

On the Reducibility of Surgered 3-Manifolds

Ying-Qing Wu

The reducibility of surgered manifolds has attracted much attention in the past years. (See for example [2, 3, 4, 5]). Let M be a connected orientable irreducible 3-manifold with torus T as a boundary component. Suppose that γ_1, γ_2 are two slopes on T with geometric intersection number $\Delta = \Delta(\gamma_1, \gamma_2)$. Let $M(\gamma_i)$ be the manifold obtained by gluing a solid torus J_i to M so that the boundary of a meridian disc has slope γ_i . An interesting unsolved problem is the *reducibility conjecture*, which says that if both $M(\gamma_1)$ and $M(\gamma_2)$ are reducible, then $\Delta \leq 1$. There is some strong evidence for this conjecture. For example, Gordon and Luecke observed that if $\Delta \geq 2$, then both $M(\gamma_1)$ and $M(\gamma_2)$ will be the connected sum of two lens spaces. Especially, the conjecture is true if either M is noncompact or it has more than one boundary component. In the general case, Gordon and Litherland [3] proved that Δ can not be greater than 4. In this paper we will prove

Theorem 1 *If $M(\gamma_1)$ and $M(\gamma_2)$ are both reducible, then $\Delta \leq 2$.*

This rules out the possibility of $\Delta = 3, 4$. A nontrivial example of $\Delta = 1$ was presented in [3]. (Surgery on cable knots in reducible manifolds gives “trivial” examples). The possibility of $\Delta = 2$ remains a challenging open problem.

The author is indebted to Cameron Gordon and John Luecke for many stimulating conversations and various kinds of help during his visit to Austin.

1 Notations and Definitions

Suppose $M(\gamma_i)$ is reducible. Let S_i be a reducing sphere of $M(\gamma_i)$. Isotope S_1 so that $S_1 \cap J_1 = v_1 \cup \dots \cup v_{n_1}$ is a disjoint union of meridian discs, where v_i is subscripted so that v_1, \dots, v_{n_1} appear consecutively in J_1 . Similarly, we have $S_2 \cap J_2 = w_1 \cup \dots \cup w_{n_2}$. We choose S_i so that n_i is minimal. By an isotopy of S_2 , we may assume that ∂v_i intersects

∂w_j at Δ points for all i, j . Thus when we travel around ∂v_i , we will consecutively meet $\partial w_1, \partial w_2, \dots, \partial w_{n_2}, \dots, \partial w_1, \dots, \partial w_{n_2}$ (repeated Δ times).

Let $P_i = S_i \cap M$. It is a planar surface with n_i boundary components. Since n_i is minimized, P_i is incompressible and boundary incompressible. By a further isotopy of S_2 (fixing $\cup w_j$), we may assume that P_1 and P_2 are in general position, and $P_1 \cap P_2$ has the minimal number of components. A standard innermost disc argument then guarantees that no circle component of $P_1 \cap P_2$ bounds a disc in either P_1 or P_2 . Since P_i is boundary incompressible, no arc in $P_1 \cap P_2$ can be boundary parallel in P_i .

Let $\Gamma_1 = (\cup v_i) \cup \{\text{arc components of } P_1 \cap P_2\}$. Γ_1 is considered a graph in S_1 : It has the discs v_i as its ‘‘fat’’ vertices, and the arcs in $P_1 \cap P_2$ as its edges. There are Δn_2 edges incident to each vertex v_i of Γ_1 . If e is such an edge, and an end of e is in $\partial v_i \cap \partial w_j$, then we give this end of e the label j . In this way, each end of each edge in Γ_1 has a label. When we travel around ∂v_i in some direction, the labels appear in the order $1, 2, \dots, n_2, \dots, 1, \dots, n_2$ (repeated Δ times). The labels are considered to be a mod n_2 number. Thus, for example, $n_2 + 1$ is the same label as 1.

In the same way we can define $\Gamma_2 = (\cup w_j) \cup \{\text{arc components of } P_1 \cap P_2\}$ as a graph in S_2 , and we label the ends of edges of Γ_2 in a similar way. Since the edges in Γ_i are arcs in $P_1 \cap P_2$, each edge e in Γ_1 can also be considered as an edge in Γ_2 . Note that if in Γ_1 an edge e is incident to v_i and has label j at that end, then in Γ_2 it is incident to w_j and has label i at that end.

Given an orientation to S_i and K_i (the central curve of J_i), we can refer to $+$ or $-$ vertex, according to the sign of its intersection with K_i . Two vertices are *parallel* if they have the same sign. Otherwise they are *antiparallel*. Since M is orientable, we have the following

Parity rule: An edge e connects parallel vertices in Γ_1 if and only if it connects antiparallel vertices in Γ_2 .

A pair of edges $\{e_1, e_2\}$ in Γ_i is called an *S-cycle* if it is a Scharlemann cycle of length 2. That is, e_1, e_2 are adjacent parallel edges connecting a pair of parallel vertices in Γ_i , and have the same two labels at their ends. Note that in this case the two labels are successive, and we call them the labels of the S-cycle.

A set of four parallel edges $\{e_1, e_2, e_3, e_4\}$ is called an *extended S-cycle* if $\{e_2, e_3\}$ is an S-cycle, and e_i is adjacent to e_{i+1} , $i = 1, 2, 3$. (I.e, e_i and e_{i+1} are parallel, and there are

no edges between them.)

2 Proof of Theorem 1

By [3], Theorem 1 is true if n_1 or n_2 is less than 4. So we assume that $n_i \geq 4$. Suppose that $\{e_1, e_2\}$ is an S-cycle in Γ_2 with labels $\{r, r+1\}$. Then on the other graph Γ_1 , the two edges e_1, e_2 connect the vertex v_r to v_{r+1} .

Lemma 2.1 *If there is a disc B in S_1 such that $e_1 \cup e_2 \cup v_r \cup v_{r+1} \subset B$, then $|B \cap J_1| \geq (n_1/2) + 1$.*

Proof Let V be the part of J_1 between v_r and v_{r+1} . Then a regular neighborhood N of $V \cup B$ is a solid torus. Let D be the disc in P_2 bounded by e_1, e_2 and two arcs α, β on ∂P_2 . Then in $M(\gamma_1)$, the curve ∂D is contained in $V \cup B$, and intersects a meridian disc of N twice in the same direction. So a regular neighborhood of $V \cup B \cup D$ is a projective space P (See [1, P280] for details). If $B \cap \Gamma_1$ has k vertices, then $P \cap J_1$ has $k - 1$ components (since the vertices v_r and v_{r+1} are connected by V). Thus ∂P is a reducing sphere, and $\partial P \cap J_1$ has $2(k - 1)$ components. By the minimality of n_1 , $2(k - 1) \geq n_1$. Therefore $k \geq (n_1/2) + 1$ □

Lemma 2.2 *If Γ_2 has two S-cycles $\{e_1, e_2\}$ and $\{e'_1, e'_2\}$ with labels $\{r, r+1\}$ and $\{s, s+1\}$ respectively, then $\{r, r+1\} = \{s, s+1\}$.*

Proof If $\{r, r+1\} \neq \{s, s+1\}$, then on Γ_1 , the complement of $(e_1 \cup e_2 \cup v_r \cup v_{r+1}) \cup (e'_1 \cup e'_2 \cup v_s \cup v_{s+1})$ has three components D_1, D_2 and A such that $\partial \overline{D}_1 \subset e_1 \cup e_2 \cup v_r \cup v_{r+1}$, and $\partial \overline{D}_2 \subset e'_1 \cup e'_2 \cup v_s \cup v_{s+1}$, where \overline{D}_i is the closure of D_i . (A is an annulus or a disc, depending on whether $\{r, r+1\} \cap \{s, s+1\} = \emptyset$). Since $D_1 \cup D_2$ does not contain $\{v_r, v_{r+1}, v_s, v_{s+1}\}$, it contains at most $n_1 - 3$ vertices of Γ_1 . Thus one of the D_i , say D_1 , contains at most $(n_1 - 3)/2$ vertices of Γ_1 . Let $D = D_1 \cup e_1 \cup e_2 \cup v_r \cup v_{r+1}$. Then $|D \cap J_1| \leq (n_1 + 1)/2$, contradicting Lemma 2.1. □

Lemma 2.3 Γ_2 has no extended S-cycles.

Proof Let $\{e_1, e_2, e_3, e_4\}$ be an extended S-cycle on Γ_2 such that $\{e_2, e_3\}$ has labels $\{r, r+1\}$. Then, since the labels on ∂w_i are successive, both e_1 and e_4 have labels $r - 1$

and $r + 2$ on their ends. Thus on Γ_1 , e_2, e_3 connect v_r to v_{r+1} , and e_1, e_4 connect v_{r-1} to v_{r+2} . Since we have assumed that $n_i \geq 4$, the two sets $C_1 = e_1 \cup e_4 \cup v_{r-1} \cup v_{r+2}$ and $C_2 = e_2 \cup e_3 \cup v_r \cup v_{r+1}$ are disjoint in S_1 . So we can choose disjoint discs B_1, B_2 on S_1 such that $B_i \supset C_i$, and B_i has boundary disjoint from the vertices of Γ_1 . By Lemma 2.1, $|B_2 \cap J_1| > n_1/2$. Hence $|B_1 \cap J_1| < n_1/2$. The rest of the proof is similar to that of Lemma 2.1: Choose V to be the part of J_1 which is between v_{r-1} and v_{r+2} , and contains v_r and v_{r+1} . Let D be the disc on P_2 bounded by e_1, e_4 and two other arcs α and β on ∂P_2 . Then $\partial D \subset \partial V \cup B_1$. Note that $\text{Int}D \cap P_1 \subset e_2 \cup e_3 \subset B_2$. Thus $\text{Int}D \cap (V \cup B_1) = \emptyset$. Now it is easy to see that a regular neighborhood N of $B_1 \cup V \cup D$ is a projective space, and $|\partial N \cap J_1| < n_1$, contradicting the minimality of n_1 . \square

Lemma 2.4 Γ_2 can not have more than $(n_1/2) + 1$ parallel edges connecting a pair of parallel vertices.

Proof Suppose e_1, \dots, e_t are parallel edges connecting w_a to w_b , where $t = n_1/2 + 2$ if n_1 is even, and $t = (n_1 - 1)/2 + 2$ if n_1 is odd. Then there is an S-cycle within them. (See [1, Cor.2.6.7]). Suppose that $\{e_i, e_{i+1}\}$ is an S-cycle. If $i \neq 1, t - 1$, then $\{e_{i-1}, e_i, e_{i+1}, e_{i+2}\}$ would be an extended S-cycle, contradicting lemma 2.3.

Now suppose $i = 1$. By relabeling v_i if necessary, we may assume that e_i has label i at w_a . Since $\{e_1, e_2\}$ form an S-cycle, e_1 has label 2 at w_b , e_2 has label 1 at w_b . Thus e_i has label $n_1 - i + 3$ at w_b for $i \geq 3$. If n_1 were odd, e_t would have label t at both ends, contradicting the parity rule. If n_1 were even, e_{t-1} would have label $n_1 - ((t - 1) - 3) = t$ at w_b , and e_t would have label $t - 1$ at w_b . Thus $\{e_{t-1}, e_t\}$ would be an S-cycle with labels $\{t - 1, t\}$. So we have two S-cycles with different set of labels, contradicting lemma 2.2. The proof of the case $i = t - 1$ is similar. \square

Lemma 2.5 Suppose that e', e'' are two parallel edges connecting a pair of parallel vertices. If they have a label r in common, then they form an S-cycle.

Proof Let $e' = e_1, e_2, \dots, e_k = e''$ be the successive parallel edges between e' and e'' . The edges e', e'' cannot both have label r at the same vertex, otherwise $k \geq n_1 + 1$, contradicting Lemma 2.4. So suppose e' (resp. e'') has label r at w_a (resp. w_b). We assume that e_2 has label $r + 1$ at w_a . (The other case is similar). Then e_i has label $r + i - 1$ at w_a . Since the vertices are parallel, the label of e_{k-i} at w_b is $r + i$. Now k must be even,

otherwise the edge $e_{(k+1)/2}$ has the same label $r + (k - 1)/2$ at both ends, contradicting the parity rule. Let $t = k/2$. Then $\{e_t, e_{t+1}\}$ is an S-cycle, because the two edges are adjacent and both have labels $\{r + t - 1, r + t\}$ at their ends. If $k > 2$, we would have an extended S-cycle $\{e_{t-1}, e_t, e_{t+1}, e_{t+2}\}$, contradicting Lemma 2.3. Therefore, $k = 2$, and $\{e', e''\}$ is an S-cycle. \square

Lemma 2.6 *One of Γ_1 and Γ_2 satisfies*

Property (*): *Each vertex is incident to an edge connecting it to an antiparallel vertex.*

Proof If Γ_1 does not have property (*), then there is a vertex v_r such that each edge incident to it will connect it to a parallel vertex. By the parity rule, for each vertex w_i of Γ_2 , all the edges incident to w_i with label r will connect w_i to antiparallel vertices. Thus Γ_2 has Property (*). \square

Now we suppose that Γ_2 has Property (*). Let Γ'_2 be the subgraph of Γ_2 consisting of edges connecting parallel vertices. A component F' of Γ'_2 is called an extremal component if there is a disc D such that $D \cap \Gamma'_2 = F'$. In this case $F = D \cap \Gamma_2$ is a graph in D . If e is an edge in Γ_2 connecting a vertex of F' to an antiparallel vertex, then $e \cap D$ is an edge of F connecting that vertex to ∂D . Such an edge is called a boundary edge of F . Property (*) means that each vertex of F belongs to a boundary edge.

The reduced graph \overline{F} of F is defined to be the graph obtained from F by choosing one edge from each family of parallel edges. Define the *valency* of a vertex to be the number of edges incident to it.

Lemma 2.7 *Let Γ be a graph in a disk D with no trivial loops or parallel edges, such that every vertex of Γ belongs to a boundary edge. Then either Γ has only one vertex, or there are at least two vertices of valency at most 3, each of which belongs to a single boundary edge.*

This follows immediately from the proof of [1, Lemma 2.6.5]. In particular, it is true for the graph \overline{F} in D .

Lemma 2.8 *Suppose $\Delta \geq 3$. If w_r is a vertex of \overline{F} which has valency at most 3, then Γ_1 has an S-cycle with r as one of its labels.*

Proof The hypothesis implies that in Γ_2 , there are at most two families of parallel edges connecting w_r to parallel vertices, and if there are two, then they are successive. By Lemma 5, each family has at most $(n_1/2) + 1$ edges. So there are at most $n_1 + 2$ successive edges connecting w_r to parallel vertices, and all the others connect w_r to antiparallel vertices. On Γ_1 it means

(**): *Except for at most two vertices, each vertex v_i is incident to at most one edge that has label r at v_i and connects v_i to an antiparallel vertex. For each exceptional vertex, there are at most two such edges.*

Denote by K the subgraph of Γ_1 consisting of edges which connects parallel vertices and has one end labeled r . Let E be an extremal component of K and let D be a disc in S_1 such that $D \cap K = E$. Let E_1 be the subgraph of Γ_1 consisting of edges which has one end at a vertex of E and has label r at that vertex. Restricting this graph to D , we get a graph $H = E_1 \cap D$ in D . The disc D can be chosen so that no edges of H have both ends in ∂D . The interior edges are just the edges of E , while a boundary edge corresponds to an edge in Γ_1 which connects a vertex of E to an antiparallel vertex, and has label r at the end in E . By (**) above, at most two vertices of H have two boundary edges. We want to show that H has a pair of parallel inner edges.

To prove this, we shrink the boundary of D into one point. Then H becomes a connected graph H' in a sphere S^2 . Note that among all the edges connecting to a certain vertex of H , just Δ of them have label r at that vertex. Thus if H has v vertices, then H' has $v + 1$ vertices and Δv edges. Denote by f_1 the number of faces bounded by 2 edges, and by f_2 the number of the other faces (which must be bounded by at least 3 edges because H' has no trivial loops). Then $2f_1 + 3f_2 \leq 2(\text{number of edges}) = 2\Delta v$. Thus, $f_2 \leq (2\Delta v - 2f_1)/3$. So we have

$$\begin{aligned} 2 &= \chi(S^2) = (v + 1) - (\Delta v) + (f_1 + f_2) \\ &\leq 1 + (1 - \Delta/3)v + f_1/3 \leq 1 + f_1/3 \end{aligned}$$

It follows that $f_1 \geq 3$. Since H has at most two pairs of parallel boundary edges, it must have at least one pair of parallel interior edges. The conclusion now follows from Lemma 2.5. \square

Proof of Theorem 1. Suppose $\Delta \geq 3$. By Lemma 2.8, for each extremal component F' of Γ'_2 , and each vertex w_r of valency at most 3 in the corresponding graph \overline{F} , there is an S-cycle in Γ_1 with r as a label. By Property (*), Γ'_2 is disconnected. So there are at least

two extremal components. By Lemma 2.7, if F' has more than one vertex, then there are two vertices of valency at most 3 in \overline{F} . Therefore, if either Γ'_2 has more than 2 extremal components, or one of the extremal components has more than one vertex, then we can find 3 different labels, each of which is a label of some S-cycles in Γ_1 . But since an S-cycle has only two labels, this would imply that there exist two S-cycles with different pair of labels, contradicting Lemma 2.2. So we suppose that $\{w_r\}, \{w_s\}$ are the only extremal components of Γ'_2 . Since we have assumed $n_i \geq 4$, there must be some other components in Γ'_2 . Each of these components will separate $\{w_r\}$ from $\{w_s\}$, for otherwise we could find another extremal component. It follows that in Γ_2 , there is no arc connecting w_r to w_s . Now suppose $\{e_1, e_2\}$ (resp. $\{e'_1, e'_2\}$) is an S-cycle in Γ_1 with r (resp. s) as a label. Since e_i can not connect w_r to w_s in Γ_2 , we conclude that s is not a label of e_i . Thus these two S-cycles have different sets of labels, again a contradiction to Lemma 2.2. Therefore we have $\Delta \leq 2$, and the theorem follows. \square

University of Texas at Austin, Austin, TX 78712

and

Nanjing Normal University, Nanjing, P.R.CHINA

References

- [1] M. Culler, C. Gordon, J. Luecke, P. Shalen, Dehn surgery on knots, *Ann. Math.* **125** (1987) 237–300.
- [2] D. Gabai, Foliations and surgery on knots, *Bull. Amer. Math. Soc.*, **15** (1986) 83–87.
- [3] C. Gordon, R. Litherland, Incompressible planar surfaces in 3-manifolds, *Topology Appl.* **18** (1984) 121–144.
- [4] C. Gordon, J. Luecke, Only integral Dehn surgery can yield reducible manifolds, *Math. Proc. Camb. Phil. Soc.* **102** (1987) 97–101.
- [5] M. Scharlemann, Producing reducible 3-manifolds by surgery on a knot, *Topology* (1990) **29** (1990) 481–500.