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Two algorithms for two-phase Stefan type problems

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Abstract. In this paper, the relaxation algorithm and two Uzawa type algorithms for solving discretized variational inequalities arising from the two-phase Stefan type problem are proposed. An analysis of their convergence is presented and the upper bounds of the convergence rates are derived. Some numerical experiments are shown to demonstrate that for the second Uzawa algorithm which is an improved version of the first Uzawa algorithm, the convergence rate is uniformly bounded away from 1 if τh^{-2} is kept bounded, where τ is the time step size and h the space mesh size.

§1 Introduction

In this paper we present and analyze a relaxation algorithm and two Uzawa type algorithms for solving finite dimensional variational inequalities that result from discretization of moving boundary problems of two-phase Stefan type. Following [4, 12], we consider the following degenerate parabolic initial-boundary value problem:

$$\frac{\partial e}{\partial t} - \Delta u = f,\ e \in H(u) \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

$$e(0) = e^0 \in H(u^0) \quad \text{in } \Omega, \tag{1.2}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,T). \tag{1.3}$$

The function H is multi-valued,

$$H(z) = \begin{cases} a_1(z-\theta_0) - s_1 & \text{if } z < \theta_0, \\ [-s_1, s_2] & \text{if } z = \theta_0, \\ a_2(z-\theta_0) + s_2 & \text{if } z > \theta_0 \end{cases} \tag{1.4}$$

with fixed $\theta_0 \in \mathbf{R}$, positive constants a_1 and a_2, and nonnegative constants s_1 and s_2.

Let $0 = t_0 < t_1 < \cdots < t_M = T$ be a partition of the time interval $[0,T]$. The backward Euler time discretization of the degenerate parabolic initial-boundary value problem (1.1)–(1.3)

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leads to the successive solution of elliptic differential inclusion problem:

$$b^m + \tau_m \Delta u^m \in H(u^m) \quad \text{in } \Omega, \tag{1.5}$$

$$u^m = 0 \quad \text{on } \partial\Omega, \tag{1.6}$$

where $\tau_m = t_m - t_{m-1}$, $b^m = H^{m-1} + \tau_m f(t_m)$ with an appropriately chosen $H^{m-1} \in H(u^{m-1})$. We focus on the numerical solution of the semi-discrete problem (1.5)–(1.6).

For notational convenience, from now on, we omit the index m in our discussion of the problem (1.5)–(1.6), which is equivalent to the following unconstrained subdifferentiable convex optimization problem: Find $u \in H_0^1(\Omega)$ such that

$$\mathcal{F}(u) + \phi(u) \le \mathcal{F}(v) + \phi(v) \quad \forall\, v \in H_0^1(\Omega), \tag{1.7}$$

where the quadratic functional $\mathcal{F}$,

$$\mathcal{F}(v) = \frac{1}{2} a(v, v) - (b, v), \tag{1.8}$$

is induced by a continuous, symmetric and $H_0^1(\Omega)$-elliptic bilinear form $a(\cdot,\cdot)$ associated with the elliptic operator $-\tau\Delta$. The convex functional ϕ is of the form

$$\phi(v) = \int_\Omega \Phi(v(x))dx, \tag{1.9}$$

and is generated by a scalar convex function Φ with $\partial\Phi = H$,

$$\Phi(z) = \begin{cases} \frac{1}{2} a_1 (z-\theta_0)^2 - s_1(z-\theta_0) & \text{if } z \le \theta_0, \\ \frac{1}{2} a_2 (z-\theta_0)^2 + s_2(z-\theta_0) & \text{if } z > \theta_0. \end{cases} \tag{1.10}$$

It is well known that (1.7) can be equivalently rewritten as an elliptic variational inequality of the second kind: Find $u \in H_0^1(\Omega)$ such that

$$a(u, v-u) + \phi(v) - \phi(u) \ge (b, v-u) \quad \forall\, v \in H_0^1(\Omega). \tag{1.11}$$

Well-posedness of the problems (1.7) and (1.11) follows from a standard result of elliptic variational inequalities (cf. [6-8, 11]).

Let $\{\mathcal{T}_h\}_h$ be a regular family of triangulations of $\overline{\Omega}$, and let $V_h \subset H_0^1(\Omega)$ be the finite element space of linear elements associated with the triangulation $\mathcal{T}_h$. We use n for the dimension of the space V_h and $\{x_i\}_{i=1}^n$ the finite element nodes. Denote by $\{\lambda_i\}_{i=1}^n$ the standard linear element basis functions satisfying $\lambda_i \in V_h$ and $\lambda_i(x_j) = \delta_{ij}$. Then we approximate (1.7) by the problem: Find $u_h \in V_h$ such that

$$\mathcal{F}(u_h) + \phi_h(u_h) \le \mathcal{F}(v_h) + \phi_h(v_h) \quad \forall\, v \in V_h, \tag{1.12}$$

where ϕ_h is an approximation of the functional ϕ by the finite element interpolation on $\mathcal{T}_h$, i.e.,

$$\phi_h(v_h) = \sum_{i=1}^n \Phi(v_h(x_i)) \int_\Omega \lambda_i(x)\, dx. \tag{1.13}$$

The optimization problem (1.12) is also uniquely solvable and can be reformulated as a discrete variational inequality: Find $u_h \in V_h$ such that

$$a(u_h, v_h - u_h) + \phi_h(v_h) - \phi_h(u_h) \ge (b, v_h - u_h) \quad \forall\, v_h \in V_h. \tag{1.14}$$

Convergence and error estimates of the approximation (1.14) to (1.11) are discussed in several references, see e.g. [4-7, 11].

In this paper, we consider iterative algorithms for the discretized variational inequality (1.14). We notice that SOR type method as well as relaxation and over-relaxation methods have been studied for solving discrete variational inequalities of the form (1.14) ([4, 6]). These methods are based on the use of Gauss-Seidel or SOR ideas to solve the discrete optimization problem (1.12). One can find convergence proofs of the algorithms in [4, 6], although there is no information on convergence rate. In [16] and [17], the convergence rate for elliptic variational inequalities of second kind is discussed. Here we will first apply the method to our problem and obtain the convergence rate. The single-grid relaxation typically suffers from rapidly deteriorating convergence rates when the number of unknowns becomes large. In literature one can also find some other extended relaxation methods, for example, the multigrid method [12-14] and a globally damped version multilevel method[10].

The Uzawa algorithm is a predictor-corrector type method for solving nonlinear problems or problems with more than one unknown variable. It was first proposed by the group of Uzawa in 1958[1] in applying the gradient method to the minimization problem of dual functional of the Stokes problem. The most attractive features of the Uzawa or inexact Uzawa algorithm are its simplicity and robustness. See [2, 6, 8-9, 16]. For the variational inequality problem (1.14), however, there is no Uzawa-type algorithm so far. In this paper, we first propose a relaxation algorithm. Then we present and analyze a standard form Uzawa algorithm for solving problem (1.14). We propose and study an improved Uzawa algorithm by merging some part of the non-differentiable term into the bilinear form. We prove convergence and investigate the convergence rate for each algorithm. We also include numerical results to show the good performance of the improved Uzawa algorithm.

§2 The relaxation algorithm

We express the problem (1.12) in algebraic form: Find $\boldsymbol{u} = (u_1, u_2, \cdots, u_n)^T \in \mathbf{R}^n$ such that

$$J(\boldsymbol{u}) \le J(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} = (v_1, v_2, \cdots, v_n)^T \in \mathbf{R}^n, \tag{2.1}$$

where

$$J(\boldsymbol{v}) = \frac{1}{2}(A\boldsymbol{v}, \boldsymbol{v}) - (\boldsymbol{b}, \boldsymbol{v}) + \sum_{i=1}^{n} \alpha_i \Phi(v_i). \tag{2.2}$$

Here, the stiffness matrix A is symmetric and positive definite with entries $a_{i,j} = a(\lambda_i, \lambda_j)$, the load vector $\boldsymbol{b}$ has components $b_i = (b, \lambda_i)$ and $\alpha_i = \int_\Omega \lambda_i(x)\, dx > 0$.

We first consider the following relaxation algorithm.

Algorithm 2.1. Given the initial guess $\boldsymbol{u}^0 \in \mathbf{R}^n$, for $k = 0, 1, 2, \cdots$, compute $\boldsymbol{u}^{k+1}$, component by component, as follows,

$$\begin{cases} J(u_1^{k+1}, \cdots, u_{i-1}^{k+1}, u_i^{k+1}, u_{i+1}^k, \cdots, u_n^k), \\ = \min\limits_{v \in \mathbf{R}^1} J(u_1^{k+1}, \cdots, u_{i-1}^{k+1}, v, u_{i+1}^k, \cdots, u_n^k), \quad i = 1, 2, \cdots, n. \end{cases} \tag{2.3}$$

Let

$$J_i(v) = \frac{1}{2}a_{ii}v^2 + \Big(\sum_{j=1}^{i-1} a_{ij}u_j^{k+1} + \sum_{j=i+1}^{n} a_{ij}u_j^k - b_i\Big)v + \alpha_i\Phi(v). \tag{2.4}$$

From (2.3)-(2.4) we can see that for $i = 1, 2, \cdots, n$, u_i^{k+1} satisfies

$$J_i(u_i^{k+1}) = \min_{v\in\mathbf{R}^1} J_i(v). \tag{2.5}$$

Then we can derive an explicit formulation of u_i^{k+1},

$$u_i^{k+1} = \frac{a_1\theta_0 + s_1\alpha_i + p_i^k}{a_i i + a_1\alpha_i}, \quad \text{if } u_i^{k+1} \le \theta_0, \tag{2.6}$$

$$u_i^{k+1} = \frac{a_2\theta_0 - s_2\alpha_i + p_i^k}{a_i i + a_2\alpha_i}, \quad \text{if } u_i^{k+1} > \theta_0, \tag{2.7}$$

and $u_i^{k+1} = \theta_0$ if (2.6),(2.7) do not hold. Here

$$p_i^k = b_i - \sum_{j=1}^{i-1} a_{ij}u_j^{k+1} - \sum_{j=i+1}^{n} a_{ij}u_j^k. \tag{2.8}$$

From (2.3)-(2.5) it is easy to see that u_i^{k+1} satisfies the following inequality:

$$a_{ii}u_i^{k+1}(v - u_i^{k+1}) - p_i^k(v - u_i^{k+1}) + \alpha_i\Phi(v) - \alpha_i\Phi(u_i^{k+1}) \ge 0. \tag{2.9}$$

Following the method of [17], we will prove the convergence of the algorithm and obtain the convergence rate. Denote the error $e_k = J(\boldsymbol{u}^k) - J(\boldsymbol{u})$, then

$$\begin{aligned}
e_k - e_{k+1} &= J(\boldsymbol{u}^k) - J(\boldsymbol{u}^{k+1}) \\
&= \sum_{i=1}^{n}[J(u_1^{k+1}, \cdots, u_{i-1}^{k+1}, u_i^k, \cdots, u_n^k) - J(u_1^{k+1}, \cdots, u_i^{k+1}, u_{i+1}^k, \cdots, u_n^k)] \\
&= \sum_{i=1}^{n}[J_i(u_i^k) - J_i(u_i^{k+1})] \\
&= \sum_{i=1}^{n}[\frac{1}{2}a_{ii}(u_i^k - u_i^{k+1})^2 + a_{ii}u_i^{k+1}(u_i^k - u_i^{k+1}) + p_i^k(u_i^k - u_i^{k+1}) + \alpha_i\Phi(u_i^k) - \alpha_i\Phi(u_i^{k+1})] \\
&\ge \tfrac{1}{2}\sum_{i=1}^{n} a_{ii}(u_i^k - u_i^{k+1})^2,
\end{aligned}$$

i.e.,

$$e_k - e_{k+1} \ge c_{0,A}\|u^k - u^{k+1}\|^2, \tag{2.10}$$

where the constant $c_{0,A}$ (as well as $c_{1,A}, c_{2,A}$ below) depends only on matrix A.

Obviously, (2.1) is equivalent to the variational form: Find $\boldsymbol{u} = (u_1, u_2, \cdots, u_n)^T \in \mathbf{R}^n$ such that

$$(A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + \sum_{i=1}^{n}\alpha_i\Phi(v_i) - \sum_{i=1}^{n}\alpha_i\Phi(u_i) \ge (\boldsymbol{b}, \boldsymbol{v} - \boldsymbol{u}) \quad \forall\, \boldsymbol{v} = (v_1, v_2, \cdots, v_n)^T \in \mathbf{R}^n. \tag{2.11}$$

Then we can obtain

$$\begin{aligned}(A(\boldsymbol{u}^{k+1}-\boldsymbol{u}),\boldsymbol{u}^{k+1}-\boldsymbol{u}) &= (A\boldsymbol{u}^{k+1},\boldsymbol{u}^{k+1}-\boldsymbol{u})+\sum_{i=1}^{n}\alpha_i(\Phi(u_i^{k+1})-\Phi(u_i)) \\ &\quad -[(A\boldsymbol{u},\boldsymbol{u}^{k+1}-\boldsymbol{u})+\sum_{i=1}^{n}\alpha_i(\Phi(u_i^{k+1})-\Phi(u_i))] \\ &\le (A\boldsymbol{u}^{k+1}-\boldsymbol{b},\boldsymbol{u}^{k+1}-\boldsymbol{u})+\sum_{i=1}^{n}\alpha_i(\Phi(u_i^{k+1})-\Phi(u_i)).\end{aligned} \tag{2.12}$$

Taking $v=u_i$ in (2.9) and using the inequality (2.12), we can obtain

$$\begin{aligned}(A(\boldsymbol{u}^{k+1}-\boldsymbol{u}),\boldsymbol{u}^{k+1}-\boldsymbol{u}) &\le \sum_{i=1}^{n}[(A\boldsymbol{u}^{k+1}-\boldsymbol{b})_i-a_{ii}u_i^{k+1}+p_i^k](u_i-u_i^{k+1}) \\ &\le \sum_{i=1}^{n}(\sum_{j=i+1}^{n}a_{ij}(u_j^{k+1}-u_j^k))(u_i^{k+1}-u_i).\end{aligned}$$

Then $(A(\boldsymbol{u}^{k+1}-\boldsymbol{u}),\boldsymbol{u}^{k+1}-\boldsymbol{u})\le c_{1,A}\|\boldsymbol{u}^{k+1}-\boldsymbol{u}^k\|\,\|\boldsymbol{u}^{k+1}-\boldsymbol{u}\|$, thus

$$\|\boldsymbol{u}^{k+1}-\boldsymbol{u}\|\le c_{2,A}\,\|\boldsymbol{u}^{k+1}-\boldsymbol{u}^k\|. \tag{2.13}$$

On the other hand, similarly we have

$$\begin{aligned}e_{k+1} &= J(\boldsymbol{u}^{k+1})-J(\boldsymbol{u})=\tfrac{1}{2}(A\boldsymbol{u}^{k+1},\boldsymbol{u}^{k+1})-\tfrac{1}{2}(A\boldsymbol{u},\boldsymbol{u})-(\boldsymbol{b},\boldsymbol{u}^{k+1}-\boldsymbol{u}) \\ &\quad +\sum_{i=1}^{n}\alpha_i(\Phi(u_i^{k+1})-\Phi(u_i))\le c_{3,A}\|\boldsymbol{u}^{k+1}-\boldsymbol{u}\|^2+c_{4,A}\|\boldsymbol{u}^{k+1}-\boldsymbol{u}^k\|^2.\end{aligned}$$

Then

$$e_{k+1}\le c_{3,A}c_{2,A}^2\|\boldsymbol{u}^{k+1}-\boldsymbol{u}^k\|^2+c_{4,A}\|\boldsymbol{u}^{k+1}-\boldsymbol{u}^k\|^2\le\frac{c_{3,A}c_{2,A}^2+c_{4,A}}{c_{0,A}}(e_k-e_{k+1}). \tag{2.14}$$

We have the relation $e_{k+1}\le\rho e_k$ with $\rho=\dfrac{\tilde{c_A}}{1+\tilde{c_A}}<1$ and $\tilde{c_A}=(c_{3,A}c_{2,A}^2+c_{4,A})/c_{0,A}$.

Thus we prove $e_k\to 0$ as $k\to\infty$. So the convergence rate is ρ dependent on matrix A. As showed in [17], it depends on the condition number of A. Next we will propose another so-called Uzawa type algorithm and analyze it.

§3 Uzawa algorithms

To present the Uzawa algorithm, we first introduce a new variable vector $\boldsymbol{p}\in\mathbf{R}^n$ based on the formulation (2.1),(2.2) and (2.11) by

$$p_i=-\frac{1}{\alpha_i}(A\boldsymbol{u}-\boldsymbol{b})_i,\quad i=1,2,\cdots,n. \tag{3.1}$$

Then we know from (2.11) that

$$p_i=-\frac{1}{\alpha_i}((A\boldsymbol{u})_i-b_i)\in\partial\Phi(u_i), \tag{3.2}$$

where the sub-differentiable $\partial\Phi(z)=H(z)$ is given in (1.4).

For any positive parameter $r > 0$, we construct a mapping from $\mathbf{R}$ to $\mathbf{R}$ as follows:

$$\mathcal{P}_r(z) = \begin{cases} \dfrac{a_1}{a_1+r}z - \dfrac{r}{a_1+r}s_1 & \text{if } z < -s_1, \\ z & \text{if } -s_1 \le z \le s_2, \\ \dfrac{a_2}{a_2+r}z + \dfrac{r}{a_2+r}s_2 & \text{if } z > s_2. \end{cases} \tag{3.3}$$

Define a diagonal $n \times n$ matrix D_α with diagonal elements $d_{i,i} = \alpha_i$.

We consider the following Uzawa algorithm.

Algorithm 3.1. Given the initial guess $\boldsymbol{p}^0 \in \mathbf{R}^n$ and $r_i > 0$, for $k = 0, 1, 2, \cdots$, compute $\boldsymbol{u}^{k+1}$ and $\boldsymbol{p}^{k+1}$ from

$$\begin{cases} A\boldsymbol{u}^{k+1} = \boldsymbol{b} - D_\alpha \boldsymbol{p}^k, \\ p_i^{k+1} = \mathcal{P}_{r_i}(p_i^k + r_i(u_i^{k+1} - \theta_0)), \quad i = 1, 2, \cdots, n. \end{cases} \tag{3.4}$$

To establish a convergence result, we introduce some preparatory lemmas.

Lemma 3.2. For any $r > 0$, the mapping $\mathcal{P}_r$ is non-expansive, i.e.,

$$|\mathcal{P}_r(z_1) - \mathcal{P}_r(z_2)| \le |z_1 - z_2| \quad \forall\, z_1, z_2 \in \mathbf{R}. \tag{3.5}$$

This result follows by the observation that the function $\mathcal{P}_r$ defined in (3.3) is Lipschitz continuous with the Lipschitz continuity constant 1.

Lemma 3.3. Suppose that $\boldsymbol{u}$ is the solution of the optimization problem (2.1) and $\boldsymbol{p}$ is defined by (3.1). Then

$$\mathcal{P}_{r_i}(p_i + r_i(u_i - \theta_0)) = p_i, \quad i = 1, 2, \cdots, n. \tag{3.6}$$

Proof. When $u_i < \theta_0$, from (3.2) and (1.4) we have $p_i = a_1(u_i - \theta_0) - s_1$. Thus $p_i + r(u_i - \theta_0) < -s_1$, and

$$\mathcal{P}_{r_i}(p_i + r_i(u_i - \theta_0)) = \frac{a_1}{a_1 + r_i}(p_i + r_i(u_i - \theta_0)) - \frac{r_i}{a_1 + r_i}s_1 = a_1(u_i - \theta_0) - s_1 = p_i.$$

A similar argument works for the case $u_i > \theta_0$.

For $u_i = \theta_0$, $p_i + r_i(u_i - \theta_0) = p_i \in [-s_1, s_2]$. Then $\mathcal{P}_{r_i}(p_i + r_i(u_i - \theta_0)) = p_i$.

For simplicity, we choose

$$r_i = \frac{r}{\alpha_i}, \quad i = 1, 2, \cdots, n. \tag{3.7}$$

Recall the 2-norm $\|\cdot\|$ and inner product $(\cdot, \cdot)$ on $\mathbf{R}^n$:

$$\|\boldsymbol{v}\| = \Big(\sum_{i=1}^n v_i^2\Big)^{\frac{1}{2}}, \quad (\boldsymbol{u}, \boldsymbol{v}) = \sum_{i=1}^n u_i v_i, \quad \boldsymbol{u}, \boldsymbol{v} \in \mathbf{R}^n.$$

Now we show a convergence result for the Uzawa algorithm.

Theorem 3.4. Let $\boldsymbol{u}$ be the solution of the optimization problem (2.1) and $\boldsymbol{p}$ be defined by (3.1), Suppose $(\boldsymbol{u}^k, \boldsymbol{p}^k)$ is the iterative solution defined by the Uzawa Algorithm 3.1. Denote the k-th error

$$\boldsymbol{e}^k = \boldsymbol{u} - \boldsymbol{u}^k, \quad \epsilon_i^k = \alpha_i(p_i - p_i^k), \quad k = 0, 1, 2, \cdots. \tag{3.8}$$

Then for $0 < r < 2\lambda_{\min}(A)$, the Uzawa algorithm converges and

$$\|A\boldsymbol{e}^k\| \le \rho^k \|A\boldsymbol{e}^0\|, \quad \|\epsilon^k\| \le \rho^k \|\epsilon^0\|, \quad k = 1, 2, \cdots \tag{3.9}$$

where $\rho = \rho(I - rA^{-1}) < 1$.

Proof. By Lemmas 3.2 and 3.3, we have for all components,

$$\begin{aligned}|\epsilon_i^{k+1}| &= \alpha_i|p_i - \mathcal{P}_{r_i}(p_i^k + r_i(u_i^{k+1} - \theta_0))| \\ &= \alpha_i|\mathcal{P}_{r_i}(p_i + r_i(u_i - \theta_0)) - \mathcal{P}_{r_i}(p_i^k + r_i(u_i^{k+1} - \theta_0))| \\ &\leq \alpha_i|p_i - p_i^k + r_i(u_i - u_i^{k+1})| = |\epsilon_i^k + re_i^{k+1}|.\end{aligned}$$

By the first equation of (3.4) we have $A\boldsymbol{e}^{k+1} = -\epsilon^k$. Then we obtain

$$\|\epsilon^{k+1}\|^2 \leq (\epsilon^k + re^{k+1}, \epsilon^k + re^{k+1}) = (\epsilon^k - rA^{-1}\epsilon^k, \epsilon^k - rA^{-1}\epsilon^k) \leq [\rho(I - rA^{-1})]^2\|\epsilon^k\|^2.$$

Thus

$$\|\epsilon^{k+1}\| \leq \rho(I - rA^{-1})\|\epsilon^k\|. \tag{3.10}$$

As matrix A is symmetric and positive definite, $1/\lambda_{\max}(A) \leq \lambda(A^{-1}) \leq 1/\lambda_{\min}(A)$. So when $0 < r < 2\lambda_{\min}(A)$, we have $\rho(I - rA^{-1}) < 1$. We then get the second estimate of (3.9). The first estimate of (3.9) follows from (3.10) and the relation $A\boldsymbol{e}^{k+1} = -\epsilon^k$. In particular, we have the convergence.

We observe that the optimal choice for r is

$$r_{\text{opt}} = \frac{2}{1/\lambda_{\min}(A) + 1/\lambda_{\max}(A)} = \frac{2\lambda_{\min}(A)\lambda_{\max}(A)}{\lambda_{\min}(A) + \lambda_{\max}(A)}. \tag{3.11}$$

Correspondingly, the optimal convergence rate is

$$\rho_{\text{opt}} = \frac{\lambda_{\max}(A) - \lambda_{\min}(A)}{\lambda_{\max}(A) + \lambda_{\min}(A)} = \frac{\kappa(A) - 1}{\kappa(A) + 1}, \tag{3.12}$$

where $\kappa(A) = \lambda_{\max}(A)/\lambda_{\min}(A)$ is the condition number of A.

Generally, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ are unknown. Nevertheless, usually it is possible to find an upper bound for $\lambda_{\max}(A)$ and a lower bound for $\lambda_{\min}(A)$. From Theorem 3.4, we see that the optimal convergence rate depends on the upper and lower bounds of the eigenvalues of matrix A. As the stiff matrix A is associated with a discretization of the operator $-\tau\Delta$, $\kappa(A) = O(h^{-2})$ or bigger, h being the space mesh size. Then the optimal convergence rate is at most $1 - O(h^2)$, which deteriorates rapidly with the mesh refinement. We can use the inexact Uzawa method to Algorithm 3.1, see [3] for variational inequalities of the second kind. Here we will not analyze this case. In the following, we propose an improved Uzawa algorithm and show its better convergence rate.

Let us choose a positive constant $a_0 \leq \min(a_1, a_2)$. We still use A for the stiffness matrix of the bilinear form $a(\cdot,\cdot)$ associated with operator $-\tau\Delta$. We rewrite the problem (2.1) in the form:

Find $\boldsymbol{u} = (u_1, u_2, \cdots, u_n)^T \in \mathbf{R}^n$ such that

$$J(\boldsymbol{u}) \leq J(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} = (v_1, v_2, \cdots, v_n)^T \in \mathbf{R}^n, \tag{3.13}$$

$$J(\boldsymbol{v}) = \frac{1}{2}(\tilde{A}\boldsymbol{v}, \boldsymbol{v}) - (\tilde{\boldsymbol{b}}, \boldsymbol{v}) + \sum_{i=1}^{n} \alpha_i \tilde{\Phi}(v_i), \tag{3.14}$$

with $\tilde{A} = A + a_0 D_\alpha$, $\tilde{b}_i = b_i + a_0\theta_0\alpha_i$ and

$$\tilde{\Phi}(z) = \begin{cases} \frac{1}{2}(a_1 - a_0)(z - \theta_0)^2 - s_1(z - \theta_0) & \text{if } z \leq \theta_0, \\ \frac{1}{2}(a_2 - a_0)(z - \theta_0)^2 + s_2(z - \theta_0) & \text{if } z > \theta_0, \end{cases} \tag{3.15}$$

Then we apply the Uzawa Algorithm 3.1 to the formulation (3.13)–(3.15).

Algorithm 3.5. Given the initial guess $\boldsymbol{p}^0 \in \mathbf{R}^n$ and $r_i > 0$, for $k = 0, 1, 2, \cdots$, compute $\boldsymbol{u}^{k+1}$ and $\boldsymbol{p}^{k+1}$ from

$$\begin{cases} \tilde{A}u^{k+1} = \tilde{\boldsymbol{b}} - D_\alpha \boldsymbol{p}^k, \\ p_i^{k+1} = \tilde{\mathcal{P}}_{r_i}(p_i^k + r_i(u_i^{k+1} - \theta_0)), \quad i = 1, 2, \cdots, n, \end{cases} \tag{3.16}$$

where

$$\tilde{\mathcal{P}}_r(z) = \begin{cases} \dfrac{a_1 - a_0}{a_1 - a_0 + r} z - \dfrac{r}{a_1 - a_0 + r} s_1 & \text{if } z < -s_1, \\ z & \text{if } -s_1 \le z \le s_2, \\ \dfrac{a_2 - a_0}{a_2 - a_0 + r} z + \dfrac{r}{a_2 - a_0 + r} s_2 & \text{if } z > s_2. \end{cases} \tag{3.17}$$

Theorem 3.6. Assume the family of the finite element triangulations is quasi-uniform. Then there exists a constant $c_0 > 0$ independent of the discretization parameters τ and h such that

$$\kappa(\tilde{A}) \le c_0(1 + \tau h^{-2}). \tag{3.18}$$

Therefore, if τh^{-2} is kept bounded, then the optimal convergence rate of Algorithm 3.5 is uniformly bounded away from 1.

Proof. Obviously the matrix $\tilde{A}$ is symmetric and positive definite. Let us bound its smallest and largest eigenvalues. Since the finite element triangulations are quasi-uniform, for some positive constants $0 < c_1 \le c_2 < \infty$, depending on the continuity and ellipticity constants of the bilinear form, we have (cf. [15, p. 195])

$$c_1 \tau h^d (\boldsymbol{v}, \boldsymbol{v}) \le (A\boldsymbol{v}, \boldsymbol{v}) \le c_2 \tau h^d (1 + h^{-2}) (\boldsymbol{v}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbf{R}^n.$$

From the definition of α_i, we know that there are two constants $0 < c_3 \le c_4 < \infty$ such that

$$c_3 h^d \le \alpha_i \le c_4 h^d, \quad i = 1, 2, \cdots, n.$$

Then we obtain the following:

$$(c_1\tau + a_0 c_3) h^d (\boldsymbol{v}, \boldsymbol{v}) \le (\tilde{A}\boldsymbol{v}, \boldsymbol{v}) \le \left[c_2 \tau (1 + h^{-2}) + a_0 c_4\right] h^d (\boldsymbol{v}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbf{R}^n.$$

Therefore,

$$\lambda_{\min}(\tilde{A}) \ge (c_1 \tau + a_0 c_3) h^d, \quad \lambda_{\max}(\tilde{A}) \le \left[c_2 \tau (1 + h^{-2}) + a_0 c_4\right] h^d.$$

Consequently, we have the bound (3.18) for the condition number of $\tilde{A}$. When τh^{-2} is kept bounded, the condition number $\kappa(\tilde{A})$ is uniformly bounded and the optimal convergence rate of Algorithm 3.5 is uniformly bounded away from 1.

We observe that even if the assumption τh^{-2} being kept bounded is violated, the condition number of $\tilde{A}$ is still substantially smaller than that of A (by a factor of τ).

§4 Numerical experiments

In this section we only test the Algorithms 3.1 and 3.5. Let $\Omega = (0, 1) \times (0, 1)$ and $\mathcal{T}_h$ be the uniform triangulation of Ω into $2n^2$ triangles, $h = 1/n$. We consider the example problem studied in[12, 14] with

$$H(z)=\begin{cases}2z & \text{if } z<0,\\ [0,1] & \text{if } z=0,\\ 3z+1 & \text{if } z>0.\end{cases} \tag{4.1}$$

Then the stiff matrix $A=(\tau/h^2)A_0$, A_0 is the discretizing Poisson operator with stencil

$$\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}.$$

So the eigenvalues of matrix A_0 are

$$\lambda(A_0)=4\sin^2(ih\pi/2)+4\sin^2(jh\pi/2),\quad i,j=1,2,\cdots,n.$$

Thus $2\pi^2h^2<\lambda(A_0)<8$.

Uzawa Algorithm 3.1 becomes:

Given the initial guess $\boldsymbol{p}^0\in\mathbf{R}^n$ and $r>0$, for $k=0,1,2,\cdots$, compute $\boldsymbol{u}^{k+1}$ and $\boldsymbol{p}^{k+1}$ from

$$\begin{cases}\frac{\tau}{h^2}A_0\boldsymbol{u}^{k+1}=\boldsymbol{b}-\boldsymbol{p}^k,\\ p_i^{k+1}=\mathcal{P}_r(p_i^k+ru_i^{k+1}),\quad i=1,2,\cdots,n,\end{cases} \tag{4.2}$$

where

$$\mathcal{P}_r(z)=\begin{cases}\dfrac{2}{2+r}z & \text{if } z<0,\\ z & \text{if } 0\le z\le 1,\\ \dfrac{3}{3+r}z+\dfrac{r}{3+r} & \text{if } z>1.\end{cases} \tag{4.3}$$

By Theorem 3.4, we require $r<2(\frac{\tau}{h^2})(2\pi^2h^2)=4\pi^2\tau$. To test the convergence rate, we let $\boldsymbol{b}=\mathbf{0}$ and compute the ratios $\|\boldsymbol{p}^k\|/\|\boldsymbol{p}^{k-1}\|$ with a random initial value. It turns out that different initial values lead to slightly different numerical convergence rates. Some numerical convergence rates of the Uzawa algorithm (4.2) are shown in Tables 1.

Table 1 Numerical Convergence Rate ($\tau=0.0125$, $h=0.1$)

iteration number	$r=0.1$	$r=0.2$	$r=0.3$	$r=0.4$
1	0.9313	0.8756	0.8196	0.7836
2	0.9353	0.8795	0.8307	0.7940
3	0.9375	0.8817	0.8370	0.8012
4	0.9384	0.8835	0.8407	0.8040
5	0.9391	0.8852	0.8440	0.8082
10	0.9414	0.8918	0.8569	0.8266
20	0.9444	0.9050	0.8900	0.8913
50	0.9535	0.9499	0.9235	0.9010
100	0.9708	0.9516	0.9286	0.9074
200	0.9754	0.9541	0.9322	0.9101
300	0.9764	0.9549	0.9328	0.9103
400	0.9767	0.9556	0.9332	0.9103
500	0.9770	0.9560	0.9337	0.9103

The improved Uzawa Algorithm 3.5 reads ($a_0=1$):

Given the initial guess $\boldsymbol{p}^0 \in \mathbf{R}^n$ and $r > 0$, for $k = 0, 1, 2, \cdots$, compute $\boldsymbol{u}^{k+1}$ and $\boldsymbol{p}^{k+1}$ from

$$\begin{cases} (\frac{\tau}{h^2} A_0 + I)\boldsymbol{u}^{k+1} = \boldsymbol{b} - \boldsymbol{p}^k, \\ p_i^{k+1} = \mathcal{P}_r(p_i^k + r u_i^{k+1}), \quad i = 1, 2, \cdots, n, \end{cases} \tag{4.4}$$

with

$$\mathcal{P}_r(z) = \begin{cases} \dfrac{1}{1+r} z & \text{if } z < 0, \\ z & \text{if } 0 \le z \le 1, \\ \dfrac{2}{2+r} z + \dfrac{r}{2+r} & \text{if } z > 1. \end{cases} \tag{4.5}$$

Convergence requirement is $r < 2\lambda_{\min}(\frac{\tau}{h^2} A_0 + I) \le 2 + 4\pi^2\tau$. Some numerical convergence rates of the Uzawa algorithm (4.4) are shown in Tables 2 . We observe clearly the improved convergence rate of the algorithm (4.4) over the algorithm (4.2).

We have performed many experiments of convergence rate for various τ and h using algorithm (4.4). The convergence rates are almost the same as Table 2. For example, for $\tau = 0.003125$, $h = 0.05$ we have Table 3 .

Table 2 Numerical Convergence Rate ($\tau = 0.0125$, $h = 0.1$)

iteration number	$r = 1.0$	$r = 1.5$	$r = 1.8$	$r = 2.0$
1	0.5517	0.4212	0.3530	0.3775
2	0.6192	0.5263	0.4814	0.4077
3	0.6821	0.6028	0.5446	0.5296
4	0.7207	0.6299	0.5803	0.5088
5	0.7347	0.6512	0.6012	0.5984
10	0.7727	0.7001	0.6394	0.6231
20	0.7931	0.7125	0.6572	0.6321
50	0.8107	0.7240	0.6692	0.6371
100	0.8115	0.7295	0.6761	0.6416
200	0.8118	0.7303	0.6772	0.6419
300	0.8118	0.7304	0.6772	0.6419
400	0.8118	0.7304	0.6772	0.6419
500	0.8118	0.7304	0.6772	0.6419

Table 3 Numerical Convergence Rate ($\tau = 0.003125$, $h = 0.05$)

iteration number	$r = 1.0$	$r = 1.5$	$r = 1.8$	$r = 2.0$
1	0.5165	0.4105	0.3918	0.3560
2	0.5937	0.5127	0.4343	0.4127
3	0.6585	0.5829	0.5660	0.5571
4	0.6962	0.6127	0.5842	0.5689
5	0.7133	0.6371	0.6018	0.5859
10	0.7641	0.6802	0.6229	0.5967
20	0.7917	0.7025	0.6464	0.6188
50	0.8058	0.7168	0.6614	0.6364
100	0.8100	0.7180	0.6763	0.6412
200	0.8112	0.7185	0.6771	0.6419
300	0.8115	0.7186	0.6771	0.6419
400	0.8116	0.7187	0.6771	0.6419
500	0.8116	0.7188	0.6771	0.6419

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