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Minimization principle in study of a Stokes hemivariational inequality[☆]

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A B S T R A C T

In this paper, an equivalent minimization principle is established for a hemivariational inequality of the stationary Stokes equations with a nonlinear slip boundary condition. Under certain assumptions on the data, it is shown that there is a unique minimizer of the minimization problem, and furthermore, the mixed formulation of the Stokes hemivariational inequality has a unique solution. The proof of the result is based on basic knowledge of convex minimization. For comparison, in the existing literature, the solution existence and uniqueness result for the Stokes hemivariational inequality is proved through the notion of pseudomonotonicity and an application of an abstract surjectivity result for pseudomonotone operators, in which an additional linear growth condition is required on the subdifferential of a super-potential in the nonlinear slip boundary condition.

Numerous publications can be found on studies of boundary value problems for viscous incompressible fluid flows subject to slip or leak boundary conditions of friction type. Some sample references include [1–7]. In these references, the slip or leak boundary conditions are modelled by monotone relations and the weak formulations are variational inequalities governed by the Stokes or Navier–Stokes equations. When the slip or leak boundary conditions involve non-monotone relations, the corresponding problems are hemivariational inequalities. In the literature, some papers can be found on analysis of Stokes or Navier–Stokes hemivariational inequalities, e.g., [8–11]. In these references, as in most other references on hemivariational inequalities, the solution existence and uniqueness are proved by applying abstract surjectivity results for pseudomonotone operators. In this paper, we establish a minimization principle for the Stokes hemivariational inequality studied in [8]. Through the minimization principle, we improve the solution existence and uniqueness result presented in [8] by removing a linear growth assumption on the

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generalized subdifferential of the super-potential in the slip boundary condition. We note that for elliptic hemivariational inequalities, minimization principles are established in [12], and improved solution existence and uniqueness results are shown in [13].

In description of hemivariational inequalities, we need the notions of the generalized directional derivative and generalized subdifferential in the sense of Clarke [14]. Recall that for a locally Lipschitz continuous functional $\Psi\colon V \to \mathbb{R}$ defined on a real Banach space V, the generalized (Clarke) directional derivative of Ψ at $u \in V$ in the direction $v \in V$ is defined by

$$\Psi^0(u;v) := \limsup_{w\to u,\, \lambda\downarrow 0} \frac{\Psi(w+\lambda v) - \Psi(w)}{\lambda},$$

whereas the generalized subdifferential of Ψ at $u \in V$ is

$$\partial \Psi(u) := \left\{ \eta \in V^* \mid \Psi^0(u;v) \geq \langle \eta, v\rangle \ \forall\, v \in V \right\}.$$

Basic properties of the generalized directional derivative and the generalized subdifferential can be found in [14]. We record below some of the basic properties needed in this paper.

Proposition 1. *If $\Psi\colon V \to \mathbb{R}$ is locally Lipschitz continuous and convex, then the subdifferential $\partial\Psi(u)$ at any $u \in V$ in the sense of Clarke coincides with the convex subdifferential $\partial\Psi(u)$.*

Let $\Psi, \Psi_1, \Psi_2\colon V \to \mathbb{R}$ be locally Lipschitz functions. Then $\partial(\lambda\,\Psi)(u) = \lambda\,\partial\Psi(u)$ for all $\lambda \in \mathbb{R}$ and all $u \in V$. Moreover, the inclusion

$$\partial(\Psi_1 + \Psi_2)(u) \subseteq \partial\Psi_1(u) + \partial\Psi_2(u) \quad \forall\, u \in V \tag{1}$$

holds, or equivalently,

$$(\Psi_1 + \Psi_2)^0(u;v) \leq \Psi_1^0(u;v) + \Psi_2^0(u;v) \quad \forall\, u, v \in V. \tag{2}$$

We will consider a fluid flow problem in a Lipschitz domain $\Omega \subset \mathbb{R}^d$ ($d \leq 3$ in applications). The boundary $\partial\Omega$ is split into two non-trivial parts: $\partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}$ with $|\Gamma_0| > 0$, $|\Gamma_1| > 0$, and $\Gamma_0 \cap \Gamma_1 = \emptyset$. Since $\partial\Omega$ is Lipschitz continuous, the unit outward normal vector $\boldsymbol{n} = (n_1, \ldots, n_d)^T$ is defined a.e. on $\partial\Omega$. For a vector-valued function $\boldsymbol{u}$ on the boundary, its normal and tangential components are $u_n = \boldsymbol{u}\cdot\boldsymbol{n}$ and $\boldsymbol{u}_\tau = \boldsymbol{u} - u_n\boldsymbol{n}$, respectively. With the velocity field $\boldsymbol{u}$ and the pressure p, we define the strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)$ and the stress tensor $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\nu\boldsymbol{\varepsilon}(\boldsymbol{u})$, where $\boldsymbol{I}$ is the identity matrix. We call $\sigma_n = \boldsymbol{n}\cdot\boldsymbol{\sigma}\boldsymbol{n}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{n} - \sigma_n\boldsymbol{n}$ the normal and tangential components of $\boldsymbol{\sigma}$ on the boundary.

Denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. We adopt the summation convention over a repeated index. The indices i and j run between 1 and d. The canonical inner products and norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are

$$\begin{aligned}
&\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i, \quad \|\boldsymbol{v}\|_{\mathbb{R}^d} = (\boldsymbol{v}\cdot\boldsymbol{v})^{1/2} \quad \text{for all } \boldsymbol{u} = (u_i),\ \boldsymbol{v} = (v_i) \in \mathbb{R}^d,\\
&\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\sigma}\|_{\mathbb{S}^d} = (\boldsymbol{\sigma} : \boldsymbol{\sigma})^{1/2} \quad \text{for all } \boldsymbol{\sigma} = (\sigma_{ij}),\ \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d.
\end{aligned}$$

We consider the Stokes problem

$$-\operatorname{div}(2\nu\boldsymbol{\varepsilon}(\boldsymbol{u})) + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{3}$$

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{4}$$

with the following boundary conditions

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_0, \tag{5}$$

$$u_n = 0, \quad -\boldsymbol{\sigma}_\tau \in \partial\psi(\boldsymbol{u}_\tau) \quad \text{on } \Gamma_1. \tag{6}$$

Here, the super-potential $\psi : \mathbb{R}^d \to \mathbb{R}$ is assumed locally Lipschitz and $\partial\psi$ is the subdifferential of ψ in the sense of Clarke, $\nu > 0$ is the viscosity coefficient, and $\boldsymbol{f}$ is a given function. The relation (6) is known as a slip boundary condition. The first part $u_n = 0$ indicates that the fluid cannot pass through Γ_1 outside the domain, and the second part represents a friction condition for the friction $\boldsymbol{\sigma}_\tau$ in terms of the tangential velocity $\boldsymbol{u}_\tau$.

The function spaces for the velocity variable and the pressure variable are

$$V = \left\{\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0,\, v_n = 0 \text{ on } \Gamma_1\right\}, \tag{7}$$

$$Q = \left\{q \in L^2(\Omega) \mid (q,1)_\Omega = 0\right\}. \tag{8}$$

Since $|\Gamma_0| > 0$, Korn's inequality holds (cf. [15, p. 79]): for a constant $c_1 > 0$ depending only on Ω and Γ_0,

$$\|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^d)} \le c_1 \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{L^2(\Omega;\mathbb{S}^d)} \quad \forall\, \boldsymbol{v} \in V. \tag{9}$$

Thus $\|\boldsymbol{\varepsilon}(\cdot)\|_{L^2(\Omega;\mathbb{S}^d)}$ defines a norm and is equivalent to the standard $H^1(\Omega;\mathbb{R}^d)$-norm on V. We use the norm $\|\cdot\|_V = \|\boldsymbol{\varepsilon}(\cdot)\|_{L^2(\Omega;\mathbb{S}^d)}$ on V. By the Sobolev trace theorem, we have the inequality

$$\|\boldsymbol{v}_\tau\|_{L^2(\Gamma_1;\mathbb{R}^d)} \le \lambda_0^{-1/2} \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V, \tag{10}$$

where $\lambda_0 > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in V, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx = \lambda \int_{\Gamma_1} \boldsymbol{u}_\tau \cdot \boldsymbol{v}_\tau\, ds \quad \forall\, \boldsymbol{v} \in V. \tag{11}$$

We assume $\boldsymbol{f} \in V^*$. Introduce the following bilinear and linear forms:

$$a(\boldsymbol{u},\boldsymbol{v}) = 2\nu \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx \quad \forall\, \boldsymbol{u},\boldsymbol{v} \in V, \tag{12}$$

$$b(\boldsymbol{v},q) = -\int_\Omega q \,\mathrm{div} \boldsymbol{v}\, dx \quad \forall\, \boldsymbol{v} \in V,\, q \in Q, \tag{13}$$

$$\langle \boldsymbol{f},\boldsymbol{v}\rangle = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx \quad \forall\, \boldsymbol{v} \in V. \tag{14}$$

Concerning the super-potential $\psi : \mathbb{R}^d \to \mathbb{R}$, we assume the following properties:

$H(\psi)$. ψ is locally Lipschitz on $\mathbb{R}^d$ and for a constant $\alpha_\psi \ge 0$,

$$\psi^0(\boldsymbol{\xi}_1;\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1) + \psi^0(\boldsymbol{\xi}_2;\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \le \alpha_\psi \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|^2_{\mathbb{R}^d} \quad \forall\, \boldsymbol{\xi}_1,\boldsymbol{\xi}_2 \in \mathbb{R}^d. \tag{15}$$

It is known [16] that (15) is equivalent to

$$(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \ge -\alpha_\psi \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|^2_{\mathbb{R}^d} \quad \forall \boldsymbol{\xi}_i \in \mathbb{R}^d,\, \boldsymbol{\eta}_i \in \partial\psi(\boldsymbol{\xi}_i),\, i = 1,2. \tag{16}$$

Remark 2. In [8], the following property is also assumed on ψ for the solution existence and uniqueness, due to the need to apply an abstract surjectivity result on pseudomonotonic operators.

$$\|\boldsymbol{\eta}\|_{\mathbb{R}^d} \le c_0 + c_1 \|\boldsymbol{\xi}\|_{\mathbb{R}^d} \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^d, \boldsymbol{\eta} \in \partial\psi(\boldsymbol{\xi}) \text{ with } c_0, c_1 \ge 0.$$

Such a condition is assumed in other existing references on hemivariational inequalities, e.g., [16].

In this paper, we prove the solution existence and uniqueness without this assumption.

Now we consider the functional $\Psi : L^2(\Gamma_1;\mathbb{R}^d) \to \mathbb{R}$ defined by

$$\Psi(\boldsymbol{v}) = \int_{\Gamma_1} \psi(\boldsymbol{v})\, ds, \quad \boldsymbol{v} \in L^2(\Gamma_1;\mathbb{R}^d). \tag{17}$$

From the proof of Theorem 4.20 in [16], we have the following result.

Lemma 3. *Assume that $\psi\colon \mathbb{R}^d \to \mathbb{R}$ has the properties $H(\psi)$. Then the functional Ψ defined by (17) is locally Lipschitz on $L^2(\Gamma_1;\mathbb{R}^d)$ and*

$$\Psi^0(\boldsymbol{u};\boldsymbol{v}) \le \int_{\Gamma_1} \psi^0(\boldsymbol{u}_\tau;\boldsymbol{v}_\tau)\,ds \quad \forall\,\boldsymbol{u},\boldsymbol{v}\in V. \tag{18}$$

Combining (15) and (18), we have, for $\boldsymbol{v}_1,\boldsymbol{v}_2\in V$,

$$\begin{aligned}\Psi^0(\boldsymbol{v}_1;\boldsymbol{v}_2-\boldsymbol{v}_1)+\Psi^0(\boldsymbol{v}_2;\boldsymbol{v}_1-\boldsymbol{v}_2) &\le \int_{\Gamma_1}\left[\psi^0(\boldsymbol{v}_{1,\tau};\boldsymbol{v}_{2,\tau}-\boldsymbol{v}_{1,\tau})+\psi^0(\boldsymbol{v}_{2,\tau};\boldsymbol{v}_{1,\tau}-\boldsymbol{v}_{2,\tau})\right]ds\\ &\le \alpha_\psi\int_{\Gamma_1}|\boldsymbol{v}_{1,\tau}-\boldsymbol{v}_{2,\tau}|^2 ds.\end{aligned}$$

Then apply (10) to obtain

$$\Psi^0(\boldsymbol{v}_1;\boldsymbol{v}_2-\boldsymbol{v}_1)+\Psi^0(\boldsymbol{v}_2;\boldsymbol{v}_1-\boldsymbol{v}_2)\le \alpha_\psi\lambda_0^{-1}\|\boldsymbol{v}_1-\boldsymbol{v}_2\|_V^2 \quad \forall\,\boldsymbol{v}_1,\boldsymbol{v}_2\in V. \tag{19}$$

Equivalently,

$$\langle\boldsymbol{\eta}_1-\boldsymbol{\eta}_2,\boldsymbol{v}_1-\boldsymbol{v}_2\rangle \ge -\alpha_\psi\lambda_0^{-1}\|\boldsymbol{v}_1-\boldsymbol{v}_2\|_V^2 \quad \forall\,\boldsymbol{v}_i\in V,\ \boldsymbol{\eta}_i\in\partial\Psi(\boldsymbol{v}_i),\ i=1,2. \tag{20}$$

The mixed weak formulation of problems (3)–(6) is as follows [8].

Problem 4. Find $(\boldsymbol{u},p)\in V\times Q$ such that

$$a(\boldsymbol{u},\boldsymbol{v})+b(\boldsymbol{v},p)+\int_{\Gamma_1}\psi^0(\boldsymbol{u}_\tau;\boldsymbol{v}_\tau)\,ds \ge \langle\boldsymbol{f},\boldsymbol{v}\rangle \quad \forall\,\boldsymbol{v}\in V, \tag{21}$$

$$b(\boldsymbol{u},q)=0 \quad \forall\,q\in Q. \tag{22}$$

It is possible to eliminate the unknown pressure $p\in Q$ to deduce a reduced weak formulation. For this purpose, we introduce a subspace of V:

$$\tilde V = \{\boldsymbol{v}\in V \mid \operatorname{div}\boldsymbol{v}=0 \text{ in } \Omega\}. \tag{23}$$

Problem 5. Find $\boldsymbol{u}\in\tilde V$ such that

$$a(\boldsymbol{u},\boldsymbol{v})+\int_{\Gamma_1}\psi^0(\boldsymbol{u}_\tau;\boldsymbol{v}_\tau)\,ds \ge \langle\boldsymbol{f},\boldsymbol{v}\rangle \quad \forall\,\boldsymbol{v}\in\tilde V. \tag{24}$$

Let us study Problem 5 through an equivalent minimization principle. At this point, we recall a sufficient and necessary condition on strong convexity of a locally Lipschitz continuous function, characterized by the strong monotonicity of its generalized subdifferential in the sense of Clarke, cf. [17, Proposition 3.1].

Lemma 6. *Let V be a real Banach space and let $g\colon V\to\mathbb{R}$ be locally Lipschitz continuous. Then g is strongly convex on V with a constant $\alpha>0$, i.e.,*

$$g(\lambda\,u+(1-\lambda)\,v)\le\lambda\,g(u)+(1-\lambda)\,g(v)-\alpha\,\lambda\,(1-\lambda)\,\|u-v\|_V^2 \quad \forall\,u,v\in V,\ \forall\,\lambda\in[0,1],$$

if and only if ∂g is strongly monotone on V with a constant $2\,\alpha$, i.e.,

$$\langle\xi-\eta,u-v\rangle\ge 2\,\alpha\,\|u-v\|_V^2 \quad \forall\,u,v\in V,\ \xi\in\partial g(u),\ \eta\in\partial g(v).$$

A proof of the next result can be found in [12].

Proposition 7. *Let V be a real Hilbert space and let $g\colon V \to \mathbb{R}$ be a locally Lipschitz continuous and strongly convex functional on V with a constant $\alpha > 0$. Then there exist two constants $\overline{c}_0$ and $\overline{c}_1$ such that*

$$g(v) \geq \alpha \|v\|_V^2 + \overline{c}_0 + \overline{c}_1 \|v\|_V \quad \forall\, v \in V. \tag{25}$$

Consequently, $g(\cdot)$ is coercive on V.

Define an operator $A\colon V \to V^*$ by

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = a(\boldsymbol{u}, \boldsymbol{v}) \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V. \tag{26}$$

Note that

$$\langle A\boldsymbol{v}, \boldsymbol{v}\rangle = a(\boldsymbol{v}, \boldsymbol{v}) = 2\,\nu\, \|\boldsymbol{v}\|_V^2 \quad \forall\, \boldsymbol{v} \in V. \tag{27}$$

Then we introduce an energy functional

$$E(\boldsymbol{v}) = \frac{1}{2}\, a(\boldsymbol{v}, \boldsymbol{v}) + \Psi(\boldsymbol{v}) - \langle \boldsymbol{f}, \boldsymbol{v}\rangle, \quad \boldsymbol{v} \in \tilde{V}, \tag{28}$$

and consider a minimization problem related to Problem 5.

Problem 8. Find $\boldsymbol{u} \in \tilde{V}$ such that

$$E(\boldsymbol{u}) = \inf \left\{ E(\boldsymbol{v}) \mid \boldsymbol{v} \in \tilde{V} \right\}.$$

Theorem 9. *Assume $H(\psi)$ and $\alpha_\psi < 2\,\nu\,\lambda_0$. Then for any $\boldsymbol{f} \in V^*$, Problem 8 has a unique solution.*

Proof. Obviously, $E\colon \tilde{V} \to \mathbb{R}$ is locally Lipschitz continuous, and by (1),

$$\partial E(\boldsymbol{v}) \subset A\boldsymbol{v} + \partial \Psi(\boldsymbol{v}) - \boldsymbol{f}. \tag{29}$$

For any $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \tilde{V}$ and any $\boldsymbol{\zeta}_i \in \partial E(\boldsymbol{v}_i)$, $i = 1, 2$, write

$$\boldsymbol{\zeta}_i = A\boldsymbol{v}_i + \boldsymbol{\eta}_i - \boldsymbol{f}, \quad \boldsymbol{\eta}_i \in \partial \Psi(\boldsymbol{v}_i),\ i = 1, 2.$$

Apply (27) and (20),

$$\langle \boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2\rangle = a(\boldsymbol{v}_1 - \boldsymbol{v}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2) + \langle \boldsymbol{\eta}_1 - \boldsymbol{\eta}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2\rangle \geq \left(2\nu - \alpha_\psi \lambda_0^{-1}\right) \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V^2.$$

Hence, by Lemma 6, the energy functional E is strongly convex on $\tilde{V}$. Moreover, by Proposition 7, E is coercive on $\tilde{V}$. It is well known (e.g. [18, Subsection 3.3.2]) that such a functional E has a unique minimizer $\boldsymbol{u}$ on $\tilde{V}$, i.e., Problem 8 has a unique solution. ■

Theorem 10. *Assume $H(\psi)$, $\alpha_\psi < 2\,\nu\,\lambda_0$, and $\boldsymbol{f} \in V^*$. Then the unique solution of Problem 8 is also the unique solution of Problem 5.*

Proof. For the solution $\boldsymbol{u}$ of Problem 8, we have

$$E^0(\boldsymbol{u}; \boldsymbol{v}) \geq 0 \quad \forall\, \boldsymbol{v} \in \tilde{V}.$$

Similar to (29), by (2),

$$E^0(\boldsymbol{u}; \boldsymbol{v}) \leq a(\boldsymbol{u}, \boldsymbol{v}) + \Psi^0(\boldsymbol{u}; \boldsymbol{v}) - \langle \boldsymbol{f}, \boldsymbol{v}\rangle,$$

we see that the solution $\boldsymbol{u}$ of Problem 8 satisfies

$$a(\boldsymbol{u}, \boldsymbol{v}) + \Psi^0(\boldsymbol{u}; \boldsymbol{v}) \geq \langle \boldsymbol{f}, \boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in \tilde{V}.$$

By (18), this inequality implies (24). Thus, a solution of Problem 8 is also a solution of Problem 5. Uniqueness of a solution to Problem 5 can be shown by a standard procedure. Therefore, both Problems 5 and 8 have the same unique solution. ■

Finally, since Problems 4 and 5 are equivalent ([8, Theorem 3.4]), we have the next result concerning the unique solvability of Problem 4.

Theorem 11. *Assume $H(\psi)$ and $\alpha_\psi < 2\,\nu\,\lambda_0$. Then for any $\boldsymbol{f} \in V^*$, Problem 4 has a unique solution $(\boldsymbol{u}, p) \in V \times Q$.*

References

[1] H. Fujita, A mathematical analysis of motions of viscous incompressible fluid under leak or slip boundary conditions, RIMS Kôkyûroku 888 (1994) 199–216.
[2] H. Fujita, H. Kawarada, Variational inequalities for the Stokes equation with boundary conditions of friction type, GAKUTO Int. Ser. Math. Sci. Appl. 11 (1998) 15–33.
[3] T. Kashiwabara, On a finite element approximation of the Stokes equations under a slip boundary condition of the friction type, Japan J. Indust. Appl. Math. 30 (2013) 227–261.
[4] C. Le Roux, Steady Stokes flows with threshold slip boundary conditions, Math. Models Methods Appl. Sci. 15 (2005) 1141–1168.
[5] C. Le Roux, A. Tani, Steady solutions of the Navier–Stokes equations with threshold slip boundary conditions, Math. Methods Appl. Sci. 30 (2007) 595–624.
[6] Y. Li, R. An, Semi-discrete stabilized finite element methods for Navier–Stokes equations with nonlinear slip boundary conditions based on regularization procedure, Numer. Math. 117 (2011) 1–36.
[7] F. Saidi, Non-Newtonian Stokes flow with frictional boundary conditions, Math. Model. Anal. 12 (2007) 483–495.
[8] C. Fang, K. Czuprynski, W. Han, X.L. Cheng, X. Dai, Finite element method for a stationary Stokes hemivariational inequality with slip boundary condition, IMA J. Numer. Anal. 40 (2020) 2696–2716.
[9] C. Fang, W. Han, S. Migórski, M. Sofonea, A class of hemivariational inequalities for nonstationary Navier–Stokes equations, Nonlinear Anal. RWA 31 (2016) 257–276.
[10] S. Migórski, A note on optimal control problem for a hemivariational inequality modeling fluid flow, Discrete Contin. Dyn. Syst. (Supplement) (2013) 545–554.
[11] S. Migórski, A. Ochal, Hemivariational inequalities for stationary Navier–Stokes equations, J. Math. Anal. Appl. 306 (2005) 197–217.
[12] W. Han, Minimization principles for elliptic hemivariational inequalities, Nonlinear Anal. RWA 54 (2020) #103114.
[13] W. Han, A revisit of elliptic variational–hemivariational inequalities, Numer. Funct. Anal. Optim. 42 (2021) http://dx.doi.org/10.1080/01630563.2021.1881541, in press.
[14] F.H. Clarke, Optimization and Nonsmooth Analysis, Wiley, New York, 1983.
[15] J. Nečas, I. Hlavaček, Mathematical Theory of Elastic and Elastoplastic Bodies: An Introduction, Elsevier, Amsterdam, 1981.
[16] S. Migórski, A. Ochal, M. Sofonea, Nonlinear Inclusions and Hemivariational Inequalities, in: Models and Analysis of Contact Problems, Advances in Mechanics and Mathematics, vol. 26, Springer, New York, 2013.
[17] L. Fan, S. Liu, S. Gao, Generalized monotonicity and convexity of non-differentiable functions, J. Math. Anal. Appl. 279 (2003) 276–289.
[18] K. Atkinson, W. Han, Theoretical Numerical Analysis: A Functional Analysis Framework, Third ed., Springer, New York, 2009.