Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 15, No. 1-2/2023

ON A NEW CLASS OF MIXED HEMIVARIATIONAL-VARIATIONAL INEQUALITIES*

W. Han†

Dedicated to Dr. Dan Tiba on the occasion of his 70^{th} anniversary

Abstract

In this paper, we study a new class of mixed hemivariational-variational inequalities in which both the non-smooth convex functional and the non-smooth non-convex functional can depend on two arguments. We present solution existence and uniqueness results. Then, we apply the theoretical results on a mixed hemivariational-variational inequality in the study of a stationary incompressible flow of Bingham type fluid subject to non-smooth non-monotone slip boundary condition.

MSC: 49J40, 35J50, 76A05, 76D03.

keywords: Mixed hemivariational-variational inequality, well-posedness, Banach fixed-point, Bingham type fluid

*Accepted for publication in revised form on April 26, 2023. The work was supported by Simons Foundation Collaboration, USA Grants, No. 850737, and by the European Union's Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie Grant Agreement No. 823731 CONMECH.

†E-mail: weimin-han@uiowa.edu. Department of Mathematics, University of Iowa, Iowa City, IA 52242-1410, USA

1 Introduction

Hemivariational inequalities arise in applications of problems involving non-smooth, non-monotone and set-valued relations among physical quantities. Since the pioneering work of Panagiotopoulos in early 1980s ([17]), there has been extensive research on modeling, analysis, numerical solution and applications of hemivariational inequalities. For recent representative references, one is referred to [18] for well-posedness analysis of hemivariational inequalities, and to [12] for a survey of numerical analysis of hemivariational inequalities.

Mixed formulations are useful in the numerical solution of problems with certain constraints. A standard example of a mixed formulation is its use in the treatment of the incompressibility constraint for fluid flow problems. Mixed formulations are also useful in the development of efficient numerical methods for the computation of physical quantities other than the original unknown variable of the underlying partial differential equations. The reference [4] provides a comprehensive coverage of mixed finite element methods for solving a variety of boundary value problems through their mixed formulations. Mixed finite element methods have also been applied to solve mixed hemivariational inequalities of the Stokes equations ([6]) and Navier–Stokes equations ([9]).

In [11], well-posedness analysis is provided for the following elliptic mixed hemivariational-variational inequality:

Problem 1 *Find* $(u, p) \in K_V \times K_Q$ *such that*

$$\begin{aligned} &\langle Au, v-u\rangle + b(v-u,p) + \Phi(u,v) - \Phi(u,u) + \Psi^0(u; v-u) \\ &\qquad \geq \langle f, v-u\rangle \quad \forall\, v \in K_V, \qquad (1) \\ &b(u, q-p) \leq 0 \quad \forall\, q \in K_Q. \qquad (2) \end{aligned}$$

In Problem 1, K_V and K_Q are closed convex sets in Hilbert spaces, A is a Lipschitz continuous and strongly monotone operator, b is a continuous bilinear form, Φ is continuous and convex with respect to its second argument, Ψ is locally Lipschitz continuous and Ψ^0 denotes its generalized directional derivative in the sense of Clarke, and f is a given linear continuous functional. Precise descriptions of the assumptions on the data will be given in Section 2. For applications, the term $\Psi^0(u; v-u)$ in (1) is usually replaced by $I_\Delta(\psi^0(\gamma_\psi u; \gamma_\psi v - \gamma_\psi u))$, where I_Δ stands for the operator of integration over Δ which can be the spatial domain of the application problem, or a subdomain, or a part of the boundary of the spatial domain,

ψ is a real-valued function, and γ_ψ is a linear operator mapping functions defined on the spatial domain of the problem to functions defined on Δ. The corresponding elliptic mixed hemivariational-variational inequality is of the following form:

Problem 2 *Find* $(u,p) \in K_V \times K_Q$ *such that*

$$\begin{aligned}&\langle Au, v-u\rangle + b(v-u,p) + \Phi(u,v) - \Phi(u,u) + I_\Delta(\psi^0(\gamma_\psi u; \gamma_\psi v - \gamma_\psi u))\\ &\qquad \geq \langle f, v-u\rangle \quad \forall\, v \in K_V, \qquad (3)\\ &b(u, q-p) \leq 0 \quad \forall\, q \in K_Q. \qquad (4)\end{aligned}$$

Elliptic mixed hemivariational-variational inequalities of special types of Problem 1 or Problem 2 have been studied in other papers. For example, Problem 2 with $\Phi \equiv 0$ was studied in [13], Problem 1 with $\Phi \equiv 0$ was studied in [3, 15]. The results in [3] were extended in [2] to a problem of the form of Problem 1 where $\Phi(u,v) \equiv \Phi(v)$ depends on only one variable. Note that in these references, the main theoretical tools for proving solution existence are rather complicated fixed-point principles for set-valued mappings. In comparison, in [11], well-posedness of Problems 1 and 2 is shown through the use of more elementary knowledge of functional analysis, starting with the analysis of related saddle-point formulations in [10], and is thus more accessible by applied mathematicians and engineers.

The rest of the paper is organized as follows. In Section 2, we review preliminary materials needed later; in particular, we recall results from [11] on well-posedness of Problems 1 and 2. In Section 3, we provide a well-posedness analysis of a new class of elliptic mixed hemivariational-variational inequalities that are more general than Problems 1 and 2 in that the locally Lipschitz continuous functions Ψ and ψ are allowed to depend on two arguments. In Section 4, we study a Bingham type fluid flow problem by applying the theoretical results proved in Section 3.

2 Preliminaries

In this section, we review some basic notions and recall well-posedness results on Problems 1 and 2 from [11]. In describing hemivariational inequalities, we need the notions of the generalized directional derivative and generalized subdifferential in the sense of Clarke for a locally Lipschitz continuous function ([5]). Let $\Psi\colon V \to \mathbb{R}$ be a locally Lipschitz continuous functional defined on a real Banach space V. The generalized (Clarke) directional derivative of Ψ at an element $u \in V$ in the direction $v \in V$ is defined by

$$\Psi^0(u;v) := \limsup_{w\to u,\,\lambda\downarrow 0} \frac{\Psi(w+\lambda\, v) - \Psi(w)}{\lambda}.$$

Given $\Psi^0(u;v)$, the generalized subdifferential of Ψ at $u \in V$ is defined by

$$\partial\Psi(u) := \left\{\eta \in V^* \mid \Psi^0(u;v) \geq \langle \eta, v\rangle \;\forall\, v \in V\right\}.$$

Conversely, given $\partial\Psi(u)$, the generalized directional derivative can be determined through the formula

$$\Psi^0(u;v) = \max\left\{\langle u^*, v\rangle \mid u^* \in \partial\Psi(u)\right\} \quad \forall\, u, v \in V.$$

If the locally Lipschitz continuous function $\Psi\colon V \to \mathbb{R}$ is convex, then the subdifferential $\partial\Psi(u)$ at any $u \in V$ in the sense of Clarke coincides with the convex subdifferential $\partial\Psi(u)$. In this sense, the notion of the Clarke subdifferential is a generalization of the notion of the convex subdifferential to non-convex functions. We recall here some basic properties of the generalized directional derivative and the generalized subdifferential. For all $\lambda \in \mathbb{R}$ and all $u \in V$, we have

$$\partial(\lambda\,\Psi)(u) = \lambda\,\partial\Psi(u).$$

For locally Lipschitz functions $\Psi_1, \Psi_2\colon V \to \mathbb{R}$, the inclusion

$$\partial(\Psi_1 + \Psi_2)(u) \subset \partial\Psi_1(u) + \partial\Psi_2(u) \quad \forall\, u \in V \tag{5}$$

holds. This inclusion is equivalent to the inequality

$$(\Psi_1 + \Psi_2)^0(u;v) \leq \Psi_1^0(u;v) + \Psi_2^0(u;v) \quad \forall\, u, v \in V. \tag{6}$$

Detailed discussions of properties of the generalized directional derivative and the generalized subdifferential for locally Lipschitz continuous functionals can be found in several references, e.g. [5, 14].

As in [10, 11], we will assume V and Q are real Hilbert spaces, with dual spaces V^* and Q^*. The symbol $\langle\cdot,\cdot\rangle$ denotes the duality pairing between V^* and V, or between Q^* and Q; it should be clear from the context which duality pairing is meant by $\langle\cdot,\cdot\rangle$. Let $K_V \subset V$ and $K_Q \subset Q$. Given operators and functionals $A\colon V \to V^*$, $b\colon V \times Q \to \mathbb{R}$, $\Phi\colon V \times V \to \mathbb{R}$, $\Psi\colon V \times V \to \mathbb{R}$, and $f \in V^*$, we consider Problems 1 and 2. We will use the following conditions on the problem data.

- $H(K_V)$: V is a real Hilbert space, $K_V \subset V$ is non-empty, closed and convex.
- $H(K_Q)$: Q is a real Hilbert space, $K_Q \subset Q$ is non-empty, closed and convex.
- $H(A)$: $A\colon V \to V^*$ is Lipschitz continuous and strongly monotone.
- $H(b)$: $b : V \times Q \to \mathbb{R}$ is bilinear and bounded.
- $H(\Phi)_2$: $\Phi\colon V \times V \to \mathbb{R}$; for any $u \in V$, $\Phi(u,\cdot)\colon V \to \mathbb{R}$ is convex and continuous, and there exists a constant $\alpha_\Phi \geq 0$ such that

$$\begin{aligned}&\Phi(u_1,v_2) - \Phi(u_1,v_1) + \Phi(u_2,v_1) - \Phi(u_2,v_2)\\ &\qquad \leq \alpha_\Phi \|u_1 - u_2\|_V \|v_1 - v_2\|_V \quad \forall\, u_1,u_2,v_1,v_2 \in V.\end{aligned} \tag{7}$$

- $H(\Psi)_1$: $\Psi\colon V \to \mathbb{R}$ is locally Lipschitz continuous, and there exists a constant $\alpha_\Psi \geq 0$ such that

$$\Psi^0(v_1; v_2 - v_1) + \Psi^0(v_2; v_1 - v_2) \leq \alpha_\Psi \|v_1 - v_2\|_V^2 \quad \forall\, v_1,v_2 \in V. \tag{8}$$

- $H(f)$: $f \in V^*$.

We use $M_A > 0$ for the Lipschitz constant of A:

$$\|Av_1 - Av_2\|_{V^*} \leq M_A \|v_1 - v_2\|_V \quad \forall\, v_1,v_2 \in V, \tag{9}$$

and use $m_A > 0$ for the strong monotonicity constant:

$$\langle Av_1 - Av_2, v_1 - v_2\rangle \geq m_A \|v_1 - v_2\|_V^2 \quad \forall\, v_1,v_2 \in V. \tag{10}$$

We denote by $M_b > 0$ for the boundedness constant of the bilinear form b:

$$|b(v,q)| \leq M_b \|v\|_V \|q\|_Q \quad \forall\, v \in V,\ q \in Q. \tag{11}$$

Assumption $H(b)$ allows us to define an operator $B \in \mathcal{L}(V;Q^*)$ by the relation

$$\langle Bv, q\rangle = b(v,q) \quad \forall\, v \in V,\ q \in Q.$$

The subscript 2 in $H(\Phi)_2$ reflects the fact that this is an assumption on Φ for the case where Φ has two arguments; similarly, $H(\Psi)_1$ is an assumption on Ψ where Ψ is a one-argument function. In $H(\Phi)_2$, the convex function $\Phi\colon V \to \mathbb{R}$ is assumed to be continuous, instead of l.s.c. As is explained in

[7, 8], there is no loss of generality with the stronger assumption of continuity for a vast majority of applications.

When K_Q is an unbounded set in Q and K_V is a subspace of V, we will assume additionally that the bilinear form $b(\cdot,\cdot)$ satisfies an inf-sup condition: there exists a constant $\alpha_b > 0$ such that

$$\sup_{0\neq v\in K_V} \frac{b(v,q)}{\|v\|_V} \ge \alpha_b \|q\|_Q \quad \forall\, q\in Q. \tag{12}$$

The following result is proved in [11].

Theorem 3 *Assume $H(K_V)$, $H(K_Q)$, $H(A)$, $H(b)$, $H(\Phi)_2$, $H(\Psi)_1$, $H(f)$, and $\alpha_\Phi + \alpha_\Psi < m_A$. Moreover, assume either*

> *K_Q is bounded,*

or

> *K_Q is unbounded; K_V is a subspace of V; for any $u\in V$, $\Phi(u,\cdot)\colon V\to\mathbb{R}$ is Lipschitz continuous on K_V; there exist non-negative constants c_0, c_1 such that*
>
> $$\|\partial\Psi(v)\|_{V^*} \le c_0 + c_1\|v\|_V \quad \forall\, v\in V; \tag{13}$$
>
> *and the inf-sup condition* (12) *holds.*

Then, Problem 1 has a solution $(u,p)\in K_V\times K_Q$ and the first component u of the solution is unique. Moreover, u depends Lipschitz continuously on f.

Note that the assumption (13) is a short-hand notation for the property

$$\|\eta\|_{V^*} \le c_0 + c_1\|v\|_V \quad \forall\, v\in V,\ \eta\in\partial\Psi(v),$$

and it is equivalent to

$$\left|\Psi^0(u;v)\right| \le (c_0 + c_1\|u\|_V)\,\|v\|_V \quad \forall\, u,v\in V.$$

For Problem 2, assume $\gamma_\psi v$ is an $\mathbb{R}^m$-valued function for $v\in V$, for some positive integer m. In applications in solid mechanics or fluid mechanics, the operator γ_ψ is either the normal trace operator and then $m=1$, or the tangential component trace operator and then $m=d$, d being the dimension of the spatial domain of the problem. Let us introduce the following assumption on the function ψ.

- $H(\psi)_1$: $\gamma_\psi \in \mathcal{L}(V; L^2(\Delta; \mathbb{R}^m))$; $\psi\colon \Delta \times \mathbb{R}^m \to \mathbb{R}$; $\psi(\cdot, z)$ is measurable on Δ for all $z \in \mathbb{R}^m$; there exists $z_0 \in L^2(\Delta; \mathbb{R}^m)$ such that $\psi(\cdot, z_0(\cdot)) \in L^1(\Delta)$; $\psi(\boldsymbol{x}, \cdot)$ is locally Lipschitz continuous on $\mathbb{R}^m$ for a.e. $\boldsymbol{x} \in \Delta$; and for some non-negative constants c and α_ψ, a.e. on Δ,

$$|\partial\psi(\cdot, z)| \le c\,(1 + |z|_{\mathbb{R}^m}) \quad \forall\, z \in \mathbb{R}^m, \tag{14}$$

$$\psi^0(z_1; z_2 - z_1) + \psi^0(z_2; z_1 - z_2) \le \alpha_\psi |z_1 - z_2|^2_{\mathbb{R}^m} \quad \forall\, z_1, z_2 \in \mathbb{R}^m. \tag{15}$$

To simplify the notation, we usually write $\psi(z)$, $\partial\psi(z)$ and $\psi^0(z_1; z_2)$ to mean $\psi(\cdot, z)$, $\partial\psi(\cdot, z)$ and $\psi^0(\cdot, z_1; z_2)$. Denote by $c_\Delta > 0$ the smallest constant in the inequality

$$I_\Delta(|\gamma_\psi v|^2_{\mathbb{R}^m}) \le c_\Delta^2 \|v\|_V^2 \quad \forall\, v \in V. \tag{16}$$

Define the functional

$$\Psi(v) = I_\Delta(\psi(\gamma_\psi v)), \quad v \in V. \tag{17}$$

Then under the assumption $H(\psi)_1$, similar to the results and arguments in [14, Section 3.3], it can be shown that $\Psi(\cdot)$ is well-defined and locally Lipschitz on V, and

$$\Psi^0(u; v) \le I_\Delta(\psi^0(\gamma_\psi u; \gamma_\psi v)) \quad \forall\, u, v \in V. \tag{18}$$

Thus, (15) and (16) imply that for any $v_1, v_2 \in V$,

$$\begin{aligned}
&\Psi^0(v_1; v_2 - v_1) + \Psi^0(v_2; v_1 - v_2)\\
&\qquad \le I_\Delta(\psi^0(\gamma_\psi v_1; \gamma_\psi v_2 - \gamma_\psi v_1) + \psi^0(\gamma_\psi v_2; \gamma_\psi v_1 - \gamma_\psi v_2))\\
&\qquad \le I_\Delta(\alpha_\psi |\gamma_\psi (v_1 - v_2)|^2_{\mathbb{R}^m})\\
&\qquad \le \alpha_\psi c_\Delta^2 \|v_1 - v_2\|_V^2,
\end{aligned} \tag{19}$$

i.e., (8) is satisfied with $\alpha_\Psi = \alpha_\psi c_\Delta^2$. Moreover, (13) follows from (14) and (18). The following result is found in [11].

Theorem 4 *Assume $H(K_V)$, $H(K_Q)$, $H(A)$, $H(b)$, $H(\Phi)_2$, $H(\psi)_1$, $H(f)$, and $\alpha_\Phi + \alpha_\psi c_\Delta^2 < m_A$. Moreover, assume either*

K_Q is bounded,

or

K_Q is unbounded; K_V is a subspace of V; for any $u \in V$, $\Phi(u, \cdot)\colon V \to \mathbb{R}$ is Lipschitz continuous on K_V; and the inf-sup condition (12) *holds.*

Then, Problem 2 has a solution $(u, p) \in K_V \times K_Q$ and the first component u of a solution is unique. Moreover, u depends Lipschitz continuously on f.

3 New elliptic mixed hemivariational-variational inequalities

We first consider an elliptic mixed hemivariational-quasivariational inequality that is more general than Problem 1.

Problem 5 *Find* $(u,p) \in K_V \times K_Q$ *such that*

$$\begin{aligned} &\langle Au, v-u\rangle + b(v-u,p) + \Phi(u,v) - \Phi(u,u) + \Psi^0(u,u;v-u) \\ &\qquad \geq \langle f, v-u\rangle \quad \forall\, v \in K_V, && (20)\\ &b(u,q-p) \leq 0 \quad \forall\, q \in K_Q. && (21)\end{aligned}$$

Here, for a two-argument function $\Psi(w,u)$, the symbol $\Psi^0(w,u;v)$ stands for the generalized directional derivative of Ψ with respect to its second argument u in the direction v. In the study of Problem 5, we modify $H(\Psi)_1$ to $H(\Psi)_2$; the subscript 2 in $H(\Psi)_2$ reminds the reader that this is a condition for the case where Ψ depends on two variables.

- $H(\Psi)_2$: $\Psi\colon V \times V \to \mathbb{R}$ is locally Lipschitz continuous with respect to its second argument, and there exist two constants $\alpha_{\Psi,1}, \alpha_{\Psi,2} \geq 0$ such that

$$\begin{aligned} &\Psi^0(w_1,v_1;v_2-v_1) + \Psi^0(w_2,v_2;v_1-v_2) \\ &\qquad \leq \alpha_{\Psi,1}\|v_1-v_2\|_V^2 + \alpha_{\Psi,2}\|w_1-w_2\|_V\|v_1-v_2\|_V \\ &\qquad\ \ \forall\, w_1,w_2,v_1,v_2 \in V. && (22)\end{aligned}$$

Theorem 6 *Assume* $H(K_V)$, $H(K_Q)$, $H(A)$, $H(b)$, $H(\Phi)_2$, $H(\Psi)_2$, $H(f)$, *and*

$$\alpha_\Phi + \alpha_{\Psi,1} + \alpha_{\Psi,2} < m_A. \tag{23}$$

Moreover, assume either

> K_Q *is bounded,*

or

> K_Q *is unbounded;* K_V *is a subspace of* V*; for any* $u \in V$, $\Phi(u,\cdot)\colon V \to \mathbb{R}$ *is Lipschitz continuous on* K_V*; there exists a constant* c *such that*
>
> $$\left|\Psi^0(w,u;v)\right| \leq c\,(1+\|w\|_V + \|u\|_V)\,\|v\|_V \quad \forall\, w,u,v \in V;$$
>
> *and the inf-sup condition* (12) *holds.*

Then, Problem 5 has a solution $(u, p) \in K_V \times K_Q$ and the first component u of a solution is unique. Moreover, the solution component u depends Lipschitz continuously on f.

Proof. We use a fixed-point argument based on the result of Theorem 3. For any $w \in K_V$, introduce an auxiliary problem of finding $(u, p) \in K_V \times K_Q$ such that

$$\begin{aligned} &\langle Au, v-u\rangle + b(v-u,p) + \Phi(u,v) - \Phi(u,u) + \Psi^0(w,u;v-u) \\ &\qquad \geq \langle f, v-u\rangle \quad \forall\, v \in K_V, \qquad (24)\\ &b(u, q-p) \leq 0 \quad \forall\, q \in K_Q. \qquad (25)\end{aligned}$$

Under the stated assumptions, we can apply Theorem 3 to conclude that there is a pair $(u, p) \in K_V \times K_Q$ satisfying (24)–(25) and u is unique. This allows us to define a mapping $P\colon K_V \to K_V$ by the relation

$$P(w) = u.$$

Let us show that $P\colon K_V \to K_V$ is a contraction. For any $w_1, w_2 \in K_V$, denote $u_1 = P(w_1)$, $u_2 = P(w_2)$. By the definition of the mapping P, there exist $p_1, p_2 \in K_Q$ such that

$$\begin{aligned} &\langle Au_1, v-u_1\rangle + b(v-u_1,p_1) + \Phi(u_1,v) - \Phi(u_1,u_1) + \Psi^0(w_1,u_1;v-u_1) \\ &\qquad \geq \langle f, v-u_1\rangle \quad \forall\, v \in K_V, \qquad (26)\\ &b(u_1, q-p_1) \leq 0 \quad \forall\, q \in K_Q \qquad (27)\end{aligned}$$

and

$$\begin{aligned} &\langle Au_2, v-u_2\rangle + b(v-u_2,p_2) + \Phi(u_2,v) - \Phi(u_2,u_2) + \Psi^0(w_2,u_2;v-u_2) \\ &\qquad \geq \langle f, v-u_2\rangle \quad \forall\, v \in K_V, \qquad (28)\\ &b(u_2, q-p_2) \leq 0 \quad \forall\, q \in K_Q. \qquad (29)\end{aligned}$$

We take $v = u_2$ in (26), $v = u_1$ in (28), and add the two resulting inequalities to obtain

$$\begin{aligned} \langle Au_1 - Au_2, u_1 - u_2\rangle \leq\; & -b(u_1-u_2, p_1-p_2) \\ & + \Psi^0(w_1,u_1;u_2-u_1) + \Psi^0(w_2,u_2;u_1-u_2) \\ & + \Phi(u_1,u_2) - \Phi(u_1,u_1) + \Phi(u_2,u_1) - \Phi(u_2,u_2).\end{aligned}$$

From (27) and (29),

$$\begin{aligned} b(u_1, p_2-p_1) &\leq 0,\\ b(u_2, p_1-p_2) &\leq 0.\end{aligned}$$

Hence,

$$-b(u_1 - u_2, p_1 - p_2) = b(u_1, p_2 - p_1) + b(u_2, p_1 - p_2) \leq 0. \tag{30}$$

Then,

$$\begin{aligned}\langle Au_1 - Au_2, u_1 - u_2\rangle \leq \Psi^0(w_1, u_1; u_2 - u_1) + \Psi^0(w_2, u_2; u_1 - u_2)\\ + \Phi(u_1, u_2) - \Phi(u_1, u_1) + \Phi(u_2, u_1) - \Phi(u_2, u_2).\end{aligned} \tag{31}$$

Thus, by making use of the assumptions $H(A)$, $H(\Psi)_2$ and $H(\Phi)$, we derive from (31) that

$$\begin{aligned}m_A\|u_1 - u_2\|_V^2 \leq \alpha_{\Psi,1}\|u_1 - u_2\|_V^2 + \alpha_{\Psi,2}\|w_1 - w_2\|_V\|u_1 - u_2\|_V\\ + \alpha_\Phi\|u_1 - u_2\|_V^2.\end{aligned}$$

Then,

$$\|u_1 - u_2\|_V \leq \lambda\,\|w_1 - w_2\|_V, \quad \lambda := \frac{\alpha_{\Psi,2}}{m_A - \alpha_{\Psi,1} - \alpha_\Phi}.$$

Note that the smallness assumption (23) implies that $\lambda < 1$. So the mapping $P\colon K_V \to K_V$ is contractive. By the Banach fixed-point theorem ([1, Section 5.1]), the mapping P has a unique fixed-point $u \in K_V$. From $u = P(u)$, it is easy to see that for some $p \in K_Q$, the pair $(u, p) \in K_V \times K_Q$ is a solution of Problem 5. Moreover, the first component u of the solution of Problem 5 is unique due to the uniqueness of the fixed-point of the mapping P.

Now we prove the Lipschitz continuous dependence property of the solution on the right side. Let $f_1, f_2 \in V^*$, and let (u_1, p_1) and (u_2, p_2) be corresponding solutions of Problem 5. Then,

$$\begin{aligned}&\langle Au_1, v - u_1\rangle + b(v - u_1, p_1) + \Phi(u_1, v) - \Phi(u_1, u_1) + \Psi^0(u_1, u_1; v - u_1)\\ &\qquad \geq \langle f_1, v - u_1\rangle \quad \forall\, v \in K_V, \end{aligned}\tag{32}$$

$$b(u_1, q - p_1) \leq 0 \quad \forall\, q \in K_Q \tag{33}$$

and

$$\begin{aligned}&\langle Au_2, v - u_2\rangle + b(v - u_2, p_2) + \Phi(u_2, v) - \Phi(u_2, u_2) + \Psi^0(u_2, u_2; v - u_2)\\ &\qquad \geq \langle f_2, v - u_2\rangle \quad \forall\, v \in K_V, \end{aligned}\tag{34}$$

$$b(u_2, q - p_2) \leq 0 \quad \forall\, q \in K_Q. \tag{35}$$

We take $v = u_2$ in (32), $v = u_1$ in (34), and add the two resulting inequalities to obtain

$$
\begin{aligned}
\langle Au_1 - Au_2, u_1 - u_2\rangle \le\ & -b(u_1 - u_2, p_1 - p_2) \\
& + \Psi^0(u_1, u_1; u_2 - u_1) + \Psi^0(u_2, u_2; u_1 - u_2) \\
& + \Phi(u_1, u_2) - \Phi(u_1, u_1) + \Phi(u_2, u_1) - \Phi(u_2, u_2) \\
& + \langle f_1 - f_2, u_1 - u_2\rangle. \qquad (36)
\end{aligned}
$$

Similar to (30), we deduce from (33) and (35) that

$$-b(u_1 - u_2, p_1 - p_2) = b(u_1, p_2 - p_1) + b(u_2, p_1 - p_2) \le 0.$$

Then, from (36) with the use of $H(A)$, $H(\Psi)_2$ and $H(\Phi)_2$,

$$
\begin{aligned}
m_A\|u_1 - u_2\|_V^2 \le\ & (\alpha_{\Psi,1} + \alpha_{\Psi,2})\,\|u_1 - u_2\|_V^2 + \alpha_\Phi\|u_1 - u_2\|_V^2 \\
& + \|f_1 - f_2\|_{V^*}\|u_1 - u_2\|_V.
\end{aligned}
$$

Therefore,

$$\|u_1 - u_2\|_V \le \frac{1}{m_A - (\alpha_\Phi + \alpha_{\Psi,1} + \alpha_{\Psi,2})}\,\|f_1 - f_2\|_{V^*},$$

implying the Lipschitz continuous dependence of u on f. ■

For applications in contact and fluid mechanics, the term $\Psi^0(u, u; v - u)$ in (20) is typically replaced by the integral of $\psi^0(\gamma_1 u, \gamma_2 u; \gamma_2 v - \gamma_2 u)$ over a set Δ, where $\gamma_1 \in \mathcal{L}(V; L^2(\Delta; \mathbb{R}^{m_1}))$, $\gamma_2 \in \mathcal{L}(V; L^2(\Delta; \mathbb{R}^{m_2}))$ for some positive integers m_1 and m_2, and $\psi\colon \Delta \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ is a given function that is locally Lipschitz continuous with respect to its last argument. Thus, we consider the following variant of Problem 5.

Problem 7 *Find $(u, p) \in K_V \times K_Q$ such that*

$$
\begin{aligned}
& \langle Au, v - u\rangle + b(v - u, p) + \Phi(u, v) - \Phi(u, u) + I_\Delta(\psi^0(\gamma_1 u, \gamma_2 u; \gamma_2 v - \gamma_2 u)) \\
& \quad \ge \langle f, v - u\rangle \quad \forall\, v \in K_V, \qquad (37) \\
& b(u, q - p) \le 0 \quad \forall\, q \in K_Q. \qquad (38)
\end{aligned}
$$

On the function ψ, we introduce the following assumption.

- $H(\psi)_2$: For $i = 1, 2$, $\gamma_i \in \mathcal{L}(V; L^2(\Delta; \mathbb{R}^{m_i}))$, m_1 and m_2 being positive integers; $\psi\colon \Delta \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$; $\psi(\cdot, z_1, z_2)$ is measurable on Δ for all $z_1 \in \mathbb{R}^{m_1}$ and $z_2 \in \mathbb{R}^{m_2}$; there exists $z_0 \in L^2(\Delta; \mathbb{R}^{m_2})$ such that for any $z \in L^2(\Delta;\mathbb{R}^{m_1})$, $\psi(\cdot, z(\cdot), z_0(\cdot)) \in L^1(\Delta)$; $\psi(\boldsymbol{x}, z_1, \cdot)$ is locally Lipschitz continuous on $\mathbb{R}^{m_2}$ for any $z_1 \in \mathbb{R}^{m_1}$ and a.e. $\boldsymbol{x} \in \Delta$; and for some non-negative constants c and α_{ψ_1}, α_{ψ_2}, a.e. on Δ,

$$|\partial\psi(\cdot, z_1, z_2)| \le c\,(1 + |z_1|_{\mathbb{R}^{m_1}} + |z_2|_{\mathbb{R}^{m_2}}) \quad \forall\, z_1 \in \mathbb{R}^{m_1},\ z_2 \in \mathbb{R}^{m_2}, \tag{39}$$

$$\begin{aligned}&\psi^0(w_1, z_1; z_2 - z_1) + \psi^0(w_2, z_2; z_1 - z_2)\\ &\quad \le \alpha_{\psi,1}|z_1 - z_2|^2_{\mathbb{R}^{m_2}} + \alpha_{\psi,2}|w_1 - w_2|_{\mathbb{R}^{m_1}}|z_1 - z_2|_{\mathbb{R}^{m_2}}\\ &\qquad \forall\, w_1, w_2 \in \mathbb{R}^{m_1},\ z_1, z_2 \in \mathbb{R}^{m_2}.\end{aligned} \tag{40}$$

For the well-posedness of Problem 7, we have the following analogue of Theorem 6.

Theorem 8 *Assume $H(K_V)$, $H(K_Q)$, $H(A)$, $H(b)$, $H(\Phi)_2$, $H(\psi)_2$, $H(f)$, and*

$$\alpha_\Phi + \alpha_{\psi,1}\|\gamma_2\|^2 + \alpha_{\psi,2}\|\gamma_1\|\,\|\gamma_2\| < m_A. \tag{41}$$

Moreover, assume either

K_Q is bounded,

or

K_Q is unbounded; K_V is a subspace of V; for any $u \in V$, $\Phi(u,\cdot)\colon V \to \mathbb{R}$ is Lipschitz continuous on K_V; there exists a constant c such that

$$\left|\psi^0(w, u; v)\right| \le c\,(1 + |w|_{\mathbb{R}^{m_1}} + |u|_{\mathbb{R}^{m_2}})\,|v|_{\mathbb{R}^{m_2}} \quad \forall\, w \in \mathbb{R}^{m_1},\ u, v \in \mathbb{R}^{m_2};$$

and the inf-sup condition (12) *holds.*

Then, Problem 7 has a solution $(u, p) \in K_V \times K_Q$ and the first component u of a solution is unique. Moreover, the solution component u depends Lipschitz continuously on f.

Proof. Let

$$\Psi(w, u) = I_\Delta(\psi(\gamma_1 w, \gamma_2 u)), \quad w, u \in V.$$

Then by [14, Theorem 3.47], $\Psi\colon V \times V \to \mathbb{R}$ is locally Lipschitz continuous with respect to its second argument, and

$$\Psi^0(w, u; v) \le I_\Delta(\psi^0(\gamma_1 u, \gamma_2 u; \gamma_2 v)). \tag{42}$$

So

$$
\begin{aligned}
&\Psi^0(w_1, v_1; v_2 - v_1) + \Psi^0(w_2, v_2; v_1 - v_2) \\
&\qquad \le I_\Delta(\psi^0(\gamma_1 w_1, \gamma_2 v_1; \gamma_2 v_2 - \gamma_2 v_1) + \psi^0(\gamma_1 w_2, \gamma_2 v_2; \gamma_2 v_1 - \gamma_2 v_2)) \\
&\qquad \le I_\Delta(\alpha_{\psi,1} |\gamma_2(v_1 - v_2)|^2_{\mathbb{R}^{m_2}} + \alpha_{\psi,2} |\gamma_1(w_1 - w_2)|_{\mathbb{R}^{m_1}} |\gamma_2(v_1 - v_2)|_{\mathbb{R}^{m_2}}) \\
&\qquad \le \alpha_{\psi,1} \|\gamma_2\|^2 \|v_1 - v_2\|^2_V + \alpha_{\psi,2} \|\gamma_1\| \, \|\gamma_2\| \, \|w_1 - w_2\|_V \|v_1 - v_2\|_V.
\end{aligned}
$$

Thus, $H(\Psi)_2$ is valid with $\alpha_{\Psi,1} = \alpha_{\psi,1} \|\gamma_2\|^2$ and $\alpha_{\Psi,2} = \alpha_{\psi,2} \|\gamma_1\| \, \|\gamma_2\|$. Also,

$$
\begin{aligned}
\Psi^0(w, u; v) &\le I_\Delta(c\,(1 + |\gamma_1 w|_{\mathbb{R}^{m_1}} + |\gamma_2 u|_{\mathbb{R}^{m_2}})\, |\gamma_2 v|_{\mathbb{R}^{m_2}}) \\
&\le c\,(1 + \|w\|_V + \|u\|_V)\, \|v\|_V.
\end{aligned}
$$

Applying Theorem 6, we know that Problem 5 has a solution $(u, p) \in K_V \times K_Q$ and the first component u of a solution is unique. By (42), $(u, p) \in K_V \times K_Q$ is also a solution of Problem 7.

Uniqueness of the solution component u, as well as the Lipschitz continuous dependence of u on f, can be proved similarly as that in the proof of Theorem 6. ■

The fixed-point argument in the proof of Theorems 6 and 8 naturally leads to a convergent iterative procedure for approximating the solutions of Problems 5 and 7 by solving a sequence of problems in the form of Problem 1 or Problem 2. Take Problem 7 as an example. We can introduce the following iterative method:

Initialization. Choose an arbitrary $u_0 \in K_V$.

Iteration. For $n \ge 1$, find $(u_n, p_n) \in K_V \times K_Q$ such that

$$
\begin{aligned}
&\langle A u_n, v - u_n \rangle + b(v - u_n, p_n) + \Phi(u_n, v) - \Phi(u_n, u_n) \\
&\qquad + I_\Delta(\psi^0(\gamma_1 u_{n-1}, \gamma_2 u_n; \gamma_2 v - \gamma_2 u_n)) \ge \langle f, v - u_n \rangle \quad \forall\, v \in K_V, \qquad (43) \\
&b(u_n, q - p_n) \le 0 \quad \forall\, q \in K_Q. \qquad (44)
\end{aligned}
$$

Then under the assumptions stated in Theorem 8, we have the convergence ([1, Section 5.1]):

$$
\lim_{n\to\infty} \|u_n - u\|_V = 0,
$$

where u is the first solution component of Problem 7.

4 Application in study of a Bingham type fluid flow

In this section we consider a mixed hemivariational-variational inequality that models a Bingham type fluid flow under non-smooth non-monotone slip boundary condition. Such a problem is studied in [20] in which the existence of a solution is shown by applying an abstract surjectivity result for pseudomonotone operators, and the solution existence result concerns about the unknown velocity field only. The results developed in the previous section allows us to conclude the solution existence for both the velocity field $\boldsymbol{u}$ and the pressure field p, and the uniqueness of $\boldsymbol{u}$.

Let $\Omega \subset \mathbb{R}^d$ be the domain occupied by the fluid, $d \le 3$. Assume Ω is a Lipschitz domain so that the unit outward normal vector $\boldsymbol{\nu} = (\nu_1, \cdots, \nu_d)^T$ exists a.e. on the boundary $\partial\Omega$. For an $\mathbb{R}^d$-valued function $\boldsymbol{u}$ on the boundary, its normal and tangential components are $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$ and $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}$, respectively. Denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. For an $\mathbb{S}^d$-valued function $\boldsymbol{\sigma}$ on the boundary, we call $\sigma_\nu = \boldsymbol{\nu} \cdot \boldsymbol{\sigma}\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$ the normal and tangential components of $\boldsymbol{\sigma}$ on the boundary. We have the identities:

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_\nu v_\nu + \boldsymbol{u}_\tau \cdot \boldsymbol{v}_\tau, \tag{45}$$

$$(\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{v} = \sigma_\nu v_\nu + \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau. \tag{46}$$

We adopt the summation convention over a repeated index. The indices i and j range between 1 and d. The canonical inner products and norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \quad |\boldsymbol{v}| \equiv |\boldsymbol{v}|_{\mathbb{R}^d} = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2} \quad \text{for all } \boldsymbol{u} = (u_i),\ \boldsymbol{v} = (v_i) \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad |\boldsymbol{\sigma}| \equiv |\boldsymbol{\sigma}|_{\mathbb{S}^d} = (\boldsymbol{\sigma} : \boldsymbol{\sigma})^{1/2} \quad \text{for all } \boldsymbol{\sigma} = (\sigma_{ij}),\ \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d.$$

Denote by $\boldsymbol{u}$ the velocity field of the fluid. The quantity $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)/2$ is the deformation rate tensor or the rate of deformation tensor. In the Bingham type fluid considered in this paper, the constitutive relation is of the form

$$\boldsymbol{S} \in 2\,\mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) + \partial(g\,|\boldsymbol{\varepsilon}(\boldsymbol{u})|) \quad \text{in } \Omega \tag{47}$$

between the extra stress tensor $\boldsymbol{S} = (S_{ij})\colon \Omega \to \mathbb{S}^d$ and the velocity strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u})$. Here $\mu > 0$ is the viscosity coefficient, $g \ge 0$ is the plasticity threshold, and ∂ stands for the convex subdifferential. The relation (47) can be equivalently written as

$$\boldsymbol{S} = 2\,\mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\xi}, \quad \boldsymbol{\xi} \in \partial(g\,|\boldsymbol{\varepsilon}(\boldsymbol{u})|) \quad \text{in } \Omega. \tag{48}$$

By the definition of the convex subdifferential, $\boldsymbol{\xi} \in \partial(g\,|\boldsymbol{\varepsilon}(\boldsymbol{u})|)$ is equivalent to

$$|\boldsymbol{\xi}| \le g, \quad \boldsymbol{\xi} = g\,\frac{\boldsymbol{\varepsilon}(\boldsymbol{u})}{|\boldsymbol{\varepsilon}(\boldsymbol{u})|} \text{ if } \boldsymbol{\varepsilon}(\boldsymbol{u}) \neq \boldsymbol{0}.$$

For $g = 0$ and $\mu(\cdot)$ a positive constant, the constitutive relation (47) reduces to that for a Newtonian fluid. The total stress tensor is

$$\boldsymbol{\sigma} = \boldsymbol{S}(\boldsymbol{\varepsilon}(\boldsymbol{u})) - p\,\boldsymbol{I}, \tag{49}$$

where p is the pressure variable, and $\boldsymbol{I}$ is the $d \times d$ identity matrix.

The Stokes equations for the fluid flow is

$$-\mathrm{div}\boldsymbol{S} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{50}$$

where the divergence of $\boldsymbol{S}$ is a vector-valued function $\mathrm{div}\boldsymbol{S} = (S_{ij,j})\colon \Omega \to \mathbb{R}^d$, $\boldsymbol{f}$ is a given density function. The fluid is assumed to be incompressible,

$$\mathrm{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{51}$$

where $\mathrm{div}\boldsymbol{u} = u_{i,i}$ is a scalar-valued function. To describe the boundary conditions, we split the boundary Γ as

$$\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_S}$$

such that Γ_D and Γ_S are non-empty, mutually disjoint open subsets of Γ. We specify the homogeneous Dirichlet boundary condition on Γ_D:

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_D; \tag{52}$$

and a no-leak slip boundary condition of friction type on Γ_S:

$$u_\nu = 0, \quad -\boldsymbol{\sigma}_\tau \in k_\tau(|\boldsymbol{u}_\tau|)\,\partial\psi_\tau(\boldsymbol{u}_\tau) \quad \text{on } \Gamma_S. \tag{53}$$

The classical formulation of the problem is to find a velocity field $\boldsymbol{u}$ and a pressure field p such that (47) and (50)–(53) are satisfied. The problem will be studied in a mixed weak formulation. For convenience, we will use the same symbols for the function spaces, operators and functionals as in the previous sections for the abstract problems, e.g., V, Q, A, b, Φ, ψ. We will use the space

$$V = \left\{\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D,\ v_\nu = 0 \text{ on } \Gamma_S\right\} \tag{54}$$

for the velocity variable. The trace of $\boldsymbol{v} \in V$ on the boundary is denoted by the same symbol $\boldsymbol{v}$. Since $|\Gamma_D| > 0$, Korn's inequality holds (cf. [16, p. 79]): for some constant $c > 0$,

$$\|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^d)} \leq c\,\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{L^2(\Omega;\mathbb{S}^d)} \quad \forall\, \boldsymbol{v} \in V.$$

Consequently, the expression $\|\boldsymbol{v}\|_V = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{L^2(\Omega;\mathbb{S}^d)}$ defines a norm on V, which is equivalent to the standard norm $\|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^d)}$ on V.

Let $\lambda_0 > 0$ be the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in V, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx = \lambda \int_{\Gamma_S} \boldsymbol{u}_\tau \cdot \boldsymbol{v}_\tau\,ds \quad \forall\, \boldsymbol{v} \in V.$$

Then we have the trace inequality

$$\|\boldsymbol{v}_\tau\|_{L^2(\Gamma_\tau;\mathbb{R}^d)} \leq \lambda_0^{-1/2} \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V. \tag{55}$$

The space for the pressure variable is

$$Q = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) \mid I_\Omega(q) = 0 \right\}. \tag{56}$$

Recall that $I_\Omega(q)$ stands for the integral of q over Ω.

To derive the weak formulation, assume there exist $\boldsymbol{u} \in V$ and $p \in Q$, sufficiently smooth, that satisfy the equations (47) and (50)–(53). Let $\boldsymbol{v} \in V$ be an arbitrary function, also sufficiently smooth. We multiply the equation (50) by $\boldsymbol{v}$ and integrate over Ω:

$$-\int_\Omega \mathrm{div}\boldsymbol{S} \cdot \boldsymbol{v}\,dx + \int_\Omega (\nabla p) \cdot \boldsymbol{v}\,dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,dx. \tag{57}$$

Integrate by part,

$$\begin{aligned} \int_\Omega \mathrm{div}\boldsymbol{S} \cdot \boldsymbol{v}\,dx &= \int_\Gamma (\boldsymbol{S}\boldsymbol{\nu}) \cdot \boldsymbol{v}\,ds - \int_\Omega \boldsymbol{S} : \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx, \\ \int_\Omega (\nabla p) \cdot \boldsymbol{v}\,dx &= \int_\Gamma p\,\boldsymbol{v} \cdot \boldsymbol{\nu}\,ds - \int_\Omega p\,\mathrm{div}\boldsymbol{v}\,dx. \end{aligned}$$

Then from (57),

$$\int_\Gamma -(\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{v}\,ds + \int_\Omega [\boldsymbol{S} : \boldsymbol{\varepsilon}(\boldsymbol{v}) - p\,\mathrm{div}\boldsymbol{v}]\,dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,dx. \tag{58}$$

Applying the homogeneous boundary value conditions from (52) and (53), with the use of (45) amd (46), we derive from (58) that

$$\int_{\Gamma_S} -\boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau ds + \int_\Omega [\boldsymbol{S} : \boldsymbol{\varepsilon}(\boldsymbol{v}) - p\,\mathrm{div}\boldsymbol{v}]\,dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,dx. \tag{59}$$

Then apply the second part of the boundary condition from (53),

$$\int_{\Gamma_S} -\boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau ds \le \int_{\Gamma_S} k_\tau(|\boldsymbol{u}_\tau|)\, \psi_\tau^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\, ds.$$

So from (59) with $\boldsymbol{v}$ replaced by $(\boldsymbol{v}-\boldsymbol{u})$, we have

$$\int_\Omega \boldsymbol{S} : \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})\, dx - \int_\Omega p\, \mathrm{div}(\boldsymbol{v}-\boldsymbol{u})\, dx + \int_{\Gamma_S} k_\tau(|\boldsymbol{u}_\tau|)\, \psi_\tau^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau - \boldsymbol{u}_\tau)\, ds \\ \ge \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v}-\boldsymbol{u})\, dx.$$

Now make use of (48),

$$\begin{aligned}\int_\Omega \boldsymbol{S} : \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})\, dx \\ &= 2\,\mu \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})\, dx + \int_\Omega \boldsymbol{\xi} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})\, dx \\ &\le 2\,\mu \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})\, dx + \int_\Omega g\,(|\boldsymbol{\varepsilon}(\boldsymbol{v})| - |\boldsymbol{\varepsilon}(\boldsymbol{u})|)\, dx.\end{aligned}$$

Summarizing, we arrive at the following mixed hemivariational-variational inequality: find $\boldsymbol{u} \in V$ and $p \in Q$ such that

$$\langle A\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}\rangle + b(\boldsymbol{v}-\boldsymbol{u}, p) + \Phi(\boldsymbol{v}) - \Phi(\boldsymbol{u}) + I_{\Gamma_S}(k_\tau(|\boldsymbol{u}_\tau|)\, \psi_\tau^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau - \boldsymbol{u}_\tau)) \\ \ge \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v}-\boldsymbol{u})\, dx \quad \forall\, \boldsymbol{v} \in V, \tag{60}$$

$$b(\boldsymbol{u}, q) = 0 \quad \forall\, q \in Q, \tag{61}$$

where the operator $A\colon V \to V^*$, the bilinear form $b\colon V \times Q \to \mathbb{R}$, and the functional $\Phi\colon V \to \mathbb{R}$ are defined by

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = 2\,\mu \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V, \tag{62}$$

$$b(\boldsymbol{v}, q) = \int_\Omega q\, \mathrm{div}\boldsymbol{v}\, dx \quad \forall\, \boldsymbol{v} \in V,\ q \in Q, \tag{63}$$

$$\Phi(\boldsymbol{v}) = \int_\Omega g\, |\boldsymbol{\varepsilon}(\boldsymbol{v})|\, dx \quad \forall\, \boldsymbol{v} \in V. \tag{64}$$

Note that the operator $A\colon V \to V^*$ is Lipschitz continuous and strongly monotone, with the respective constants $M_A = 2\,\mu$ and $m_A = 2\,\mu$. The bilinear form $b\colon V \times Q \to \mathbb{R}$ is bilinear and bounded, and the inf-sup condition is valid ([19]):

$$\sup_{\boldsymbol{v} \in V_0} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_V} \ge \alpha_b \|q\|_Q \quad \forall\, q \in Q, \tag{65}$$

where $V_0 = H_0^1(\Omega)^d$. For the functional defined by (64), $\Phi(\boldsymbol{u},\boldsymbol{v}) = \Phi(\boldsymbol{v})$ depends on only one argument; obviously, $\Phi\colon V \to \mathbb{R}$ is convex and continuous, and (7) holds with $\alpha_\Phi = 0$. We now introduce conditions on k_τ and ψ_τ.

- $H(k_\tau)$: $k_\tau\colon \Gamma_S \times \mathbb{R}_+ \to \mathbb{R}$; $k_\tau(\cdot, r)$ is measurable on Γ_τ for all $r \in \mathbb{R}_+$; $k_\tau(\boldsymbol{x},\cdot)$ is Lipschitz continuous with a Lipschitz constant L_{k_τ} for a.e. $\boldsymbol{x} \in \Gamma_S$; and there exist constants $0 < k_{\tau,0} \le k_{\tau,1}$ such that $k_{\tau,0} \le k_\tau(\boldsymbol{x}, r) \le k_{\tau,1}$ for all $r \in \mathbb{R}_+$ and a.e. $\boldsymbol{x} \in \Gamma_S$.

- $H(\psi_\tau)$: $\psi_\tau\colon \Gamma_S \times \mathbb{R}^d \to \mathbb{R}$; $\psi_\tau(\cdot,\boldsymbol{z})$ is measurable on Γ_S for all $\boldsymbol{z} \in \mathbb{R}^d$; there exists $\boldsymbol{z}_0 \in L^2(\Gamma_S;\mathbb{R}^d)$ such that $\psi_\tau(\cdot,\boldsymbol{z}_0(\cdot)) \in L^1(\Gamma_S)$; $\psi_\tau(\boldsymbol{x},\cdot)$ is Lipschitz continuous with a Lipschitz constant L_{ψ_τ} for a.e. $\boldsymbol{x} \in \Gamma_S$; and there exists a constant $m_{\psi_\tau} \ge 0$ such that
$$\psi_\tau^0(\boldsymbol{x},\boldsymbol{z}_1;\boldsymbol{z}_2-\boldsymbol{z}_1) + \psi_\tau^0(\boldsymbol{x},\boldsymbol{z}_2;\boldsymbol{z}_1-\boldsymbol{z}_2) \le m_{\psi_\tau}|\boldsymbol{z}_1-\boldsymbol{z}_2|^2_{\mathbb{R}^d}$$
$$\forall\, \boldsymbol{z}_1,\boldsymbol{z}_2 \in \mathbb{R}^d,\ \text{a.e. } \boldsymbol{x} \in \Gamma_S. \tag{66}$$

Since $\psi_\tau(\boldsymbol{x},\cdot)$ is Lipschitz continuous with a Lipschitz constant L_{ψ_τ}, we have
$$|\psi_\tau^0(\boldsymbol{x},\boldsymbol{z}_1;\boldsymbol{z}_2)| \le L_{\psi_\tau}|\boldsymbol{z}_2|_{\mathbb{R}^d} \quad \forall\, \boldsymbol{z}_1,\boldsymbol{z}_2 \in \mathbb{R}^d,\ \text{a.e. } \boldsymbol{x} \in \Gamma_S. \tag{67}$$

Theorem 9 *Assume $H(k_\tau)$, $H(\psi_\tau)$, and*
$$(L_{k_\tau}L_{\psi_\tau} + k_{\tau,1}m_{\psi_\tau})\,\lambda_0^{-1} < 2\,\mu. \tag{68}$$
Then there is a solution $(\boldsymbol{u},p) \in V\times Q$ to the problem (60)–(61); the solution component $\boldsymbol{u}$ is unique and it depends Lipschitz continuously on $\boldsymbol{f}$.

Proof. The problem defined by (60)–(61) is of the form Problem 7 with the space V defined in (54), the space Q defined in (56), $K_V = V$, $K_Q = Q$, $\Delta = \Gamma_S$, $m_1 = m_2 = d$, and γ_1 and γ_2 are the trace operator of the tangential component on Γ_S:
$$\gamma_1\boldsymbol{v} = \gamma_2\boldsymbol{v} = \boldsymbol{v}_\tau|_{\Gamma_\tau} \quad \forall\, \boldsymbol{v} \in V.$$

We have verified most of the conditions stated in Theorem 8, and let us verify the rest of the conditions. We have
$$\|\gamma_1\| = \|\gamma_2\| = \lambda_0^{-1/2}.$$
The condition (40) is verified as follows:
$$\psi(\boldsymbol{w},\boldsymbol{z}) = k_\tau(|\boldsymbol{w}|)\,\psi_\tau(\boldsymbol{z}),$$
$$\psi^0(\boldsymbol{w},\boldsymbol{z}_1;\boldsymbol{z}_2) = k_\tau(|\boldsymbol{w}|)\,\psi_\tau^0(\boldsymbol{z}_1;\boldsymbol{z}_2).$$

Write

$$\begin{aligned}
&\psi^0(\boldsymbol{w}_1, \boldsymbol{z}_1; \boldsymbol{z}_2 - \boldsymbol{z}_1) + \psi^0(\boldsymbol{w}_2, \boldsymbol{z}_2; \boldsymbol{z}_1 - \boldsymbol{z}_2) \\
&\quad = k_\tau(|\boldsymbol{w}_1|)\, \psi_\tau^0(\boldsymbol{z}_1; \boldsymbol{z}_2 - \boldsymbol{z}_1) + k_\tau(|\boldsymbol{w}_2|)\, \psi_\tau^0(\boldsymbol{z}_2; \boldsymbol{z}_1 - \boldsymbol{z}_2) \\
&\quad = k_\tau(|\boldsymbol{w}_1|) \left[\psi_\tau^0(\boldsymbol{z}_1; \boldsymbol{z}_2 - \boldsymbol{z}_1) + \psi_\tau^0(\boldsymbol{z}_2; \boldsymbol{z}_1 - \boldsymbol{z}_2)\right] \\
&\qquad + \left[k_\tau(|\boldsymbol{w}_2|) - k_\tau(|\boldsymbol{w}_1|)\right] \psi_\tau^0(\boldsymbol{z}_2; \boldsymbol{z}_1 - \boldsymbol{z}_2),
\end{aligned}$$

and bound the expression by

$$k_{\tau,1} m_{\psi_\tau} |\boldsymbol{z}_1 - \boldsymbol{z}_2|_{\mathbb{R}^d}^2 + L_{k_\tau} L_{\psi_\tau} |\boldsymbol{w}_1 - \boldsymbol{w}_2|_{\mathbb{R}^d} |\boldsymbol{z}_1 - \boldsymbol{z}_2|_{\mathbb{R}^d}.$$

So (40) is satisfied with $\alpha_{\psi_1,1} = k_{\tau,1} m_{k_\tau}$ and $\alpha_{\psi_1,2} = L_{k_\tau} L_{\psi_\tau}$. Hence, the smallness condition (41) for the problem (60)–(61) is (68).

By applying Theorem 8, we know that the problem (60)–(61) has a solution $(\boldsymbol{u}, p) \in V \times Q$, $\boldsymbol{u}$ being unique and depending Lipschitz continuously on $\boldsymbol{f}$. ■

In general, it seems difficult to have/prove the uniqueness of the solution component p. We explore the uniqueness of p next.

Proposition 1 *Keep the assumptions stated in Theorem 9. Let* $(\boldsymbol{u}, p) \in V \times Q$ *be a solution of the problem* (60)–(61)*. Suppose* $|\boldsymbol{\varepsilon}(\boldsymbol{u})| \ge c_0$ *for some constant* $c_0 > 0$*, a.e. on* Ω*. Then the solution component* $p \in Q$ *is unique.*

Proof. Suppose $(\boldsymbol{u}, p_1), (\boldsymbol{u}, p_2) \in V \times Q$ both solve the problem. In (60) with p replaced by p_1, we substitute $\boldsymbol{v} \in V$ by $\boldsymbol{u} \pm \boldsymbol{v}$, $\boldsymbol{v} \in V_0$, to obtain

$$\begin{aligned}
\langle A\boldsymbol{u}, \boldsymbol{v}\rangle + b(\boldsymbol{v}, p_1) + \Phi(\boldsymbol{u} + \boldsymbol{v}) - \Phi(\boldsymbol{u}) &\ge \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx, \\
-\langle A\boldsymbol{u}, \boldsymbol{v}\rangle - b(\boldsymbol{v}, p_1) + \Phi(\boldsymbol{u} - \boldsymbol{v}) - \Phi(\boldsymbol{u}) &\ge -\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx.
\end{aligned}$$

Then,

$$\begin{aligned}
b(\boldsymbol{v}, p_1) &\ge \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx + \Phi(\boldsymbol{u}) - \Phi(\boldsymbol{u} + \boldsymbol{v}) - \langle A\boldsymbol{u}, \boldsymbol{v}\rangle, \\
b(\boldsymbol{v}, p_1) &\le \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx + \Phi(\boldsymbol{u} - \boldsymbol{v}) - \Phi(\boldsymbol{u}) - \langle A\boldsymbol{u}, \boldsymbol{v}\rangle.
\end{aligned}$$

The same inequalities hold with p_1 replaced by p_2. Thus, for the term

$$b(\boldsymbol{v}, p_1 - p_2) = b(\boldsymbol{v}, p_1) - b(\boldsymbol{v}, p_2),$$

we find the bound

$$|b(\boldsymbol{v}, p_1 - p_2)| \le \Phi(\boldsymbol{u}+\boldsymbol{v}) + \Phi(\boldsymbol{u}-\boldsymbol{v}) - 2\,\Phi(\boldsymbol{u}) \quad \forall\, \boldsymbol{v} \in V_0. \tag{69}$$

For any $\lambda > 0$ and any $\boldsymbol{v} \in C_0^\infty(\Omega)^d$, we replace $\boldsymbol{v}$ by $\lambda\,\boldsymbol{v}$ in (69) to obtain

$$|b(\boldsymbol{v}, p_1 - p_2)| \le \frac{g}{\lambda} \int_\Omega \left(|\boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda\,\boldsymbol{\varepsilon}(\boldsymbol{v})| + |\boldsymbol{\varepsilon}(\boldsymbol{u}) - \lambda\,\boldsymbol{\varepsilon}(\boldsymbol{v})| - 2\,|\boldsymbol{\varepsilon}(\boldsymbol{u})|\right) dx. \tag{70}$$

For $\lambda > 0$ small, we have

$$\begin{aligned} |\boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda\,\boldsymbol{\varepsilon}(\boldsymbol{v})| &= |\boldsymbol{\varepsilon}(\boldsymbol{u})| + \lambda\,\frac{\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})}{|\boldsymbol{\varepsilon}(\boldsymbol{u})|} + O(\lambda^2), \\ |\boldsymbol{\varepsilon}(\boldsymbol{u}) - \lambda\,\boldsymbol{\varepsilon}(\boldsymbol{v})| &= |\boldsymbol{\varepsilon}(\boldsymbol{u})| - \lambda\,\frac{\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})}{|\boldsymbol{\varepsilon}(\boldsymbol{u})|} + O(\lambda^2). \end{aligned}$$

Then from (70),

$$|b(\boldsymbol{v}, p_1 - p_2)| \le O(\lambda) \quad \forall\, \boldsymbol{v} \in C_0^\infty(\Omega)^d.$$

We now take the limit $\lambda \to 0+$ to obtain

$$b(\boldsymbol{v}, p_1 - p_2) = 0 \quad \forall\, \boldsymbol{v} \in C_0^\infty(\Omega)^d.$$

Since $C_0^\infty(\Omega)^d$ is dense in V_0, we deduce that

$$b(\boldsymbol{v}, p_1 - p_2) = 0 \quad \forall\, \boldsymbol{v} \in V_0.$$

Apply the inequality (65),

$$\alpha_b \|p_1 - p_2\|_Q \le \sup_{\boldsymbol{v}\in V_0} \frac{b(\boldsymbol{v}, p_1 - p_2)}{\|\boldsymbol{v}\|_V} = 0.$$

Therefore, $p_1 = p_2$, i.e., the solution component p is unique. ∎

We can introduce the following iteration procedure:
Initialization. Choose an arbitrary $\boldsymbol{u}_0 \in V$.
Iteration. For $n \ge 1$, find $(\boldsymbol{u}_n, p_n) \in V \times Q$ such that

$$\begin{aligned} &\langle A\boldsymbol{u}_n, \boldsymbol{v} - \boldsymbol{u}_n\rangle + b(\boldsymbol{v} - \boldsymbol{u}_n, p_n) + \Phi(\boldsymbol{v}) - \Phi(\boldsymbol{u}_n) \\ &\qquad + I_{\Gamma_S}(k_\tau(|\boldsymbol{u}_{n-1,\tau}|)\,\psi_\tau^0(\boldsymbol{u}_{n,\tau}; \boldsymbol{v}_\tau - \boldsymbol{u}_{n,\tau})) \ge \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v} - \boldsymbol{u}_n)\,dx \quad \forall\, \boldsymbol{v} \in V, \\ &b(\boldsymbol{u}_n, q) = 0 \quad \forall\, q \in Q. \end{aligned}$$

Then, under the assumptions stated in Theorem 9, we have the convergence

$$\boldsymbol{u}_n \to \boldsymbol{u} \quad \text{in } V.$$

References

[1] K. Atkinson and W. Han. *Theoretical Numerical Analysis: A Functional Analysis Framework*, third edition. Springer, New York, 2009.

[2] Y. Bai, S. Migórski, and S. Zeng. Well-posedness of a class of generalized mixed hemivariational-variational inequalities. *Nonlinear Analysis: Real World Applications* **48**:424–444, 2019.

[3] Y. Bai, S. Migórski, and S. Zeng. A class of generalized mixed variational-hemivariational inequalities I: Existence and uniqueness results. *Computers and Mathematics with Applications* **79**:2897–2911, 2020.

[4] D. Boffi, F. Brezzi, and M. Fortin. *Mixed Finite Element Methods and Applications*. Springer, Heidelberg, 2013.

[5] F. H. Clarke. *Optimization and Nonsmooth Analysis*. Wiley, New York, 1983.

[6] C. Fang, K. Czuprynski, W. Han, X.L. Cheng, and X. Dai. Finite element method for a stationary Stokes hemivariational inequality with slip boundary condition. *IMA J. Numer. Anal.* **40**:2696–2716, 2020.

[7] W. Han. Minimization principles for elliptic hemivariational inequalities. *Nonlinear Analysis: Real World Applications* **54**:103114, 2020.

[8] W. Han. A revisit of elliptic variational-hemivariational inequalities. *Numerical Functional Analysis and Optimization* **42**:371–395, 2021.

[9] W. Han, K. Czuprynski, and F. Jing. Mixed finite element method for a hemivariational inequality of stationary Navier-Stokes equations. *Journal of Scientific Computing* **89**:8, 2021.

[10] W. Han and A. Matei. Minimax principles for elliptic mixed hemivariational-variational inequalities. *Nonlinear Analysis: Real World Applications* **64**:103448, 2022.

[11] W. Han and A. Matei. Well-posedness of a general class of elliptic mixed hemivariational-variational inequalities. *Nonlinear Analysis: Real World Applications* **66**:103553, 2022.

[12] W. Han and M. Sofonea. Numerical analysis of hemivariational inequalities in contact mechanics. *Acta Numerica* **28**:175–286, 2019.

[13] A. Matei. A mixed hemivariational-variational problem and applications. *Computers and Mathematics with Applications* **77**:2989–3000, 2019.

[14] S. Migorski, A. Ochal, and M. Sofonea. *Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems.* Springer, 2013.

[15] S. Migórski, Y. Bai, and S. Zeng. A class of generalized mixed variational-hemivariational inequalities II: Applications. *Nonlinear Analysis: Real World Applications* **50**:633–650, 2019.

[16] J. Nečas and I. Hlaváček. *Mathematical Theory of Elastic and Elasto-Plastic Bodies: An Introduction.* Elsevier Scientific Publishing Company, Amsterdam, Oxford, New York, 1981.

[17] P. D. Panagiotopoulos. Nonconvex energy functions, hemivariational inequalities and substationary principles. *Acta Mechanica* **42**:160–183, 1983.

[18] M. Sofonea and S. Migórski. *Variational-Hemivariational Inequalities with Applications.* CRC Press, Boca Raton, FL, 2018.

[19] R. Temam. *Navier-Stokes Equations.* North-Holland, Amsterdam, 1979.

[20] J. Zhao, J. He, S. Migórski, and S. Dudek. An inverse problem for Bingham type fluids, *J. Comp. Appl. Math.* **404**:113906, 2022.