Taylor & Francis
Taylor & Francis Group

Numerical Functional Analysis and Optimization

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/lnfa20

A Revisit of Elliptic Variational-Hemivariational Inequalities

Weimin Han

NUMERICAL FUNCTIONAL ANALYSIS AND OPTIMIZATION
2021, VOL. 42, NO. 4, 371–395
https://doi.org/10.1080/01630563.2021.1881541

A Revisit of Elliptic Variational-Hemivariational Inequalities

Weimin Han

Department of Mathematics & Program in Applied Mathematical and Computational Sciences (AMCS), University of Iowa, Iowa City, Iowa, USA

ABSTRACT

In this paper, we provide an alternative approach to establish the solution existence and uniqueness for elliptic variational-hemivariational inequalities. The new approach is based on elementary results from functional analysis, and thus removes the need of the notion of pseudomonotonicity and the dependence on surjectivity results for pseudomonotone operators. This makes the theory of elliptic variational-hemivariational inequalities more accessible to applied mathematicians and engineers. In addition, equivalent minimization principles are further explored for particular elliptic variational-hemivariational inequalities. Representative examples from contact mechanics are discussed to illustrate application of the theoretical results.

ARTICLE HISTORY
Received 21 January 2021
Accepted 22 January 2021

KEYWORDS
Existence; hemivariational inequality; uniqueness; variational-hemivariational inequality

AMS SUBJECT CLASSIFICATIONS
35J87; 49K20

1. Introduction

Hemivariational inequalities are useful in modeling and studying problems in physical sciences and engineering that involve non-smooth, non-monotone and multi-valued relations among different physical quantities. Research on hemivariational inequalities started in early 1980s (cf. [1]), and has attracted much attention in the research community, especially in the recent years. The mathematical theory of hemivariational inequalities can be found in some research monographs, e.g. [2–7], and in many journal articles. In applications, numerical methods have to be used to solve hemivariational inequalities. An early comprehensive reference on the numerical solution of hemivariational inequalities is [8] where convergence of finite element solutions and solution algorithms are discussed. More recently, optimal order error bounds have been derived for finite element solutions of various kinds of hemivariational inequalities, including elliptic ones, evolutionary ones, history-dependent ones, cf. [9–14] and a summarizing

CONTACT Weimin Han weimin-han@uiowa.edu Department of Mathematics & Program in Applied Mathematical and Computational Sciences (AMCS), University of Iowa, Iowa City, IA 52242-1410, USA.
The work was partially supported by the European Union's Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie grant agreement No. 823731 - CONMECH.

account in [15]. Other numerical methods can be also applied to solve hemivariational inequalities, e.g., the virtual element method is applied in [16] for the numerical solution of hemivariational inequalities in contact mechanics, where an optimal order error bound for the virtual element solutions is derived.

In [17], a well-posedness result is proved for a general elliptic variational-hemivariational inequality, cf. also [7, Section 5.4]. Let V be a reflexive Banach space with the norm $||\cdot||_V$. Denote by V^* the dual space of V, and by $\langle\cdot,\cdot\rangle$ the duality pairing between V^* and V. Let K be a nonempty closed convex subset of V, $A : V \to V^*, \tilde{\Phi} : K \times K \to \mathbb{R}$, $\Psi : X \to \mathbb{R}$, and $f \in V^*$. The problem is of the form

$$u \in K, \quad \langle Au, v-u\rangle + \tilde{\Phi}(u,v) - \tilde{\Phi}(u,u) + \Psi^0(u; v-u) \geq \langle f, v-u\rangle \quad \forall\, v \in K. \tag{1.1}$$

For the unique solvability analysis of this problem, it is assumed that (a) the operator $A : V \to V^*$ is pseudomonotone and strongly monotone with a constant $m_A > 0$; (b) $\tilde{\Phi} : K \times K \to \mathbb{R}$ is convex and l.s.c. with respect to its second argument, and for some constant $\alpha_\Phi > 0$,

$$\begin{aligned} &\tilde{\Phi}(u_1, v_2) - \tilde{\Phi}(u_1, v_1) + \tilde{\Phi}(u_2, v_1) - \tilde{\Phi}(u_2, v_2) \\ &\quad \leq \alpha_{\tilde{\Phi}} ||u_1 - u_2||\ ||v_1 - v_2|| \quad \forall\, u_1, u_2, v_1, v_2 \in K; \end{aligned}$$

(c) $\Psi : V \to \mathbb{R}$ is locally Lipschitz and there are constants $\alpha_\Psi, c_0, c_1 \geq 0$ such that

$$\Psi^0(v_1; v_2 - v_1) + \Psi^0(v_2; v_1 - v_2) \leq \alpha_\Psi ||v_1 - v_2||_V^2 \quad \forall\, v_1, v_2 \in V$$

and

$$||\eta||_{V^*} \leq c_0 + c_1 ||v||_V \quad \forall\ v \in V, \forall\ \eta \in \partial\Psi(v). \tag{1.2}$$

Here, Ψ^0 and $\partial\Psi$ denote generalized directional derivative and generalized subdifferential in the sense of Clarke (cf. Section 2 for a brief review). Then if the smallness condition $\alpha_{\tilde{\Phi}} + \alpha_\Psi < m_A$ holds, for any $f \in V^*$, the problem (1.1) has a unique solution. The proof of the statement relies on properties of pseudomonotone operators, on Banach fixed-point theorem, and more importantly, on abstract surjectivity results of pseudomonotone operators.

In this paper, we revisit the problem (1.1) from a new approach, and prove the existence of a unique solution of the problem through arguments applying only elementary results from functional analysis. Moreover, equivalent minimization principles are provided for particular elliptic variational-hemivariational inequalities where the functional $\tilde{\Phi}$ has only one independent variable. We achieve these goals at the expense of slightly stronger assumptions on the problem data from the theoretical point of view; nevertheless, the slightly stronger assumptions do not represent a serious restriction of the unique solvability results for applications. On the operator A, instead of the pseudomonotonicity, we will assume it is Lipschitz continuous. On the functional $\tilde{\Phi}$, with application problems in mind, we consider nonsmooth convex functionals of the form

$$\tilde{\Phi}(u,v) = \Phi(u,v) + I_K(v),$$

where $\Phi : V \times V \to \mathbb{R}$ is convex and l.s.c. with respect to its second argument, and I_K is the indicator function of the convex set K. Then (1.1) becomes

$$u \in K, \quad \langle Au, v-u\rangle + \Phi(u,v) - \Phi(u,u) + \Psi^0(u; v-u) \geq \langle f, v-u\rangle \quad \forall\, v \in K. \tag{1.3}$$

Note that a real-valued l.s.c. convex function on the Banach space V is continuous ([18, Corollary 2.5, p. 13]). As is known from convex analysis, local Lipschitz continuity of a convex functional is guaranteed from its boundedness on a non-empty open set; for a precise statement, cf. Lemma 2.2. So we will assume the convex function $\Phi(u,\cdot)$ to be bounded above on a non-empty open set of V, for any $u \in V$.

We note that the new feature stated in the previous paragraph makes the theory more accessible to applied mathematicians and engineers. Moreover, the availability of equivalent minimization principles for particular elliptic variational-hemivariational inequalities justifies the use of optimization tools in solving discrete systems of the inequalities.

The organization of the rest of the paper is as follows. In Section 2, basic notions and results are recalled on convex subdifferentials, the generalized directional derivative and subdifferential (or generalized gradient) in the sense of Clarke, and on their properties. The starting point of the new approach for the solution existence and uniqueness is an equivalent minimization principle for a particular elliptic variational-hemivariational inequality, which is shown in [19] and is presented in an improved form in this paper. Thus, in Section 3, we use the result on the minimization principle to establish the solution existence and uniqueness of the particular elliptic variational-hemivariational inequality. Then we proceed to prove the solution existence and uniqueness of general elliptic variational-hemivariational inequalities. In Section 4, the results in Section 3 are rephrased for elliptic variational-hemivariational inequalities in forms more convenient to use in applications or when their numerical approximation is concerned. Finally in Section 5, we discuss some representative examples from contact mechanics to illustrate applications of the solution existence and uniqueness results established in earlier sections.

2. Preliminaries

In this section, we recall some basic notions and results from convex analysis and nonsmooth analysis.

Let V be a normed space. Let $\Phi : V \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and l.s.c. (lower semicontinuous) functional. Then the (convex) subdifferential of Φ at $u \in V$ is

$$\partial\Phi(u) := \big\{\xi \in V^* | \Phi(v) - \Phi(u) \geq \langle \xi, v - u\rangle \ \forall\, v \in V\big\}.$$

If $\partial\Phi(u)$ is non-empty, then any element $\xi \in \partial\Phi(u)$ is called a subgradient of Φ at u. A general discussion of convex functionals, the convex subdifferential and its properties can be found in [18].

A proper, l.s.c. convex functional is bounded below by an affine functional (cf. [20, Lemma 11.3.5] or [21, Prop. 5.2.25]).

Lemma 2.1. *Let V be a normed space and let $\Phi : V \to \mathbb{R} \cup \{+\infty\}$ be proper, convex and l.s.c. Then there exist a continuous linear functional $\ell_\Phi \in V^*$ and a constant $c \in \mathbb{R}$ such that*

$$\Phi(v) \geq \ell_\Phi(v) + c \quad \forall\, v \in V.$$

Consequently, there exist two constants $\bar{c}_0$ and $\bar{c}_1$, not necessarily non-negative, such that

$$\Phi(v) \geq \bar{c}_0 + \bar{c}_1 \|v\|_V \quad \forall\, v \in V. \tag{2.1}$$

Regarding the continuity of convex functions, we record a result that can be derived from [18, Corollary 2.4, p. 12].

Lemma 2.2. *Let V be a normed space and let $\Phi : V \to \mathbb{R}$ be convex. Then Φ is locally Lipschitz continuous on V if and only if Φ is bounded above on a non-empty open set in V.*

We will need the notions of the generalized directional derivative and subdifferential (or generalized gradient) in the sense of Clarke ([22, 23]) to define hemivariational inequalities or variational-hemivariational inequalities. Let $\Psi : V \to \mathbb{R}$ be locally Lipschitz continuous. Then the generalized (Clarke) directional derivative of Ψ at $u \in V$ in the direction $v \in V$ is defined by

$$\Psi^0(u; v) := \limsup_{w \to u,\, \lambda \downarrow 0} \frac{\Psi(w + \lambda v) - \Psi(w)}{\lambda},$$

and the subdifferential (or generalized gradient) of Ψ at $u \in V$ is defined by

$$\partial\Psi(u) := \big\{\eta \in V^* | \Psi^0(u; v) \geq \langle \eta, v\rangle \ \forall\, v \in V\big\}.$$

Basic properties of the generalized directional derivative and the generalized gradient are recorded in the next two propositions (cf. [23, 24]).

Proposition 2.3. *Assume that $\Psi : V \to \mathbb{R}$ is a locally Lipschitz function. Then the following statements are valid.*

(i) $\Psi^0(u; \lambda\ v) = \lambda\ \Psi^0(u; v)\ \forall\ \lambda \geq 0, u, v \in V$.
(ii) $\Psi^0(u; v_1 + v_2) \leq \Psi^0(u; v_1) + \Psi^0(u; v_2)\ \forall\ v_1, v_2 \in V$.
(iii) $\Psi^0(u; v) = \max\{\langle \eta, v \rangle | \eta \in \partial\Psi(u)\}\ \forall\ u, v \in V$.
(iv) *If $u_n \to u$ and $v_n \to v$ in V, then* $\limsup \Psi^0(u_n; v_n) \leq \Psi^0(u; v)$.
(v) *For every $u \in V, \partial\Psi(u)$ is nonempty, convex, and weakly* compact in V^*.*
(vi) *If $u_n \to u$ in V, $\eta_n \in \partial\Psi(u_n)$, and $\eta_n \to \eta$ weakly* in V^*, then $\eta \in \partial\Psi(u)$.*
(vii) *If $\Psi : V \to \mathbb{R}$ is convex, then the Clark subdifferential $\partial\Psi(u)$ at any $u \in V$ coincides with the convex subdifferential $\partial\Psi(u)$.*

Because of Proposition 2.3 (vii), we use the same symbol ∂ for the subdifferential both in the sense of Clarke and in convex analysis.

Proposition 2.4. *Let $\Psi, \Psi_1, \Psi_2 : V \to \mathbb{R}$ be locally Lipschitz functions. Then:*
(i) (scalar multiples)

$$\partial(\lambda\ \Psi)(u) = \lambda\ \partial\Psi(u) \quad \forall\, \lambda \in \mathbb{R},\ u \in V. \tag{2.2}$$

(ii) (sum rules)

$$\partial(\Psi_1 + \Psi_2)(u) \subset \partial\Psi_1(u) + \partial\Psi_2(u) \quad \forall\, u \in V, \tag{2.3}$$

equivalently,

$$(\Psi_1 + \Psi_2)^0(u; v) \leq \Psi_1^0(u; v) + \Psi_2^0(u; v) \quad \forall\, u, v \in V. \tag{2.4}$$

We also recall the following Lebourg mean value theorem (cf. [24, Proposition 1.3.14]).

Theorem 2.5. *Let $u, v \in V$, and suppose Ψ is locally Lipschitz on an open set of V containing the closed line-segment $\{(1-\lambda)\ u + \lambda\ v | 0 \leq \lambda \leq 1\}$. Then there exists $\zeta \in \partial\Psi((1-\lambda)\ u + \lambda\ v)$ for some $\lambda \in (0, 1)$ such that*

$$\Psi(v) - \Psi(u) = \langle \zeta, v - u \rangle.$$

A function $\Phi : V \to \mathbb{R}$ is said to be strongly convex on V with a constant $\alpha > 0$ if

$$\Phi(\lambda\ u+(1-\lambda)\ v) \le \lambda\ \Phi(u)+(1-\lambda)\ \Phi(v)-\alpha\ \lambda\ (1-\lambda)\ ||u-v||_V^2 \\ \forall\, u,v \in V, \forall\, \lambda \in [0,1].$$

Obviously, strong convexity of a functional implies its strict convexity. The next result is a corollary of [25, Proposition 3.1], which provides a sufficient and necessary condition on strong convexity of a locally Lipschitz continuous function through the strong monotonicity of its subdifferential in the sense of Clarke.

Lemma 2.6. *Let V be a real Banach space. A locally Lipschitz continuous function Φ is strongly convex on V with a constant $\alpha>0$ if and only if $\partial\Phi$ is strongly monotone on V with a constant 2α, i.e.,*

$$\langle \xi-\eta, u-v\rangle \ge 2\alpha\ ||u-v||_V^2 \quad \forall\, u,v \in V, \xi \in \partial\Phi(u), \eta \in \partial\Phi(v).$$

From now on, we will assume V to be a real Hilbert space. As a corollary of Lemmas 2.1 and 2.6, we have the next result; its proof is given in [19].

Proposition 2.7. *Let V be a real Hilbert space, and let $\Phi : V \to \mathbb{R}$ be locally Lipschitz continuous and strongly convex with a constant $\alpha>0$. Then there exist two constants $\bar{c}_0$ and $\bar{c}_1$ such that*

$$\Phi(v) \ge \alpha\ ||v||_V^2 + \bar{c}_0 + \bar{c}_1 ||v||_V \quad \forall\, v \in V. \tag{2.5}$$

Consequently, $\Phi(\cdot)$ is coercive on V.

In this paper, studies of solution existence and uniqueness for elliptic variational-hemivariational inequalities start with minimization principles for particular ones; the minimization principles are of independent interest. For this purpose, we need the notion of a potential operator which is recalled here: an operator $A : V \to V^*$ is called a potential operator if there exists a Gâteaux differentiable functional $F_A : V \to \mathbb{R}$ such that $A = F_A'$. The functional F_A is called a potential of A. If A is hemicontinuous, then A is a potential operator if and only if

$$\int_0^1 [\langle A(t\ u), u\rangle - \langle A(t\ v), v\rangle] dt = \int_0^1 \langle A(v+t\ (u-v)), u-v\rangle\ dt \quad \forall\, u,v \in V.$$

If A is Gâteaux differentiable and the mapping $(t,s)\mapsto\langle A'(v_1+t\ v_2+s\ v_3)\ v_4, v_5\rangle$ is continuous on $[0,1]\times[0,1]$ for all $v_i \in V, 1 \le i \le 5,$ then A is potential if and only if

$$\langle A'(u)\ v, w\rangle = \langle A'(u)\ w, v\rangle \quad \forall\, u,v,w \in V. \tag{2.6}$$

If A is a potential operator, its potential can be computed from the formula

$$F_A(v) = \int_0^1 \langle A(t\ v), v\rangle\ dt; \tag{2.7}$$

any other potential of A differs from F_A by a constant.

For the special case of a linear operator, $A \in \mathcal{L}(V, V^*)$ is a potential operator if and only if it is symmetric:

$$\langle Av_1, v_2\rangle = \langle Av_2, v_1\rangle \quad \forall\ v_1, v_2 \in V. \tag{2.8}$$

Moreover, a potential functional can be chosen to be

$$F_A(v) = \frac{1}{2}\langle Av, v\rangle, \quad v \in V.$$

For a detailed discussion on potential operators, the reader is referred to [26, Section 41.3].

Throughout the paper, we will use c to denote a generic positive constant whose value may change from one place to another but it is independent of other quantities of concern.

3. Results on elliptic variational-hemivariational inequalities

We start with a unique solvability result for a particular elliptic variational-hemivariational inequality that allows an equivalent minimization principle. Then we extend the result to more general elliptic variational-hemivariational inequalities by fixed-point arguments.

An operator $A : V \to V^*$ is said to be strongly monotone with a constant $m_A > 0$ if

$$\langle Av_1 - Av_2, v_1 - v_2\rangle \geq m_A \|v_1 - v_2\|_V^2 \quad \forall\, v_1, v_2 \in V. \tag{3.1}$$

3.1. Result 1

Consider a particular elliptic variational-hemivariational inequality:

Problem 3.1. *Find $u \in K$ such that*

$$\langle Au, v-u\rangle + \Phi(v) - \Phi(u) + \Psi^0(u; v-u) \geq \langle f, v-u\rangle \quad \forall\ v \in K. \tag{3.2}$$

We introduce conditions on the problem data.

$H(K)$ V is a real Hilbert space, K is a non-empty, closed and convex set in V.

$H(A)$ $A : V \to V^*$ is Lipschitz continuous and strongly monotone with a constant $m_A > 0$.

$H(\Phi)_1$ $\Phi : V \to \mathbb{R}$ is convex and bounded above on a non-empty open set in V.

$H(\Psi)$ $\Psi : V \to \mathbb{R}$ is locally Lipschitz continuous, and for a constant $\alpha_\Psi \geq 0$,

$$\Psi^0(v_1; v_2-v_1) + \Psi^0(v_2; v_1-v_2) \leq \alpha_\Psi \|v_1-v_2\|_V^2 \quad \forall\, v_1, v_2 \in V. \tag{3.3}$$

H(f) $f \in V^*$.

The subscript 1 in $H(\Phi)_1$ reminds the reader that this is a condition for the case where Φ has one independent variable. Applying Lemma 2.2, we know that $H(\Phi)_1$ implies that $\Phi : V \to \mathbb{R}$ is convex and locally Lipschitz continuous. We note that (3.3) is equivalent to the following inequality, known as the relaxed monotonicity condition (cf. [7, p. 124]):

$$\langle \eta_1-\eta_2, v_1-v_2\rangle \geq -\alpha_\Psi \|v_1-v_2\|_V^2 \quad \forall\, v_1, v_2 \in V, \eta_1 \in \partial\Psi(v_1),\ \eta_2 \in \partial\Psi(v_2). \tag{3.4}$$

Note that in $H(\Psi)$, we do not assume a linear growth condition on the generalized gradient $\partial\Psi(\cdot)$ of the form

$$\|\partial\Psi(v)\|_{V^*} \leq c_0 + c_1\|v\|_V \quad \forall\, v \in V \tag{3.5}$$

for some non-negative constants c_0, c_1, which is assumed in other references on Problem 3.1 (cf. [17] or [7, Section 5.4]).

In the case where $A : V \to V^*$ is a potential operator, we can further consider a corresponding minimization problem. We will always denote by F_A a potential of the potential operator A.

Problem 3.2. *Find $u \in K$ such that*

$$u = \operatorname*{argmin}_{v\in K} E(v) \tag{3.6}$$

where

$$E(v) = F_A(v) + \Phi(v) + \Psi(v) - \langle f, v\rangle, \quad v \in V. \tag{3.7}$$

Theorem 3.3. *Assume H(K), H(A), $H(\Phi)_1, H(\Psi)$, H(f), and $\alpha_\Psi < m_A$. Moreover, assume $A : V \to V^*$ is a potential operator with the potential $F_A(\cdot)$. Then Problem 3.2 has a unique solution $u \in K$.*

This result is proved by applying a standard result in convex optimization, cf. e.g. [20, Theorem 3.3.12]. It can be shown that $E(\cdot)$ is locally Lipschitz and strongly convex on V; in particular, the strong convexity of $E(\cdot)$ implies the coercivity of $E(\cdot)$ on V (cf. Proposition 2.7). A detailed argument can be found in [19], and is omitted here. Note that in [19], the unnecessary condition (3.5) was not explicitly ruled out. In the literature, a condition such as $\alpha_\Psi < m_A$ is called a smallness condition.

Regarding Problem 3.1, under the additional assumption that A is a potential operator, we have the following solution existence and uniqueness result.

Theorem 3.4. *Assume H(K), H(A), $H(\Phi)_1, H(\Psi)$, H(f), $\alpha_\Psi < m_A$, and $A : V \to V^*$ is a potential operator with the potential $F_A(\cdot)$. Then Problem 3.1 has a unique solution, which is also the unique solution of Problem 3.2.*

Proof. Under the stated assumptions, by Theorem 3.3, Problem 3.2 has a unique solution $u \in K$. This solution is characterized by the relation

$$0 \in \partial(E(u) + I_K(u)) \subset \partial E(u) + \partial I_K(u).$$

For the generalized subdifferential of $E(\cdot)$, we repeatedly apply the summation rule (2.3), and note that for a locally Lipschitz continuous convex functional, the Clark subdifferential and the convex subdifferential coincide. As a result, we have

$$\partial E(v) \subset Av + \partial\Phi(v) + \partial\Psi(v) - f \quad \forall\ v \in V. \tag{3.8}$$

Hence, the solution $u \in K$ of Problem 3.2 satisfies (3.2), i.e., it is also a solution of Problem 3.1.

Uniqueness of the solution of Problem 3.1 follows by a standard argument, with the use of the smallness condition $\alpha_\Psi < m_A$. Since a similar argument will be used in the proof of Theorem 3.5 below, we omit the detail here. □

3.2. Result 2

We continue to consider Problem 3.1, but now without assuming $A : V \to V^*$ to be potential.

Theorem 3.5. *Assume H(K), H(A), $H(\Phi)_1, H(\Psi)$, H(f), and $\alpha_\Psi < m_A$. Then Problem 3.1 has a unique solution.*

Proof. We apply a fixed-point argument to prove the existence. For any $\theta > 0$, Problem 3.1 is equivalent to

$$\begin{aligned} u \in K, \quad & (u, v-u) + \theta\big[\Phi(v) - \Phi(u) + \Psi^0(u; v-u)\big] \\ & \geq (u, v-u) - \theta\ \langle Au, v-u\rangle + \theta\ \langle f, v-u\rangle \quad \forall\, v \in K. \end{aligned} \tag{3.9}$$

For an arbitrary $w \in K$, consider the auxiliary problem

$$\begin{aligned} u \in K, \quad & (u, v-u) + \theta\big[\Phi(v) - \Phi(u) + \Psi^0(u; v-u)\big] \\ & \geq (w, v-u) - \theta\ \langle Aw, v-u\rangle + \theta\ \langle f, v-u\rangle \quad \forall\ v \in K. \end{aligned} \tag{3.10}$$

Since the inner product induces a potential operator, by Theorem 3.4, for $\theta < 1/\alpha_\Psi$, the problem (3.10) has a unique solution $u \in K$. This allows us to define a mapping $P_\theta : K \to K$ by the formula $P_\theta(w) = u$. Let us show that P_θ is a contraction for $\theta>0$ sufficiently small. For this purpose, let $w_1, w_2 \in K$ be arbitrary and denote $u_1 = P_\theta(w_1)$ and $u_2 = P_\theta(w_2)$. Take $v = u_2$ in the inequality (3.10) defining u_1 to get

$$\begin{aligned}(u_1, u_2-u_1) + \theta\big[\Phi(u_2) - \Phi(u_1) + \Psi^0(u_1; u_2 - u_1)\big]\\ \geq (w_1, u_2-u_1)-\theta\ \langle Aw_1, u_2-u_1\rangle + \theta\ \langle f, u_2-u_1\rangle,\end{aligned}$$

and take $v = u_1$ in the inequality (3.10) defining u_2 to get

$$\begin{aligned}(u_2, u_1-u_2) + \theta\big[\Phi(u_1) - \Phi(u_2) + \Psi^0(u_2; u_1 - u_2)\big]\\ \geq (w_2, u_1-u_2)-\theta\ \langle Aw_2, u_1-u_2\rangle + \theta\ \langle f, u_1-u_2\rangle.\end{aligned}$$

Add the two inequalities,

$$\begin{aligned}||u_1-u_2||_V^2 \leq (w_1-w_2, u_1-u_2)-\theta\ \langle Aw_1-Aw_2, u_1-u_2\rangle\\ + \theta\big[\Psi^0(u_1; u_2 - u_1) + \Psi^0(u_2; u_1 - u_2)\big].\end{aligned}$$

Thus, applying the condition (3.3),

$$(1 - \theta\ \alpha_\Psi)||u_1-u_2||_V^2 \leq (w_1-w_2, u_1-u_2)-\theta\ \langle Aw_1-Aw_2, u_1-u_2\rangle. \tag{3.11}$$

Let $\mathcal{J} : V^* \to V$ be the Riesz mapping. Then

$$\langle Aw_1-Aw_2, u_1-u_2\rangle = (\mathcal{J}(Aw_1-Aw_2), u_1-u_2),$$

and we can rewrite (3.11) as

$$(1 - \theta\ \alpha_\Psi)||u_1-u_2||_V^2 \leq ((w_1 - w_2) - \theta\ \mathcal{J}(Aw_1 - Aw_2), u_1 - u_2).$$

Hence,

$$(1 - \theta\ \alpha_\Psi)||u_1-u_2||_V \leq ||(w_1-w_2)-\theta\ \mathcal{J}(Aw_1-Aw_2)||_V. \tag{3.12}$$

Now

$$\begin{aligned}||(w_1-w_2)-\theta\ \mathcal{J}(Aw_1-Aw_2)||_V^2 = ||w_1-w_2||_V^2-2\ \theta\ (\mathcal{J}(Aw_1 - Aw_2), w_1 - w_2)\\ + \theta^2||\mathcal{J}(Aw_1-Aw_2)||_V^2.\end{aligned}$$

Denote by L_A the Lipschitz constant of the operator A. Then

$$||\mathcal{J}(Aw_1-Aw_2)||_V^2 = ||Aw_1-Aw_2||_{V^*}^2 \leq L_A^2||w_1-w_2||_V^2.$$

Since A is assumed to be strongly monotone with the constant m_A, by (3.1),

$$(\mathcal{J}(Aw_1 - Aw_2), w_1 - w_2) = \langle Aw_1-Aw_2, w_1-w_2\rangle \geq m_A||w_1-w_2||_V^2.$$

So

$$||(w_1-w_2)-\theta\ \mathcal{J}(Aw_1-Aw_2)||_V^2 \leq \big(1 - 2\ \theta\ m_A + \theta^2 L_A^2\big)||w_1-w_2||_V^2.$$

Therefore, we derive from (3.12) that

$$||u_1-u_2||_V^2 \leq \frac{1-2\ \theta\ m_A+\theta^2 L_A^2}{(1-\theta\ \alpha_\Psi)^2}\ ||w_1-w_2||_V^2. \tag{3.13}$$

Note that

$$\frac{1-2\ \theta\ m_A+\theta^2 L_A^2}{(1-\theta\ \alpha_\Psi)^2}<1 \tag{3.14}$$

if and only if

$$\left(L_A^2-\alpha_\Psi^2\right)\theta<2(m_A-\alpha_\Psi). \tag{3.15}$$

If $L_A \leq \alpha_\Psi$, then (3.15) is valid for any $0<\theta<1/\alpha_\Psi$. If $L_A>\alpha_\Psi$, then (3.15) is valid for $0<\theta<\min\{1/\alpha_\Psi, 2\ (m_A-\alpha_\Psi)/(L_A^2-\alpha_\Psi^2)\}$. So in any case, for $\theta>0$ sufficiently small, (3.14) holds and the operator $P_\theta : K \to K$ is a contraction. By the Banach fixed-point theorem, P_θ has a unique fixed-point $u \in K : P_\theta u = u$. It is easy to see that u satisfies (3.9), or equivalently (3.2). This completes the existence part of the proof.

For uniqueness, assume there are two solutions $u_1, u_2 \in K$ to Problem 3.1. Take $v = u_2$ in (3.2) for the solution u_1, take $v = u_1$ in (3.2) for the solution u_2, and add the two resulting inequalities to obtain

$$\langle Au_1-Au_2, u_1-u_2\rangle \leq \Psi^0(u_1; u_2-u_1) + \Psi^0(u_2; u_1-u_2).$$

Then, applying the conditions (3.1) and (3.3),

$$m_A||u_1-u_2||_V^2 \leq \alpha_\Psi||u_1-u_2||_V^2.$$

Recall the smallness condition $\alpha_\Psi<m_A$. We conclude from the above inequality that $u_1 = u_2$, i.e., a solution of Problem 3.1 must be unique. □

3.3. Result 3

In this subsection, we consider a more general elliptic variational-hemivariational inequality.

Problem 3.6. *Find $u \in K$ such that*

$$\langle Au, v-u\rangle + \Phi(u, v)-\Phi(u, u) + \Psi^0(u; v-u) \geq \langle f, v-u\rangle \quad \forall\, v \in K. \tag{3.16}$$

We modify $H(\Phi)_1$ to $H(\Phi)_2$; the subscript 2 in $H(\Phi)_2$ reminds the reader that this is a condition for the case where Φ has two independent variables.

$H(\Phi)_2$ $\Phi : V \times V \to \mathbb{R}$; for any $u \in V, \Phi(u, \cdot) : V \to \mathbb{R}$ is convex and bounded above on a non-empty open set; and there exists a constant $\alpha_\Phi \geq 0$ such that

$$\Phi(u_1,v_2)-\Phi(u_1,v_1)+\Phi(u_2,v_1)-\Phi(u_2,v_2)$$
$$\le \alpha_\Phi\|u_1-u_2\|\ \|v_1-v_2\| \quad \forall\ u_1,u_2,v_1,v_2\in V. \tag{3.17}$$

Theorem 3.7. *Assume H(K), H(A), $H(\Phi)_2, H(\Psi)$, H(f), and $\alpha_\Phi+\alpha_\Psi<m_A$. Then Problem 3.6 has a unique solution.*

Proof. Once more, we use a fixed-point argument to prove the existence. By Theorem 3.5, under the stated assumptions, for any $w\in K$, the following auxiliary problem has a unique solution: find $u\in K$ such that

$$\langle Au,v-u\rangle+\Phi(w,v)-\Phi(w,u)+\Psi^0(u;v-u)\ge\langle f,v-u\rangle \quad \forall\ v\in K. \tag{3.18}$$

This defines a mapping $P:K\to K$ by the formula $P(w)=u$. Let us show that this mapping is contractive. For this purpose, let $w_1,w_2\in K$ be arbitrary, and denote $u_1=P(w_1), u_2=P(w_2)$. Then,

$$\langle Au_1,u_2-u_1\rangle+\Phi(w_1,u_2)-\Phi(w_1,u_1)+\Psi^0(u_1;u_2-u_1)\ge\langle f,u_2-u_1\rangle,$$
$$\langle Au_2,u_1-u_2\rangle+\Phi(w_2,u_1)-\Phi(w_2,u_2)+\Psi^0(u_2;u_1-u_2)\ge\langle f,u_1-u_2\rangle.$$

Add the two inequalities,

$$\begin{aligned}\langle Au_1-Au_2,u_1-u_2\rangle\le{}&\Phi(w_1,u_2)-\Phi(w_1,u_1)+\Phi(w_2,u_1)-\Phi(w_2,u_2)\\&+\Psi^0(u_1;u_2-u_1)+\Psi^0(u_2;u_1-u_2).\end{aligned} \tag{3.19}$$

Apply the conditions (3.1), (3.3) and (3.17) in (3.19),

$$m_A\|u_1-u_2\|_V^2\le\alpha_\Phi\|w_1-w_2\|_V\|u_1-u_2\|_V+\alpha_\Psi\|u_1-u_2\|_V^2,$$

or

$$\|u_1-u_2\|_V\le\frac{\alpha_\Phi}{m_A-\alpha_\Psi}\ \|w_1-w_2\|_V.$$

Note that the condition $\alpha_\Phi+\alpha_\Psi<m_A$ implies $\alpha_\Phi/(m_A-\alpha_\Psi)<1$. Thus, the mapping $P:K\to K$ is contractive. By the Banach fixed-point theorem, there is a unique fixed-point $u\in K$ of the mapping P: $u=P(u)$. It is easy to see that u is a solution of Problem 3.6.

For uniqueness, let $u_1,u_2\in K$ be two solutions of Problem 3.6. Then similar to (3.19), we have

$$\begin{aligned}\langle Au_1-Au_2,u_1-u_2\rangle\le{}&\Phi(u_1,u_2)-\Phi(u_1,u_1)+\Phi(u_2,u_1)-\Phi(u_2,u_2)\\&+\Psi^0(u_1;u_2-u_1)+\Psi^0(u_2;u_1-u_2).\end{aligned}$$

Then

$$m_A\|u_1-u_2\|_V^2\le(\alpha_\Phi+\alpha_\Psi)\|u_1-u_2\|_V^2.$$

Since $\alpha_\Phi+\alpha_\Psi<m_A$, we deduce that $u_1-u_2=0$, i.e., a solution to Problem 3.6 is unique. □

4. Some variants

In this section, we provide some variants of the theoretical results shown in the previous section.

On occasions, especially in numerical analysis of the inequalities, it is more convenient to view the functions Φ and Ψ as defined over spaces V_Φ and V_Ψ that are different from V. These spaces are connected through linear operators $\gamma_\Phi : V \to V_\Phi$ and $\gamma_\Psi : V \to V_\Psi$. Thus, instead of Problem 3.1, we consider its following variant (cf. [10]).

Problem 4.1. Find $u \in K$ such that

$$\langle Au, v-u\rangle + \Phi(\gamma_\Phi v)-\Phi(\gamma_\Phi u) + \Psi^0(\gamma_\Psi u; \gamma_\Psi v-\gamma_\Psi u) \geq \langle f, v-u\rangle \quad \forall\ v \in K. \tag{4.1}$$

In case A is a potential operator with the potential $F_A(\cdot)$, it is possible to introduce a corresponding minimization problem.

Problem 4.2. *Find $u \in K$ such that*

$$u = \operatorname*{argmin}_{v\in K} E(v) \tag{4.2}$$

where

$$E(v) = F_A(v) + \Phi(\gamma_\Phi v) + \Psi(\gamma_\Psi v)-\langle f, v\rangle. \tag{4.3}$$

We keep the assumptions $H(K)$, $H(A)$, and modify $H(\Phi)_1, H(\Psi)$ as follows. We use a prime for conditions on Φ and Ψ when they are viewed as functionals over spaces V_Φ and V_Ψ.

$H(\Phi)_1'$ V_Φ is a real Banach space; $\gamma_\Phi \in \mathcal{L}(V; V_\Phi)$; $\Phi : V_\Phi \to \mathbb{R}$ is convex and bounded above on a non-empty open set.

$H(\Psi)'$ V_Ψ is a real Banach space; $\gamma_\Psi \in \mathcal{L}(V; V_\Psi)$; $\Psi : V_\Psi \to \mathbb{R}$ is locally Lipschitz continuous, and for a non-negative constant α_Ψ,

$$\Psi^0(v_1; v_2-v_1) + \Psi^0(v_2; v_1-v_2) \leq \alpha_\Psi ||v_1-v_2||^2_{V_\Psi} \quad \forall\ v_1, v_2 \in V_\Psi. \tag{4.4}$$

We denote by $||\gamma_\Phi||$ and $||\gamma_\Psi||$ the operator norms of $\gamma_\Phi \in \mathcal{L}(V; V_\Phi)$ and $\gamma_\Psi \in \mathcal{L}(V; V_\Psi)$. By applying Theorem 3.5 and Theorem 3.4, we have the next result regarding Problem 4.1.

Theorem 4.3. *Assume H(K), H(A), $H(\Phi)_1', H(\Psi)'$, H(f) and $\alpha_\Psi ||\gamma_\Psi||^2 < m_A$. Then Problem 4.1 has a unique solution. If in addition, A is a potential operator, then Problem 4.2 has a unique solution, which is also the unique solution of Problem 4.1.*

Then we consider a variant of the more general elliptic variational-hemivariational inequality, Problem 3.6 ([10]).

Problem 4.4. *Find $u \in K$ such that*

$$\langle Au, v-u\rangle + \Phi(\gamma_\Phi u, \gamma_\Phi v) - \Phi(\gamma_\Phi u, \gamma_\Phi u) + \Psi^0(\gamma_\Psi u; \gamma_\Psi v - \gamma_\Psi u) \\ \geq \langle f, v-u\rangle \quad \forall\, v \in K. \tag{4.5}$$

We need to modify the condition on the functional Φ.

$H(\Phi)'_2$ V_Φ is a real Banach space; $\gamma_\Phi \in \mathcal{L}(V; V_\Phi)$; $\Phi : V_\Phi \times V_\Phi \to \mathbb{R}$; for any $u \in V_\Phi, \Phi(u, \cdot) : V_\Phi \to \mathbb{R}$ is convex and bounded above on a non-empty open set; and there exists a constant $\alpha_\Phi \geq 0$ such that

$$\Phi(u_1, v_2) - \Phi(u_1, v_1) + \Phi(u_2, v_1) - \Phi(u_2, v_2) \\ \leq \alpha_\Phi ||u_1 - u_2||_{V_\Phi} ||v_1 - v_2||_{V_\Phi} \quad \forall\, u_1, u_2, v_1, v_2 \in V_\Phi. \tag{4.6}$$

We can apply Theorem 3.7 to get a unique solvability result for Problem 4.4.

Theorem 4.5. *Assume H(K), H(A), $H(\Phi)'_2, H(\Psi)'$, H(f), and $\alpha_\Phi||\gamma_\Phi||^2 + \alpha_\Psi||\gamma_\Psi||^2 < m_A$. Then Problem 4.4 has a unique solution.*

Now we consider variational-hemivariational inequalities where the superpotential Ψ is defined as the integral of a function ψ. In most applications, a hemivariational inequality describes a physical problem on a spatial domain in a finite-dimensional Euclidean space $\mathbb{R}^d$. Denote by Δ the domain or its sub-domain, or its boundary or part of the boundary. Let I_Δ stand for the integration over Δ. Instead of the space V_Ψ, we will need a space V_ψ and a linear operator γ_ψ from V to V_ψ. Assume elements from the space V_ψ are $\mathbb{R}^m$-valued functions, for some positive integer m. In applications in contact mechanics, the operator γ_ψ is either the normal component trace operator or the tangential component trace operator. In the former case, $m = 1$, whereas in the latter case, $m = d$. Then related to Problem 3.1, we introduce the next problem.

Problem 4.6. *Find $u \in K$ such that*

$$\langle Au, v-u\rangle + \Phi(v) - \Phi(u) + I_\Delta(\psi^0(\gamma_\psi u; \gamma_\psi v - \gamma_\psi u)) \geq \langle f, v-u\rangle \quad \forall\, v \in K. \tag{4.7}$$

When A is a potential operator with the potential $F_A(\cdot)$, we can introduce a corresponding minimization problem.

Problem 4.7. *Find $u \in K$ such that*

$$u = \operatorname*{argmin}_{v \in K} E(v) \tag{4.8}$$

where

$$E(v) = F_A(v) + \Phi(v) + I_\Delta(\psi(\gamma_\psi v)) - \langle f, v\rangle. \tag{4.9}$$

We allow the function ψ to depend on the spatial variable $\boldsymbol{x} \in \Delta$, and then $\psi(\cdot)$ is a short-hand notation for $\psi(\boldsymbol{x}, \cdot)$. In the study of Problem 4.6 and Problem 4.7, we will replace the assumption $H(\Psi)'$ by $H(\psi)'$.

$H(\psi)'$ $\gamma_\psi \in \mathcal{L}(V; L^p(\Delta; \mathbb{R}^m))$ for a constant $p \geq 2$; $\psi : \Delta \times \mathbb{R}^m \to \mathbb{R}$; $\psi(\cdot, v)$ is measurable on Δ for all $v \in \mathbb{R}^m$; $\psi(\boldsymbol{x}, \cdot)$ is locally Lipschitz continuous on $\mathbb{R}^m$ for a.e. $\boldsymbol{x} \in \Delta$; and for some non-negative constants c and α_ψ,

$$|\partial\psi(\cdot, v)| \leq c\left(1 + |v|_{\mathbb{R}^m}^{p-1}\right) \quad \forall\ v \in \mathbb{R}^m, \text{a.e. on } \Delta, \tag{4.10}$$

$$\psi^0(v_1; v_2 - v_1) + \psi^0(v_2; v_1 - v_2) \leq \alpha_\psi |v_1 - v_2|_{\mathbb{R}^m}^2 \quad \forall\ v_1, v_2 \in \mathbb{R}^m, \text{a.e. on } \Delta. \tag{4.11}$$

Note that in $H(\psi)'$, we do not assume a linear growth condition on the Clark subdifferential $\partial\psi(\cdot, v)$:

$$|\partial\psi(v)|_{\mathbb{R}^m} \leq c\big(1 + |v|_{\mathbb{R}^m}\big) \quad \forall\ v \in \mathbb{R}^m, \text{a.e. in } \Delta$$

as would be necessary in the existing literature. Instead, we assume a less restrictive condition (4.10) (recall that $p \geq 2$). Applying Theorem 2.5, it can be shown from (4.10) that

$$|\psi(\cdot, v)| \leq c\left(1 + |v|_{\mathbb{R}^m}^{p}\right) \quad \forall\ v \in \mathbb{R}^m, \text{a.e. on } \Delta. \tag{4.12}$$

Denote by $c_\Delta > 0$ the smallest constant in the inequality

$$I_\Delta(|\gamma_\psi v|_{\mathbb{R}^m}^2) \leq c_\Delta^2 \|v\|_V^2 \quad \forall\ v \in V. \tag{4.13}$$

With the above preparation, we derive from Theorem 3.5 the next result.

Theorem 4.8. *Assume H(K), H(A), $H(\Phi)_1, H(\psi)'$, H(f), and $\alpha_\psi c_\Delta^2 < m_A$. Then Problem 4.6 has a unique solution. If in addition, $A : V \to V^*$ is a potential operator with the potential $F_A(\cdot)$, then Problem 4.7 has a unique solution which is also the unique solution of Problem 4.6.*

Proof. Introduce an auxiliary functional

$$\Psi(v) = I_\Delta(\psi(\gamma_\psi v)), \quad v \in V.$$

By an argument similar to the one used in proving Theorem 3.47 in [3], it can be proved that under the assumption $H(\psi)'$, the functional $\Psi(\cdot)$ is well defined and is locally Lipschitz continuous over V, and

$$\Psi^0(u; v) \leq I_\Delta(\psi^0(\gamma_\psi u; \gamma_\psi v)) \quad \forall\ u, v \in V. \tag{4.14}$$

For $v_1, v_2 \in V$, by using (4.11) and (4.13),

$$\begin{aligned}\Psi^0(v_1;v_2-v_1)+\Psi^0(v_2;v_1-v_2) &\le I_\Delta(\psi^0(\gamma_\psi v_1;\gamma_\psi v_2-\gamma_\psi v_1)+\psi^0(\gamma_\psi v_2;\gamma_\psi v_1-\gamma_\psi v_2))\\ &\le I_\Delta(\alpha_\psi|\gamma_\psi(v_1-v_2)|^2_{\mathbb{R}^m})\\ &\le \alpha_\psi c_\Delta^2||v_1-v_2||_V^2,\end{aligned}$$

i.e., (3.3) is satisfied with $\alpha_\Psi = \alpha_\psi c_\Delta^2$. Hence, we can apply Theorem 3.5 to conclude that the auxiliary problem

$$u\in K,\ \ \langle Au,v-u\rangle+\Phi(v)-\Phi(u)+\Psi^0(u;v-u)\ge\langle f,v-u\rangle\ \ \forall v\in K$$

has a unique solution $u \in K$. Thanks to (4.14), this solution u is a solution of Problem 4.6. Uniqueness of the solution of Problem 4.6 is proved similar to that in the proof of Theorem 3.5. Moreover, in case A is a potential operator, it is meaningful to consider Problem 4.7 which also has $u \in K$ as its unique solution. □

Consider a problem more general than Problem 4.6.

Problem 4.9. *Find $u \in K$ such that*

$$\langle Au,v-u\rangle+\Phi(u,v)-\Phi(u,u)+I_\Delta(\psi^0(\gamma_\psi u;\gamma_\psi v-\gamma_\psi u))\ge\langle f,v-u\rangle\ \ \forall\ v\in K. \tag{4.15}$$

Based on Theorem 3.7 and the argument leading to Theorem 4.8, we have the next result.

Theorem 4.10. *Assume H(K), H(A), $H(\Phi)_2, H(\psi)'$, H(f), and $\alpha_\Phi + \alpha_\psi c_\Delta^2 < m_A$. Then Problem 4.9 has a unique solution.*

Finally, we consider the particular case of hemivariational inequalities, i.e., the case where $\Phi \equiv 0$. Corresponding to Problem 3.1 and Problem 3.2, we have

Problem 4.11. *Find $u \in K$ such that*

$$\langle Au, v-u\rangle + \Psi^0(u; v-u) \ge \langle f, v-u\rangle \quad \forall\ v \in K. \tag{4.16}$$

Problem 4.12. *Find $u \in K$ such that*

$$u = \operatorname*{argmin}_{v\in K} E(v) \tag{4.17}$$

where

$$E(v) = F_A(v) + \Psi(v) - \langle f, v\rangle. \tag{4.18}$$

As a corollary of Theorem 3.5 and Theorem 3.4, we have the next result concerning Problem 4.11 and Problem 4.12.

Theorem 4.13. *Assume H(K), H(A), H(Ψ), H(f), and $\alpha_\Psi < m_A$. Then Problem 4.11 has a unique solution. If in addition, $A : V \to V^*$ is a potential operator with the potential $F_A(\cdot)$, then Problem 4.12 has a unique solution which is also the unique solution of Problem 4.11.*

As another particular case, consider Problem 4.6 and Problem 4.7 with $\Phi = 0$.

Problem 4.14. *Find $u \in K$ such that*

$$\langle Au, v - u\rangle + I_\Delta(\psi^0(\gamma_\psi u; \gamma_\psi v - \gamma_\psi u)) \geq \langle f, v - u\rangle \quad \forall\ v \in K. \tag{4.19}$$

Problem 4.15. *Find $u \in K$ such that*

$$u = \underset{v \in K}{\operatorname{argmin}}\, E(v) \tag{4.20}$$

where

$$E(v) = F_A(v) + I_\Delta(\psi(\gamma_\psi v)) - \langle f, v\rangle. \tag{4.21}$$

The following result is a special case of Theorem 4.8.

Theorem 4.16. *Assume H(K), H(A), H(ψ)′, H(f), and $\alpha_\psi c_\Delta^2 < m_A$. Then Problem 4.14 has a unique solution. If in addition, $A : V \to V^*$ is a potential operator with the potential $F_A(\cdot)$, then Problem 4.15 has a unique solution which is also the unique solution of Problem 4.14.*

Note that we can get similar results for further special cases where $K = V$. Since it is trivial to deduce results for such further special cases, we omit the detail.

5. Applications in contact mechanics

We illustrate applications of the theoretical results developed in previous sections on some static contact problems between an elastic material and a foundation. Let Ω represent the reference configuration of the elastic body, assumed to be open, bounded and connected in $\mathbb{R}^d$ $(d \leq 3)$ with a Lipschitz continuous boundary $\Gamma = \partial\Omega$. Then the unit outward normal vector $\boldsymbol{\nu}$ is defined a.e. on Γ.

The displacement is $\mathbb{R}^d$-valued, whereas the stress and strain tensors take values in the space $\mathbb{S}^d$ of second order symmetric tensors on $\mathbb{R}^d$. Over the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$, we use "$\cdot$" and "$|\cdot|$" for the canonical inner products and the induced norms. The linearized strain tensor associated with a displacement field $\boldsymbol{u}$ is denoted by $\boldsymbol{\varepsilon}(\boldsymbol{u})$. For a vector field $\boldsymbol{v}$, we use $v_\nu := \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau := \boldsymbol{v} - v_\nu \boldsymbol{\nu}$ for the normal and tangential components of $\boldsymbol{v}$ on Γ.

Similarly, for the stress field $\boldsymbol{\sigma}$, its normal and tangential components on the boundary are defined as $\sigma_\nu := (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau := \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$, respectively.

Let $\boldsymbol{f}_0$ be the density of the total body force acting on Ω. Then the equilibrium equation of the body is

$$\text{Div}\ \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega. \tag{5.1}$$

To describe boundary conditions for the contact problems, we partition the boundary Γ into three disjoint and measurable parts Γ_1, Γ_2 and Γ_3 such that meas $(\Gamma_1)>0$ and meas $(\Gamma_3)>0$. We assume the body is fixed on Γ_1, is subject to the action of a surface traction of density $\boldsymbol{f}_2$ on Γ_2, and is in potential contact on Γ_3 with the rigid foundation. Thus, the homogeneous displacement boundary condition is specified on Γ_1:

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{5.2}$$

and the traction boundary condition is specified on Γ_2:

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2. \tag{5.3}$$

Different contact conditions on Γ_3 will lead to different hemivariational inequalities.

To study the contact problems, we will use the space

$$V = \left\{ \boldsymbol{v} \in H^1(\Omega; \mathbb{R}^d) | \boldsymbol{v} = 0 \text{ a.e. on } \Gamma_1 \right\} \tag{5.4}$$

and its subspace/subset for the displacement fields. Since meas $(\Gamma_1)>0$, Korn's inequality (cf. [27]) implies that V is a Hilbert space with the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_V := \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\ dx, \quad \boldsymbol{u}, \boldsymbol{v} \in V,$$

and the associated norm $\|\cdot\|_V$ is equivalent to the standard $H^1(\Omega; \mathbb{R}^d)$-norm over V. For $\boldsymbol{v} \in V$, we use the same symbol $\boldsymbol{v}$ for its trace on Γ. We use $Q = L^2(\Omega; \mathbb{S}^d)$ as the space for the stress and strain fields; this is a Hilbert space with the canonical inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_Q := \int_\Omega \sigma_{ij}(\boldsymbol{x})\ \tau_{ij}(\boldsymbol{x})\ dx$$

and the associated norm $\|\cdot\|_Q$. From Sobolev embedding theorems and trace theorems, we have

$$H^1(\Omega) \subset L^q(\Gamma_3) \quad \forall\ q<\infty, \text{ if } d = 2$$

and

$$H^1(\Omega) \subset L^4(\Gamma_3) \quad \text{if } d = 3.$$

Let

$$p \in [2, \infty) \text{ if } d = 2, \quad p = 4 \text{ if } d = 3. \tag{5.5}$$

Then $V \subset L^p(\Gamma_3; \mathbb{R}^d)$.

We assume on the densities of body forces and surface tractions that

$$\boldsymbol{f}_0 \in L^2(\Omega; \mathbb{R}^d), \quad \boldsymbol{f}_2 \in L^2(\Gamma_2; \mathbb{R}^d) \tag{5.6}$$

and define $\boldsymbol{f} \in V^*$ by the formula

$$\langle \boldsymbol{f}, \boldsymbol{v} \rangle_{V^* \times V} = (\boldsymbol{f}_0, \boldsymbol{v})_{L^2(\Omega;\mathbb{R}^d)} + (\boldsymbol{f}_2, \boldsymbol{v})_{L^2(\Gamma_2;\mathbb{R}^d)} \quad \forall\ \boldsymbol{v} \in V. \tag{5.7}$$

The elastic constitutive law is of the form

$$\boldsymbol{\sigma} = \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})) \quad \text{in } \Omega. \tag{5.8}$$

Here $\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}))$ is a short-hand notation for $\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}(\boldsymbol{u}))$, i.e., we allow the non-homogeneity of the material. The elasticity operator $\mathcal{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is assumed to have the following properties:

$$\begin{cases} \text{(a) there exists } L_{\mathcal{F}}>0 \text{ such that for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{a .e. } \boldsymbol{x} \in \Omega, \\ \qquad |\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)| \le L_{\mathcal{F}} |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|; \\ \text{(b) there exists } m_{\mathcal{F}}>0 \text{ such that for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega, \\ \qquad (\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{F}} |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2; \\ \text{(c) } \mathcal{F}(\cdot, \boldsymbol{\varepsilon}) \text{is measurable on } \Omega, \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d; \\ \text{(d) } \mathcal{F}(\boldsymbol{x}, 0) = 0 \text{ a.e.} \boldsymbol{x} \in \Omega. \end{cases} \tag{5.9}$$

Define an operator $A : V \to V^*$ by

$$\langle A(\boldsymbol{u}), \boldsymbol{v} \rangle = (\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q, \quad \boldsymbol{u}, \boldsymbol{v} \in V. \tag{5.10}$$

Recall from (2.6) that the operator A is potential if and only if

$$\langle A'(\boldsymbol{u})\boldsymbol{v}, \boldsymbol{w} \rangle = \langle A'(\boldsymbol{u})\boldsymbol{w}, \boldsymbol{v} \rangle \quad \forall\ \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V. \tag{5.11}$$

As an example of a potential operator A defined by (5.10), consider a nonlinear Hencky material. The stress-strain relation is (cf. [28])

$$\boldsymbol{\sigma} = k_0(\text{tr } \boldsymbol{\varepsilon}(\boldsymbol{u}))\ \boldsymbol{I} + \phi(|\boldsymbol{\varepsilon}^D(\boldsymbol{u})|^2)\ \boldsymbol{\varepsilon}^D(\boldsymbol{u}),$$

where $k_0>0$ is a material coefficient, $\boldsymbol{I}$ is the identity tensor of the second order, $\phi : \mathbb{R} \to \mathbb{R}$ is a constitutive function, and $\boldsymbol{\varepsilon}^D(\boldsymbol{u}) = \boldsymbol{\varepsilon}(\boldsymbol{u}) - (1/d)\ \text{tr}\ \boldsymbol{\varepsilon}(\boldsymbol{u})\ \boldsymbol{I}$ is the deviatoric part of the strain tensor. So the elasticity operator is

$$\mathcal{F}(\boldsymbol{\varepsilon}) = k_0(\text{tr } \boldsymbol{\varepsilon})\ \boldsymbol{I} + \phi(|\boldsymbol{\varepsilon}^D|^2)\ \boldsymbol{\varepsilon}^D. \tag{5.12}$$

The function ϕ is assumed to be piecewise continuously differentiable, and there exist positive constants c_1, c_2, d_1 and d_2, such that for $\xi \ge 0$,

$$\begin{aligned}\phi(\xi) &\le d_1,\\ -c_1 &\le \phi'(\xi) \le 0,\\ c_2 &\le \phi(\xi) + 2\ \phi'(\xi)\ \xi \le d_2.\end{aligned}$$

Then for the operator A defined in (5.10), it can be shown that

$$\begin{aligned}\langle A'(\boldsymbol{u})\boldsymbol{v}, \boldsymbol{w}\rangle = \int_\Omega [&k_0 \mathrm{tr}\ \boldsymbol{\varepsilon}(\boldsymbol{v})\ \mathrm{tr}\ \boldsymbol{\varepsilon}(\boldsymbol{w}) + \phi(|\boldsymbol{\varepsilon}^D(\boldsymbol{u})|^2)\ \boldsymbol{\varepsilon}^D(\boldsymbol{v}) \cdot \boldsymbol{\varepsilon}^D(\boldsymbol{w})\\ &+ 2\ \phi'(|\boldsymbol{\varepsilon}^D(\boldsymbol{u})|^2)\big(\boldsymbol{\varepsilon}^D(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}^D(\boldsymbol{v})\big)\big(\boldsymbol{\varepsilon}^D(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}^D(\boldsymbol{w})\big)]dx.\end{aligned}$$

Thus (5.11) is satisfied and the operator A is potential. By (2.7),

$$F_A(\boldsymbol{v}) = \frac{k_0}{2}\int_\Omega |\mathrm{tr}\ \boldsymbol{\varepsilon}(\boldsymbol{v})|^2 dx + \frac{1}{2}\int_\Omega |\boldsymbol{\varepsilon}^D(\boldsymbol{v})|^2 \int_0^1 \phi(t^2|\boldsymbol{\varepsilon}^D(\boldsymbol{v})|^2)\ dt\ dx.$$

As another example of a potential operator A defined by (5.10), consider a linearly elastic material. The elastic constitutive law is

$$\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{5.13}$$

where $\mathbb{C} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is the elasticity operator. As usual in the literature, we assume $\mathbb{C} = (C_{ijkl})_{1\le i,j,k,l\le d}$ is symmetric, bounded, and pointwise stable:

$$C_{ijkl} = C_{jikl} = C_{klij}, \quad 1 \le i,j,k,l \le d, \tag{5.14}$$

$$C_{ijkl} \in L^\infty(\Omega), \quad 1 \le i,j,k,l \le d, \tag{5.15}$$

$$(\mathbb{C}\boldsymbol{\epsilon}) \cdot \boldsymbol{\epsilon} \ge m_{\mathbb{C}}|\boldsymbol{\epsilon}|^2, \quad m_{\mathbb{C}}>0, \forall\ \boldsymbol{\epsilon} \in \mathbb{S}^d. \tag{5.16}$$

It is easy to see that the operator A defined in (5.10) corresponding to (5.13) is potential, and

$$F_A(\boldsymbol{v}) = \frac{1}{2}\ (\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q.$$

We now turn to discussions of three contact problems.

Example 5.1. We first consider a frictional contact problem with normal compliance ([7, 11]). The contact boundary conditions are

$$-\sigma_\nu \in \partial\psi_\nu(u_\nu), \quad |\boldsymbol{\sigma}_\tau| \le F_b(u_\nu), \quad -\boldsymbol{\sigma}_\tau = F_b(u_\nu)\frac{\boldsymbol{u}_\tau}{|\boldsymbol{u}_\tau|} \text{ if } \boldsymbol{u}_\tau \ne 0, \quad \text{on } \Gamma_3. \tag{5.17}$$

For the friction bound $F_b : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$, assume

$$\left\{\begin{array}{ll} \text{(a)} & \text{there exists } L_{F_b}>0 \text{ such that for all } z_1, z_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3,\\ & \quad |F_b(\boldsymbol{x}, z_1) - F_b(\boldsymbol{x}, z_2)| \le L_{F_b}|z_1 - z_2|;\\ \text{(b)} & F_b(\cdot, z) \text{ is measurable on } \Gamma_3, \text{for all } z \in \mathbb{R};\\ \text{(c)} & F_b(\boldsymbol{x}, z) \ge 0 \text{ for z} \in \mathbb{R} \text{ and } F_b(\boldsymbol{x}, 0) = 0, \text{a.e. } \boldsymbol{x} \in \Gamma_3. \end{array}\right. \tag{5.18}$$

For the potential function $\psi_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$, assume

$$\begin{cases} \text{(a) } \psi_\nu(\cdot, z) \text{ is measurable on } \Gamma_3 \text{ for all } z \in \mathbb{R} \text{and } \psi_\nu(\cdot, 0) \in L^1(\Gamma_3); \\ \text{(b) } \psi_\nu(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_3; \\ \text{(c) } |\partial\psi_\nu(\boldsymbol{x}, z)| \le c\big(1 + |z|^{p-1}\big) \text{ for all } \mathrm{z} \in \mathbb{R}, \text{a.e.} \boldsymbol{x} \in \Gamma_3; \\ \text{(d) } \psi_\nu^0(\boldsymbol{x}, z_1; z_2 - z_1) + \psi_\nu^0(\boldsymbol{x}, z_2; z_1 - z_2) \le \alpha_{\psi_\nu} |z_1 - z_2|^2 \\ \qquad \text{for a.e. } \boldsymbol{x} \in \Gamma_3, \text{ all } \mathrm{z}_1, \mathrm{z}_2 \in \mathbb{R} \text{ with } \alpha_{\psi_\nu} \ge 0. \end{cases} \tag{5.19}$$

Note that the commonly required linear growth condition $|\partial\psi_\nu(\boldsymbol{x}, z)| \le c(1 + |z|)$ for a.e. $\boldsymbol{x} \in \Gamma_3, \forall\ z \in \mathbb{R}$, is relaxed to (5.19)(c) for p specified in (5.5). Let $\gamma_\psi \boldsymbol{v} = v_\nu$ for $\boldsymbol{v} \in V$; then $\gamma_\psi \in \mathcal{L}(V, L^p(\Gamma_3))$. By applying Theorem 2.5, we can derive the bound

$$|\psi_\nu(\boldsymbol{x}, z)| \le c\big(1 + |z|^p\big) \quad \forall\ z \in \mathbb{R}, \text{a.e. } \boldsymbol{x} \in \Gamma_3.$$

Therefore, the superpotential $\int_{\Gamma_3} \psi_\nu(v_\nu)\ da$ is well-defined for $\boldsymbol{v} \in V$.

The weak formulation of the contact problem (5.8), (5.1)–(5.3) and (5.17) is a variational-hemivariational inequality: find a displacement field $\boldsymbol{u} \in V$ such that

$$\begin{aligned} &(\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + \textstyle\int_{\Gamma_3} F_b(u_\nu)(|\boldsymbol{v}_\tau| - |\boldsymbol{u}_\tau|) da \\ &\quad + \textstyle\int_{\Gamma_3} \psi_\nu^0(u_\nu; v_\nu - u_\nu)\ da \ge \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u} \rangle_{V^* \times V} \quad \forall\ \boldsymbol{v} \in V. \end{aligned} \tag{5.20}$$

Let $\lambda_0 > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in V, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\ dx = \lambda \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}\ da \quad \forall \boldsymbol{v} \in V.$$

By Theorem 4.10, the problem (5.20) has a unique solution $\boldsymbol{u} \in V$ under the assumptions (5.9), (5.18), (5.19) and the smallness condition $L_{F_b} + \alpha_{\psi_\nu} < \lambda_0 m_{\mathcal{F}}$. □

Example 5.2. In this example, we consider a bilateral frictional contact problem between an elastic body and a rigid foundation (e.g. [12]). The contact boundary conditions are

$$u_\nu = 0, \quad -\boldsymbol{\sigma}_\tau \in \partial\psi_\tau(\boldsymbol{u}_\tau) \quad \text{on } \Gamma_3. \tag{5.21}$$

The feature of bilateral contact is reflected by the condition $u_\nu = 0$. We assume the potential function $\psi_\tau : \Gamma_3 \times \mathbb{R}^d \to \mathbb{R}$ has the following properties.

$$\begin{cases} \text{(a) } \psi_\tau(\cdot, \boldsymbol{z}) \text{ is measurable on } \Gamma_3 \text{ for all } \boldsymbol{z} \in \mathbb{R}^d \text{ and } \psi_\tau(\cdot, 0) \in L^1(\Gamma_3); \\ \text{(b) } \psi_\tau(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R}^d \text{ for a.e. } \boldsymbol{x} \in \Gamma_3; \\ \text{(c) } |\partial\psi_\tau(\boldsymbol{x}, \boldsymbol{z})| \le c\big(1 + |\boldsymbol{z}|^{p-1}\big) \text{forall } \boldsymbol{z} \in \mathbb{R}^d, \text{a.e. } \boldsymbol{x} \in \Gamma_3; \\ \text{(d) } \psi_\tau^0(\boldsymbol{x}, \boldsymbol{z}_1; \boldsymbol{z}_2 - \boldsymbol{z}_1) + \psi_\tau^0(\boldsymbol{x}, \boldsymbol{z}_2; \boldsymbol{z}_1 - \boldsymbol{z}_2) \le \alpha_{\psi_\tau} |\boldsymbol{z}_1 - \boldsymbol{z}_2|^2 \\ \qquad \text{for a.e. } \boldsymbol{x} \in \Gamma_3, \text{ all } \boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathbb{R}^\mathrm{d} \text{ with } \alpha_{\psi_\tau} \ge 0. \end{cases} \tag{5.22}$$

Note that the commonly required condition (e.g. [12])

$$|\partial\psi_\tau(\boldsymbol{x},\boldsymbol{z})| \le c(1+|\boldsymbol{z}|)\text{for a.e. } \boldsymbol{x}\in\Gamma_3, \forall\ \boldsymbol{z}\in\mathbb{R}^{\mathrm{d}}$$

is replaced by the less restrictive condition (5.22)(c), where the exponent p is specified in (5.5).

With V from (5.4), we define the subspace

$$V_1 = \{\boldsymbol{v}\in V | v_\nu = 0 \text{ on } \Gamma_3\}. \tag{5.23}$$

Let $\gamma_{\psi_\tau} : V \to L^p(\Gamma_3;\mathbb{R}^d)$ be the tangential component trace operator: $\gamma_{\psi_\tau}\boldsymbol{v} = \boldsymbol{v}_\tau$ for $\boldsymbol{v}\in V$. Then $\gamma_{\psi_\tau}\in\mathcal{L}(V;L^p(\Gamma_3;\mathbb{R}^d))$. Let $c_\tau = \lambda_1^{-1/2}$ where $\lambda_1{>}0$ is the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u}\in V_1, \qquad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\ dx = \lambda\int_{\Gamma_3}\boldsymbol{u}_\tau\cdot\boldsymbol{v}_\tau da \quad \forall\ \boldsymbol{v}\in V_1.$$

Then

$$||\boldsymbol{v}_\tau||_{L^2(\Gamma_3;\mathbb{R}^d)} \le c_\tau||\boldsymbol{v}||_V \quad \forall\ \boldsymbol{v}\in V_1. \tag{5.24}$$

The weak formulation of the contact problem defined by (5.8), (5.1)–(5.3) and (5.21) is to find a displacement field $\boldsymbol{u}\in V_1$ such that

$$(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + \int_{\Gamma_3}\psi_\tau^0(\boldsymbol{u}_\tau;\boldsymbol{v}_\tau)\ da \ge \langle \boldsymbol{f},\boldsymbol{v}\rangle_{V^*\times V} \quad \forall\ \boldsymbol{v}\in V_1. \tag{5.25}$$

By Theorem 4.16, this problem has a unique solution under the stated assumptions on the problem data and $\alpha_{\psi_\tau}{<}\lambda_1 m_{\mathcal{F}}$.

When (5.11) is satisfied, the problem (5.25) admits an equivalent minimization principle:

$$E(\boldsymbol{u}) \le E(\boldsymbol{v}) \quad \forall\,\boldsymbol{v}\in V_1,$$

where the energy functional is defined by the formula

$$E(\boldsymbol{v}) = F_A(\boldsymbol{v}) + \int_{\Gamma_3}\psi_\tau(\boldsymbol{v}_\tau)\ da - \langle\boldsymbol{f},\boldsymbol{v}\rangle_{V^*\times V}.$$

In particular, in the case (5.13) for a linearly elastic material, the energy functional is

$$E(v) = \frac{1}{2}\ (\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + \int_{\Gamma_3}\psi_\tau(\boldsymbol{v}_\tau)\ da - \langle\boldsymbol{f},\boldsymbol{v}\rangle_{V^*\times V}.$$

This completes the second example. □

Example 5.3. Consider a static frictionless unilateral contact problem between an elastic body and a rigid foundation ([12]). On the contact boundary Γ_3, we assume the contact is frictionless:

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0} \qquad \text{on } \Gamma_3, \tag{5.26}$$

and satisfies the unilateral relations

$$u_\nu \le g, \sigma_\nu + \xi_\nu \le 0, (u_\nu - g)(\sigma_\nu + \xi_\nu) = 0, \xi_\nu \in \partial\psi_\nu(u_\nu) \quad \text{on } \Gamma_3. \tag{5.27}$$

The relations (5.27) model a frictionless contact with a foundation made of a rigid body covered by a layer made of elastic material. Penetration is restricted by the relation $u_\nu \le g$, where g is non-negative valued and it represents the thickness of the elastic layer. When there is penetration and the normal displacement does not reach the bound g, the contact is described by a multi-valued normal compliance condition: $-\sigma_\nu = \xi_\nu \in \partial\psi_\nu(u_\nu)$. We assume the potential function $\psi_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ has the properties stated in (5.19).

With V from (5.4), we define the set

$$K = \{\boldsymbol{v} \in V | v_\nu \le g \text{ on } \Gamma_3\}. \tag{5.28}$$

We define γ_ψ to be the normal component trace operator as in Example 5.1. Let $c_\nu = \lambda_2^{-1/2}$ where $\lambda_2 > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in V, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\ dx = \lambda \int_{\Gamma_3} u_\nu v_\nu da \quad \forall\ \boldsymbol{v} \in V.$$

Then

$$\|v_\nu\|_{L^2(\Gamma_3)} \le c_\nu \|\boldsymbol{v}\|_V \quad \forall\ \boldsymbol{v} \in V. \tag{5.29}$$

The weak formulation of the contact problem defined by (5.8), (5.1)–(5.3), (5.26) and (5.27) is to find a displacement field $\boldsymbol{u} \in K$ such that

$$(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}))_Q + \int_{\Gamma_3} \psi^0(u_\nu; v_\nu - u_\nu)\ da \ge \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u} \rangle_{V^* \times V} \quad \forall\ \boldsymbol{v} \in K. \tag{5.30}$$

By Theorem 4.16, this hemivariational inequality has a unique solution under the stated assumptions on the problem data and $\alpha_\psi < \lambda_2 m_{\mathcal{F}}$.

When (5.11) is satisfied, the problem (5.30) admits an equivalent minimization principle:

$$E(\boldsymbol{u}) \le E(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in V_1,$$

where the energy functional is defined by the formula

$$E(\boldsymbol{v}) = F_A(\boldsymbol{v}) + \int_{\Gamma_3} \psi(v_\nu)\ da - \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{V^* \times V}.$$

In particular, in the case (5.13) for a linearly elastic material, the energy functional is

$$E(v) = \frac{1}{2} \ (\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + \int_{\Gamma_3} \psi(v_\nu) \ da - \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{V^* \times V}.$$

This concludes the third example. □

References

[1] Panagiotopoulos, P. D. (1983). Nonconvex energy functions, hemivariational inequalities and substationary principles. *Acta Mech*. 42:160–183.

[2] Carl, S., Le, V. K., Motreanu, D. (2007). *Nonsmooth Variational Problems and Their Inequalities: Comparison Principles and Applications*. New York, NY: Springer.

[3] Migórski, S., Ochal, A., Sofonea, M. (2013). *Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems*. Advances in Mechanics and Mathematics 26. New York, NY: Springer.

[4] Motreanu, D., Panagiotopoulos, P.D. (1999). *Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities and Applications*. Boston, Dordrecht, London: Kluwer Academic Publishers.

[5] Naniewicz, Z., Panagiotopoulos, P. D. (1995). *Mathematical Theory of Hemivariational Inequalities and Applications*. New York, NY: Dekker.

[6] Panagiotopoulos, P. D. (1993). *Hemivariational Inequalities, Applications in Mechanics and Engineering*. Berlin, Germany: Springer-Verlag.

[7] Sofonea, M., Migórski, S. (2018). *Variational-Hemivariational Inequalities with Applications*. Boca Raton, FL: CRC Press.

[8] Haslinger, J., Miettinen, M., Panagiotopoulos, P.D. (1999). *Finite Element Method for Hemivariational Inequalities. Theory, Methods and Applications*. Boston, Dordrecht, London: Kluwer Academic Publishers.

[9] Barboteu, M., Bartosz, K., Han, W., Janiczko, T. (2015). Numerical analysis of a hyperbolic hemivariational inequality arising in dynamic contact. *SIAM J. Numer. Anal*. 53(1):527–550. DOI: 10.1137/140969737.

[10] Han, W. (2018). Numerical analysis of stationary variational-hemivariational inequalities with applications in contact mechanics. *Math. Mech. Solids*. 23(3): 279–293. DOI: 10.1177/1081286517713342.

[11] Han, W., Migórski, S., Sofonea, M. (2014). A class of variational-hemivariational inequalities with applications to frictional contact problems. *SIAM J. Math. Anal*. 46(6):3891–3912. DOI: 10.1137/140963248.

[12] Han, W., Sofonea, M., Barboteu, M. (2017). Numerical analysis of elliptic hemivariational inequalities. *SIAM J. Numer. Anal*. 55(2):640–663. DOI: 10.1137/16M1072085.

[13] Han, W., Sofonea, M., Danan, D. (2018). Numerical analysis of stationary variational-hemivariational inequalities. *Numer. Math*. 139(3):563–592. DOI: 10.1007/s00211-018-0951-9.

[14] Xu, W., Huang, Z., Han, W., Chen, W., Wang, C. (2019). Numerical analysis of history-dependent variational-hemivariational inequalities with applications in contact mechanics. *J. Comput. Appl. Math*. 351:364–377. DOI: 10.1016/j.cam.2018.08.046.

[15] Han, W., Sofonea, M. (2019). Numerical analysis of hemivariational inequalities in contact mechanics. *Acta Numer*. 28:175–286. DOI: 10.1017/S0962492919000023.

[16] Feng, F., Han, W., Huang, J. (2019). Virtual element method for elliptic hemivariational inequalities. *J. Sci. Comput*. 81(3):2388–2412. DOI: 10.1007/s10915-019-01090-2.

[17] Migórski, S., Ochal, A., Sofonea, M. (2017). A class of variational-hemivariational inequalities in reflexive Banach spaces. *J. Elast*. 127(2):151–178. DOI: 10.1007/s10659-016-9600-7.

[18] Ekeland, I., Temam, R. (1976). *Convex Analysis and Variational Problems*. Amsterdam, the Netherlands: North-Holland.

[19] Han, W. (2020). Minimization principles for elliptic hemivariational inequalities. *Nonlinear Anal. Real World Appl*. 54:103114. DOI: 10.1016/j.nonrwa.2020.103114.

[20] Atkinson, K., Han, W. (2009). *Theoretical Numerical Analysis: A Functional Analysis Framework*, 3rd ed. New York, NY: Springer.

[21] Denkowski, Z., Migórski, S., Papageorgiou, N.S. (2003). *An Introduction to Nonlinear Analysis: Theory*. Boston, Dordrecht, London, New York: Kluwer Academic/Plenum Publishers.

[22] Clarke, F. H. (1975). Generalized gradients and applications. *Trans. Amer. Math. Soc*. 205:247–262. DOI: 10.1090/S0002-9947-1975-0367131-6.

[23] Clarke, F. H. (1983). *Optimization and Nonsmooth Analysis*. New York, NY: Wiley.

[24] Gasiński, L., Papageorgiou, N. S. (2005). *Nonsmooth Critical Point Theory and Nonlinear Boundary Value Problems*. Boca Raton, FL: Chapman & Hall/CRC.

[25] Fan, L., Liu, S., Gao, S. (2003). Generalized monotonicity and convexity of non-differentiable functions. *J. Math. Anal. Appl*. 279(1):276–289. DOI: 10.1016/S0022-247X(03)00009-X.

[26] Zeidler, E. (1986). *Nonlinear Functional Analysis and its Applications. III: Variational Methods and Optimization*. New York, NY: Springer-Verlag.

[27] Duvaut, G., Lions, J.-L. (1976). *Inequalities in Mechanics and Physics*. Berlin, Germany: Springer-Verlag.

[28] Zeidler, E. (1988). *Nonlinear Functional Analysis and its Applications. IV: Applications to Mathematical Physics*. New York, NY: Springer-Verlag.