ∎

The main result of the section is to show the convergence $u_\varepsilon \to u$. In the convergence proof, we will use the next result.

PROPOSITION 3.3. *Let* $V_1 \subset V_0$ *be two normed spaces with the norms* $\|\cdot\|_1$ *and* $\|\cdot\|_0$, *let* K_1 *be a nonempty set in* V_1, *and let* K_0 *be the closure of* K_1 *in* V_0. *Assume* $a_1 > a_2 > \cdots > a_n > \cdots$ *is a sequence of decreasing positive numbers with* $a_n \to 0$. *Then for any* $v \in K_0$, *there exist a constant* $M > 0$ *and a sequence* $\{v_n\} \subset K_1$ *such that* $v_n \to v$ *in* V_0 *and* $a_n \|v_n\|_1 \le M$ *for any* n.

Proof. Since K_1 is dense in K_0, for $v \in K_0$, there is a sequence $\{\tilde{v}_n\} \subset K_1$ such that $\tilde{v}_n \to v$ in V_0. If $\{\|\tilde{v}_n\|_1\}$ is bounded, the result is true by taking $v_n = \tilde{v}_n$ and $M = a_1 \sup_n \|\tilde{v}_n\|_1$.

Suppose $\{\|\tilde{v}_n\|_1\}$ is not bounded. Without loss of generality and by resorting to a subsequence if necessary, we may assume $\tilde{v}_1 \neq 0$, $\{\|\tilde{v}_n\|_1\}$ is a nondecreasing sequence, and $\|\tilde{v}_n\|_1 \to \infty$. Take $M = a_1\|\tilde{v}_1\|_1$ and let $v_1 = \tilde{v}_1$. For $n \geq 1$, let $m(n)$ be the largest integer such that $a_n\|\tilde{v}_{m(n)}\|_1 \leq M$ and let $v_n = \tilde{v}_{m(n)}$. Obviously, $m(n) \to \infty$ as $n \to \infty$. Then the result holds with the sequence $\{v_n\}$. ∎

THEOREM 3.4. *Keep the assumptions stated in Theorem* 3.1. *As* $\varepsilon \to 0$, *we have*

$$\|u_\varepsilon - u\|_0 \to 0, \tag{3.18}$$

$$\sqrt{\varepsilon}\,\|u_\varepsilon\|_1 \to 0. \tag{3.19}$$

Proof. First we show that $\{u_\varepsilon\}$ is bounded in V_0 and $\{\sqrt{\varepsilon}\,u_\varepsilon\}$ is bounded in V_1. From the conditions (3.5) and (3.3) on A_1 and A_0, we have for any $v \in K_1$,

$$\varepsilon\, m_1\|u_\varepsilon - v\|_1^2 + (m_0 - \varepsilon\, m_2)\,\|u_\varepsilon - v\|_0^2 \leq \varepsilon\,\langle A_1u_\varepsilon - A_1v, u_\varepsilon - v\rangle_1 + \langle A_0u_\varepsilon - A_0v, u_\varepsilon - v\rangle_0. \tag{3.20}$$

By (3.16),

$$\varepsilon\,\langle A_1u_\varepsilon, u_\varepsilon - v\rangle_1 + \langle A_0u_\varepsilon, u_\varepsilon - v\rangle_0 \leq \Phi(u_\varepsilon, v) - \Phi(u_\varepsilon, u_\varepsilon) + \tilde{J}^0(u_\varepsilon; v - u_\varepsilon) - \langle f, v - u_\varepsilon\rangle_0.$$

Then,

$$\varepsilon\, m_1\|u_\varepsilon - v\|_1^2 + (m_0 - \varepsilon\, m_2)\,\|u_\varepsilon - v\|_0^2 \leq \varepsilon\,\langle A_1v, v - u_\varepsilon\rangle_1 + \langle A_0v, v - u_\varepsilon\rangle_0 + \Phi(u_\varepsilon, v) - \Phi(u_\varepsilon, u_\varepsilon) + \tilde{J}^0(u_\varepsilon; v - u_\varepsilon) - \langle f, v - u_\varepsilon\rangle_0. \tag{3.21}$$

Use (3.6) with $v_1 = u_\varepsilon$, $v_2 = u$, $v_3 = u_\varepsilon$, and $v_4 = v$,

$$\Phi(u_\varepsilon, v) - \Phi(u_\varepsilon, u_\varepsilon) + \Phi(u, u_\varepsilon) - \Phi(u, v) \leq m_\Phi\|u_\varepsilon - u\|_0\|u_\varepsilon - v\|_0,$$

to obtain

$$\Phi(u_\varepsilon, v) - \Phi(u_\varepsilon, u_\varepsilon) \leq m_\Phi\|u_\varepsilon - u\|_0\|u_\varepsilon - v\|_0 + \Phi(u, v) - \Phi(u, u_\varepsilon). \tag{3.22}$$

Since $\Phi(u, \cdot)$ is convex, we have the lower bound

$$\Phi(u, u_\varepsilon) \geq c_2 + c_3\|u_\varepsilon\|_0$$

for some constants c_2 and c_3 depending on u (cf. [1, p. 433]). This implies

$$-\,\Phi(u, u_\varepsilon) \leq c\,(1 + \|u_\varepsilon - v\|_0)\,. \tag{3.23}$$

By (3.12),

$$\tilde{J}^0(u_\varepsilon; v - u_\varepsilon) \leq J^0(u_\varepsilon; v - u_\varepsilon), \tag{3.24}$$

and by (3.8),

$$J^0(u_\varepsilon; v - u_\varepsilon) \leq m_J\|u_\varepsilon - v\|_0^2 - J^0(v; u_\varepsilon - v). \tag{3.25}$$

We have, by making use of (3.7),

$$-J^0(v; u_\varepsilon - v) \le (c_0 + c_1\|v\|_0)\,\|u_\varepsilon - v\|_0. \tag{3.26}$$

Use (3.22)–(3.26) in (3.21) to obtain

$$\begin{aligned}\varepsilon\, m_1\|u_\varepsilon - v\|_1^2 + (m_0 - \varepsilon\, m_2)\,\|u_\varepsilon - v\|_0^2 \le{}& \varepsilon\,\langle A_1 v, v - u_\varepsilon\rangle_1 + \langle A_0 v, v - u_\varepsilon\rangle_0\\ &+ m_\Phi\|u_\varepsilon - u\|_0\|u_\varepsilon - v\|_0 + \Phi(u, v)\\ &+ c\,(1 + \|u_\varepsilon - v\|_0) + m_J\|u_\varepsilon - v\|_0^2\\ &+ (c_0 + c_1\|v\|_0)\,\|u_\varepsilon - v\|_0\\ &- \langle f, v - u_\varepsilon\rangle_0.\end{aligned} \tag{3.27}$$

In the rest of the proof, we will use repeatedly the modified Cauchy–Schwarz inequality

$$a\,b \le \delta\, a^2 + \frac{1}{4\,\delta}\, b^2 \quad \forall\, a, b \in \mathbb{R},\ \delta > 0.$$

We bound each term of the right side of the inequality (3.27)

$$\varepsilon\,\langle A_1 v, v - u_\varepsilon\rangle_1 \le \frac{\varepsilon\, m_1}{2}\,\|u_\varepsilon - v\|_1^2 + \frac{\varepsilon}{2\, m_1}\,\|A_1 v\|_{1*}^2,$$

and for any $\delta > 0$,

$$\begin{aligned}\langle A_0 v, v - u_\varepsilon\rangle_0 &\le \delta\,\|u_\varepsilon - v\|_0^2 + \frac{1}{4\,\delta}\,\|A_0 v\|_{0*}^2,\\ m_\Phi\|u_\varepsilon - u\|_0\|u_\varepsilon - v\|_0 &\le m_\Phi\|u_\varepsilon - v\|_0^2 + m_\Phi\|u - v\|_0\|u_\varepsilon - v\|_0\\ &\le (m_\Phi + \delta)\,\|u_\varepsilon - v\|_0^2 + \frac{m_\Phi^2}{4\,\delta}\,\|u - v\|_0^2,\\ c\,\|u_\varepsilon - v\|_0 &\le \delta\,\|u_\varepsilon - v\|_0^2 + \frac{c^2}{4\,\delta},\\ (c_0 + c_1\|v\|_0)\,\|u_\varepsilon - v\|_0 &\le \delta\,\|u_\varepsilon - v\|_0^2 + \frac{(c_0 + c_1\|v\|_0)^2}{4\,\delta},\\ -\langle f, v - u_\varepsilon\rangle_0 &\le \delta\,\|u_\varepsilon - v\|_0^2 + \frac{\|f\|_{0*}^2}{4\,\delta}.\end{aligned}$$

Thus,

$$\frac{1}{2}\,\varepsilon\, m_1\|u_\varepsilon - v\|_1^2 + (m_0 - \varepsilon\, m_2 - m_\Phi - m_J - 5\,\delta)\,\|u_\varepsilon - v\|_0^2 \le c$$

for a constant c depending on δ, $\|v\|_1$ and other problem data, but independent of ε. Let $\varepsilon \le (m_0 - m_\Phi - m_J)\,/(2\,\max\{m_2, 1\})$. Choose $\delta > 0$ sufficiently small to get

$$\varepsilon\,\|u_\varepsilon - v\|_1^2 + \|u_\varepsilon - v\|_0^2 \le c \tag{3.28}$$

for a constant $c > 0$ independent of ε. Then with a particular $v \in K_1$ chosen and fixed, we have

$$\sqrt{\varepsilon}\,\|u_\varepsilon\|_1 \le c, \quad \|u_\varepsilon\|_0 \le c. \tag{3.29}$$

Since V_0 is reflexive, there exists an element $w \in V_0$ such that for a subsequence of the solutions, still denoted as $\{u_\varepsilon\}$, we have the weak convergence

$$u_\varepsilon \rightharpoonup w \quad \text{in } V_0. \tag{3.30}$$

Moreover, since convexity and closedness imply weak closedness, the weak limit $w \in K_0$.

Next, let us prove the strong convergence

$$u_\varepsilon \to w \quad \text{in } V_0. \tag{3.31}$$

According to Proposition 3.3, we can find $w_\varepsilon \in K_1$ such that $\varepsilon^{1/4}\|w_\varepsilon\|_1 \le \bar{c}_0$ for some constant $\bar{c}_0$ and

$$\|w_\varepsilon - w\|_0 \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.32}$$

Thus, (3.31) will follow if we can prove

$$\|w_\varepsilon - u_\varepsilon\|_0 \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.33}$$

Use (3.21) with $v = w_\varepsilon$,

$$\begin{aligned} \varepsilon\, m_1\|u_\varepsilon - w_\varepsilon\|_1^2 + (m_0 - \varepsilon\, m_2)\, \|u_\varepsilon - w_\varepsilon\|_0^2 \\ \le \varepsilon\, \langle A_1 w_\varepsilon, w_\varepsilon - u_\varepsilon\rangle_1 + \langle A_0 w_\varepsilon, w_\varepsilon - u_\varepsilon\rangle_0 + \Phi(u_\varepsilon, w_\varepsilon) - \Phi(u_\varepsilon, u_\varepsilon) \\ + \tilde{J}^0(u_\varepsilon; w_\varepsilon - u_\varepsilon) - \langle f, w_\varepsilon - u_\varepsilon\rangle_0. \end{aligned} \tag{3.34}$$

Let us bound each term on the right side of (3.34). First, by (3.2),

$$\|A_1 w_\varepsilon\|_{1*} \le \|A_1 w_\varepsilon - A_1 0\|_{1*} + \|A_1 0\|_{1*} \le L_1\|w_\varepsilon\|_1 + \|A_1 0\|_{1*}.$$

So

$$\varepsilon\, \langle A_1 w_\varepsilon, w_\varepsilon - u_\varepsilon\rangle_1 \le \varepsilon\, \|A_1 w_\varepsilon\|_{1*}\|w_\varepsilon - u_\varepsilon\|_1 \le \frac{1}{2}\,\varepsilon\, m_1\|u_\varepsilon - w_\varepsilon\|_1^2 + c\,\varepsilon\left(\|w_\varepsilon\|_1^2 + 1\right).$$

Write

$$\langle A_0 w_\varepsilon, w_\varepsilon - u_\varepsilon\rangle_0 = \langle A_0 w_\varepsilon - A_0 w, w_\varepsilon - u_\varepsilon\rangle_0 + \langle A_0 w, w_\varepsilon - u_\varepsilon\rangle_0$$

and note that due to (3.4), for any $\delta > 0$, there is a constant $c > 0$ depending on δ such that

$$\begin{aligned} \langle A_0 w_\varepsilon - A_0 w, w_\varepsilon - u_\varepsilon\rangle_0 &\le L_0\|w_\varepsilon - w\|_0\|w_\varepsilon - u_\varepsilon\|_0 \\ &\le \delta\, \|w_\varepsilon - u_\varepsilon\|_0^2 + c\, \|w_\varepsilon - w\|_0^2. \end{aligned}$$

By (3.22) with $v = w_\varepsilon$ and with u replaced by w,

$$\Phi(u_\varepsilon, w_\varepsilon) - \Phi(u_\varepsilon, u_\varepsilon) \le m_\Phi\|u_\varepsilon - w\|_0\|u_\varepsilon - w_\varepsilon\|_0 + \Phi(w, w_\varepsilon) - \Phi(w, u_\varepsilon).$$

Moreover, for any $\delta > 0$,

$$\begin{aligned} m_\Phi\|u_\varepsilon - w\|_0\|u_\varepsilon - w_\varepsilon\|_0 &\le m_\Phi\left(\|u_\varepsilon - w_\varepsilon\|_0 + \|w_\varepsilon - w\|_0\right)\|u_\varepsilon - w_\varepsilon\|_0 \\ &\le (m_\Phi + \delta)\, \|u_\varepsilon - w_\varepsilon\|_0^2 + c\, \|w_\varepsilon - w\|_0^2 \end{aligned}$$

with a constant $c > 0$ depending on δ. Note that using (3.12) and (2.7),

$$\tilde{J}^0(u_\varepsilon; w_\varepsilon - u_\varepsilon) \le J^0(u_\varepsilon; w_\varepsilon - u_\varepsilon) \le J^0(u_\varepsilon; w - u_\varepsilon) + J^0(u_\varepsilon; w_\varepsilon - w).$$

By (3.7) and (3.29),

$$J^0(u_\varepsilon; w_\varepsilon - w) \le (c_0 + c_1\|u_\varepsilon\|_0)\, \|w_\varepsilon - w\|_0 \le c\, \|w_\varepsilon - w\|_0.$$

By (3.25) with $v = w$,

$$J^0(u_\varepsilon; w - u_\varepsilon) \le m_J \|u_\varepsilon - w\|_0^2 - J^0(w; u_\varepsilon - w).$$

On the first term on the right side, for any $\delta > 0$, there is a constant c depending on δ such that

$$\begin{aligned} m_J \|u_\varepsilon - w\|_0^2 &\le m_J \left(\|u_\varepsilon - w_\varepsilon\|_0 + \|w_\varepsilon - w\|_0\right)^2 \\ &\le (m_J + \delta)\, \|u_\varepsilon - w_\varepsilon\|_0^2 + c\, \|w_\varepsilon - w\|_0^2. \end{aligned}$$

So from (3.34), we derive the inequality

$$\begin{aligned} \frac{1}{2}\, \varepsilon\, m_1 \|u_\varepsilon - w_\varepsilon\|_1^2 + (m_0 - m_\Phi - m_J - \varepsilon\, m_2 - 3\,\delta)\, \|u_\varepsilon - w_\varepsilon\|_0^2 \\ \le c\, \varepsilon \left(\|w_\varepsilon\|_1^2 + 1\right) + \langle A_0 w, w_\varepsilon - u_\varepsilon\rangle_0 + \Phi(w, w_\varepsilon) - \Phi(w, u_\varepsilon) \\ + c\, \|w_\varepsilon - w\|_0 + c\, \|w_\varepsilon - w\|_0^2 - J^0(w; u_\varepsilon - w) - \langle f, w_\varepsilon - u_\varepsilon\rangle_0. \end{aligned} \tag{3.35}$$

As $\varepsilon \to 0$, $\varepsilon\, \|w_\varepsilon\|_1^2 \to 0$ since $\varepsilon^{1/4} \|w_\varepsilon\|_1$ is uniformly bounded. The term $\langle A_0 w, w_\varepsilon - u_\varepsilon\rangle_0 \to 0$ since $w_\varepsilon - u_\varepsilon \rightharpoonup 0$ in V_0. Moreover, from the assumption (H_{Φ,V_0}), we know that $\Phi(w, \cdot)$ is continuous and weakly l.s.c. on V_0; thus,

$$\begin{aligned} &\Phi(w, w_\varepsilon) \to \Phi(w, w), \\ &\limsup\, [-\Phi(w, u_\varepsilon)] \le -\Phi(w, w). \end{aligned}$$

The term $\|w_\varepsilon - w\|_0 \to 0$ by (3.32). For any $\xi \in \partial J(w)$,

$$-J^0(w; u_\varepsilon - w) \le -\langle \xi, u_\varepsilon - w\rangle \to 0,$$

and so

$$\limsup \left[-J^0(w; u_\varepsilon - w)\right] \le 0.$$

The term $\langle f, w_\varepsilon - u_\varepsilon\rangle_0 \to 0$ since $w_\varepsilon - u_\varepsilon \rightharpoonup 0$ in V_0. In conclusion, from (3.35) with $\delta > 0$ sufficiently small, we have

$$\limsup_{\varepsilon \to 0} \left[\varepsilon\, \|u_\varepsilon - w_\varepsilon\|_1^2 + \|u_\varepsilon - w_\varepsilon\|_0^2\right] \le 0.$$

Thus, (3.33) and therefore the strong convergence (3.31) holds. Moreover, from the above inequality and $\sqrt{\varepsilon}\, w_\varepsilon \to 0$ in V_1, we have (3.19).

Finally, we prove that the limit w is a solution of (3.17). Note that by Theorem 2.4, (3.16) is equivalent to

$$\begin{aligned} u_\varepsilon \in K_1, \quad \varepsilon\, \langle A_1 v, v - u_\varepsilon\rangle_1 + \langle A_0 v, v - u_\varepsilon\rangle_0 + \Phi(u_\varepsilon, v) - \Phi(u_\varepsilon, u_\varepsilon) + \tilde{J}^0(v; v - u_\varepsilon) \\ \ge \langle f, v - u_\varepsilon\rangle_0 \quad \forall\, v \in K_1. \end{aligned}$$

Thanks to (3.12), $u_\varepsilon \in K_1$ satisfies the relation

$$\begin{aligned} \varepsilon\, \langle A_1 v, v - u_\varepsilon\rangle_1 + \langle A_0 v, v - u_\varepsilon\rangle_0 + \Phi(u_\varepsilon, v) - \Phi(u_\varepsilon, u_\varepsilon) + J^0(v; v - u_\varepsilon) \\ \ge \langle f, v - u_\varepsilon\rangle_0 \quad \forall\, v \in K_1. \end{aligned} \tag{3.36}$$

Then we use the strong convergence (3.31) and (3.19), as well as the property (2.9), to show from (3.36) that $w \in K_0$ satisfies the inequality

$$\langle A_0 v, v - w\rangle_0 + \Phi(w, v) - \Phi(w, w) + J^0(v; v - w) \ge \langle f, v - w\rangle_0 \quad \forall\, v \in K_0,$$

where we make use of (3.22) with u replaced by w:

$$\Phi(u_\varepsilon, v) - \Phi(u_\varepsilon, u_\varepsilon) \le m_\Phi \|u_\varepsilon - w\|_0 \|u_\varepsilon - v\|_0 + \Phi(w, v) - \Phi(w, u_\varepsilon).$$

Applying Theorem 2.4 again, we conclude that w is a solution of (3.17). By the uniqueness of a solution of the problem (3.17), we have $w = u$. Since the limit is unique, the limiting relations (3.18) and (3.19) hold for any $\varepsilon \to 0$. ∎

We have an improved convergence result under an additional assumption on the solution regularity of the problem (3.17).

THEOREM 3.5. *Keep the assumptions stated in Theorem* 3.1. *If additionally* $u \in K_1$, *then as* $\varepsilon \to 0$,

$$\|u_\varepsilon - u\|_0 = O(\sqrt{\varepsilon}), \tag{3.37}$$

$$\|u_\varepsilon - u\|_1 \to 0. \tag{3.38}$$

Proof. We let $v = u$ in (3.16), $v = u_\varepsilon$ in (3.17), and add the two inequalities to obtain

$$\begin{aligned} &\varepsilon \langle A_1 u_\varepsilon - A_1 u, u_\varepsilon - u\rangle_1 + \langle A_0 u_\varepsilon - A_0 u, u_\varepsilon - u\rangle_0 \\ &\quad \le \varepsilon \langle A_1 u, u - u_\varepsilon\rangle_1 + J^0(u_\varepsilon, u - u_\varepsilon) + J^0(u; u_\varepsilon - u) \\ &\qquad + \Phi(u_\varepsilon, u) - \Phi(u_\varepsilon, u_\varepsilon) + \Phi(u, u_\varepsilon) - \Phi(u, u), \end{aligned}$$

where again (3.12) is applied. Thus, applying (3.3), (3.5), (3.6), and (3.8),

$$\begin{aligned} \varepsilon\, m_1 \|u_\varepsilon - u\|_1^2 + (m_0 - \varepsilon\, m_2)\, \|u_\varepsilon - u\|_0^2 &\le \varepsilon \langle A_1 u, u - u_\varepsilon\rangle_1 \\ &\quad + m_\Phi \|u_\varepsilon - u\|_0^2 + m_J \|u_\varepsilon - u\|_0^2, \end{aligned}$$

and then

$$\varepsilon\, m_1 \|u_\varepsilon - u\|_1^2 + (m_0 - \varepsilon\, m_2 - m_\Phi - m_J)\, \|u_\varepsilon - u\|_0^2 \le \varepsilon \langle A_1 u, u - u_\varepsilon\rangle_1. \tag{3.39}$$

By bounding the right side of (3.39) with

$$\varepsilon \langle A_1 u, u - u_\varepsilon\rangle_1 \le \frac{\varepsilon\, m_1}{2}\, \|u_\varepsilon - u\|_1^2 + \frac{\varepsilon}{2\, m_1}\, \|A_1 u\|_{1*}^2,$$

we derive from (3.39) that

$$\begin{aligned} &(m_0 - \varepsilon\, m_2 - m_\Phi - m_J)\, \|u_\varepsilon - u\|_0^2 \le \frac{\varepsilon}{2\, m_1}\, \|A_1 u\|_1^2, \\ &\|u_\varepsilon - u\|_1^2 \le \frac{1}{m_1^2}\, \|A_1 u\|_{1*}^2. \end{aligned}$$

Hence, (3.37) holds and $\{u_\varepsilon\}$ is bounded in V_1. Since V_1 is reflexive, a subsequence $\{u_{\varepsilon'}\}$ weakly converges in V_1. Since $u_\varepsilon \to u$ in V_0, the weak limit of $\{u_{\varepsilon'}\}$ equals u. Because the limit u is unique, we conclude that $u_\varepsilon \rightharpoonup u$ in V_1 as $\varepsilon \to 0$. By (3.39) again, as $\varepsilon \to 0$,

$$m_1 \|u_\varepsilon - u\|_1^2 \le \langle A_1 u, u - u_\varepsilon\rangle_1 \to 0,$$

i.e., (3.38) holds. ∎

4. Singular perturbations of hemivariational and variational inequalities. We first consider the special case of singular perturbation of "pure" hemivariational inequalities which are obtained when we set $\Phi \equiv 0$. Thus, the singularly perturbed hemivariational inequality is of the form

$$u_\varepsilon \in K_1, \quad \varepsilon \langle A_1 u_\varepsilon, v - u_\varepsilon \rangle_1 + \langle A_0 u_\varepsilon, v - u_\varepsilon \rangle_0 + \tilde{J}^0(u_\varepsilon; v - u_\varepsilon) \geq \langle f, v - u_\varepsilon \rangle_0 \quad \forall\, v \in K_1, \tag{4.1}$$

whereas the limiting problem is

$$u \in K_0, \quad \langle A_0 u, v - u \rangle_0 + J^0(u; v - u) \geq \langle f, v - u \rangle_0 \quad \forall\, v \in K_0. \tag{4.2}$$

By Theorems 3.4 and 3.5, we have the following result.

Corollary 4.1. *Assume* (H_{K_1}), (H_{K_0}), (H_{K_1,K_0}), (H_{A_1}), (H_{A_0}), (H_{J,V_0}), (H_{f,V_0}), *and* $m_J < m_0$. *Then the problem* (4.1) *has a unique solution* u_ε *if* $\varepsilon\, m_2 \leq m_0 - m_J$, *and* (4.2) *has a unique solution* u. *Moreover, as* $\varepsilon \to 0$,

$$\|u_\varepsilon - u\|_0 \to 0, \quad \sqrt{\varepsilon}\, \|u_\varepsilon\|_1 \to 0,$$

and if in addition $u \in K_1$, *then*

$$\|u_\varepsilon - u\|_0 = O(\sqrt{\varepsilon}), \quad \|u_\varepsilon - u\|_1 \to 0.$$

We then specialize to the case of singular perturbation of pure variational inequalities by setting $J \equiv 0$. The singularly perturbed variational inequality is

$$u_\varepsilon \in K_1, \quad \varepsilon \langle A_1 u_\varepsilon, v - u_\varepsilon \rangle_1 + \langle A_0 u_\varepsilon, v - u_\varepsilon \rangle_0 + \Phi(u_\varepsilon, v) - \Phi(u_\varepsilon, u_\varepsilon) \geq \langle f, v - u_\varepsilon \rangle_0 \quad \forall\, v \in K_1. \tag{4.3}$$

The corresponding limiting problem is

$$u \in K_0, \quad \langle A_0 u, v - u \rangle_0 + \Phi(u, v) - \Phi(u, u) \geq \langle f, v - u \rangle_0 \quad \forall\, v \in K_0. \tag{4.4}$$

By Theorems 3.4 and 3.5, we have the following result.

Corollary 4.2. *Assume* (H_{K_1}), (H_{K_0}), (H_{K_1,K_0}), (H_{A_1}), (H_{A_0}), (H_{Φ,V_0}), (H_{f,V_0}), *and* $m_\Phi < m_0$. *Then the problem* (4.3) *has a unique solution* u_ε *if* $\varepsilon\, m_2 \leq m_0$, *and* (4.4) *has a unique solution* u. *Moreover, as* $\varepsilon \to 0$,

$$\|u_\varepsilon - u\|_0 \to 0, \quad \sqrt{\varepsilon}\, \|u_\varepsilon\|_1 \to 0,$$

and if in addition $u \in K_1$, *then*

$$\|u_\varepsilon - u\|_0 = O(\sqrt{\varepsilon}), \quad \|u_\varepsilon - u\|_1 \to 0.$$

Finally, consider the special case where $J \equiv 0$ and $\Phi \equiv 0$. The singularly perturbed variational inequality is

$$u_\varepsilon \in K_1, \quad \varepsilon \langle A_1 u_\varepsilon, v - u_\varepsilon \rangle_1 + \langle A_0 u_\varepsilon, v - u_\varepsilon \rangle_0 \geq \langle f, v - u_\varepsilon \rangle_0 \quad \forall\, v \in K_1, \tag{4.5}$$

whereas the corresponding limiting problem is

$$u \in K_0, \quad \langle A_0 u, v - u \rangle_0 \geq \langle f, v - u \rangle_0 \quad \forall\, v \in K_0. \tag{4.6}$$

We deduce from either Corollaries 4.1 or 4.2 the following known result for singular perturbation of variational inequalities.

COROLLARY 4.3. *Assume* (H_{K_1}), (H_{K_0}), (H_{K_1,K_0}), (H_{A_1}), (H_{A_0}), *and* (H_{f,V_0}). *Then the problem* (4.5) *has a unique solution* u_ε *if* $\varepsilon\, m_2 \le m_0$, *and* (4.6) *has a unique solution* u. *Moreover, as* $\varepsilon \to 0$,

$$\|u_\varepsilon - u\|_0 \to 0, \quad \sqrt{\varepsilon}\, \|u_\varepsilon\|_1 \to 0,$$

and if in addition $u \in K_1$, *then*

$$\|u_\varepsilon - u\|_0 = O(\sqrt{\varepsilon}), \quad \|u_\varepsilon - u\|_1 \to 0.$$

5. An example: Plate bending over an obstacle. We illustrate the application of the results from previous sections in the study of a mathematical model in plate bending. Following [27, section 1:4], we consider the bending of a plate that occupies a region $\{(\boldsymbol{x}, z) \mid \boldsymbol{x} \in \Omega,\, -h/2 < z < h/2\}$, where $\boldsymbol{x} = (x_1, x_2)$, $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain, and $h > 0$ is the thickness of the plate which is small compared to the size of Ω in both x_1- and x_2-directions. The plate is assumed to be made of an elastic isotropic and homogeneous material, and it is subject to the action of a vertical load only. The unknown variable is the vertical displacement of the plate $u(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega$. In the linearized theory of the plate bending, the total potential energy is

$$\tilde{\mathcal{E}}(u) = \frac{D}{2\,h} \int_\Omega |\Delta u|^2 dx + \frac{T}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega P\, u\, dx,$$

where $D = h^3 E/[12\,(1-\nu^2)]$ is the stiffness coefficient (modulus of flexural rigidity) of the plate, $E > 0$ is Young's modulus, $\nu \in (0, 0.5)$ is Poisson's ratio, $T > 0$ is the constant absolute value of stress per unit surface area, and P is the density of external vertical forces per unit surface area. Denoting $f = P/T$ and $\varepsilon = D/(h\,T)$, we consider the scaled energy functional

$$\mathcal{E}(u) = \frac{\varepsilon}{2} \int_\Omega |\Delta u|^2 dx + \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega f\, u\, dx.$$

Note that $\varepsilon = O(h^2)$ is small.

With the classical clamped boundary condition or the simply supported boundary condition on $\Gamma = \partial\Omega$, minimizing the energy functional $\mathcal{E}(u)$ leads to a fourth order partial differential equation

$$\varepsilon\, \Delta^2 u - \Delta u = f \quad \text{in } \Omega. \tag{5.1}$$

For the clamped boundary condition, the function space for the displacement u is $V = H_0^2(\Omega)$, whereas for the simply supported boundary condition, the function space for the displacement u is $V = H^2(\Omega) \cap H_0^1(\Omega)$. We note in passing that the conforming finite element method is studied in [28] for solving the singular perturbation problem of the partial differential equation (5.1) (in a slightly more general form) with the clamped boundary condition.

We now assume the plate lies over an obstacle with the height function $z = \psi(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega$. Then the constraint set is

$$K = \{v \in V \mid v \ge \psi \text{ in } \Omega\},$$

where it is assumed that

$$\psi \in H^1(\Omega), \quad \psi \le 0 \text{ a.e. on } \Gamma. \tag{5.2}$$

We consider the case where the vertical force consists of two parts:

$$f = f_0 + f_1, \quad f_0 \in L^2(\Omega),\ -f_1 \in \partial j(u). \tag{5.3}$$

For situations with the force decomposition as in (5.3), see the discussions in [25]; see also [10] on the numerical analysis of a related hemivariational inequality. We assume the function $j\colon \mathbb{R} \to \mathbb{R}$ has the following properties:

$$\begin{cases} \text{(a) } j \text{ is locally Lipschitz continuous;} \\ \text{(b) there exist constants } c_0,\ c_1 \in \mathbb{R} \text{ such that} \\ \qquad |\partial j(r)| \le \bar{c}_0 + \bar{c}_1 |r| \quad \forall\, r \in \mathbb{R}; \\ \text{(c) there exists a constant } m_j \text{ such that} \\ \qquad j^0(r_1; r_2 - r_1) + j^0(r_2; r_1 - r_2) \le m_j |r_1 - r_2|^2 \quad \forall\, r_1, r_2 \in \mathbb{R}. \end{cases} \tag{5.4}$$

For definiteness, in the following we consider the case with the clamped boundary condition. Then

$$V_1 = H_0^2(\Omega), \tag{5.5}$$

$$V_0 = H_0^1(\Omega), \tag{5.6}$$

$$K_1 = \{v \in V_1 \mid v \ge \psi \text{ in } \Omega\}, \tag{5.7}$$

$$K_0 = \{v \in V_0 \mid v \ge \psi \text{ in } \Omega\}. \tag{5.8}$$

By Poincaré's inequality (cf. [1, Example 7.3.16]), $\|v\|_0 = \|\nabla v\|_{L^2(\Omega)^d}$ defines a norm on V_0 and it is equivalent to the canonical norm $\|v\|_{H^1(\Omega)}$ over V_0. Moreover, $\|v\|_1 = \|\Delta v\|_{L^2(\Omega)}$ defines a norm over V_1 and it is equivalent to $\|v\|_{H^2(\Omega)}$ over V_1 [3, Theorem 6.8-1]. The operators $A_1\colon V_1 \to V_1$ and $A_0\colon V_0 \to V_0$ are defined as follows:

$$\langle A_1 u, v\rangle_1 = \int_\Omega \Delta u\, \Delta v\, dx, \quad u, v \in V_1, \tag{5.9}$$

$$\langle A_0 u, v\rangle_0 = \int_\Omega \nabla u \cdot \nabla v\, dx, \quad u, v \in V_0. \tag{5.10}$$

Moreover,

$$J(v) = \int_\Omega j(v)\, dx, \quad v \in V_0, \tag{5.11}$$

$$\langle f, v\rangle_0 = \int_\Omega f_0 v\, dx, \quad v \in V_0. \tag{5.12}$$

It is easy to see that (3.2) holds with $L_1 = 1$, (3.3) holds with $m_1 = 1$ and $m_2 = 0$, (3.4) holds with $L_0 = 1$, and (3.5) holds with $m_0 = 1$.

Then the obstacle plate bending problem is

$$u_\varepsilon \in K_1, \quad \varepsilon \langle A_1 u_\varepsilon, v - u_\varepsilon\rangle_1 + \langle A_0 u_\varepsilon, v - u_\varepsilon\rangle_0 + \int_\Omega j^0(u_\varepsilon; v - u_\varepsilon)\, dx \ge \langle f, v - u_\varepsilon\rangle_0 \quad \forall\, v \in K_1 \tag{5.13}$$

and the limiting problem is

$$u \in K_0, \quad \langle A_0 u, v - u\rangle_0 + \int_\Omega j^0(u; v - u)\, dx \ge \langle f, v - u\rangle_0 \quad \forall\, v \in K_0. \tag{5.14}$$

Recall that $j\colon \mathbb{R}\to\mathbb{R}$ is regular (in the sense of Clarke) if for any $r\in\mathbb{R}$, the directional derivatives $j'_+(r)$ and $j'_-(r)$ exist, and $j'_+(r)=j^0(r;1)$, $j'_-(r)=j^0(r;-1)$. Under the additional assumption that $j\colon\mathbb{R}\to\mathbb{R}$ is regular, we have (cf. [21, Theorem 3.47])

$$J^0(u;v)=\int_\Omega j^0(u;v)\,dx,\quad u,v\in V_0. \tag{5.15}$$

In this case, (5.13) is equivalent to

$$\begin{aligned} u_\varepsilon\in K_1,\quad \varepsilon\,\langle A_1u_\varepsilon,v-u_\varepsilon\rangle_1+\langle A_0u_\varepsilon,v-u_\varepsilon\rangle_0+J^0(u_\varepsilon;v-u_\varepsilon)\\ \ge\langle f,v-u_\varepsilon\rangle_0\quad\forall\,v\in K_1, \end{aligned} \tag{5.16}$$

whereas (5.14) is equivalent to

$$u\in K_0,\quad \langle A_0u,v-u\rangle_0+J^0(u;v-u)\ge\langle f,v-u\rangle_0\quad\forall\,v\in K_0. \tag{5.17}$$

Let us apply Corollary 4.1. It is easy to verify the conditions (H_{K_1}), (H_{K_0}), (H_{K_1,K_0}), (H_{A_1}), (H_{A_0}), (H_{J,V_0}), and (H_{f,V_0}). For $v_1,v_2\in V_0$,

$$\begin{aligned} J^0(v_1;v_2-v_1)+J^0(v_2;v_1-v_2)&=\int_\Omega\left[j^0(v_1;v_2-v_1)+j^0(v_2;v_1-v_2)\right]dx\\ &\le m_j\int_\Omega|v_1-v_2|^2\,dx\le m_j\lambda_0^{-1}\int_\Omega|\nabla(v_1-v_2)|^2\,dx, \end{aligned}$$

where $\lambda_0>0$ is the smallest eigenvalue of the eigenvalue problem

$$\begin{aligned} -\Delta u&=\lambda\,u\quad\text{in }\Omega,\\ u&=0\quad\text{on }\Gamma. \end{aligned}$$

Thus, $m_J=m_j\lambda_0^{-1}$ and the smallness condition $m_J<1$ is

$$m_j<\lambda_0. \tag{5.18}$$

Under this condition, we know that both problems have a unique solution and as $\varepsilon\to 0$, we have the convergence

$$\|u_\varepsilon-u\|_0\to 0,\quad \sqrt{\varepsilon}\,\|u_\varepsilon\|_1\to 0, \tag{5.19}$$

and if $u\in K_1$,

$$\|u_\varepsilon-u\|_0=O(\sqrt{\varepsilon}),\quad \|u_\varepsilon-u\|_1\to 0. \tag{5.20}$$

Without assuming $j\colon\mathbb{R}\to\mathbb{R}$ is regular, we proceed as follows. We still assume (5.18). Since

$$J^0(u;v)\le\int_\Omega j^0(u;v)\,dx,\quad u,v\in V_0,$$

the solution of (5.16) is also a solution of (5.13). Moreover, the uniqueness of a solution of the problem (5.13) can be proved similarly to that for the solution uniqueness of the problem (5.16). In the same way, it can be proved that the problem (5.14) has a unique solution. Moreover, the proofs of Theorems 3.4 and 3.5 can be repeated with $J^0(u;v)$ and $\tilde{J}^0(u;v)$ replaced by $\int_\Omega j^0(u;v)\,dx$. Then we again conclude the convergence results (5.19) and if $u\in K_1$, also (5.20).

We summarize the above discussions in the form of a theorem.

THEOREM 5.1. *Assume* $f_0 \in L^2(\Omega)$, (5.2), (5.4), *and* (5.18). *Then for the solution* u_ε *of the problem* (5.13) *and the solution* u *of the problem* (5.14), *we have the convergence* (5.19). *Moreover, if* $u \in K_1$, *then* (5.20) *holds.*

For the case of the simply supported boundary condition, the definition (5.5) of the space V_1 needs to be changed to $V_1 = H^2(\Omega) \cap H_0^1(\Omega)$ and the set K_1 is defined by (5.7) with this V_1, but the rest of the definitions (5.6)–(5.12) are unchanged. The hemivariational inequalities under the consideration are still of the forms (5.13) and (5.14). We claim that in this case, the statement of Theorem 5.1 is still valid if we assume that Ω is smooth or convex. According to [1, p. 301], if Ω is smooth or convex, $\|v\|_1 = \|\Delta v\|_{L^2(\Omega)}$ defines a norm over $V_1 = H^2(\Omega) \cap H_0^1(\Omega)$ and it is equivalent to $\|v\|_{H^2(\Omega)}$ over V_1. The rest of the arguments leading to Theorem 5.1 remain the same.

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