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The Best Constant in a Trace Inequality in H^1

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ABSTRACT

The best possible constant in the trace inequality

$$c_0 \cdot \|v\|^2_{L^1(\Gamma_1)} \leq \|\nabla v\|^2_{L^2(\Omega)} + k \cdot \|v\|^2_{L^2(\Omega)}, \qquad \forall\, v \in H^1(\Omega)$$

is shown to be given by a quantity in terms of the solution of an elliptic boundary value problem, where, $\Omega \subset R^N$ is a Lipschitz domain, Γ_1 is a measurable subset of $\partial\Omega$, and $k > 0$ is fixed.

Trace inequality in the form:

$$c \cdot \|v\|^2_{L^2(\partial\Omega)} \leq \|\nabla v\|^2_{L^2(\Omega)} + k \cdot \|v\|^2_{L^2(\Omega)}, \qquad \forall\, v \in H^1(\Omega) \tag{1}$$

has been extensively studied, where, $\Omega \subset R^N$ is a Lipschitz domain, $k > 0$ is a fixed constant. The best possible constant c can be determined by solving an associated eigenvalue problem. Various estimates on c are available, cf. [3] and references therein.

In this paper, we consider a different kind of trace inequality:

$$c_0 \cdot \|v\|^2_{L^1(\Gamma_1)} \leq \|\nabla v\|^2_{L^2(\Omega)} + k \cdot \|v\|^2_{L^2(\Omega)}, \qquad \forall v \in H^1(\Omega) \tag{2}$$

where, Γ_1 is a measurable subset of $\partial\Omega$. It turns out that the determination of the best possible constant c_0 in the inequality (2) is much simpler than that of c in (1). We will show that c_0 can be obtained by a quantity in terms of the solution of an associated elliptic boundary value problem.

Note that, if v vanishes on a positive measure subset of $\partial\Omega$, then we may drop the second term $k\|v\|^2_{L^2(\Omega)}$ on the right hand side of the inequality (2). In this case, the best constant in the modified inequality can be obtained by a quantity in terms of the solution of a mixed boundary value problem for the Laplace equation in Ω (cf. [2]).

The organization of the paper is as follows. First we prove that there is a nonzero function $u \in H^1(\Omega)$ such that both sides of the inequality (2) are equal. Next, we formally derive a boundary value problem for u. Then, we give an expression for c_0 in terms of the solution u of the boundary value problem. Finally, we examine a one-dimensional example.

Obviously, c_0 can be rewritten as

$$c_0 = \inf_{v \in H^1(\Omega)} \frac{\|\nabla v\|^2_{L^2(\Omega)} + k \cdot \|v\|^2_{L^2(\Omega)}}{\|v\|^2_{L^1(\Gamma_1)}}. \tag{3}$$

Lemma. There exists a $0 \neq u \in H^1(\Omega)$, such that:

$$c_0 = \frac{\|\nabla u\|^2_{L^2(\Omega)} + k \cdot \|u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^1(\Gamma_1)}}. \tag{4}$$

Proof. We choose a sequence $\{u_n\} \subset H^1(\Omega)$, such that:

$$\|\nabla u_n\|^2_{L^2(\Omega)} + k \cdot \|u_n\|^2_{L^2(\Omega)} = 1, \tag{5}$$

$$\|u_n\|^{-2}_{L^1(\Gamma_1)} \to c_0 \qquad \text{as } n \to \infty. \tag{6}$$

From the boundedness of $\{u_n\}$ in $H^1(\Omega)$, and the compactness of the embedding

$$H^1(\Omega) \subset L^1(\Gamma_1),$$

we obtain a subsequence $\{u_{n_k}\}$ and $u \in H^1(\Omega)$, such that:

$$u_{n_k} \rightharpoonup u \qquad \text{weakly in } H^1(\Omega), \tag{7}$$

$$u_{n_k} \to u \qquad \text{strongly in } L^1(\Gamma_1). \tag{8}$$

Thus,

$$\|\nabla u\|^2_{L^2(\Omega)} + k \cdot \|u\|^2_{L^2(\Omega)} \leq \liminf_{k\to\infty}\{\|\nabla u_{n_k}\|^2_{L^2(\Omega)} + k \cdot \|u_{n_k}\|^2_{L^2(\Omega)}\} = 1,$$

$$\|u\|^2_{L^1(\Gamma_1)} = \lim_{k\to\infty} \|u_{n_k}\|^2_{L^1(\Gamma_1)} = 1/c_0.$$

Therefore,

$$\frac{\|\nabla u\|^2_{L^2(\Omega)} + k \cdot \|u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^1(\Gamma_1)}} \leq c_0. \tag{9}$$

However, by (3), the above inequality is an equality.

Now we formally derive a boundary value problem for u. For any $v \in H^1(\Omega)$, we consider a real variable function:

$$f(t) = \frac{\|\nabla u + t\,v\|^2_{L^2(\Omega)} + k \cdot \|u + t\,v\|^2_{L^2(\Omega)}}{\|u + t\,v\|^2_{L^1(\Gamma_1)}}, \tag{10}$$

which attains its minimum at 0. For the time being, we assume u is such a "nice" nonnegative function that f is differentiable at $t = 0$, and

$$f'(0) = \frac{2}{\|u\|^2_{L^1(\Gamma_1)}} \cdot \left\{ \int_\Omega (\nabla u \nabla v + kuv) - \frac{\|\nabla u\|^2_{L^2(\Omega)} + k \cdot \|u\|^2_{L^2(\Omega)}}{\|u\|_{L^1(\Gamma_1)}} \int_{\Gamma_1} v \right\} = 0. \tag{11}$$

Hence, if we normalize u so that:

$$\|\nabla u\|^2_{L^2(\Omega)} + k \cdot \|u\|^2_{L^2(\Omega)} = \|u\|_{L^1(\Gamma_1)} \tag{12}$$

then u is the solution of:

$$\begin{aligned} -\Delta u + ku &= 0 && \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= 1 && \text{on } \Gamma_1, \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \Gamma_2, \end{aligned} \tag{13}$$

where, $\Gamma_2 = \partial\Omega \backslash \overline{\Gamma}_1$, which is empty if $\overline{\Gamma}_1 = \partial\Omega$.

Obviously, the sclution u of (13) satisfies (12). Therefore, the best constant would be:

$$\begin{aligned} c_0 &= \frac{\|\nabla u\|^2_{L^2(\Omega)} + k \cdot \|u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^1(\Gamma_1)}} \\ &= \|u\|^{-1}_{L^1(\Gamma_1)} \\ &= \left(\|\nabla u\|^2_{L^2(\Omega)} + k \cdot \|u\|^2_{L^2(\Omega)}\right)^{-1}. \end{aligned}$$

The main result of the paper is:

Theorem. Let u be the solution of the elliptic boundary value problem (13). Then, the best constant c_0 is given by:

$$c_0 = \|u\|^{-1}_{L^1(\Gamma_1)}, \tag{14}$$

or,

$$c_0 = \left(\|\nabla u\|^2_{L^2(\Omega)} + k \cdot \|u\|^2_{L^2(\Omega)}\right)^{-1}. \tag{15}$$

Proof. For any $v \in H^1(\Omega)$, we have (cf. [1]):

$$\nabla|v| = \begin{cases} \nabla v, & \text{if } v > 0, \\ 0, & \text{if } v = 0, \\ -\nabla v, & \text{if } v < 0. \end{cases} \tag{16}$$

Hence, $|v| \in H^1(\Omega)$, and:

$$\|\nabla|v|\|_{L^2(\Omega)} \leq \|\nabla v\|_{L^2(\Omega)}. \tag{17}$$

Now,

$$\begin{aligned} \|v\|_{L^1(\Gamma_1)} &= \int_{\partial\Omega} \frac{\partial u}{\partial n} |v| \\ &= \int_{\Omega} \nabla u \nabla |v| + k\, u\, |v| \\ &\leq \left\{\|\nabla u\|^2_{L^2(\Omega)} + k\, \|u\|^2_{L^2(\Omega)}\right\}^{1/2} \left\{\|\nabla|v|\|^2_{L^2(\Omega)} + k\, \|v\|^2_{L^2(\Omega)}\right\}^{1/2} \\ &\leq (1/\sqrt{c_0}) \left\{\|\nabla v\|^2_{L^2(\Omega)} + k\|v\|^2_{L^2(\Omega)}\right\}^{1/2} \end{aligned}$$

i.e.,

$$c_0 \cdot ||v||^2_{L^1(\Gamma_1)} \leq ||\nabla v||^2_{L^2(\Omega)} + k \cdot ||v||^2_{L^2(\Omega)}.$$

The proof of the Theorem is completed.

Let us examine a one-dimensional trace inequality.

Example. Consider the best possible constant of the inequality:

$$c_0 \cdot ||v||^2_{L^1(\Gamma_1)} \leq ||\nabla v||^2_{L^2(0,l)} + k \cdot ||v||^2_{L^2(0,l)}, \qquad \forall\, v \in H^1(0,l).$$

If $\Gamma_1 = \partial\Omega = \{0\} \cup \{l\}$, then the problem (13) is:

$$-u'' + k\,u = 0 \quad \text{in } (0,l)$$
$$u'(l) = 1$$
$$u'(0) = -1$$

the solution of which is

$$u(x) = \frac{\mathrm{ch}\big(\sqrt{k}\,x - \sqrt{k}\,l/2\big)}{\sqrt{k}\,\mathrm{sh}\big(\sqrt{k}\,l/2\big)}.$$

Hence,

$$c_0 = \frac{1}{|u(l)| + |u(0)|} = \frac{\sqrt{k}}{2}\,\mathrm{th}\Big(\frac{\sqrt{k}\,l}{2}\Big),$$

and we have the optimal inequality:

$$\frac{\sqrt{k}}{2}\,\mathrm{th}\Big(\frac{\sqrt{k}\,l}{2}\Big) \cdot \big\{|v(l)| + |v(0)|\big\}^2 \leq ||\nabla v||^2_{L^2(0,l)} + k \cdot ||v||^2_{L^2(0,l)}, \qquad \forall\, v \in H^1(0,l).$$

If $\Gamma_1 = \{l\}$, then $\Gamma_2 = \{0\}$, and the problem (13) is:

$$-u'' + k\,u = 0 \quad \text{in } (0,l)$$
$$u'(l) = 1$$
$$u'(0) = 0$$

with the solution:

$$u(x) = \frac{\mathrm{ch}\big(\sqrt{k}\,x\big)}{\sqrt{k}\,\mathrm{sh}\big(\sqrt{k}\,l\big)}.$$

Hence,

$$c_0 = |u(l)|^{-1} = \sqrt{k}\,\mathrm{th}(\sqrt{k}\,l),$$

and we have the optimal inequality:

$$\sqrt{k}\,\mathrm{th}(\sqrt{k}\,l)\cdot|v(l)|^2 \le \|\nabla v\|^2_{L^2(0,l)} + k\cdot\|v\|^2_{L^2(0,l)}, \qquad \forall\, v \in H^1(0,l).$$

Remark. For a general domain in R^N, it is usually impossible to have an analytic expression for the solution of the problem (13). In such a case, one may solve (13) numerically, and obtain a numerical value of the best possible constant.

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