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Numerical analysis of history-dependent variational-hemivariational inequalities

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Abstract In this paper, numerical analysis is carried out for a class of history-dependent variational-hemivariational inequalities by arising in contact problems. Three different numerical treatments for temporal discretization are proposed to approximate the continuous model. Fixed-point iteration algorithms are employed to implement the implicit scheme and the convergence is proved with a convergence rate independent of the time step-size and mesh grid-size. A special temporal discretization is introduced for the history-dependent operator, leading to numerical schemes for which the unique solvability and error bounds for the temporally discrete systems can be proved without any restriction on the time step-size. As for spatial approximation, the finite element method is applied and an optimal order error estimate for the linear element solutions is provided under appropriate regularity assumptions. Numerical examples are presented to illustrate the theoretical results.

Keywords variational-hemivariational inequality, Clarke subdifferential, history-dependent operator, fixed-point iteration, optimal order error estimate, contact mechanics

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1 Introduction

The theory of variational and hemivariational inequalities plays an important role in the study of nonlinear problems arising in Contact Mechanics, Physics, Economics and Engineering. It is generally agreed that interest in variational inequalities started with a contact problem posed by Signorini in 1930s. The mathematical theory of variational inequalities relies on the properties of monotonicity, convexity and the subdifferential of a convex function. The existence and uniqueness results can be found in [3, 17, 18]. In terms of the numerical analysis for variational inequalities, the readers are referred to, e.g., [7, 8, 15]. Hemivariational inequalities as a useful generalization of variational inequalities were introduced in early 1980s by Panagiotopoulos [22]. For hemivariational inequalities, the notion of the subdifferential in the sense of Clarke [5, 6], defined for the locally Lipschitz function, plays an important role. Mathematical theory of hemivariational inequalities is documented in several research monographs (see, e.g., [4, 19, 21,

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23, 27]). A comprehensive reference on the numerical solution of hemivariational inequalities is [14] where the finite element method is applied to solve hemivariational inequalities, convergence of the numerical solution is discussed, and solution algorithms are proposed and tested. More recently, there has been extensive research effort on optimal order error estimation and general convergence analysis of numerical solutions for hemivariational inequalities (see, e.g., [2, 9, 10, 12, 13], and the survey paper [11]).

Variational-hemivariational inequalities are a particular family of hemivariational inequalities, having a special structure that includes both convex and nonconvex functionals. Such inequalities arise naturally in mathematical models for many contact problems (see [27] and the references therein). A class of history-dependent variational-hemivariational inequalities with convex constraint is studied in [26]. The novel structure of the inequalities involves a history-dependent operator, unilateral constraint and two nondifferential functions, one of which is convex and the other may be nonconvex. Existence, uniqueness and continuous dependence results are shown on the inequalities, and are applied to the study of a quasistatic frictionless contact problem. Numerical approximations of the history-dependent variational-hemivariational inequalities are the topic of [28], where the second-order accuracy for temporal discretization is achieved by using the trapezoidal rule to approximate the history-dependent term. The spatial discretization is done by using the linear finite element and an optimal order error estimate is proved. Note that for the numerical method studied in [28], a restriction on the time step-size is needed to ensure the unique solvability of the numerical solution. In this paper, we develop new numerical methods to solve the history-dependent variational-hemivariational inequalities with the property that no restriction on the time step-size is needed for the unique solvability of the numerical solution. Specifically, we use a partial trapezoidal rule to approximate the history-dependent operator, i.e., we modify the trapezoidal rule by applying the left-point rectangular rule for the sub-integral over the last time sub-interval. Consequently, the history-dependent term is treated explicitly without loss of accuracy. This explicit treatment of the history-dependent term eliminates the need for a restriction on the time step-size. Although the explicit treatment is given in the history-dependent term, other implicit terms in the numerical scheme remain. We provide a fixed-point iterative algorithm to implement the implicit scheme and prove convergence of the iterative scheme, with a convergence rate independent of the time step-size and the mesh grid-size. In addition, we propose two more schemes to solve the history-dependent variational-hemivariational inequalities. One is of first-order and the other is of second-order with a slightly stringent small condition compared with that of the other two schemes. For all the three schemes, optimal order error estimates with linear finite elements for spatial approximation are shown.

The rest of the paper is organized as follows. In Section 2, we review some preliminary materials on functional analysis and present the history-dependent variational-hemivariational inequality problem. In Section 3, we propose three temporally semi-discrete schemes to approximate the continuous problem and error estimates are established. The corresponding fully discrete schemes are provided in Section 4, and the error estimates are derived for the discrete problems with or without convex constraints. To implement the second-order implicit scheme, in Section 5 we describe a fixed-point iterative process and prove that the iteration converges linearly with a convergence rate independent of the time step-size and mesh grid-size. Then in Section 6 we apply the theoretical results developed in the previous sections in the numerical solution of a viscoelastic contact problem and obtain an optimal order error estimate for the linear finite element solutions under appropriate solution regularity assumptions. In Section 7 we report results from simulation tests, focusing on the numerical evidence of the convergence orders.

2 Preliminaries

In this section we recall some notation, definitions and preliminary materials. Then we present a class of history-dependent variational-hemivariational inequalities introduced in [26].

For the normed spaces X and X_j, let X^* and X_j^* be their topological duals, and write $\|\cdot\|_X$, $\|\cdot\|_{X_j}$, $\|\cdot\|_{X^*}$ and $\|\cdot\|_{X_j^*}$ for their norms. The duality pairing between X and X^*, $\langle\cdot,\cdot\rangle_{X^*\times X}$, is usually simply written as $\langle\cdot,\cdot\rangle$. Similarly, the duality pairing between X_j^* and X_j, $\langle\cdot,\cdot\rangle_{X_j^*\times X_j}$, is usually written as $\langle\cdot,\cdot\rangle_{X_j}$.

For a convex function $\varphi: X \to \mathbb{R}\cup\{+\infty\}$, the subset $\partial\varphi(x)$ of X^*,

$$\partial\varphi(x) = \{x^* \in X^* \mid \varphi(v) - \varphi(x) \geqslant \langle x^*, v-x\rangle_{X^*\times X},\ \forall\, v \in X\}$$

is called the subdifferential (see [24]) of φ. If $\partial\varphi(x)$ is non-empty, any element $x^* \in \partial\varphi(x)$ is called a subgradient of φ at x. Let $\phi: X \to \mathbb{R}$ be a locally Lipschitz function. The generalized (Clarke) directional derivative of ϕ at x in the direction $v \in X$ is defined by (see [6])

$$\phi^0(x;v) = \limsup_{y\to x,\,\lambda\downarrow 0} \frac{\phi(y+\lambda v) - \phi(y)}{\lambda}.$$

The generalized gradient (subdifferential) of ϕ at x is a subset of the dual space X^* given by

$$\partial\phi(x) = \{\xi \in X^* \mid \phi^0(x;v) \geqslant \langle \xi, v\rangle_{X^*\times X},\ \forall\, v \in X\}.$$

An operator $A: X \to X^*$ is pseudomonotone (see [19]) if it is bounded and $u_n \to u$ weakly in X together with $\limsup_n \langle Au_n, u_n - u\rangle_{X^*\times X} \leqslant 0$ imply

$$\langle Au, u-v\rangle_{X^*\times X} \leqslant \liminf_n \langle Au_n, u_n - v\rangle_{X^*\times X}, \quad \forall\, v \in X.$$

Next, we turn to some preliminary materials on function spaces and related operators. Following the standard notation, we denote by $\mathbb{N}$ the set of positive integers, $\mathbb{R}_+ = [0,+\infty)$ the set of nonnegative real numbers, $C(\mathbb{R}_+;X)$ and $C^1(\mathbb{R}_+;X)$ the spaces of continuous and continuously differentiable functions from $\mathbb{R}_+$ to X, respectively. It is well known that if X is a Banach space, $C(\mathbb{R}_+;X)$ can be organized in a canonical way as a Fréchet space, i.e., it is a complete metric space in which the corresponding topology is induced by a countable family of seminorms. Furthermore, $x_k \to x$ in $C(\mathbb{R}_+;X)$ as $k\to\infty$ if and only if $\max_{r\in[0,n]} \|x_k(r) - x(r)\|_X \to 0$ as $k \to \infty$ for all $n \in \mathbb{N}$.

Let two normed spaces X and Y be given. Following [25], an operator $\mathcal{S}: C(\mathbb{R}_+;X) \to C(\mathbb{R}_+;Y)$ is called history-dependent if for any $n \in \mathbb{N}$, there exists an $s_n > 0$ such that for all $t \in [0,n]$,

$$\|(\mathcal{S}u_1)(t) - (\mathcal{S}u_2)(t)\|_Y \leqslant s_n \int_0^t \|u_1(s) - u_2(s)\|_X ds, \quad \forall\, u_1, u_2 \in C(\mathbb{R}_+;X). \tag{2.1}$$

Now we are in a position to introduce the variational-hemivariational inequalities. Let X, X_j and Y be normed spaces and $K \subset X$. Given the operators $A: X \to X^*$, $\mathcal{S}: C(\mathbb{R}_+;X) \to C(\mathbb{R}_+;Y)$, $\gamma_j: X \to X_j$ and the functions $\varphi: Y\times K\times K \to \mathbb{R}$, $j: X_j \to \mathbb{R}$, we consider the following problem (see [26,28]).

Problem 2.1. *Find $u \in C(\mathbb{R}_+;K)$ such that for all $t \in \mathbb{R}_+$,*

$$\begin{aligned}&\langle Au(t), v-u(t)\rangle + \varphi((\mathcal{S}u)(t), u(t), v) - \varphi((\mathcal{S}u)(t), u(t), u(t))\\ &\quad + j^0(\gamma_j u(t); \gamma_j v - \gamma_j u(t)) \geqslant \langle f(t), v-u(t)\rangle, \quad \forall\, v \in K.\end{aligned} \tag{2.2}$$

In the study of Problem 2.1, the following hypotheses are adopted (see [26,28]):

$$X \text{ is a reflexive Banach space, } K \text{ is a closed and convex subset of } X \text{ with } 0 \in K, \tag{2.3}$$

$$\begin{cases} X_j \text{ is a Banach space, } \gamma_j \in \mathcal{L}(X;X_j), \text{ there exists } c_j > 0 \text{ such that}\\ \quad \|\gamma_j v\|_{X_j} \leqslant c_j \|v\|_X, \quad \forall\, v \in X,\end{cases} \tag{2.4}$$

$$\begin{cases} A: X \to X^* \text{ is an operator such that}\\ \quad \text{(a) } A \text{ is Lipschitz continuous with a Lipschitz constant } L_A > 0;\\ \quad \text{(b) } A \text{ is strongly monotone, i.e., there exists } m_A > 0 \text{ such that}\\ \qquad \langle Av_1 - Av_2, v_1 - v_2\rangle \geqslant m_A \|v_1 - v_2\|_X^2, \quad \forall\, v_1, v_2 \in X,\end{cases} \tag{2.5}$$

$$
\begin{cases}
\varphi : Y \times K \times K \to \mathbb{R} \text{ is a function such that} \\
\quad \text{(a) } \varphi(y,u,\cdot) : K \to \mathbb{R} \text{ is convex and lower semi-continuous on } K, \forall\, y \in Y, \forall\, u \in K; \\
\quad \text{(b) there exists } \alpha_\varphi > 0 \text{ and } \beta_\varphi > 0 \text{ such that} \\
\qquad \varphi(y_1,u_1,v_2) - \varphi(y_1,u_1,v_1) + \varphi(y_2,u_2,v_1) - \varphi(y_2,u_2,v_2) \\
\qquad\quad \leqslant \alpha_\varphi \|u_1 - u_2\|_X \|v_1 - v_2\|_X + \beta_\varphi \|y_1 - y_2\|_Y \|v_1 - v_2\|_X \\
\qquad\qquad \forall\, y_1, y_2 \in Y, \quad \forall\, u_1, u_2, v_1, v_2 \in K,
\end{cases} \tag{2.6}
$$

$$
\mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y) \text{ is a history-dependent operator,} \tag{2.7}
$$

$$
\begin{cases}
j : X_j \to \mathbb{R} \text{ is a function such that} \\
\quad \text{(a) } j \text{ is locally Lipschitz;} \\
\quad \text{(b) } \|\partial j(z)\|_{X_j^*} \leqslant c_0 + c_1 \|z\|_{X_j}, \ \forall\, z \in X_j \text{ with } c_0, c_1 \geqslant 0; \\
\quad \text{(c) there exists } \alpha_j > 0 \text{ such that} \\
\qquad j^0(z_1; z_2 - z_1) + j^0(z_2; z_1 - z_2) \leqslant \alpha_j \|z_1 - z_2\|_{X_j}^2, \quad \forall\, z_1, z_2 \in X_j.
\end{cases} \tag{2.8}
$$

$$
f \in C(\mathbb{R}_+; X^*), \tag{2.9}
$$

$$
\alpha_\varphi + \alpha_j c_j^2 < m_A. \tag{2.10}
$$

The space X_j is introduced for convenience of error estimation for the discrete problems. For a specific contact problem, X_j can be the space of square integrable functions over the contact boundary and $\gamma_j : X \to X_j$ is the corresponding trace operator. For a locally Lipschitz function j, (2.8)(c) is equivalent to the following relaxed monotonicity condition:

$$
\langle \partial j(z_1) - \partial j(z_2), z_1 - z_2 \rangle \geqslant -\alpha_j \|z_1 - z_2\|_{X_j}^2, \quad \forall\, z_1, z_2 \in X_j.
$$

The unique solvability of Problem 2.1 has been shown in [28] under the conditions (2.3)–(2.10). We consider the following form of the operator $\mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y)$ (see [25]):

$$
(\mathcal{S}v)(t) = R\bigg(\int_0^t q(t,s) v(s) ds + a_S \bigg), \quad \forall\, v \in C(\mathbb{R}_+; X), \quad \forall\, t \in \mathbb{R}_+, \tag{2.11}
$$

where $R \in \mathcal{L}(X; Y)$, $q \in C(\mathbb{R}_+ \times \mathbb{R}_+; \mathcal{L}(X))$, $a_S \in X$. It can be shown that the operator $\mathcal{S}$ given by (2.11) is a history-dependent operator.

3 Temporally semi-discrete approximations

In [28], a second-order numerical scheme is provided to approximate the continuous Problem 2.1 with a restriction on the time step-size. In this section, we handle the history-dependent term in a different manner, and propose three temporally discrete schemes for solving Problem 2.1 without any restriction on the time step-size. Moreover, we derive the corresponding convergence results. Below we use C to represent a positive constant independent of time step-size and mesh grid-size. We use the standard notation for Sobolev spaces (see [1]).

For a fixed $T \in \mathbb{R}_+$, we split the time interval $I = [0, T]$ by uniform partitions. Given a positive integer N, let $k = T/N$ be the time step-size, and denote by $t_n = nk$, $0 \leqslant n \leqslant N$, the nodes. We comment that all the discussions below can be extended to the case with non-uniform partitions of the time interval. For a continuous function v of the temporal variable t, we write $v_j = v(t_j)$, $0 \leqslant j \leqslant N$. For a discretization of the history-dependent operator $\mathcal{S}$ in (2.11), we employ a modified trapezoidal rule to approximate the integral $\int_0^{t_n} q(t,s) v(s) ds$ in the sense that on the last sub-interval $[t_{n-1}, t_n]$, the left-point rectangular rule is applied. Recall the trapezoidal rule

$$
\int_0^{t_n} Z(s) ds \approx \frac{k}{2} Z(t_0) + k \sum_{j=1}^{n-1} Z(t_j) + \frac{k}{2} Z(t_n). \tag{3.1}
$$

The approximation of $\mathcal{S}_n := \mathcal{S}(t_n)$ can be defined as follows:

$$\mathcal{S}^k_{n,L} v := R\bigg(\frac{k}{2} q(t_n, t_0) v_0 + k \sum_{j=1}^{n-1} q(t_n, t_j) v_j + \frac{k}{2} q(t_n, t_{n-1}) v_{n-1} + a_S\bigg). \tag{3.2}$$

Using arguments similar to that in [16, Section 3], for $v \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}_+; X)$ and $q \in C^1(\mathbb{R}_+ \times \mathbb{R}_+; \mathcal{L}(X))$, we have

$$\|\mathcal{S}^k_{n,L} v - \mathcal{S}_n v\| \leqslant Ck \|v\|_{W^{1,\infty}(I;X)}, \tag{3.3}$$

and for $v \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}_+; X)$ and $q \in C^2(\mathbb{R}_+ \times \mathbb{R}_+; \mathcal{L}(X))$,

$$\|\mathcal{S}^k_{n,L} v - \mathcal{S}_n v\| \leqslant C\, k^2 \|v\|_{W^{2,\infty}(I;X)}. \tag{3.4}$$

Remark 3.1. The choice of the operator $\mathcal{S}^k_{n,L}$ used to approximate $\mathcal{S}_n$ is not unique. For example, we may choose

$$\hat{\mathcal{S}}^k_n v := R\bigg(\frac{k}{2} q(t_n, t_0) v_0 + k \sum_{j=1}^{n-1} q(t_n, t_j) v_j + \frac{k}{2} (2q(t_n, t_{n-1}) v_{n-1} - q(t_n, t_{n-2}) v_{n-2}) + a_S\bigg),$$

which defines another second-order accurate approximation of $\mathcal{S}_n$, or choose

$$\widetilde{\mathcal{S}}^k_n v := R\bigg(k \sum_{j=0}^{n-1} q(t_n, t_j) v_j + a_S\bigg),$$

which is a first-order accurate approximation.

We note that the following weak formulation is equivalent to Problem 2.1.

Problem 3.2. *Find $u \in C(\mathbb{R}_+; K)$ such that for all $t \in \mathbb{R}_+$,*

$$\begin{aligned} &\langle Au(t), v - u(t)\rangle + \varphi((\mathcal{S}u)(t), u(t), v) - \varphi((\mathcal{S}u)(t), u(t), u(t)) \\ &\quad + j^0(\gamma_j u(t); \gamma_j v - \gamma_j u(t)) + \langle j_c(\gamma_j u(t)), \gamma_j v - \gamma_j u(t)\rangle_{X_j} \\ &\quad\quad \geqslant \langle f(t), v - u(t)\rangle + \langle j_c(\gamma_j u(t)), \gamma_j v - \gamma_j u(t)\rangle_{X_j}, \quad \forall\, v \in K. \end{aligned} \tag{3.5}$$

In [28], j_c is chosen as the differential of a quadratic function $\frac{\alpha}{2}\|u\|^2_{X_j}$. In this paper, we discuss about j_c in a more general framework. Assume

$$\begin{cases} j_c : X_j \to X_j^* \quad \text{is a linear operator such that} \\ \quad \text{(a) } \|j_c(z)\|_{X_j^*} \leqslant \alpha_c \|z\|_{X_j}, \ \forall\, z \in X_j; \\ \quad \text{(b) } \langle j_c(z), z\rangle_{X_j} \geqslant \alpha_j \|z\|^2_{X_j}, \ \forall\, z \in X_j. \end{cases} \tag{3.6}$$

The operator j_c can be regarded as a convexification of j^0 in the sense that

$$\begin{aligned} &j^0(z_1; z_2 - z_1) + j^0(z_2; z_1 - z_2) + \langle j_c(z_1), z_2 - z_1\rangle_{X_j} + \langle j_c(z_2), z_1 - z_2\rangle_{X_j} \\ &\quad \leqslant \alpha_j \|z_1 - z_2\|^2_{X_j} + \langle j_c(z_1 - z_2), z_2 - z_1\rangle_{X_j} \\ &\quad \leqslant 0, \end{aligned} \tag{3.7}$$

where the last equality follows from (3.6)(b).

3.1 A first-order temporally semi-discrete scheme

The first-order temporally semi-discrete scheme for Problem 2.1 is the following.

Problem 3.3. *Find a discrete solution $u^k := \{u^k_n\}_{n=0}^N \subset K$ such that*

$$\begin{aligned} &\langle Au^k_n, v - u^k_n\rangle + \varphi(\mathcal{S}^k_{n,L} u^k, u^k_{n-1}, v) - \varphi(\mathcal{S}^k_{n,L} u^k, u^k_{n-1}, u^k_n) \\ &\quad + j^0(\gamma_j u^k_n; \gamma_j v - \gamma_j u^k_n) + \langle j_c(\gamma_j u^k_n), \gamma_j v - \gamma_j u^k_n\rangle_{X_j} \\ &\quad\quad \geqslant \langle f_n, v - u^k_n\rangle + \langle j_c(\gamma_j u^k_{n-1}), \gamma_j v - \gamma_j u^k_n\rangle_{X_j}, \quad \forall\, v \in K. \end{aligned} \tag{3.8}$$

Remark 3.4. Note that the approximation $\mathcal{S}_{n,L}^k u^k$ for the history-dependent operator does not involve information on the current numerical solution u_n^k, and the second argument of φ is explicitly treated, which is important for numerical implementation. The function φ appeared in (3.8) is convex with respect to the unknown variable (the third argument) according to assumption (2.6). Moreover, j_c plays the role to convexify the function j, i.e., $j^0(\gamma_j u_n^k; \gamma_j v - \gamma_j u_n^k) + \langle j_c(\gamma_j u_n^k), \gamma_j v - \gamma_j u_n^k\rangle_{X_j}$ becomes the directional derivative of a convex function. Therefore, convex optimization techniques could be applied to solve the inequality (3.8) and the unique solvability of Problem 3.3 can be obtained without the constraint (2.10) by applying results on the elliptic variational-hemivariational inequality (see [20]). Specifically, the operator T_1 defined by $T_1 v = Av + \partial j(v) + j_c(v)$ is bounded, coercive and pseudomonotone, and the function $\varphi(v)$ can be extended to X, denoted as $\widetilde{\varphi}(v)$ with $\widetilde{\varphi}(v) = +\infty$ for any $v \in X\backslash K$. In this way, the operator T_2 with $T_2 v = \partial\widetilde{\varphi}(v)$ is maximal monotone. Hence, Problem 3.3 has a unique solution.

Remark 3.5. The choice of j_c is not unique. The critical point is that j_c should be "convex" enough to have the non-convexity of j^0 under control, i.e., the inequality (3.7) is required. On the other hand, we can split j^0 in another way, e.g.,

$$\langle j_c(\gamma_j u_n^k), \gamma_j v - \gamma_j u_n^k\rangle_{X_j} + (j^0(\gamma_j u_{n-1}^k; \gamma_j v - \gamma_j u_n^k) - \langle j_c(\gamma_j u_{n-1}^k), \gamma_j v - \gamma_j u_n^k\rangle_{X_j})$$

could be used to approximate $j^0(\gamma_j u_n^k; \gamma_j v - \gamma_j u_n^k)$. In this way, the inequality (3.8) becomes a convex problem with linear operators, for which efficient numerical algorithms are available.

According to the statement in Remark 3.4, we have the following unique solvability result for Problem 3.3.

Theorem 3.6. *Under the conditions* (2.3)–(2.9) *and* (3.6), *the semi-discrete Problem* 3.3 *is uniquely solvable.*

For error estimation, we first introduce some auxiliary techniques.

Lemma 3.7. *Let* $\{a_n\}$ *be a nonnegative sequence satisfying*

$$a_n \leqslant b_0 + C_1 k \sum_{j=0}^{n-1} a_j + \theta_1 a_{n-1} + \theta_2 a_{n-2}, \quad \forall\, n \geqslant 2,$$

where a_0, a_1, b_0, θ_1, θ_2 *and* C_1 *are nonnegative constants and* $0 \leqslant \theta_1 + \theta_2 < 1$. *Then*

$$a_n \leqslant \left(\frac{b_0}{1-\theta_1-\theta_2} + \frac{C_1 k(a_0+a_1)}{1-\theta_1-\theta_2} + \theta_1 a_1 + \theta_2 a_0\right)\left(1 + \frac{C_1 k}{1-\theta_1-\theta_2}\right)^{n-2}. \tag{3.9}$$

Proof. For convenience, let

$$\bar{\alpha} := \frac{b_0}{1-\theta_1-\theta_2} + \frac{C_1 k(a_0+a_1)}{1-\theta_1-\theta_2} + \theta_1 a_1 + \theta_2 a_0.$$

We prove the result with an induction. For $n = 2$, we have the following bound:

$$a_2 \leqslant b_0 + C_1 k(a_1 + a_0) + \theta_1 a_1 + \theta_2 a_0 \leqslant \bar{\alpha}.$$

Thus, (3.9) holds for $n = 2$. Assume that for $n \leqslant m$,

$$a_n \leqslant \bar{\alpha}\left(1 + \frac{C_1 k}{1-\theta_1-\theta_2}\right)^{n-2}.$$

Then for $n = m + 1$,

$$\begin{aligned} a_{m+1} &\leqslant b_0 + C_1 k \sum_{j=0}^{m} a_j + \theta_1 a_m + \theta_2 a_{m-1} \\ &\leqslant b_0 + C_1 k\left[a_0 + a_1 + \bar{\alpha}\sum_{j=2}^{m}\left(1 + \frac{C_1 k}{1-\theta_1-\theta_2}\right)^{j-2}\right] \end{aligned}$$

$$
\begin{aligned}
&+ (\theta_1+\theta_2)\bar{\alpha}\left(1+\frac{C_1k}{1-\theta_1-\theta_2}\right)^{m-2}\\
&= b_0 + C_1k(a_0+a_1) + \bar{\alpha}\cdot C_1k\frac{(1+\frac{C_1k}{1-\theta_1-\theta_2})^{m-1}-1}{C_1k}(1-\theta_1-\theta_2)\\
&+ (\theta_1+\theta_2)\bar{\alpha}\left(1+\frac{C_1k}{1-\theta_1-\theta_2}\right)^{m-2}\\
&\leqslant \bar{\alpha}\left(1+\frac{C_1k}{1-\theta_1-\theta_2}\right)^{m-1}(1-\theta_1-\theta_2) + (\theta_1+\theta_2)\bar{\alpha}\left(1+\frac{C_1k}{1-\theta_1-\theta_2}\right)^{m-2}\\
&\leqslant \bar{\alpha}\left(1+\frac{C_1k}{1-\theta_1-\theta_2}\right)^{m-2}(1+C_1k) \leqslant \bar{\alpha}\left(1+\frac{C_1k}{1-\theta_1-\theta_2}\right)^{m-1},
\end{aligned}
$$

where we use the fact that

$$b_0 + C_1k(a_0+a_1) - (1-\theta_1-\theta_2)\bar{\alpha} \leqslant 0.$$

This completes the proof. □

Corollary 3.8. *Assume that $\{a_n\}$ is a nonnegative sequence satisfying*

$$a_n \leqslant b_0 + C_2k\sum_{j=0}^{n-1} a_j + \theta_1 a_{n-1}, \quad \forall\, n \geqslant 1,$$

where a_0, b_0, θ_1 and C_2 are nonnegative constants and $\theta_1 < 1$. Then

$$a_n \leqslant \left(\frac{b_0}{1-\theta_1} + \frac{C_2k}{1-\theta_1}a_0 + \theta_1 a_0\right)\left(1+\frac{C_2k}{1-\theta_1}\right)^{n-1}. \tag{3.10}$$

Lemma 3.9. *Assume e_0, e_1 and e_2 are nonnegative numbers such that*

$$e_0^2 \leqslant e_1e_0 + e_2^2. \tag{3.11}$$

Then

$$e_0 \leqslant e_1 + e_2. \tag{3.12}$$

Proof. From (3.11), we have

$$\left(e_0 - \frac{e_1}{2}\right)^2 \leqslant \frac{e_1^2}{4} + e_2^2 \leqslant \left(\frac{e_1}{2} + e_2\right)^2. \tag{3.13}$$

Taking the square root of both sides gives (3.12). □

We now turn to an error analysis for Problem 3.3. For convenience, we denote $\|R\| = \|R\|_{\mathcal{L}(X;Y)}$ and $\|q\| = \|q\|_{C(I\times I;\mathcal{L}(X))}$. The following smallness condition is needed instead of the original one (2.10):

$$\alpha_\varphi + \alpha_c c_j^2 < m_A. \tag{3.14}$$

Theorem 3.10. *Assume* (2.3)–(2.9), (3.6), (3.14) *and the regularity $q \in C^1(\mathbb{R}_+\times\mathbb{R}_+;\mathcal{L}(X))$, $u \in W^{1,\infty}_{\rm loc}(\mathbb{R}_+;X)$. Then for the semi-discrete solution of Problem* 3.3*, the following error bound holds*:

$$\max_{n\leqslant N}\|u_n - u_n^k\|_X \leqslant C_3k, \tag{3.15}$$

where $C_3 > 0$ is a constant independent of k.

Proof. We take $t = t_n$ in the inequality (2.2) to get

$$
\begin{aligned}
&\langle Au_n, v-u_n\rangle + \varphi(\mathcal{S}_nu, u_n, v) - \varphi(\mathcal{S}_nu, u_n, u_n)\\
&\quad + j^0(\gamma_ju_n;\gamma_jv-\gamma_ju_n) \geqslant \langle f_n, v-u_n\rangle, \quad \forall\, v\in K,
\end{aligned} \tag{3.16}
$$

where $\mathcal{S}_n u = R(\int_0^{t_n} q(t_n,s)u(s)ds + a_S)$. Let $v = u_n^k$ in (3.16),

$$\begin{aligned}&\langle Au_n, u_n^k - u_n\rangle + \varphi(\mathcal{S}_n u, u_n, u_n^k) - \varphi(\mathcal{S}_n u, u_n, u_n)\\&\quad + j^0(\gamma_j u_n; \gamma_j u_n^k - \gamma_j u_n) \geqslant \langle f_n, u_n^k - u_n\rangle.\end{aligned}\tag{3.17}$$

Taking $v = u_n$ in (3.8) yields

$$\begin{aligned}&\langle Au_n^k, u_n - u_n^k\rangle + \varphi(\mathcal{S}_{n,L}^k u^k, u_{n-1}^k, u_n) - \varphi(\mathcal{S}_{n,L}^k u^k, u_{n-1}^k, u_n^k)\\&\quad + j^0(\gamma_j u_n^k; \gamma_j u_n - \gamma_j u_n^k) + \langle j_c(\gamma_j u_n^k), \gamma_j u_n - \gamma_j u_n^k\rangle_{X_j}\\&\qquad \geqslant \langle f_n, u_n - u_n^k\rangle + \langle j_c(\gamma_j u_{n-1}^k), \gamma_j u_n - \gamma_j u_n^k\rangle_{X_j}.\end{aligned}\tag{3.18}$$

Adding (3.17) to (3.18) and employing the strong monotonicity of A, we obtain

$$\begin{aligned}m_A\|u_n - u_n^k\|_X^2 &\leqslant \langle Au_n - Au_n^k, u_n - u_n^k\rangle\\&\leqslant \varphi(\mathcal{S}_n u, u_n, u_n^k) - \varphi(\mathcal{S}_n u, u_n, u_n) + \varphi(\mathcal{S}_{n,L}^k u^k, u_{n-1}^k, u_n)\\&\quad - \varphi(\mathcal{S}_{n,L}^k u^k, u_{n-1}^k, u_n^k) + j^0(\gamma_j u_n; \gamma_j u_n^k - \gamma_j u_n)\\&\quad + j^0(\gamma_j u_n^k; \gamma_j u_n - \gamma_j u_n^k) + \langle j_c(\gamma_j u_n^k), \gamma_j u_n - \gamma_j u_n^k\rangle_{X_j}\\&\quad - \langle j_c(\gamma_j u_{n-1}^k), \gamma_j u_n - \gamma_j u_n^k\rangle_{X_j},\end{aligned}$$

which is rewritten as

$$m_A\|u_n - u_n^k\|_X^2 \leqslant E_\varphi + E_j + E_{j_c},\tag{3.19}$$

where

$$\begin{aligned}E_\varphi &= \varphi(\mathcal{S}_{n,L}^k u^k, u_{n-1}^k, u_n) - \varphi(\mathcal{S}_{n,L}^k u^k, u_{n-1}^k, u_n^k)\\&\quad + \varphi(\mathcal{S}_n u, u_n, u_n^k) - \varphi(\mathcal{S}_n u, u_n, u_n),\end{aligned}\tag{3.20}$$

$$E_{j_c} = \langle j_c(\gamma_j u_n), \gamma_j u_n - \gamma_j u_n^k\rangle_{X_j} - \langle j_c(\gamma_j u_{n-1}^k), \gamma_j u_n - \gamma_j u_n^k\rangle_{X_j},\tag{3.21}$$

$$\begin{aligned}E_j &= j^0(\gamma_j u_n; \gamma_j u_n^k - \gamma_j u_n) + j^0(\gamma_j u_n^k; \gamma_j u_n - \gamma_j u_n^k)\\&\quad + \langle j_c(\gamma_j u_n), \gamma_j u_n^k - \gamma_j u_n\rangle_{X_j} + \langle j_c(\gamma_j u_n^k), \gamma_j u_n - \gamma_j u_n^k\rangle_{X_j}.\end{aligned}\tag{3.22}$$

The term E_j can be bounded by zero from the above according to (3.7). Utilizing the regularity of u and the properties of j_c gives

$$\begin{aligned}E_{j_c} &= \langle j_c(\gamma_j u_n - \gamma_j u_{n-1}^k), \gamma_j u_n - \gamma_j u_n^k\rangle_{X_j}\\&= \langle j_c(\gamma_j u_n - \gamma_j u_{n-1}), \gamma_j u_n - \gamma_j u_n^k\rangle_{X_j}\\&\quad + \langle j_c(\gamma_j u_{n-1} - \gamma_j u_{n-1}^k), \gamma_j u_n - \gamma_j u_n^k\rangle_{X_j}\\&\leqslant \alpha_c c_j^2(k\|u\|_{W^{1,\infty}(I,X)} + \|u_{n-1} - u_{n-1}^k\|_X)\|u_n - u_n^k\|_X.\end{aligned}\tag{3.23}$$

From (2.6) we can see

$$\begin{aligned}E_\varphi &\leqslant (\alpha_\varphi\|u_n - u_{n-1}^k\|_X + \beta_\varphi\|\mathcal{S}_n u - \mathcal{S}_{n,L}^k u^k\|_Y)\|u_n - u_n^k\|_X\\&\leqslant (k\alpha_\varphi\|u\|_{W^{1,\infty}(I,X)} + \alpha_\varphi\|u_{n-1} - u_{n-1}^k\|_X\\&\quad + \beta_\varphi\|\mathcal{S}_n u - \mathcal{S}_{n,L}^k u^k\|_Y)\|u_n - u_n^k\|_X.\end{aligned}\tag{3.24}$$

From (3.3), it holds that

$$\begin{aligned}\|\mathcal{S}_n u - \mathcal{S}_{n,L}^k u^k\|_Y &\leqslant \|\mathcal{S}_n u - \mathcal{S}_{n,L}^k u\|_Y + \|\mathcal{S}_{n,L}^k u - \mathcal{S}_{n,L}^k u^k\|_Y\\&\leqslant Ck\|u\|_{W^{1,\infty}(I;X)} + \frac{3}{2}k\|R\|\|q\|\sum_{j=0}^{n-1}\|u_j - u_j^k\|_X.\end{aligned}\tag{3.25}$$

From (3.19) and (3.23)–(3.25), we obtain

$$
\begin{aligned}
m_A\|u_n - u_n^k\|_X \leqslant{} & Ck\|u\|_{W^{1,\infty}(I;X)} + \frac{3}{2}k\beta_\varphi\|R\|\|q\|\sum_{j=0}^{n-1}\|u_j - u_j^k\|_X \\
& + (\alpha_\varphi + \alpha_c c_j^2)\|u_{n-1} - u_{n-1}^k\|_X.
\end{aligned} \tag{3.26}
$$

By applying Corollary 3.8, we have

$$
\begin{aligned}
\|u_n - u_n^k\|_X \leqslant{} & \Bigg(k\frac{C\|u\|_{W^{1,\infty}(I;X)}}{m_A - \alpha_\varphi - \alpha_c c_j^2} + \Bigg(\frac{\frac{3}{2}k\beta_\varphi\|R\|\|q\|}{m_A - \alpha_\varphi - \alpha_c c_j^2} \\
& + \frac{\alpha_\varphi + \alpha_c c_j^2}{m_A}\Bigg)\|u_0 - u_0^k\|_X\Bigg) \cdot \Bigg(1 + k\frac{\frac{3}{2}\beta_\varphi\|R\|\|q\|}{m_A - \alpha_\varphi - \alpha_c c_j^2}\Bigg)^{n-1}.
\end{aligned} \tag{3.27}
$$

Note that when $t = t_0 = 0$, the integral of history-dependent operator is zero and there is no temporally discrete error; thus $\|u_0 - u_0^k\|_X = 0$. Then,

$$
\begin{aligned}
\|u_n - u_n^k\|_X &\leqslant k\frac{C\|u\|_{W^{1,\infty}(I;X)}}{m_A - \alpha_\varphi - \alpha_c c_j^2} \cdot \Bigg(1 + k\frac{\frac{3}{2}\beta_\varphi\|R\|\|q\|}{m_A - \alpha_\varphi - \alpha_c c_j^2}\Bigg)^{n-1} \\
&\leqslant C_3 k,
\end{aligned}
$$

where

$$
C_3 = \frac{C\|u\|_{W^{1,\infty}(I;X)}}{m_A - \alpha_\varphi - \alpha_c c_j^2} \cdot \exp\Bigg\{\frac{\frac{3}{2}\beta_\varphi\|R\|\|q\|}{m_A - \alpha_\varphi - \alpha_c c_j^2}t_n\Bigg\},
$$

and the error bound (3.15) follows. □

Remark 3.11. The first-order accuracy remains valid if $\widetilde{\mathcal{S}}_n^k$ is used to approximate the history-dependent operator $\mathcal{S}$ in the temporally semi-discrete scheme (3.8).

3.2 Second-order temporally semi-discrete schemes

In this subsection, we propose and study two second-order schemes to temporally approximate Problem 2.1. The first scheme is the following.

Problem 3.12. *Find* $u^k := \{u_n^k\}_{n=0}^N \subset K$ *such that*

$$
\begin{aligned}
& \langle Au_n^k, v - u_n^k\rangle + \varphi(\mathcal{S}_{n,L}^k u^k, u_n^k, v) - \varphi(\mathcal{S}_{n,L}^k u^k, u_n^k, u_n^k) \\
& \quad + j^0(\gamma_j u_n^k; \gamma_j v - \gamma_j u_n^k) \geqslant \langle f_n, v - u_n^k\rangle, \quad \forall\, v \in K.
\end{aligned} \tag{3.28}
$$

Note that the history-dependent operator is approximated by using available numerical solution values and the current unknown value u_n^k is not involved. In this way, unlike the numerical scheme studied in [28], the semi-discrete Problem 3.12 is ensured to have a unique solution regardless of the size of the time step-size by using the same Banach fixed-point argument as in [28].

Theorem 3.13. *Under the conditions* (2.3)–(2.10)*, the semi-discrete Problem* 3.12 *has a unique solution.*

We turn to the error estimation of Problem 3.12.

Theorem 3.14. *Assume* (2.3)–(2.10) *and the regularity* $q \in C^2(\mathbb{R}_+ \times \mathbb{R}_+; \mathcal{L}(X))$, $u \in W_{\mathrm{loc}}^{2,\infty}(\mathbb{R}_+; X)$. *Then for the semi-discrete solution of Problem* 3.12*, we have the error bound*

$$
\max_{n\leqslant N}\|u_n - u_n^k\|_X \leqslant C_4 k^2, \tag{3.29}
$$

where $C_4 > 0$ *is a constant independent of* k.

Proof. Let $v = u_n$ in (3.28) to get

$$\begin{aligned}&\langle Au_n^k, u_n - u_n^k\rangle + \varphi(\mathcal{S}_{n,L}^k u^k, u_n^k, u_n) - \varphi(\mathcal{S}_{n,L}^k u^k, u_n^k, u_n^k)\\&\quad + j^0(\gamma_j u_n^k; \gamma_j u_n - \gamma_j u_n^k) \geqslant \langle f_n, u_n - u_n^k\rangle. \end{aligned} \tag{3.30}$$

Adding (3.17) to (3.30) and employing the strong monotonicity of A, we have

$$\begin{aligned} m_A\|u_n - u_n^k\|_X^2 &\leqslant \varphi(\mathcal{S}_{n,L}^k u^k, u_n^k, u_n) - \varphi(\mathcal{S}_{n,L}^k u^k, u_n^k, u_n^k)\\ &\quad + \varphi(\mathcal{S}_n u, u_n, u_n^k) - \varphi(\mathcal{S}_n u, u_n, u_n)\\ &\quad + j^0(\gamma_j u_n^k; \gamma_j u_n - \gamma_j u_n^k) + j^0(\gamma_j u_n; \gamma_j u_n^k - \gamma_j u_n)\\ &\leqslant \alpha_\varphi\|u_n - u_n^k\|_X^2 + \beta_\varphi\|\mathcal{S}_n u - \mathcal{S}_{n,L}^k u^k\|_Y\|u_n - u_n^k\|_X\\ &\quad + \alpha_j c_j^2\|u_n - u_n^k\|_X^2. \end{aligned} \tag{3.31}$$

Similar to (3.25) by using (3.4) instead, it holds that

$$\|\mathcal{S}_n u - \mathcal{S}_{n,L}^k u^k\|_Y \leqslant Ck^2\|u\|_{W^{2,\infty}(I;X)} + \frac{3}{2}k\beta_\varphi\|R\|\|q\|\sum_{j=0}^{n-1}\|u_j - u_j^k\|_X. \tag{3.32}$$

Applying (3.32) to (3.31), we have

$$\begin{aligned}\|u_n - u_n^k\|_X &\leqslant \frac{\beta_\varphi}{m_A - \alpha_\varphi - \alpha_j c_j^2}\|\mathcal{S}_n u - \mathcal{S}_{n,L}^k u^k\|_Y\\ &\leqslant Ck^2\|u\|_{W^{2,\infty}(I;X)} + \frac{\frac{3}{2}k\beta_\varphi\|R\|\|q\|}{m_A - \alpha_\varphi - \alpha_j c_j^2}\sum_{j=0}^{n-1}\|u_j - u_j^k\|_X. \end{aligned} \tag{3.33}$$

Then by Corollary 3.8,

$$\begin{aligned}\|u_n - u_n^k\|_X &\leqslant \left(k^2C\|u\|_{W^{2,\infty}(I;X)} + \frac{\frac{3}{2}k\beta_\varphi\|R\|\|q\|}{m_A - \alpha_\varphi - \alpha_j c_j^2}\|u_0 - u_0^k\|_X\right)\\ &\quad\times\left(1 + k\frac{\frac{3}{2}\beta_\varphi\|R\|\|q\|}{m_A - \alpha_\varphi - \alpha_j c_j^2}\right)^{n-1}\\ &\leqslant C_4k^2, \end{aligned} \tag{3.34}$$

where

$$C_4 = C\|u\|_{W^{2,\infty}(I,X)}\cdot\exp\left\{\frac{\frac{3}{2}\beta_\varphi\|R\|\|q\|}{m_A - \alpha_\varphi - \alpha_j c_j^2}t_n\right\}.$$

Thus the second-order error estimate (3.29) is established. □

Remark 3.15. For the numerical scheme in [28], the history-dependent operator is implicitly treated in the sense that its approximation depends on the current unknown solution component. As a result, a restriction for the time step-size of the form $k < (m_A - \alpha_\varphi - \alpha_j c_j^2)/\beta_\varphi\|R\|\|q\|$ is needed to ensure the unique solvability and for the derivation of the error bound there. In contrast, for our numerical scheme given by Problem 3.12, we have the unique solvability and error bound for an arbitrary time step-size.

Next, we modify (3.8) and give another scheme of second-order.

Problem 3.16. *Find a discrete solution* $u^k := \{u_n^k\}_{n=0}^N \subset K$ *such that*

$$\begin{aligned}&\langle Au_n^k, v - u_n^k\rangle + \varphi(\mathcal{S}_{n,L}^k u^k, 2u_{n-1}^k - u_{n-2}^k, v) - \varphi(\mathcal{S}_{n,L}^k u^k, 2u_{n-1}^k - u_{n-2}^k, u_n^k)\\&\quad + j^0(\gamma_j u_n^k; \gamma_j v - \gamma_j u_n^k) + \langle j_c(\gamma_j u_n^k), \gamma_j v - \gamma_j u_n^k\rangle_{X_j}\\&\qquad \geqslant \langle j_c(2\gamma_j u_{n-1}^k - \gamma_j u_{n-2}^k), \gamma_j v - \gamma_j u_n^k\rangle_{X_j} + \langle f_n, v - u_n^k\rangle, \quad \forall\, v \in K, \quad n \geqslant 2, \end{aligned} \tag{3.35}$$

and for $n = 1$,

$$\begin{aligned}&\langle Au_1^k, v - u_1^k\rangle + \varphi(\mathcal{S}_{1,L}^k u^k, u_1^k, v) - \varphi(\mathcal{S}_{1,L}^k u^k, u_1^k, u_1^k)\\&\quad + j^0(\gamma_j u_1^k; \gamma_j v - \gamma_j u_1^k) \geqslant \langle f_1, v - u_1^k\rangle, \quad \forall\, v \in K. \end{aligned} \tag{3.36}$$

The uniqueness and existence results for (3.35) are similar to that of Problem 3.3. As for (3.36), it can be referred to Problem 3.12. Then we have the following uniqueness and existence results for Problem 3.16.

Theorem 3.17. *Assume* (2.3)–(2.10) *and* (3.6). *Then Problem* 3.16 *has a unique solution* $u^k = \{u_n^k\}_{n=0}^N \subset K$.

Next, we derive an error bound for the semi-discrete solution of Problem 3.16. Meanwhile, a stronger constraint compared with (3.14) is needed, i.e.,

$$\alpha_\varphi + \alpha_c c_j^2 < \frac{m_A}{3}. \tag{3.37}$$

Theorem 3.18. *Assume* (2.3)–(2.9), (3.6), (3.37) *and the regularity* $q \in C^2(\mathbb{R}_+ \times \mathbb{R}_+; \mathcal{L}(X))$, $u \in W^{2,\infty}_{\rm loc}(\mathbb{R}_+; X)$. *Then for the semi-discrete solution of Problem* 3.16, *the following error bound holds*:

$$\max_{n \leqslant N} \|u_n - u_n^k\|_X \leqslant C_5 k^2, \tag{3.38}$$

where $C_5 > 0$ *is a constant independent of* k.

Proof. For $n = 1$, we have a second-order accuracy result for (3.36) by Theorem 3.14:

$$\|u_1 - u_1^k\|_X \leqslant C_4 k^2. \tag{3.39}$$

For $n \geqslant 2$, taking $v = u_n$ in (3.35), we have

$$\begin{aligned}
&\langle Au_n^k, u_n - u_n^k\rangle + \varphi(\mathcal{S}_{n,L}^k u^k, 2u_{n-1}^k - u_{n-2}^k, u_n) - \varphi(\mathcal{S}_{n,L}^k u^k, 2u_{n-1}^k - u_{n-2}^k, u_n^k)\\
&\quad + j^0(\gamma_j u_n^k; \gamma_j u_n - \gamma_j u_n^k) + \langle j_c(\gamma_j u_n^k), \gamma_j u_n - \gamma_j u_n^k\rangle_{X_j}\\
&\qquad \geqslant \langle f_n, u_n - u_n^k\rangle + \langle j_c(2\gamma_j u_{n-1}^k - \gamma_j u_{n-2}^k), \gamma_j u_n - \gamma_j u_n^k\rangle_{X_j}.
\end{aligned} \tag{3.40}$$

Combining (3.17) with (3.40) and using the strong monotonicity of A, we obtain

$$\begin{aligned}
m_A\|u_n - u_n^k\|_X^2 &\leqslant \varphi(\mathcal{S}_n u, u_n, u_n^k) - \varphi(\mathcal{S}_n u, u_n, u_n)\\
&\quad + \varphi(\mathcal{S}_{n,L}^k u^k, 2u_{n-1}^k - u_{n-2}^k, u_n) - \varphi(\mathcal{S}_{n,L}^k u^k, 2u_{n-1}^k - u_{n-2}^k, u_n^k)\\
&\quad + j^0(\gamma_j u_n; \gamma_j u_n^k - \gamma_j u_n) + j^0(\gamma_j u_n^k; \gamma_j u_n - \gamma_j u_n^k)\\
&\quad + \langle j_c(\gamma_j u_n^k), \gamma_j u_n - \gamma_j u_n^k\rangle_{X_j} - \langle j_c(2\gamma_j u_{n-1}^k - \gamma_j u_{n-2}^k), \gamma_j u_n - \gamma_j u_n^k\rangle_{X_j}\\
&=: \hat{E}_\varphi + E_j + \hat{E}_{j_c},
\end{aligned} \tag{3.41}$$

where E_j is defined in (3.22) and

$$\begin{aligned}
\hat{E}_\varphi &= \varphi(\mathcal{S}_{n,L}^k u^k, 2u_{n-1}^k - u_{n-2}^k, u_n) - \varphi(\mathcal{S}_{n,L}^k u^k, 2u_{n-1}^k - u_{n-2}^k, u_n^k)\\
&\quad + \varphi(\mathcal{S}_n u, u_n, u_n^k) - \varphi(\mathcal{S}_n u, u_n, u_n),
\end{aligned} \tag{3.42}$$

$$\hat{E}_{j_c} = \langle j_c(\gamma_j u_n), \gamma_j u_n - \gamma_j u_n^k\rangle_{X_j} - \langle j_c(2\gamma_j u_{n-1}^k - \gamma_j u_{n-2}^k), \gamma_j u_n - \gamma_j u_n^k\rangle_{X_j}. \tag{3.43}$$

We bound $\hat{E}_{j_c}$ and $\hat{E}_\varphi$ as follows:

$$\begin{aligned}
\hat{E}_{j_c} &= \langle j_c(\gamma_j u_n - 2\gamma_j u_{n-1} + \gamma_j u_{n-2}), \gamma_j u_n - \gamma_j u_n^k\rangle_{X_j}\\
&\quad + 2\langle j_c(\gamma_j u_{n-1} - \gamma_j u_{n-1}^k), \gamma_j u_n - \gamma_j u_n^k\rangle_{X_j}\\
&\quad - \langle j_c(\gamma_j u_{n-2} - \gamma_j u_{n-2}^k), \gamma_j u_n - \gamma_j u_n^k\rangle_{X_j}\\
&\leqslant \alpha_c c_j^2 (k^2\|u\|_{W^{2,\infty}(I,X)} + 2\|u_{n-1} - u_{n-1}^k\|_X\\
&\quad + \|u_{n-2} - u_{n-2}^k\|_X)\|u_n - u_n^k\|_X,
\end{aligned} \tag{3.44}$$

$$\begin{aligned}
\hat{E}_\varphi &\leqslant (k^2\alpha_\varphi\|u\|_{W^{2,\infty}(I,X)} + 2\alpha_\varphi\|u_{n-1} - u_{n-1}^k\|_X\\
&\quad + \alpha_\varphi\|u_{n-2} - u_{n-2}^k\|_X + \beta_\varphi\|\mathcal{S}_n u - \mathcal{S}_{n,L}^k u^k\|_Y)\|u_n - u_n^k\|_X.
\end{aligned} \tag{3.45}$$

From (3.41)–(3.45) and (3.32), we have

$$m_A\|u_n - u_n^k\|_X \leqslant Ck^2\|u\|_{W^{2,\infty}(I;X)} + \frac{3}{2}k\beta_\varphi\|R\|\|q\|\sum_{j=0}^{n-1}\|u_j - u_j^k\|_X \\ + (\alpha_\varphi + \alpha_c c_j^2)(2\|u_{n-1} - u_{n-1}^k\|_X + \|u_{n-2} - u_{n-2}^k\|_X). \tag{3.46}$$

Applying Lemma 3.7 to (3.46) and combining (3.39), we have

$$\begin{aligned}\|u_n - u_n^k\|_X \leqslant &\bigg(\frac{\alpha_\varphi + \alpha_c c_j^2}{m_A}(2\|u_1 - u_1^k\|_X + \|u_0 - u_0^k\|_X) \\ &+ \frac{\frac{3}{2}k\beta_\varphi\|R\|\|q\|}{m_A - 3(\alpha_\varphi + \alpha_c c_j^2)}(\|u_0 - u_0^k\|_X + \|u_1 - u_1^k\|_X) \\ &+ k^2\frac{C\|u\|_{W^{2,\infty}(I;X)}}{m_A - 3(\alpha_\varphi + \alpha_j c_j^2)}\bigg)\cdot\bigg(1 + \frac{\frac{3}{2}k\beta_\varphi\|R\|\|q\|}{m_A - 3(\alpha_\varphi + \alpha_c c_j^2)}\bigg)^{n-2} \\ \leqslant &\, C_5k^2,\end{aligned} \tag{3.47}$$

where

$$C_5 = \bigg(\frac{C\|u\|_{W^{2,\infty}(I;X)}}{m_A - 3(\alpha_\varphi + \alpha_c c_j^2)} + C_4\frac{\frac{3}{2}k\beta_\varphi\|R\|\|q\|}{m_A - 3(\alpha_\varphi + \alpha_c c_j^2)} \\ + 2C_4\frac{\alpha_\varphi + \alpha_c c_j^2}{m_A}\bigg)\cdot\exp\bigg\{\frac{\frac{3}{2}\beta_\varphi\|R\|\|q\|}{m_A - 3(\alpha_\varphi + \alpha_c c_j^2)}t_n\bigg\},$$

which leads to the error bound (3.38). □

4 Fully discrete approximation

In this section we consider fully discrete approximations of Problem 2.1 with or without constraints. The notation and assumptions follow from the previous section, and a regular family of finite element partitions $\{T^h\}$ with the mesh grid-size h is introduced for the spatial discretization. Let $X^h \subset X$ be the conforming finite element spaces. We consider internal approximations only, i.e., $K^h = X^h \cap K$ is nonempty, convex and closed.

Certainly, different fully discrete schemes can be constructed with different temporally semi-discrete schemes proposed in the previous section. We state these fully discrete schemes as follows.

Problem 4.1. *Find the discrete solution $u^{kh} := \{u_n^{kh}\}_{n=0}^N \subset K^h$ such that*

$$\langle Au_n^{kh}, v^h - u_n^{kh}\rangle + \varphi(\mathcal{S}_{n,L}^k u^{kh}, u_{n-1}^{kh}, v^h) - \varphi(\mathcal{S}_{n,L}^k u^{kh}, u_{n-1}^{kh}, u_n^{kh}) \\ + j^0(\gamma_j u_n^{kh}; \gamma_j v^h - \gamma_j u_n^{kh}) + \langle j_c(\gamma_j u_n^{kh}), \gamma_j v^h - \gamma_j u_n^{kh}\rangle_{X_j} \\ \geqslant \langle f_n, v^h - u_n^{kh}\rangle + \langle j_c(\gamma_j u_{n-1}^{kh}), \gamma_j v^h - \gamma_j u_n^{kh}\rangle_{X_j}, \quad \forall\, v^h \in K^h. \tag{4.1}$$

Problem 4.2. *Find the discrete solution $u^{kh} := \{u_n^{kh}\}_{n=0}^N \subset K^h$ such that*

$$\langle Au_n^{kh}, v^h - u_n^{kh}\rangle + \varphi(\mathcal{S}_{n,L}^k u^{kh}, u_n^{kh}, v^h) - \varphi(\mathcal{S}_{n,L}^k u^{kh}, u_n^{kh}, u_n^{kh}) \\ + j^0(\gamma_j u_n^{kh}; \gamma_j v^h - \gamma_j u_n^{kh}) \geqslant \langle f_n, v^h - u_n^{kh}\rangle, \quad \forall\, v^h \in K^h. \tag{4.2}$$

Problem 4.3. *Find the discrete solution $u^{kh} := \{u_n^{kh}\}_{n=0}^N \subset K^h$ such that*

$$\langle Au_n^{kh}, v^h - u_n^{kh}\rangle + \varphi(\mathcal{S}_{n,L}^k u^{kh}, 2u_{n-1}^{kh} - u_{n-2}^{kh}, v^h) - \varphi(\mathcal{S}_{n,L}^k u^{kh}, 2u_{n-1}^{kh} - u_{n-2}^{kh}, u_n^{kh}) \\ + j^0(\gamma_j u_n^{kh}; \gamma_j v^h - \gamma_j u_n^{kh}) + \langle j_c(\gamma_j u_n^{kh}), \gamma_j v^h - \gamma_j u_n^{kh}\rangle_{X_j} \\ \geqslant \langle f_n, v^h - u_n^{kh}\rangle + \langle j_c(2\gamma_j u_{n-1}^{kh} - \gamma_j u_{n-2}^{kh}), \gamma_j v^h - \gamma_j u_n^{kh}\rangle_{X_j}, \quad \forall\, v^h \in K^h, \quad n \geqslant 2, \tag{4.3}$$

for $n = 0, 1$ *the following scheme is used*:

$$\begin{aligned}&\langle Au_n^{kh}, v^h - u_n^{kh}\rangle + \varphi(\mathcal{S}_{n,L}^k u^{kh}, u_n^{kh}, v^h) - \varphi(\mathcal{S}_{n,L}^k u^{kh}, u_n^{kh}, u_n^{kh})\\ &\quad + j^0(\gamma_j u_n^{kh}; \gamma_j v^h - \gamma_j u_n^{kh}) \geqslant \langle f_n, v^h - u_n^{kh}\rangle, \quad \forall\, v^h \in K^h. \end{aligned}\tag{4.4}$$

In the following, we will only discuss the fully discrete Problem 4.3, since the other two fully discrete schemes can be discussed similarly. Similar to the temporally semi-discrete case, we can show that under those same conditions for the temporally semi-discrete, Problem 4.3 has a unique solution. An error bound for Problem 4.3 is given next.

Theorem 4.4. *Assume* (2.3)–(2.10), (3.6) *and* (3.37). *Under the regularity assumptions* $q \in C^2(\mathbb{R}_+ \times \mathbb{R}_+; \mathcal{L}(X))$, $u \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}_+; X)$, *we have the error bound*

$$\begin{aligned}\max_{0\leqslant n\leqslant N} \|u_n - u_n^{kh}\|_X \leqslant C_6 \max_{0\leqslant n\leqslant N} \inf_{v^h\in K^h} \{&\|u_n - v^h\|_X + \|\gamma_j u_n - \gamma_j v^h\|_{X_j}^{\frac{1}{2}}\\ &+ |E(v^h, u_n)|^{\frac{1}{2}}\} + C_6 k^2,\end{aligned}\tag{4.5}$$

where $C_6 > 0$ *is a constant independent of* k, h *and*

$$\begin{aligned}E(v^h, u_n) = {}&\langle Au_n, v^h - u_n\rangle + \varphi(\mathcal{S}_n u, u_n, v^h) - \varphi(\mathcal{S}_n u, u_n, u_n)\\ &+ j^0(\gamma_j u_n; \gamma_j v^h - \gamma_j u_n) - \langle f_n, v^h - u_n\rangle, \quad v^h \in K^h.\end{aligned}\tag{4.6}$$

Proof. First, we consider the general case of $n \geqslant 2$. To this end, we take $t = t_n$ and $v = u_n^{kh}$ in (2.2) to get

$$\begin{aligned}&\langle Au_n, u_n^{kh} - u_n\rangle + \varphi(\mathcal{S}_n u, u_n, u_n^{kh}) - \varphi(\mathcal{S}_n u, u_n, u_n)\\ &\quad + j^0(\gamma_j u_n; \gamma_j u_n^{kh} - \gamma_j u_n) \geqslant \langle f_n, u_n^{kh} - u_n\rangle.\end{aligned}\tag{4.7}$$

On the other hand,

$$\begin{aligned}\langle Au_n - Au_n^{kh}, u_n - u_n^{kh}\rangle = {}&\langle Au_n, u_n - u_n^{kh}\rangle + \langle Au_n^{kh}, u_n^{kh} - v^h\rangle\\ &+ \langle Au_n^{kh}, v^h - u_n\rangle.\end{aligned}\tag{4.8}$$

Combining (2.5)(b) with (4.3), and (4.7)–(4.8), we have

$$\begin{aligned}m_A\|u_n - u_n^{kh}\|_X^2 \leqslant {}&\varphi(\mathcal{S}_n u, u_n, u_n^{kh}) - \varphi(\mathcal{S}_n u, u_n, u_n)\\ &+ \varphi(\mathcal{S}_{n,L}^k u^{kh}, 2u_{n-1}^{kh} - u_{n-2}^{kh}, v^h) - \varphi(\mathcal{S}_{n,L}^k u^{kh}, 2u_{n-1}^{kh} - u_{n-2}^{kh}, u_n^{kh})\\ &+ \langle Au_n^{kh}, v^h - u_n\rangle - \langle f_n, v^h - u_n\rangle + j^0(\gamma_j u_n; \gamma_j u_n^{kh} - \gamma_j u_n)\\ &+ \langle j_c(\gamma_j u_n^{kh}), \gamma_j v^h - \gamma_j u_n^{kh}\rangle_{X_j} + j^0(\gamma_j u_n^{kh}; \gamma_j v^h - \gamma_j u_n^{kh})\\ &- \langle j_c(2\gamma_j u_{n-1}^{kh} - \gamma_j u_{n-2}^{kh}), \gamma_j v^h - \gamma_j u_n^{kh}\rangle_{X_j}\\ =: {}&E_{\varphi_1} + E_{\varphi_2} + \hat{E}_j + E_A + E(v^h, u_n),\end{aligned}\tag{4.9}$$

where

$$\begin{aligned}E_{\varphi_1} = {}&\varphi(\mathcal{S}_{n,L}^k u^{kh}, 2u_{n-1}^{kh} - u_{n-2}^{kh}, u_n) - \varphi(\mathcal{S}_{n,L}^k u^{kh}, 2u_{n-1}^{kh} - u_{n-2}^{kh}, u_n^{kh})\\ &+ \varphi(\mathcal{S}_n u, u_n, u_n^{kh}) - \varphi(\mathcal{S}_n u, u_n, u_n),\end{aligned}\tag{4.10}$$

$$\begin{aligned}E_{\varphi_2} = {}&\varphi(\mathcal{S}_{n,L}^k u^{kh}, 2u_{n-1}^{kh} - u_{n-2}^{kh}, v^h) - \varphi(\mathcal{S}_{n,L}^k u^{kh}, 2u_{n-1}^{kh} - u_{n-2}^{kh}, u_n)\\ &+ \varphi(\mathcal{S}_n u, u_n, u_n) - \varphi(\mathcal{S}_n u, u_n, v^h),\end{aligned}\tag{4.11}$$

$$\begin{aligned}\hat{E}_j = {}&j^0(\gamma_j u_n; \gamma_j u_n^{kh} - \gamma_j u_n) + j^0(\gamma_j u_n^{kh}; \gamma_j v^h - \gamma_j u_n^{kh})\\ &+ \langle j_c(\gamma_j u_n), \gamma_j u_n^{kh} - \gamma_j u_n\rangle_{X_j} + \langle j_c(\gamma_j u_n^{kh}), \gamma_j u_n - \gamma_j u_n^{kh}\rangle_{X_j}\\ &- j^0(\gamma_j u_n; \gamma_j v^h - \gamma_j u_n),\end{aligned}\tag{4.12}$$

$$E_A = \langle Au_n^{kh}, v^h - u_n\rangle - \langle Au_n, v^h - u_n\rangle$$

$$
\begin{aligned}
&+ \langle j_c(\gamma_j u_n^{kh}), \gamma_j v^h - \gamma_j u_n\rangle_{X_j} - \langle j_c(\gamma_j u_n), \gamma_j u_n^{kh} - \gamma_j u_n\rangle_{X_j} \\
&- \langle j_c(2\gamma_j u_{n-1}^{kh} - \gamma_j u_{n-2}^{kh}), \gamma_j v^h - \gamma_j u_n^{kh}\rangle_{X_j}. \qquad (4.13)
\end{aligned}
$$

Let us bound E_{φ_1}, E_{φ_2}, $\hat{E}_j$ and E_A in turn. Then

$$
\begin{aligned}
E_{\varphi_1} &\leqslant \alpha_\varphi \|u_n - u_n^{kh}\|_X \|u_n - 2u_{n-1}^{kh} + u_{n-2}^{kh}\|_X \\
&\quad + \beta_\varphi \|\mathcal{S}_n u - \mathcal{S}_{n,L}^{kh} u^{kh}\|_Y \|u_n - u_n^{kh}\|_X, \qquad (4.14)\\
E_{\varphi_2} &\leqslant \alpha_\varphi \|u_n - v^h\|_X \|u_n - 2u_{n-1}^{kh} + u_{n-2}^{kh}\|_X \\
&\quad + \beta_\varphi \|\mathcal{S}_n u - \mathcal{S}_{n,L}^{kh} u^{kh}\|_Y \|u_n - v^h\|_X. \qquad (4.15)
\end{aligned}
$$

Using the sub-additive property of generalized directional derivative, we have

$$
\begin{aligned}
\hat{E}_j &\leqslant j^0(\gamma_j u_n; \gamma_j u_n^{kh} - \gamma_j v^h) + j^0(\gamma_j u_n^{kh}; \gamma_j v^h - \gamma_j u_n^{kh}) - \alpha_j c_j^2 \|u_n - u_n^{kh}\|_X^2 \\
&\leqslant j^0(\gamma_j u_n; \gamma_j u_n - \gamma_j v^h) + j^0(\gamma_j u_n; \gamma_j u_n^{kh} - \gamma_j u_n) \\
&\quad + j^0(\gamma_j u_n^{kh}; \gamma_j v^h - \gamma_j u_n) + j^0(\gamma_j u_n^{kh}; \gamma_j u_n - \gamma_j u_n^{kh}) - \alpha_j c_j^2 \|u_n - u_n^{kh}\|_X^2 \\
&\leqslant (2c_0 + c_1\|\gamma_j u_n\|_{X_j} + c_1\|\gamma_j u_n^{kh}\|_{X_j})\|\gamma_j u_n - \gamma_j v^h\|_{X_j} \\
&\leqslant (2c_0 + 2c_1 c_j \|u_n\|_X)\|\gamma_j u_n - \gamma_j v^h\|_{X_j} + c_1 c_j^2 \|u_n - u_n^{kh}\|_X \|u_n - v^h\|_X. \qquad (4.16)
\end{aligned}
$$

Since A is Lipschitz continuous with a Lipschitz constant $L_A > 0$, we obtain

$$
\begin{aligned}
E_A &\leqslant L_A \|u_n - u_n^{kh}\|_X \|u_n - v^h\|_X + \langle j_c(\gamma_j u_n - 2\gamma_j u_{n-1}^{kh} + \gamma_j u_{n-2}^{kh}), \gamma_j u_n - \gamma_j u_n^{kh}\rangle_{X_j} \\
&\quad + \langle j_c(2\gamma_j u_{n-1}^{kh} - \gamma_j u_{n-2}^{kh}), \gamma_j u_n - \gamma_j v^h\rangle_{X_j} + \langle j_c(\gamma_j u_n^{kh}), \gamma_j v^h - \gamma_j u_n\rangle_{X_j}. \qquad (4.17)
\end{aligned}
$$

We have

$$
\begin{aligned}
&\langle j_c(2\gamma_j u_{n-1}^{kh} - \gamma_j u_{n-2}^{kh}), \gamma_j u_n - \gamma_j v^h\rangle_{X_j} + \langle j_c(\gamma_j u_n^{kh}), \gamma_j v^h - \gamma_j u_n\rangle_{X_j} \\
&\quad = \langle j_c(-\gamma_j u_n + 2\gamma_j u_{n-1} - \gamma_j u_{n-2}), \gamma_j u_n - \gamma_j v^h\rangle_{X_j} \\
&\qquad + \langle j_c(\gamma_j u_n - \gamma_j u_n^{kh}), \gamma_j u_n - \gamma_j v^h\rangle_{X_j} - 2\langle j_c(\gamma_j u_{n-1} - \gamma_j u_{n-1}^{kh}), \gamma_j u_n - \gamma_j v^h\rangle_{X_j} \\
&\qquad + \langle j_c(\gamma_j u_{n-2} - \gamma_j u_{n-2}^{kh}), \gamma_j u_n - \gamma_j v^h\rangle_{X_j} \\
&\quad \leqslant \alpha_c c_j^2 \|u_n - v^h\|_X (\|u_n - 2u_{n-1} + u_{n-2}\|_X + \|u_n - u_n^{kh}\|_X) \\
&\qquad + \alpha_c c_j^2 \|u_n - v^h\|_X (2\|u_{n-1} - u_{n-1}^{kh}\|_X + \|u_{n-2} - u_{n-2}^{kh}\|_X). \qquad (4.18)
\end{aligned}
$$

Together with (4.9) and (4.14)–(4.18), for $\varepsilon < m_A/3 - \alpha_\varphi - \alpha_c c_j^2$, we obtain

$$
\begin{aligned}
m_A \|u_n - u_n^{kh}\|_X^2 &\leqslant (Ck^2 \|u\|_{W^{2,\infty}(I;X)} + 2(\alpha_\varphi + \alpha_c c_j^2)\|u_{n-1} - u_{n-1}^{kh}\|_X \\
&\quad + (\alpha_\varphi + \alpha_c c_j^2)\|u_{n-2} - u_{n-2}^{kh}\|_X + \beta_\varphi \|\mathcal{S}_n u - \mathcal{S}_{n,L}^{k} u^{kh}\|_Y \\
&\quad + C\|u_n - v^h\|_X)\|u_n - u_n^{kh}\|_X + Ck^2 \|u_n - v^h\|_X \\
&\quad + ((\alpha_\varphi + \alpha_c c_j^2)(2\|u_{n-1} - u_{n-1}^{kh}\|_X + \|u_{n-2} - u_{n-2}^{kh}\|_X) \\
&\quad + \beta_\varphi \|\mathcal{S}_n u - \mathcal{S}_{n,L}^{kh} u^{kh}\|_Y^2)\|u_n - v^h\| \\
&\quad + C\|\gamma_j u_n - \gamma_j v^h\|_{X_j} + |E(v^h, u_n)|. \qquad (4.19)
\end{aligned}
$$

Applying Lemma 3.9 and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\|u_n - u_n^{kh}\|_X &\leqslant Ck^2 \|u\|_{W^{2,\infty}(I;X)} + 2(\alpha_\varphi + \alpha_c c_j^2)\|u_{n-1} - u_{n-1}^{kh}\|_X \\
&\quad + (\alpha_\varphi + \alpha_c c_j^2)\|u_{n-2} - u_{n-2}^{kh}\|_X + \frac{3}{2} k \beta_\varphi \|R\| \|q\| \sum_{j=0}^{n-1} \|u_j - u_j^{kh}\|_X \\
&\quad + C\|u_n - v^h\|_X + Ck^2 + \frac{\alpha_\varphi + \alpha_c^2 c_j^2}{m_A \varepsilon} \|u_n - v^h\|_X \\
&\quad + 2\varepsilon \|u_{n-1} - u_{n-1}^{kh}\|_X + \varepsilon \|u_{n-2} - u_{n-2}^{kh}\|_X
\end{aligned}
$$

$$+ C\|\gamma_j u_n - \gamma_j v^h\|_{X_j} + C|E(v^h, u_n)|. \tag{4.20}$$

For $n = 0$ and $n = 1$, a slight modification based on the proof of Theorem 3.14 and the above arguments give

$$\|u_0 - u_0^{kh}\|_X \leqslant \frac{C}{m_A - \alpha_\varphi - \alpha_j c_j^2}\{\|u_0 - v^h\|_X + \|\gamma_j u_0 - \gamma_j v^h\|_{X_j}^{\frac{1}{2}} + |E(v^h, u_0)|^{\frac{1}{2}}\}, \tag{4.21}$$

$$\begin{aligned}\|u_1 - u_1^{kh}\|_X \leqslant{}& \frac{C}{m_A - \alpha_\varphi - \alpha_j c_j^2}\{\|u_1 - v^h\|_X + \|\gamma_j u_1 - \gamma_j v^h\|_{X_j}^{\frac{1}{2}} \\ &+ |E(v^h, u_1)|^{\frac{1}{2}} + k^2\|u\|_{W^{2,\infty}(I;X)}\} + \frac{3}{2}k\beta_\varphi\|R\|\|q\|\|u_0 - u_0^{kh}\|_X. \end{aligned} \tag{4.22}$$

Applying Lemma 3.7 to (4.20) and combining (4.21)–(4.22), we get

$$\begin{aligned}\|u_n - u_n^{kh}\|_X \leqslant{}& \bigg(C\{\|u_n - v^h\|_X + \|\gamma_j u_n - \gamma_j v^h\|_{X_j}^{\frac{1}{2}} + |E(v^h, u_n)|^{\frac{1}{2}} + k^2\|u\|_{W^{2,\infty}(I;X)}\} \\ &+ \frac{\alpha_\varphi + \alpha_c c_j^2 + \varepsilon}{m_A}(2\|u_1 - u_1^{kh}\|_X + \|u_0 - u_0^{kh}\|_X) \\ &+ \frac{\frac{3}{2}k\beta_\varphi\|R\|\|q\|}{m_A - 3(\alpha_\varphi + \alpha_c c_j^2 + \varepsilon)}(\|u_0 - u_0^{kh}\|_X + \|u_1 - u_1^{kh}\|_X)\bigg) \\ &\times \left(1 + \frac{\frac{3}{2}k\beta_\varphi\|R\|\|q\|}{m_A - 3(\alpha_\varphi + \alpha_c c_j^2 + \varepsilon)}\right)^{n-2} \\ \leqslant{}& C_6 \max_{0\leqslant n\leqslant N}(\|u_n - v^h\|_X + \|\gamma_j u_n - \gamma_j v^h\|_{X_j}^{\frac{1}{2}} + |E(v^h, u_n)|^{\frac{1}{2}} + k^2), \end{aligned} \tag{4.23}$$

where

$$C_6 = C\|u\|_{W^{2,\infty}(I;X)} \cdot \exp\left\{\frac{\frac{3}{2}\beta_\varphi\|R\|\|q\|}{m_A - 3(\alpha_\varphi + \alpha_c c_j^2 + \varepsilon)}t_n\right\}.$$

Then we have the error bound (4.5). □

Now we consider the error estimation for the numerical solution of the discrete problem without constraint. We introduce the following assumption on φ as in [28], which allows us to simplify the error bound (4.5):

$$\begin{cases}\varphi : Y \times K \times K \to \mathbb{R} \text{ is a function such that} \\ \quad \text{there exists a constant } c_\varphi > 0 \text{ satisfying} \\ \quad \varphi(y, u, v_1) + \varphi(y, u, v_2) - 2\varphi\left(y, u, \dfrac{v_1 + v_2}{2}\right) \leqslant c_\varphi\|v_1 - v_2\|_X^2, \\ \qquad \forall\, y \in Y, \quad \forall\, u, v_1, v_2 \in K. \end{cases} \tag{4.24}$$

Theorem 4.5. *Keep the assumptions stated in Theorem* 4.4. *In addition, let* $K = X$ *and the function* φ *satisfy the assumption* (4.24). *Then the following error bound holds*:

$$\max_{0\leqslant n\leqslant N}\|u_n - u_n^{kh}\|_X \leqslant C\Big(\max_{0\leqslant n\leqslant N}\inf_{v^h\in K^h}\{\|u_n - v^h\|_X + \|\gamma_j u_n - \gamma_j v^h\|_{X_j}^{\frac{1}{2}}\} + k^2\Big). \tag{4.25}$$

Proof. We start with

$$\begin{aligned}\langle Au_n - Au_n^{kh}, u_n - u_n^{kh}\rangle &= \langle Au_n - Au_n^{kh}, u_n - v^h\rangle + \langle Au_n - Au_n^{kh}, v^h - u_n^{kh}\rangle \\ &= \langle Au_n - Au_n^{kh}, u_n - v^h\rangle + \langle Au_n, v^h - u_n\rangle \\ &\quad + \langle Au_n, u_n - u_n^{kh}\rangle + \langle Au_n^{kh}, u_n^{kh} - v^h\rangle. \end{aligned} \tag{4.26}$$

Furthermore, we replace v with $2u_n - v$ in (3.15) to get

$$\langle Au_n, u_n - v\rangle + \varphi(\mathcal{S}_n u, u_n, 2u_n - v) - \varphi(\mathcal{S}_n u, u_n, u_n)$$

$$+ j^0(\gamma_j u_n; \gamma_j u_n - \gamma_j v) \geqslant \langle f_n, u_n - v\rangle, \quad \forall\, v \in X. \tag{4.27}$$

Similarly, take $v = v^h$ in (4.27) to get

$$\begin{aligned}&\langle Au_n, u_n - v^h\rangle + \varphi(\mathcal{S}_n u, u_n, 2u_n - v^h) - \varphi(\mathcal{S}_n u, u_n, u_n)\\&\quad + j^0(\gamma_j u_n; \gamma_j u_n - \gamma_j v^h) \geqslant \langle f_n, u_n - v^h\rangle.\end{aligned} \tag{4.28}$$

Combining (2.5), (4.3), (4.7), (4.26) and (4.28), we have

$$\begin{aligned}m_A\|u_n - u_n^{kh}\|_X^2 &\leqslant \langle Au_n - Au_n^{kh}, u_n - v^h\rangle + \varphi(\mathcal{S}_n u, u_n, 2u_n - v^h)\\&\quad + \varphi(\mathcal{S}_n u, u_n, u_n^{kh}) - 2\varphi(\mathcal{S}_n u, u_n, u_n)\\&\quad + \varphi(\mathcal{S}_{n,L}^k u^{kh}, 2u_{n-1}^{kh} - u_{n-2}^{kh}, v^h) - \varphi(\mathcal{S}_{n,L}^k u^{kh}, 2u_{n-1}^{kh} - u_{n-2}^{kh}, u_n^{kh})\\&\quad + j^0(\gamma_j u_n; \gamma_j u_n - \gamma_j v^h) + j^0(\gamma_j u_n; \gamma_j u_n^{kh} - \gamma_j u_n)\\&\quad + j^0(\gamma_j u_n^{kh}; \gamma_j v^h - \gamma_j u_n^{kh}) + \langle j_c(\gamma_j u_n^{kh}), \gamma_j v^h - \gamma_j u_n^{kh}\rangle_{X_j}\\&\quad - \langle j_c(2\gamma_j u_{n-1}^{kh} - \gamma_j u_{n-2}^{kh}), \gamma_j v^h - \gamma_j u_n^{kh}\rangle_{X_j}\\&=: E_{\varphi_1} + E_{\varphi_2} + E_{\varphi_3} + \widetilde{E}_j + E_A,\end{aligned} \tag{4.29}$$

where E_{φ_1}, E_{φ_2} and E_A are the same as in (4.10), (4.11) and (4.13), respectively with their bounds (4.14), (4.15), (4.17). In addition,

$$E_{\varphi_3} = \varphi(\mathcal{S}_n u, u_n, 2u_n - v^h) + \varphi(\mathcal{S}_n u, u_n, v^h) - 2\varphi(\mathcal{S}_n u, u_n, u_n), \tag{4.30}$$

$$\begin{aligned}\widetilde{E}_j &= j^0(\gamma_j u_n; \gamma_j u_n - \gamma_j v^h) + j^0(\gamma_j u_n; \gamma_j u_n^{kh} - \gamma_j u_n)\\&\quad + j^0(\gamma_j u_n^{kh}; \gamma_j v^h - \gamma_j u_n^{kh}) - \langle j_c(\gamma_j u_n - \gamma_j u_n^{kh}), \gamma_j u_n - \gamma_j u_n^{kh}\rangle_{X_j}.\end{aligned} \tag{4.31}$$

The assumption (4.24) shows that

$$E_{\varphi_3} \leqslant C\|u_n - v^h\|_X^2. \tag{4.32}$$

Using the sub-additive property again, we obtain

$$\widetilde{E}_j \leqslant C\|\gamma_j u_n - \gamma_j v^h\|_{X_j} + C\|u_n - u_n^{kh}\|_X \|u_n - v^h\|_X. \tag{4.33}$$

(4.29) together with (4.30)–(4.33) and analogy to (4.20) give

$$\begin{aligned}m_A\|u_n - u_n^{kh}\|_X &\leqslant C\{\|u_n - v^h\|_X + \|\gamma_j u_n - \gamma_j v^h\|_{X_j}^{\frac{1}{2}}\} + C\|\mathcal{S}_n u - \mathcal{S}_{n,L}^k u^{kh}\|_Y\\&\quad + (\alpha_c c_j^2 + \alpha_\varphi + \varepsilon)(\|u_{n-2} - u_{n-2}^{kh}\|_X + 2\|u_{n-1} - u_{n-1}^{kh}\|_X).\end{aligned} \tag{4.34}$$

Similar to the constrained situation, the error bounds for $n = 0, 1$ are

$$\|u_0 - u_0^{kh}\|_X \leqslant \frac{C}{m_A - \alpha_\varphi - \alpha_c c_j^2}\{\|u_0 - v^h\|_X + \|\gamma_j u_0 - \gamma_j v^h\|_{X_j}^{\frac{1}{2}}\}, \tag{4.35}$$

$$\begin{aligned}\|u_1 - u_1^{kh}\|_X &\leqslant \frac{C}{m_A - \alpha_\varphi - \alpha_c c_j^2}\{\|u_1 - v^h\|_X + \|\gamma_j u_1 - \gamma_j v^h\|_{X_j}^{\frac{1}{2}}\\&\quad + k^2\|u\|_{W^{2,\infty}(I;X)}\} + Ck\|u_0 - u_0^{kh}\|_X.\end{aligned} \tag{4.36}$$

Combining (4.34)–(4.36), we find the following error bound by an application of Lemma 3.7:

$$\begin{aligned}\|u_n - u_n^{kh}\|_X \leqslant \Bigg(&C\{\|u_n - v^h\|_X + \|\gamma_j u_n - \gamma_j v^h\|_{X_j}^{\frac{1}{2}} + +k^2\|u\|_{W^{2,\infty}(I;X)}\}\\&+ \frac{\alpha_\varphi + \alpha_c c_j^2 + \varepsilon}{m_A}(2\|u_1 - u_1^{kh}\|_X + \|u_0 - u_0^{kh}\|_X)\\&+ \frac{Ck}{m_A - 3(\alpha_\varphi + \alpha_c c_j^2 + \varepsilon)}(\|u_0 - u_0^{kh}\|_X + \|u_1 - u_1^{kh}\|_X)\Bigg)\end{aligned}$$

$$\times\left(1+\frac{Ck}{m_A-3(\alpha_\varphi+\alpha_c c_j^2+\varepsilon)}\right)^{n-2}$$
$$\leqslant C\max_{0\leqslant n\leqslant N}(\|u_n-v^h\|_X+\|\gamma_j u_n-\gamma_j v^h\|_{X_j}^{\frac{1}{2}}+k^2). \tag{4.37}$$

Thus, the proof is completed. □

5 Numerical computation using fixed-point iteration

Notice that in Problems 3.12 and 4.2 and in the initial steps of Problems 3.16 and 4.3, the implicit discretization with respect to the unknown solution component is used. Let us discuss how to implement these numerical schemes in practice. We use a fixed-point iteration approach. We first consider the fixed-point iterations for the temporally semi-discrete schemes.

Problem 5.1. *Let TOL be a given error tolerance. For $1\leqslant n\leqslant N$, find a sequence $\{\tilde u_{n,i}^k\}\subset K$ from the iterations*

$$\begin{aligned}&\langle A\tilde u_{n,i}^k, v-\tilde u_{n,i}^k\rangle+\varphi(\mathcal{S}_{n,L}^k u^k,\tilde u_{n,i-1}^k,v)-\varphi(\mathcal{S}_{n,L}^k u^k,\tilde u_{n,i-1}^k,\tilde u_{n,i}^k)\\&\quad+j^0(\gamma_j\tilde u_{n,i}^k;\gamma_j v-\gamma_j\tilde u_{n,i}^k)+\langle j_c(\gamma_j\tilde u_{n,i}^k),\gamma_j v-\gamma_j\tilde u_{n,i}^k\rangle_{X_j}\\&\qquad\geqslant\langle f_n, v-\tilde u_{n,i}^k\rangle+\langle j_c(\gamma_j\tilde u_{n,i-1}^k),\gamma_j v-\gamma_j\tilde u_{n,i}^k\rangle_{X_j},\quad\forall\, v\in K\end{aligned}\tag{5.1}$$

until the relative error $\frac{\|\tilde u_{n,i}^k-\tilde u_{n,i-1}^k\|_X}{\|\tilde u_{n,i}^k\|_X}<TOL$; choose u_n^k to be the last iteration $\tilde u_{n,i}^k$.

In Problem 5.1, the index i refers to the i-th iterate at time level t_n. For the initialization of iteration, we may use the iterative solution from the previous step, i.e., $\tilde u_{n,0}^k=u_{n-1}^k$ for $n\geqslant 1$. Now we consider the convergence of the sequence $\{\tilde u_{n,i}^k\}$ generated by (5.1) to the solution of (3.28).

Theorem 5.2. *Assume* (2.3)–(2.10). *Then the iteration* (5.1) *converges linearly with a convergence rate $\rho=(\alpha_\varphi+\alpha_c c_j^2)/m_A$ that is independent of the time step-size k.*

Proof. Taking $v=\tilde u_{n,i}^k$ in (3.28), we have

$$\begin{aligned}&\langle Au_n^k,\tilde u_{n,i}^k-u_n^k\rangle+\varphi(\mathcal{S}_{n,L}^k u^k,u_n^k,\tilde u_{n,i}^k)-\varphi(\mathcal{S}_{n,L}^k u^k,u_n^k,u_n^k)\\&\quad+j^0(\gamma_j u_n^k;\gamma_j\tilde u_{n,i}^k-\gamma_j u_n^k)\geqslant\langle f_n,\tilde u_{n,i}^k-u_n^k\rangle.\end{aligned}\tag{5.2}$$

Taking $v=u_n^k$ in (5.1), we have

$$\begin{aligned}&\langle A\tilde u_{n,i}^k,u_n^k-\tilde u_{n,i}^k\rangle+\varphi(\mathcal{S}_{n,L}^k u^k,\tilde u_{n,i-1}^k,u_n^k)-\varphi(\mathcal{S}_{n,L}^k u^k,\tilde u_{n,i-1}^k,\tilde u_{n,i}^k)\\&\quad+j^0(\gamma_j\tilde u_{n,i}^k;\gamma_j u_n^k-\gamma_j\tilde u_{n,i}^k)+\langle j_c(\gamma_j\tilde u_{n,i}^k),\gamma_j u_n^k-\gamma_j\tilde u_{n,i}^k\rangle_{X_j}\\&\qquad\geqslant\langle f_n,u_n^k-\tilde u_{n,i}^k\rangle+\langle j_c(\gamma_j\tilde u_{n,i-1}^k),\gamma_j u_n^k-\gamma_j\tilde u_{n,i}^k\rangle_{X_j}.\end{aligned}\tag{5.3}$$

Combining (5.2) with (5.3), we have

$$\begin{aligned}\langle A\tilde u_{n,i}^k,u_n^k-\tilde u_{n,i}^k\rangle+\langle Au_n^k,\tilde u_{n,i}^k-u_n^k\rangle&\leqslant\alpha_\varphi\|u_n^k-\tilde u_{n,i}^k\|_X\|u_n^k-\tilde u_{n,i-1}^k\|_X\\&\quad+\alpha_c c_j^2\|u_n-\tilde u_{n,i}^k\|_X\|u_n^k-\tilde u_{n,i-1}^k\|_X.\end{aligned}\tag{5.4}$$

By the strong monotonicity of A and (5.4), we have the following relation:

$$m_A\|u_n^k-\tilde u_{n,i}^k\|_X\leqslant(\alpha_\varphi+\alpha_c c_j^2)\|u_n^k-\tilde u_{n,i-1}^k\|_X.\tag{5.5}$$

Therefore, the stated result is proved. □

In analogy to the temporally semi-discrete scheme, the iteration algorithm for the fully discrete scheme can be stated as follows.

Table 1 Comparison of the three temporally semi-discrete schemes

Semi-discrete problem	Numerical method	CO	Constraint
Problem 3.3	· Convex optimization	First-order	$m_A > \alpha_\varphi + \alpha_c c_j^2$
Problem 3.12	· Convex optimization · Fixed-point iteration (each step)	Second-order	$m_A > \alpha_\varphi + \alpha_j c_j^2$
Problem 3.16	· Convex optimization · Extrapolation · Fixed-point iteration (initial step)	Second-order	$m_A/3 > \alpha_\varphi + \alpha_c c_j^2$

Problem 5.3. *Let TOL be a given error tolerance. For $1 \leqslant n \leqslant N$, find a sequence $\{\tilde{u}_{n,i}^{kh}\} \subset K^h$ such that*

$$\begin{aligned}&\langle A\tilde{u}_{n,i}^{kh}, v^h - \tilde{u}_{n,i}^{kh}\rangle + \varphi(\mathcal{S}_{n,L}^k u^{kh}, \tilde{u}_{n,i-1}^{kh}, v^h) - \varphi(\mathcal{S}_{n,L}^k u^{kh}, \tilde{u}_{n,i-1}^{kh}, \tilde{u}_{n,i}^{kh})\\&\quad + j^0(\gamma_j \tilde{u}_{n,i}^{kh}; \gamma_j v^h - \gamma_j \tilde{u}_{n,i}^{kh}) + \langle j_c(\gamma_j \tilde{u}_{n,i}^{kh}), \gamma_j v^h - \gamma_j \tilde{u}_{n,i}^{kh}\rangle_{X_j}\\&\qquad \geqslant \langle f_n, v^h - \tilde{u}_{n,i}^{kh}\rangle + \langle j_c(\gamma_j \tilde{u}_{n,i-1}^{kh}), \gamma_j v^h - \gamma_j \tilde{u}_{n,i}^{kh}\rangle_{X_j}, \quad \forall\, v^h \in K^h,\end{aligned} \tag{5.6}$$

until the relative error $\frac{\|\tilde{u}_{n,i}^{kh} - \tilde{u}_{n,i-1}^{kh}\|_X}{\|\tilde{u}_{n,i}^{kh}\|_X} < TOL$; choose u_n^{kh} to be the last iteration $\tilde{u}_{n,i}^{kh}$.

The sequence $\{\tilde{u}_{n,i}^{kh}\}$ can be similarly proved to converge to the solution of (4.2).

Theorem 5.4. *Keep the assumptions in Theorem* 5.2*. Then the iteration* (5.6) *converges linearly with a convergence rate $\rho = (\alpha_\varphi + \alpha_c c_j^2)/m_A$ that is independent of the time step k and the mesh parameter h.*

So far we have proposed three types of schemes and the corresponding numerical treatments to solve Problem 2.1. Note that the difference of the schemes lies in the way the temporal discretization is done. We list the schemes and summarize their main properties in Table 1, where CO stands for convergence order.

We use the result of previous step to approximate the current step in Problem 3.3 which is easy to implement while with low accuracy. For Problem 3.16, the approximation for current step is performed with an extrapolation, thus an initial step is introduced and we employ a fixed-point iteration to solve it numerically. As a result, we obtain a second-order accuracy with stronger small condition constraint. Inspired by this fixed-point iterative procedure, we propose a new scheme in Problem 3.12, in which a fixed-point iteration is used to approximate this scheme for each step.

6 Application to a contact problem

In this section we apply the results of abstract numerical analysis in the previous sections to a particular history-dependent variational-hemivariational inequality. A viscoelastic frictionless contact model studied in [26] will be considered. For details on the model, we refer the reader to [26, 28].

Problem 6.1. *Find a displacement $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ such that for all $t \in \mathbb{R}_+$,*

$$\begin{aligned}\boldsymbol{\sigma}(t) = \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mu(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - P_{M(\kappa(\zeta(t)))}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))\\ + \int_0^t \mathscr{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds \quad in\ \ \Omega,\end{aligned} \tag{6.1}$$

$$\text{Div}\ \boldsymbol{\sigma}(t) + \boldsymbol{f_0}(t) = \mathbf{0} \quad in\ \ \Omega, \tag{6.2}$$

$$\boldsymbol{u}(t) = \mathbf{0} \quad on\ \ \Gamma_1, \tag{6.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f_2}(t) \quad on\ \ \Gamma_2, \tag{6.4}$$

$$\begin{cases} u_\nu(t) \leqslant g, \quad \sigma_\nu(t) + \xi_\nu(t) \leqslant 0, \\ (\sigma_\nu(t) + \xi_\nu(t))(u_\nu(t) - g) = 0 \quad on \ \Gamma_3, \\ \xi_\nu(t) \in \partial j_\nu(u_\nu(t)), \end{cases} \tag{6.5}$$

$$\boldsymbol{\sigma}_{\boldsymbol{\tau}}(t) = \mathbf{0} \quad on \ \Gamma_3. \tag{6.6}$$

As is standard in the literature in the area of the paper, we denote by $\mathbb{S}^d$ the space of second-order symmetric tensors on $\mathbb{R}^d$, $\boldsymbol{u} = (u_i)$, $\boldsymbol{\nu} = (\nu_i)$, $\boldsymbol{\sigma} = (\sigma_{ij})$, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)/2$ the displacement field, outward unit normal on the boundary, stress tensor and linearized strain tensor, respectively. In addition, $v_\nu := \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau := \boldsymbol{v} - v_\nu \boldsymbol{\nu}$ stand for the normal and tangential components of a vector field $\boldsymbol{v}$, $\sigma_\nu := (\boldsymbol{\sigma\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau := \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}$ represent the normal and tangential components of the stress field $\boldsymbol{\sigma}$, respectively. In (6.1) $P_{M(\kappa(\cdot))}$ denotes the projection on the Von Mises convex, $\mathscr{A}$ and $\mathscr{B}$ are the elastic and relaxation tensors, and μ is a constant. In this model, time-dependent surface tractions of density $\boldsymbol{f}_2$ and volume forces of density $\boldsymbol{f}_0$ are considered. On Γ_3, the penetration is restricted by a non-negative function g and the potential function is denoted as j_ν. The function spaces V and $\mathcal{H}$ are

$$V = \{\boldsymbol{v} = (v_i) \in H^1(\Omega; \mathbb{R}^d) \mid \boldsymbol{v} = 0 \text{ a.e. on } \Gamma_1\},$$
$$\mathcal{H} = \{\boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega; \mathbb{S}^d) \mid \tau_{ij} = \tau_{ji},\, 1 \leqslant i, j \leqslant d\}.$$

The inner products in the Hilbert spaces $\mathcal{H}$ and V are

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}(x)\tau_{ij}(x)dx, \quad (\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}}$$

and the associated norm are denoted by $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_V$. The space of fourth-order tensor fields Q_∞ is given by

$$\mathrm{Q}_\infty = \{\mathcal{E} = (\mathcal{E}_{ijkl}) \mid \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega),\, 1 \leqslant i, j, k, l \leqslant d\}.$$

We now list the assumptions on the problem data, following [26, 28]. The elasticity tensor $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is symmetric and positive. The relaxation tensor $\mathcal{B} \in C(\mathbb{R}_+; \mathrm{Q}_\infty)$ and the bound $\kappa : \mathbb{R} \to \mathbb{R}_+$ is Lipschitz continuous. The potential function $j_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ is measurable with respect to the first argument on Γ_3 for all $r \in \mathbb{R}$ and is locally Lipschitz with respect to the second argument on $\mathbb{R}$ for a.e. $\boldsymbol{x} \in \Gamma_3$; $j_\nu(\cdot, \overline{e}(\cdot))$ belongs to $L^1(\Gamma_3)$ for some $\overline{e} \in L^2(\Gamma_3)$. Besides, $|\partial j_\nu(\boldsymbol{x}, r)| \leqslant \overline{c}_0 + \overline{c}_1|r|$ for a.e. $\boldsymbol{x} \in \Gamma_3$, for all $r \in \mathbb{R}$ with $\overline{c}_0, \overline{c}_1 > 0$. In addition, there exists $\overline{\alpha}_\nu \geqslant 0$ such that for a.e. $\boldsymbol{x} \in \Gamma_3$,

$$j_\nu^0(\boldsymbol{x}, r_1; r_2 - r_1) + j_\nu^0(\boldsymbol{x}, r_2; r_1 - r_2) \leqslant \overline{\alpha}_\nu |r_1 - r_2|^2, \quad \forall\, r_1, r_2 \in \mathbb{R}.$$

For the body force and surface traction, we assume $\boldsymbol{f}_0 \in C(\mathbb{R}_+; L^2(\Omega; \mathbb{R}^d))$ and $\boldsymbol{f}_2 \in C(\mathbb{R}_+; L^2(\Gamma_2; \mathbb{R}^d))$. Let $U = \{\boldsymbol{v} \in V \mid v_\nu \leqslant g \text{ a.e. on } \Gamma_3\}$ be the set of admissible displacements. Define the function $\boldsymbol{f} : \mathbb{R}_+ \to V^*$ by

$$\langle \boldsymbol{f}(t), \boldsymbol{v}\rangle_{V^* \times V} = (\boldsymbol{f_0}(t), \boldsymbol{v})_{L^2(\Omega; \mathbb{R}^d)} + (\boldsymbol{f_2}(t), \boldsymbol{v})_{L^2(\Gamma_2; \mathbb{R}^d)}, \quad \forall\, \boldsymbol{v} \in V, \quad \forall\, t \in \mathbb{R}_+.$$

Then the weak formulation of Problem 6.1 can be described as follows.

Problem 6.2. *Find a displacement* $\boldsymbol{u} : \mathbb{R}_+ \to U$ *such that the following inequality holds*:

$$\begin{aligned} &(\mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}} + \mu(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}} \\ &\quad - \mu(P_{M(\kappa(\zeta(t)))}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}} \\ &\quad + \left(\int_0^t \mathscr{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \right)_{\mathcal{H}} \\ &\quad + \int_{\Gamma_3} j_v^0(u_\nu(t); v_\nu - u_\nu(t))d\Gamma \geqslant \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle_{V^* \times V}, \quad \forall\, \boldsymbol{v} \in U, \quad t \in \mathbb{R}_+. \end{aligned} \tag{6.7}$$

To apply the abstract results from the previous sections to the study of this contact problem, some definitions are needed. We let $\gamma_j : V \to L^2(\Gamma_3)$ be the trace operator defined by $\gamma_j \boldsymbol{v} = \boldsymbol{v}_\nu$ for $\boldsymbol{v} \in V$. In addition, we define the following operators (see [26, 28]):

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle_{V^*\times V} = (\mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + \mu(\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V, \tag{6.8}$$

$$\|y\|_Y = |r| + \|\boldsymbol{\theta}\|_{\mathcal{H}}, \quad \forall\, y = (r, \boldsymbol{\theta}) \in Y := \mathbb{R} \times \mathcal{H}, \tag{6.9}$$

$$\varphi(y, \boldsymbol{u}, \boldsymbol{v}) = -\mu(P_{M(\kappa(r))}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\boldsymbol{\theta}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \quad \forall\, y = (r, \boldsymbol{\theta}) \in Y, \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V, \tag{6.10}$$

$$(j_c(\gamma_j \boldsymbol{u}), \gamma_j \boldsymbol{v})_{L^2(\Gamma_3)} = \alpha_j(\gamma_j \boldsymbol{u}, \gamma_j \boldsymbol{v})_{L^2(\Gamma_3)}, \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V, \tag{6.11}$$

$$j(\gamma_j \boldsymbol{v}) = \int_{\Gamma_3} j_\nu(v_\nu)d\Gamma, \quad \forall\, \boldsymbol{v} \in V, \tag{6.12}$$

$$(\mathcal{S}u)(t) = \left(\int_0^t \|\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\|_{\mathcal{H}}ds, \int_0^t \mathscr{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds\right), \quad \forall\, \boldsymbol{u} \in C(\mathbb{R}_+; V). \tag{6.13}$$

Note that for j_c defined in (6.11), the constants α_c and α_j in (3.6) are equal: $\alpha_c = \alpha_j$.

The unique solvability of Problem 6.2 has been verified in [26]. Here, we consider fully discrete methods for solving Problem 6.2. Assume the domain Ω is polygonal/polyhedral with a regular family of partitions $\{\mathcal{T}^h\}$. The linear element space is constructed as follows:

$$V^h = \{\boldsymbol{v}^h \in C(\overline{\Omega})^d \mid \boldsymbol{v}^h|_{\mathcal{T}} \in \mathcal{P}_1(\mathcal{T})^d \text{ for } \mathcal{T} \in \mathcal{T}^h, \, \boldsymbol{v}^h = \boldsymbol{0} \text{ on } \Gamma_1\}$$

with $\mathcal{P}_1$ being the space of polynomials of degree no greater than one. Define

$$U^h = \{\boldsymbol{v}^h \in V^h \mid v^h_\nu \leqslant g \text{ at node points on } \Gamma_3\}.$$

Assume g is concave; then $U^h \subset U$. Thus the approximation is internal and the numerical methods for Problem 6.2 are defined as follows.

Problem 6.3. *Find a discrete displacement* $\boldsymbol{u}^{kh} := \{\boldsymbol{u}^{kh}_n\}_{n=0}^N \subset U^h$ *such that*

$$\begin{aligned}
&(\mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}^{kh}_n), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}^{kh}_n))_{\mathcal{H}} + \mu(\boldsymbol{\varepsilon}(\boldsymbol{u}^{kh}_n), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}^{kh}_n))_{\mathcal{H}}\\
&\quad - \mu(P_{M(\kappa(\widetilde{\zeta}(t_{n-1})))}\boldsymbol{\varepsilon}(\boldsymbol{u}^{kh}_{n-1}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}^{kh}_n))_{\mathcal{H}}\\
&\quad + \bigg(\frac{k}{2}\mathscr{B}(t_n - t_0)\boldsymbol{\varepsilon}(\boldsymbol{u}^{kh}_0) + k\sum_{j=1}^{n-1}\mathscr{B}(t_n - t_j)\boldsymbol{\varepsilon}(\boldsymbol{u}^{kh}_j)\\
&\quad + \frac{k}{2}\mathscr{B}(t_n - t_{n-1})\boldsymbol{\varepsilon}(\boldsymbol{u}^{kh}_{n-1}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}^{kh}_n)\bigg)_{\mathcal{H}}\\
&\quad + \int_{\Gamma_3} j^0_v(u^{kh}_{n,\nu}; v^h_\nu - u^{kh}_{n,\nu})d\Gamma + \alpha_j(u^{kh}_{n,\nu}, v^h_\nu - u^{kh}_{n,\nu})_{L^2(\Gamma_3)}\\
&\quad \geqslant \alpha_j(u^{kh}_{n-1,\nu}, v^h_\nu - u^{kh}_{n,\nu})_{L^2(\Gamma_3)} + \langle \boldsymbol{f}_n, \boldsymbol{v}^h - \boldsymbol{u}^{kh}_n\rangle_{V^*\times V}, \quad \forall\, \boldsymbol{v}^h \in U^h,
\end{aligned} \tag{6.14}$$

where

$$\widetilde{\zeta}(t_{n-1}) = \frac{k}{2}\|\boldsymbol{\varepsilon}(\boldsymbol{u}^{kh}_0)\|_{\mathcal{H}} + k\sum_{j=1}^{n-1}\|\boldsymbol{\varepsilon}(\boldsymbol{u}^{kh}_j)\|_{\mathcal{H}} + \frac{k}{2}\|\boldsymbol{\varepsilon}(\boldsymbol{u}^{kh}_{n-1})\|_{\mathcal{H}}.$$

Problem 6.4. *Find a discrete displacement* $\boldsymbol{u}^{kh} := \{\boldsymbol{u}^{kh}_n\}_{n=0}^N \subset U^h$ *such that*

$$\begin{aligned}
&(\mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}^{kh}_n), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}^{kh}_n))_{\mathcal{H}} + \mu(\boldsymbol{\varepsilon}(\boldsymbol{u}^{kh}_n), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}^{kh}_n))_{\mathcal{H}}\\
&\quad - \mu(P_{M(\kappa(\widetilde{\zeta}(t_{n-1})))}\boldsymbol{\varepsilon}(\boldsymbol{u}^{kh}_n), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}^{kh}_n))_{\mathcal{H}}\\
&\quad + \bigg(\frac{k}{2}\mathscr{B}(t_n - t_0)\boldsymbol{\varepsilon}(\boldsymbol{u}^{kh}_0) + k\sum_{j=1}^{n-1}\mathscr{B}(t_n - t_j)\boldsymbol{\varepsilon}(\boldsymbol{u}^{kh}_j)\\
&\quad + \frac{k}{2}\mathscr{B}(t_n - t_{n-1})\boldsymbol{\varepsilon}(\boldsymbol{u}^{kh}_{n-1}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}^{kh}_n)\bigg)_{\mathcal{H}}\\
&\quad + \int_{\Gamma_3} j^0_v(u^{kh}_{n,\nu}; v^h_\nu - u^{kh}_{n,\nu})d\Gamma \geqslant \langle \boldsymbol{f}_n, \boldsymbol{v}^h - \boldsymbol{u}^{kh}_n\rangle_{V^*\times V}, \quad \forall\, \boldsymbol{v}^h \in U^h.
\end{aligned} \tag{6.15}$$

Problem 6.5. *Find a discrete displacement* $\boldsymbol{u}^{kh} := \{\boldsymbol{u}_n^{kh}\}_{n=0}^N \subset U^h$ *such that*

$$\begin{aligned}
&(\mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{kh}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n^{kh}))_{\mathcal{H}} + \mu(\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{kh}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n^{kh}))_{\mathcal{H}} \\
&\quad - \mu(P_{M(\kappa(\widetilde{\zeta}(t_{n-1})))}\boldsymbol{\varepsilon}(2\boldsymbol{u}_{n-1}^{kh} - \boldsymbol{u}_{n-2}^{kh}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n^{kh}))_{\mathcal{H}} \\
&\quad + \bigg(\frac{k}{2}\mathscr{B}(t_n - t_0)\boldsymbol{\varepsilon}(\boldsymbol{u}_0^{kh}) + k\sum_{j=1}^{n-1}\mathscr{B}(t_n - t_j)\boldsymbol{\varepsilon}(\boldsymbol{u}_j^{kh}) \\
&\quad + \frac{k}{2}\mathscr{B}(t_n - t_{n-1})\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{kh}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n^{kh})\bigg)_{\mathcal{H}} \\
&\quad + \int_{\Gamma_3} j_v^0(u_{n,\nu}^{kh}; v_\nu^h - u_{n,\nu}^{kh})d\Gamma + \alpha_j(u_{n,\nu}^{kh}, v_\nu^h - u_{n,\nu}^{kh})_{L^2(\Gamma_3)} \\
&\quad \geqslant \alpha_j(2u_{n-1,\nu}^{kh} - u_{n-2,\nu}^{kh}, v_\nu^h - u_{n,\nu}^{kh})_{L^2(\Gamma_3)} + \langle \boldsymbol{f}_n, \boldsymbol{v}^h - \boldsymbol{u}_n^{kh}\rangle_{V^*\times V}, \quad \forall\, \boldsymbol{v}^h \in U^h, \quad n \geqslant 2,
\end{aligned} \tag{6.16}$$

and for $n = 1$,

$$\begin{aligned}
&(\mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_1^{kh}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_1^{kh}))_{\mathcal{H}} + \mu(\boldsymbol{\varepsilon}(\boldsymbol{u}_1^{kh}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_1^{kh}))_{\mathcal{H}} \\
&\quad - \mu(P_{M(\kappa(\widetilde{\zeta}(t_1)))}\boldsymbol{\varepsilon}(\boldsymbol{u}_1^{kh}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_1^{kh}))_{\mathcal{H}} \\
&\quad + (k\mathscr{B}(t_1 - t_0)\boldsymbol{\varepsilon}(\boldsymbol{u}_0^{kh}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_1^{kh}))_{\mathcal{H}} \\
&\quad + \int_{\Gamma_3} j_v^0(u_{1,\nu}^{kh}; v_\nu^h - u_{1,\nu}^{kh})d\Gamma \geqslant \langle \boldsymbol{f}_1, \boldsymbol{v}^h - \boldsymbol{u}_1^{kh}\rangle_{V^*\times V}, \quad \forall\, \boldsymbol{v}^h \in U^h.
\end{aligned} \tag{6.17}$$

The numerical scheme for $n = 0$ is similar to (6.17) except that the approximation for the history-dependent term is omitted.

Using arguments similar to that found in [28], we can show that under the following solution regularity: $\boldsymbol{u} \in W_{\mathrm{loc}}^{2,\infty}(\mathbb{R}_+; V)$, $\boldsymbol{\sigma} \in C(\mathbb{R}_+; H^1(\Omega; \mathbb{S}^d))$, $\boldsymbol{u} \in C(\mathbb{R}_+; H^2(\Omega; \mathbb{R}^d))$, and $u_\nu \in C(\mathbb{R}_+; \widetilde{H}^2(\Gamma_3))$, the following optimal order error bounds hold:

$$\max_{0\leqslant n\leqslant N} \|\boldsymbol{u}_n - \boldsymbol{u}_n^{kh}\|_V \leqslant C(h + k^\eta), \tag{6.18}$$

where $\eta = 1$ for Problem 6.3 and $\eta = 2$ for Problems 6.4 and 6.5.

7 Numerical results

In this section, we present some numerical results for the three fully discrete schemes stated in Problems 6.3–6.5. The same physical setting as depicted in Figure 1 is employed.

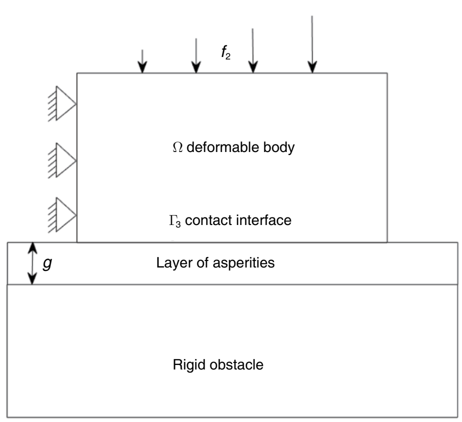

Figure 1 Initial configuration of the contact problem

Let $\Omega = (0, L_1) \times (0, L_2)$ be a rectangle with boundary Γ which is divided into three parts

$$\Gamma_1 = \{0\} \times (0, L_2), \quad \Gamma_2 = \{L_1\} \times (0, L_2) \cup [0, L_1] \times \{L_2\}, \quad \Gamma_3 = [0, L_1] \times \{0\}.$$

For a given $S > 0$, the function j_ν is defined as

$$j_\nu(\xi_\nu) = S \int_0^{|\xi_\nu|} \mu_j(s)\, ds \tag{7.1}$$

with

$$\mu_j(s) = \begin{cases} 0, & s \leqslant 0, \\ c_1 s, & 0 < s \leqslant s_1, \\ c_1 s_1 + c_2(s - s_1), & s_1 < s \leqslant s_2, \\ c_1 s_2 + c_2(s_2 - s_1) + c_3(s - s_2), & s > s_2, \end{cases} \tag{7.2}$$

where s_1, s_2, c_1, c_2 and c_3 are constants. The elasticity tensor $\mathscr{A}$ satisfies

$$(\mathscr{A}\varepsilon)_{ij} = \frac{E\kappa}{1-\kappa^2}(\varepsilon_{11} + \varepsilon_{22})\delta_{ij} + \frac{E}{1+\kappa}\varepsilon_{ij} \tag{7.3}$$

with $1 \leqslant i, j \leqslant 2$. E is the Young modulus, κ the Poisson ratio of the material and δ_{ij} denotes the Kronecker symbol. For the volume and surface forcing, we set

$$\boldsymbol{f}_0 = (0, -0.1\sin(t))N/m^2, \tag{7.4}$$

$$\boldsymbol{f}_2 = \begin{cases} (0,0)N/m & \text{on } \{L_1\} \times (0, L_2), \\ (0, -0.2\sin(t)\sin(\pi x/2))N/m & \text{on } [0, L_1] \times \{L_2\}. \end{cases} \tag{7.5}$$

7.1 Convergence tests

In this subsection, we test the convergence behavior for the three numerical schemes. The projection on the Von Mises convex is not considered in the convergence tests; thus we let $\mu = 0$ in Problems 6.3–6.5. Values of the other parameters are

$$\begin{aligned} &L_1 = 2m, \quad L_2 = 1m, \quad E = 2N/m^2, \quad \kappa = 0.3, \\ &\alpha_j = 0.5, \quad g = 0.15m, \quad S = 1N, \quad \mathscr{B}(t) = \mathrm{e}^{-t}, \quad T = 0.5, \\ &s_1 = 0.1, \quad s_2 = 0.15, \quad c_1 = 0.1, \quad c_2 = -0.1, \quad c_3 = 0.4. \end{aligned}$$

The uniform rectangular finite element partitions are introduced to numerically solve the above problem. The numerical solution with $h = k = 1/256$ is used as the "reference" solution in computing numerical solution errors, and the temporal and spatial convergence orders in the H^1 norm will be shown.

7.1.1 *First-order scheme*

In Tables 2 and 3, we present the temporal and spatial convergence orders of first-order scheme respectively, and the first-order accuracy in both time and space are shown.

7.1.2 *Second-order scheme by fixed-point iteration*

In Tables 4 and 5, we present the temporal and spatial convergence orders of second-order fixed-point iteration scheme, respectively, and the second-order accuracy in time, first-order in space are shown.

In addition, we compute the H_1 errors for different mesh grid-sizes. Two refinement paths are taken to be $k^2 = h$ and $k = h$. The results are displayed in Table 6 and the first-order accuracy is shown for both the two refinement paths in Figure 2, which indicates the second-order convergence order in time.

Table 2 Convergence orders with spatial step-size fixed for first-order scheme

h	k	$\|u(\cdot,T)-u_N^{kh}\|_1$	Order
1/256	1/4	9.82316E−3	–
1/256	1/8	2.39681E−3	2.0351
1/256	1/12	1.29335E−3	1.5215
1/256	1/16	9.51587E−4	1.0667
1/256	1/32	4.49031E−4	1.0835
1/256	1/64	1.93357E−4	1.2155

Table 3 Convergence orders with temporal step-size fixed for first-order scheme

h	k	$\|u(\cdot,T)-u_N^{kh}\|_1$	Order
1/8	1/256	1.81905E−2	–
1/16	1/256	1.01388E−2	0.8433
1/32	1/256	5.51935E−3	0.8773
1/64	1/256	2.92633E−3	0.9154

Table 4 Convergence orders with spatial step-size fixed for second-order scheme by fixed-point iteration

h	k	$\|u(\cdot,T)-u_N^{kh}\|_1$	Order
1/256	1/4	2.30136E−3	–
1/256	1/8	6.06211E−4	1.9246
1/256	1/12	2.75085E−4	1.9487
1/256	1/16	1.54881E−4	1.9967
1/256	1/32	4.16384E−5	1.8952

Table 5 Convergence order of the errors with temporal step-size fixed for second-order scheme by fixed-point iteration

h	k	$\|u(\cdot,T)-u_N^{kh}\|_1$	Order
1/8	1/256	1.81822E−2	–
1/16	1/256	1.01334E−2	0.8434
1/32	1/256	5.51570E−3	0.8775
1/64	1/256	2.92399E−3	0.9156

Table 6 Comparison of the H_1 errors in the refinement path $k^2=h$ and $k=h$ for second-order scheme by fixed-point iteration

Mesh grid-size	$\|u(\cdot,T)-u_N^{kh}\|_1$		Difference between the front two
	$k^2=h$	$k=h$	
$h=1/16$	1.03714E−2	1.01335E−2	2.38E−4
$h=1/36$	4.66049E−3	4.55114E−3	1.09E−4
$h=1/64$	2.98351E−3	2.92393E−3	5.96E−5
$h=1/100$	1.44761E−3	1.39882E−3	4.88E−5
$h=1/144$	8.58858E−4	8.17035E−4	4.18E−5
$h=1/196$	4.00877E−4	3.49632E−4	5.12E−5

7.1.3 *Second-order scheme with extrapolation*

In Tables 7 and 8, we present the temporal and spatial convergence orders of second-order scheme with extrapolation, respectively, and the second-order accuracy in time, first-order in space are shown.

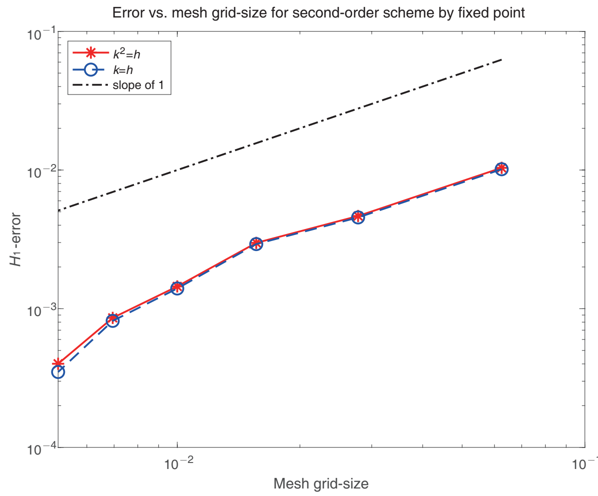

Figure 2 (Color online) The loglog plot of H_1 errors with h =1/16, 1/36, 1/64, 1/100, 1/144, 1/196 for second-order fixed-point scheme

Table 7 Convergence orders with spatial step-size fixed for second-order scheme by fixed-point iteration

h	k	$\|u(\cdot,T)-u_N^{kh}\|_1$	Order
1/256	1/4	1.02222E−2	–
1/256	1/8	1.15624E−3	3.1442
1/256	1/12	3.53015E−4	2.9261
1/256	1/16	2.41930E−4	1.3135
1/256	1/32	5.74370E−5	2.0745

Table 8 Convergence orders with temporal step-size fixed for second-order scheme with extrapolation

h	k	$\|u(\cdot,T)-u_N^{kh}\|_1$	order
1/8	1/256	1.81823E−2	–
1/16	1/256	1.01335E−2	0.8434
1/32	1/256	5.51576E−3	0.8775
1/64	1/256	2.92403E−3	0.9156

Table 9 Comparison of the H_1 errors in the refinement path $k^2=h$ and $k=h$ for second-order scheme with extrapolation

Mesh grid-size	$\|u(\cdot,T)-u_N^{kh}\|_1$		Difference between the front two
	$k^2=h$	$k=h$	
$h=1/16$	1.42988E−2	1.01343E−2	4.16E−3
$h=1/36$	6.89462E−3	4.55108E−3	2.34E−3
$h=1/64$	3.13943E−3	2.92396E−3	2.15E−4
$h=1/100$	1.49765E−3	1.39884E−3	9.88E−5
$h=1/144$	8.90577E−4	8.17048E−4	7.35E−5
$h=1/196$	3.93167E−4	3.49639E−4	4.35E−5

In addition, we compute the H_1 errors for different mesh grid-sizes. Two refinement paths are taken to be $k^2=h$ and $k=h$. The results are displayed in Table 9 and the first-order accuracy is shown for both the two refinement paths in figure 3, which indicates the second-order convergence order in time.

In Figure 4, the normal displacement on the boundary Γ_3 at time $T=0.5$ for the three numerical schemes is shown, from which we can see, the maximum penetration is reached as the forcing increased.

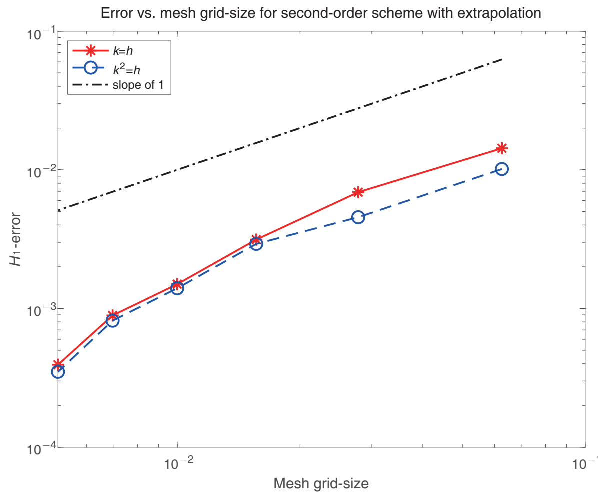

Figure 3 (Color online) The plot of H_1 errors with h =1/16, 1/36, 1/64, 1/100, 1/144, 1/196 for second-order scheme with extrapolation

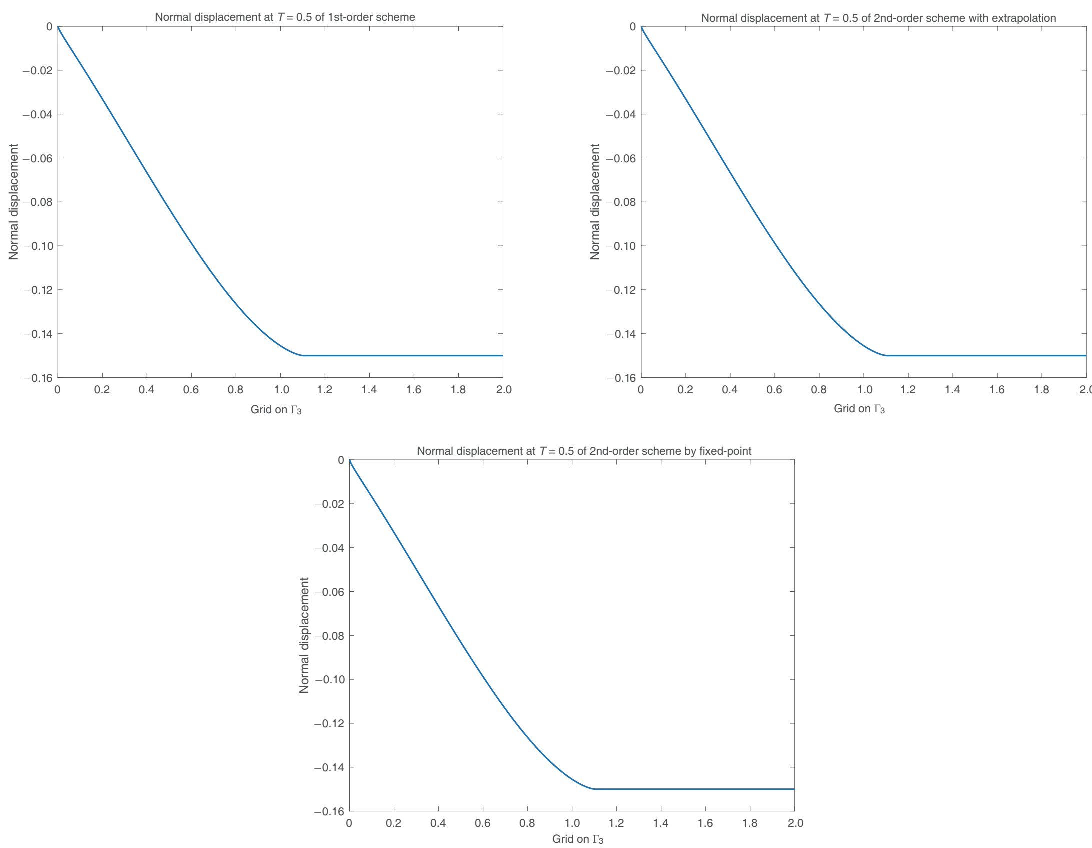

Figure 4 (Color online) Normal displacement on Γ_3 at time $T = 0.5$ of three numerical schemes

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